Patterns in permutations and diagrams

with applications to Stanley symmetric functions and Schubert calculus

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Combinatorics, Number Theory, and Sage

High Level Goals.

- Find some applications of quasisymmetric functions and permutation patterns in terms of Whittaker functions, multiple Dirichlet series, Eisenstein series, automorphic forms, etc.

- Learn/Expand new Sage tools for quasisymmetric function expansions (Bandlow-Berg-Saliola).

- Learn/Expand new Sage tools for permutation pattern recognition (Magnusson-Úlfarsson).

Possible path. via Stanley symmetric functions and Schubert calculus.

Outline

1. Symmetric Functions and Quasisymmetric Functions
2. Stanley Symmetric Functions
3. 3 properties of SSF's characterized by permutation patterns
4. Applications to Schubert calculus and Liu's conjecture
5. Sage Demo

Based on joint work with Brendan Pawlowski at the University of Washington.

Tale of Two Rings

Power Series Ring.: $\mathbb{Z}[[\boldsymbol{X}]]$ over a finite or countably infinite alphabet $\boldsymbol{X} = \{\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_n}\}$ or $\boldsymbol{X} = \{\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots\}$.

Two subrings. of $\mathbb{Z}[[\boldsymbol{X}]]$:

- Symmetric Functions (SYM)
- Quasisymmetric Functions (QSYM)

Ring of Symmetric Functions

Defn. $f(x_1, x_2, \dots) \in \mathbb{Z}[[X]]$ is a *symmetric function* if for all i

$$f(\dots, x_i, x_{i+1}, \dots) = f(\dots, x_{i+1}, x_i, \dots).$$

Example. $x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + \dots$

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Defn. $f(x_1, x_2, \dots) \in \mathbb{Z}[[X]]$ is a *quasisymmetric function* if

$$\mathrm{coef}(f; x_1^{\alpha_1} x_2^{\alpha_2} \dots x_k^{\alpha_k}) = \mathrm{coef}(f; x_a^{\alpha_1} x_b^{\alpha_2} \dots x_c^{\alpha_k})$$

for all $1 < a < b < \cdots < c$.

Example. $f(X) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + \dots$

Why study SYM and QSYM?

- Symmetric Functions (SYM): Used in representation theory, combinatorics, algebraic geometry over past 200+ years. And now in number theory!

- Quasisymmetric Functions (QSYM): 0-Hecke algebra representation theory, Hopf dual of NSYM=non-commutative symmetric functions, Schubert calculus.

- QSYM now in Sage!

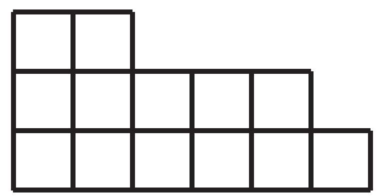
Monomial Basis of SYM

Defn. A *partition* of a number n is a weakly decreasing sequence of positive integers

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$$

such that $n = \sum \lambda_i = |\lambda|$.

Partitions can be visualized by their *Ferrers diagram*

$(6, 5, 2) \longrightarrow$ 

Defn/Thm. The *monomial symmetric functions*

$$m_\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k} + x_2^{\lambda_1} x_1^{\lambda_2} \cdots x_k^{\lambda_k} + \textbf{all other perms of vars}$$

form a basis for SYM_n = homogeneous symmetric functions of degree n.

Fact. $\dim SYM_n = p(n)$ = number of partitions of n.

Monomial Basis of QSYM

Defn. A *composition* of a number n is a sequence of positive integers

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$$

such that $n = \sum \alpha_i = |\alpha|$.

Defn/Thm. The *monomial quasisymmetric functions*

$$M_\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} + x_2^{\alpha_1} x_3^{\alpha_2} \cdots x_{k+1}^{\alpha_k} + \text{all other shifts}$$

form a basis for $QSYM_n$ = homogeneous quasisymmetric functions of deg n.

