# Reduced words and a formula of Macdonald

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Based on joint work with:
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# Outline

# Permutations

Permutations are fundamental objects in mathematics, computer science, game theory, economics, physics, chemistry and biology.

**Notation.**

- $S_n$ is the symmetric group of permutations on $n$ letters.
- $w \in S_n$ is a bijection from $[n] := \{1, 2, \ldots, n\}$ to itself denoted in *one-line notation* as $w = [w(1), w(2), \ldots, w(n)]$.
- $s_i = (i \leftrightarrow i+1) =$ adjacent transposition for $1 \leq i < n$.

**Example.** $w = [3, 4, 1, 2, 5] \in S_5$ and $s_4 = [1, 2, 3, 5, 4] \in S_5$.

$ws_4 = [3, 4, 1, 5, 2]$ and $s_4 w = [3, 5, 1, 2, 4]$.

# Permutations

## Presentation of the Symmetric Group.

**Fact.** $S_n$ is generated by $s_1, s_2, \ldots, s_{n-1}$ with relations

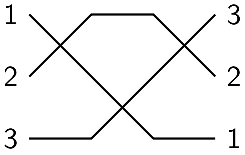
$$
\begin{aligned}
&s_i s_i = 1 \\
&(s_i s_j)^2 = 1 \text{ if } |i-j| > 1 \\
&(s_i s_{i+1})^3 = 1
\end{aligned}
$$

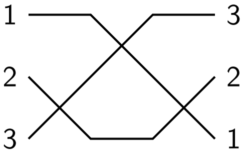
For each $w \in S_n$, there is some expression $w = s_{a_1} s_{a_2} \cdots s_{a_p}$.
If $p$ is minimal, then

- $\ell(w) =$ *length of w* $=$ p,
- $s_{a_1} s_{a_2} \cdots s_{a_p}$ is a *reduced expression* for $w$,
- $a_1 a_2 \ldots a_p$ is a *reduced word* for $w$.

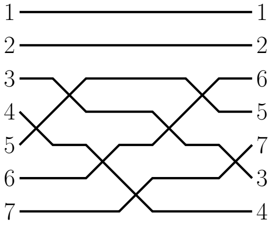
# Reduced Words and Reduced Wiring Diagrams

**Example.** 121 and 212 are reduced words for $[3, 2, 1]$.




**Example.** 4356435 is a reduced word for $[1, 2, 6, 5, 7, 3, 4] \in S_7$.



# Reduced Words

**Key Notation.** $R(w)$ is the set of all reduced words for $w$.

**Example.** $R([3,2,1]) = \{121, 212\}$.

**Example.** $R([4,3,2,1])$ has 16 elements:

| | | | |
|---|---|---|---|
| 321323 | 323123 | 232123 | 213213 |
| 231213 | 321232 | 132132 | 312132 |
| 132312 | 312312 | 123212 | 213231 |
| 231231 | 212321 | 121321 | 123121 |

**Example.** $R([5,4,3,2,1])$ has 768 elements.

# Counting Reduced Words

**Question.** How many reduced words are there for $w$?

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**Theorem.**(Stanley, 1984) For $w_0^n := [n, n-1, \ldots, 2, 1] \in S_n$,

$$|R(w_0^n)| = \frac{\binom{n}{2}!}{1^{n-1}\ 3^{n-2}\ 5^{n-3}\ \cdots (2n-3)^1}.$$

Observation: The right side is equal to the number of standard Young tableaux of staircase