Fact. $\dim QSYM_n$ = number of compositions of $n = 2^{n-1}$.

Monomial Basis of QSYM

Fact. $\dim QSYM_n =$ number of compositions of $n = 2^{n-1}$.

Bijection:

$$\begin{aligned}(\alpha_1, \alpha_2, \ldots, \alpha_k) \longrightarrow \{&\alpha_1,\\ &\alpha_1 + \alpha_2,\\ &\alpha_1 + \alpha_2 + \alpha_3,\\ &\ldots\\ &\alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}\end{aligned}$$

Counting Partitions

Asymptotic Formula:. (Hardy-Ramanujan)

$$p(n) \approx \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$$

Schur basis for SYM

Let $X = \{x_1, x_2, \ldots, x_m\}$ be a finite alphabet.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ and $\lambda_p = 0$ for $p > k$.

Defn. The following are equivalent definitions for the Schur functions $S_\lambda(X)$:

1. $S_\lambda = \frac{\det(x_i^{\lambda_j + m - j})}{\det(x_i^j)}$ with indices $1 \leq i, j \leq m$.
2. $S_\lambda = \sum x^T$ summed over all *column strict tableaux* T *of shape* λ.

Defn. T is *column strict* if entries strictly increase along columns and weakly increase along rows.

Example. A column strict tableau of shape $(5, 3, 1)$

$$T = \begin{array}{|c|c|c|c|c|} \cline{1-1} 7 \\ \cline{1-3} 4 & 7 & 7 \\ \hline 2 & 2 & 3 & 4 & 8 \\ \hline \end{array} \qquad x^T = x_2^2 x_3 x_4^2 x_7^3 x_8$$

Multiplying Schur Functions

Littlewood-Richardson Coefficients.

$$S_\lambda(X) \cdot S_\mu(X) = \sum_{|\nu|=|\lambda|+|\mu|} c^{\nu}_{\lambda,\mu} S_\nu(X)$$

$c^{\nu}_{\lambda,\mu} = \#$ skew tableaux of shape ν/λ such that $x^T = x^\mu$ and the reverse reading word is a lattice word.

Example. If $\nu = (4,3,2)$, $\lambda = (2,1)$, $\lambda = (3,2,1)$ then

$$\begin{array}{|c|c|cc} \hline 2 & 3 & \multicolumn{2}{c}{} \\ \hline \multicolumn{1}{c|}{} & 1 & \multicolumn{1}{c|}{2} & \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} \\ \cline{3-4} \end{array} \qquad \text{readingword} = 231211$$

Fundamental basis for QSYM

Defn. Let $A \subset [p-1] = \{1, 2, \ldots, p-1\}$.
The fundamental quasisymmetric function

$$F_A(X) = \sum x_{i_1} \cdots x_{i_p}$$

summed over all $1 \leq i_1 \leq \ldots \leq i_p$ such that $i_j < i_{j+1}$ whenever $j \notin A$.

Example. $F_{++-+} = x_1x_1x_1x_2x_2 + x_1x_2x_2x_3x_3 + x_1x_2x_3x_4x_5 + \ldots$

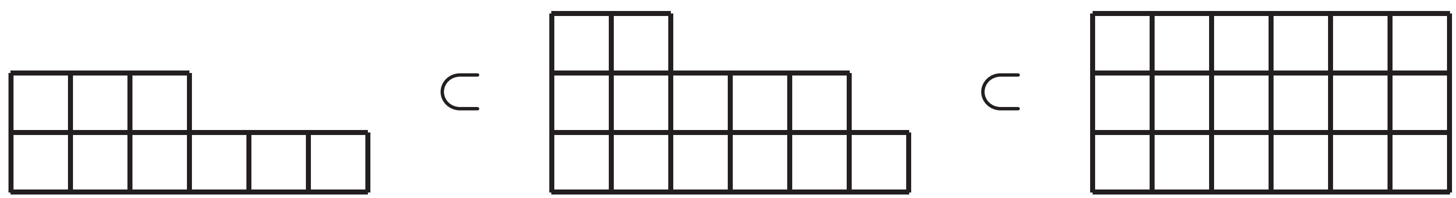
Here $++-+ = \{1, 2, 4\} \subset \{1, 2, 3, 4\}$.

Other bases of QSYM. quasi Schur basis (Haglund-Luoto-Mason-vanWilliger
matroid friendly basis (Luoto)

A Poset on Partitions

Defn. A *partial order* or a *poset* is a reflexive, anti-symmetric, and transitive relation on a set.

Defn. *Young's Lattice* on all partitions is the poset defined by the relation $\boldsymbol{\lambda} \subset \boldsymbol{\mu}$ if the Ferrers diagram for $\boldsymbol{\lambda}$ fits inside the Ferrers diagram for $\boldsymbol{\mu}$.



Defn. A standard tableau $\boldsymbol{T}$ of shape $\boldsymbol{\lambda}$ is a saturated chain in Young's lattice from $\emptyset$ to $\boldsymbol{\lambda}$.

Example. $\boldsymbol{T} = \begin{array}{|c|c|c|c|c|} \hline 7 \\ \cline{1-1} 4 & 5 & 9 \\ \cline{1-3} 1 & 2 & 3 & 6 & 8 \\ \hline \end{array}$

Schur functions

Thm. (Gessel,1984) For all partitions λ,

$$S_\lambda(X) = \sum F_{D(T)}(X)$$

summed over all standard tableaux T of shape λ.

Defn. The descent set of T, denoted $D(T)$, is the set of indices i such that $i+1$ appears northwest of i.

Example. Expand $S_{(3,2)}$ in the fundamental basis

$$\begin{array}{|c|c|}\hline 4 & 5 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 3 & 5 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 3 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 2 & 5 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 2 & 4 \\ \hline \end{array}$$
$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline \end{array}$$

$$S_{(3,2)}(X) = F_{++-+}(X) + F_{+-+-}(X) + F_{+-++}(X) + F_{-++-}(X) + F_{-+-+}$$

Macdonald Polynomials

Defn/Thm. (Macdonald 1988, Haiman-Haglund-Loehr, 2005)

$$\widetilde{H}_\mu(X;q,t) = \sum_{w \in S_n} q^{inv_\mu(w)} t^{maj_\mu(w)} F_{D(w^{-1})}$$

where $D(w)$ is the descent set of w in one-line notation.

Thm. (Haiman) Expanding $\widetilde{H}_\mu(X;q,t)$ into Schur functions

$$\widetilde{H}_\mu(X;q,t) = \sum_i \sum_j \sum_{|\lambda|=|\mu|} c_{i,j,\lambda} q^i t^j S_\lambda,$$

the coefficients $c_{i,j,\lambda}$ are all non-negative integers.

$\Longrightarrow$ Macdonald polynomials are *Schur positive*,

Open I. Find a “nice” combinatorial algorithm to compute $c_{i,j,\lambda}$ showing these are non-negative integers.

Lascoux-Leclerc-Thibon Polynomials

Defn. Let $\bar{\mu} = (\mu^{(1)}, \mu^{(1)}, \ldots, \mu^{(k)})$ be a list of partitions.

$$LLT_{\bar{\mu}}(X;q) = \sum q^{inv_{\mu}(T)} F_{D(w^{-1})}$$

summed over all bijective fillings w of $\bar{\mu}$ where each $\mu^{(i)}$ filled with rows and columns increasing. Each w is recorded as the permutation given by the content reading word of the filling.

Thm. For all $\bar{\mu} = (\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(k)})$

1. $LLT_{\bar{\mu}}(X;q)$ is symmetric. (Lascoux-Leclerc-Thibon)

Lascoux-Leclerc-Thibon Polynomials

Open II. Find a "nice" combinatorial algorithm to compute the expansion coefficients for LLT's to Schurs.

Known. Each $\widetilde{H}_\mu(X; q, t)$ expands as a positive sum of LLT's so Open II implies Open I. (Haiman-Haglund-Loehr)

Plethysm of Schur Functions

Defn. Given $f, g \in \mathbb{Z}_+[[X]]$ with $g = x^\alpha + x^\beta + x^\gamma + \ldots$, the *Plethysm* of f, g is

$$f[g] = f(x^\alpha, x^\beta, x^\gamma, \ldots)$$

Thm.[Loehr-Warrington (2012)] For all partitions λ, μ and compositions α, the plethysm

$$s_\lambda[F_\alpha] = \sum_{A \in M(\lambda,\alpha)} F_{D(w(A))}$$

$$s_\lambda[s_\mu] = \sum_{A \in M(\lambda,\mu)} F_{D(w(A))}$$

Stanley symmetric functions

Background.

- Every permutation w can be written as a product of adjacent transpositions $s_i = (i, i+1)$.
- A minimal length expression for w is said to be *reduced*.
- Let $R(w)$ be the set of all sequences $\mathbf{a} = (a_1, \ldots, a_p)$ such that $w = s_{a_1} \cdots s_{a_p}$ is reduced.

Def. For $w \in S_n$, the *Stanley symmetric function* is

$$F_w = \sum_{\mathbf{a} \in R(w)} F_{A(\mathbf{a})}$$

where $A(\mathbf{a})$ is the set of positions i where $a_i < a_{i+1}$.

Stanley symmetric functions

Background.

- Every permutation w can be written as a product of adjacent transpositions $s_i = (i, i+1)$.
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Def. For $w \in S_n$, the *Stanley symmetric function* is

$$F_w = \sum_{\mathbf{a} \in R(w)} F_{A(\mathbf{a})} = \lim_{m \to \infty} \mathfrak{S}_{1^m \times w}.$$

where $\mathfrak{S}_w$ is a Schubert polynomial and $1 \times w = [1, w_1 + 1, \ldots, w_n + 1]$.

Stanley symmetric functions

Thm. [Stanley, Edelman-Greene] F_w is symmetric and has Schur expansion:

$$F_w = \sum_{\lambda} a_{\lambda,w} S_{\lambda}, \qquad a_{\lambda,w} \in \mathbb{N}.$$

Cor. $|R(w)| = \sum_{\lambda} a_{\lambda,w} f^{\lambda}$ where f^{λ} is the number of standard tableaux of shape λ.

Nice cases.

1. If $w = [n, n-1, \ldots, 1] = w_0$ then $F_w = S_{\delta}$ where δ is the staircase shape with $n-1$ rows.
2. $F_w = s_{\lambda(w)}$ iff w is 2143-avoiding iff w is *vexillary*.

Vexillary Permutations

Def. A permutation is *vexillary* iff $F_w = s_{\lambda(w)}$ iff w is 2143-avoiding.

Properties.

- Schubert polynomial is a flagged Schur function (Wachs).
- Kazhdan-Lusztig polynomials have a combinatorial formula (Lascoux-Schützenb
- The enumeration is the same as 1234-avoiding permutations (Gessel).
- Easy to find a uniformly random reduced expression using Robinson-Schensted-Knuth correspondence and the hook-walk algorithm (Greene-Nijenhuis-Wilf).

Generalizing Vexillary Permutations

Def. A permutation is *k-vexillary* iff $F_w = \sum a_{\lambda,w} s_\lambda$ and $\sum_{a_{\lambda,w}} \leq k$.

Example. $F_{214365} = S_{(3)} + 2S_{(2,1)} + S_{(1,1,1)}$

so $\mathbf{214365}$ is 4-vexillary, but not 3-vexillary.

Generalizing Vexillary Permutations

Def. A permutation is *k-vexillary* iff $F_w = \sum a_{\lambda,w} s_\lambda$ and $\sum_{a_{\lambda,w}} \leq k$.

Thm. (Billey-Pawlowski) A permutation w is k-vexillary iff w avoids a finite set of patterns V_k for all $k \in \mathbb{N}$.

$k = 1$	$V_1 = \{2143\}$,
$k = 2$	$\lvert V_2 \rvert = 35$, all in $S_5 \cup S_6 \cup S_7 \cup S_8$
$k = 3$	$\lvert V_3 \rvert = 91$, all in $S_5 \cup S_6 \cup S_7 \cup S_8$
$k = 4$	conjecture $\lvert V_4 \rvert = 2346$, all in $S_5 \cup \cdots \cup S_{12}$.

Generalizing Vexillary Permutations

Def. A permutation is *k-vexillary* iff $F_w = \sum a_{\lambda,w} s_\lambda$ and $\sum_{a_{\lambda,w}} \leq k$.

Properties.

- 2-vex perms have easy expansion: $F_w = S_{\lambda(w)} + S_{\lambda(w^{-1})'}$.
- 3-vex perms are multiplicity free: $F_w = S_{\lambda(w)} + S_\mu + S_{\lambda(w^{-1})'}$ for some μ between first and second shape in dominance order.
- 3-vex perms have a nice essential set.

Outline of Proof

Thm. (Billey-Pawlowski) A permutation w is k-vexillary iff w avoids a finite set of patterns V_k for all $k \in \mathbb{N}$.

Proof.

1. (James-Peel) Use generalized Specht modules S^D for $D \in \mathbb{N} \times \mathbb{N}$.
2. (Kraśkiewicz, Reiner-Shimozono) For $D(w)$=diagram of permutation w,
$$S^{D(w)} = \bigoplus (S^{\lambda})^{a_{\lambda,w}}.$$
3. Compare Lascoux-Schützenberger transition tree and James-Peel moves.
4. If w contains v as a pattern, then the James-Peel moves used to expand $S^{D(v)}$ into irreducibles will also apply to $D(w)$ in a way that respects shape inclusion and multiplicity.

Another permutation filtration

Def. A permutation w is *multiplicity free* if F_w has a multiplicity free Schur expansion.

Def. A permutation w is *k-multiplicity bounded* if $\langle F_w, S_\lambda \rangle \leq k$ for all partitions λ.

Cor. If w is k-multiplicity bounded and w contains v as a pattern, then v is k-multiplicity bounded for all k.

Conjecture. The multiplicity free permutations are characterized by 198 pattern up through S_{11}.

Motivation

Let $D \subset \mathbb{N} \times \mathbb{N}$. Let $S^D = \bigoplus (S^\lambda)^{c_{\lambda,D}}$ expanded into irreducibles.

In the Grassmannian $Gr(k,n)$, consider the row spans of the matrices

$$\{(I_k|A) : A \in M_{k\times(n-k)}, A_{ij} = 0 \text{ if } (i,j) \in D\}.$$

Let Ω_D be the closure of this set in $Gr(k,n)$. Let σ_D be the cohomology class associated to this variety.

Liu's Conjecture. The Schur expansion of $\sigma_D = \sum c_{\lambda,D} S_\lambda$.

True for "forests" (Liu) and permutation diagrams (Knutson-Lam-Speyer, Pawlowski)

Summary of Conjectures/Goals

Conjectures.

1. The 4-vexillary permutations are characterized by $\mathbf{2346}$ patterns in $\boldsymbol{S_{12}}$.
2. The multiplicity free permutations are characterized by 198 pattern up through $\boldsymbol{S_{11}}$.
3. Liu's conjecture: The Schur expansion of $\boldsymbol{\sigma_D = \sum c_{\lambda,D} S_\lambda}$.

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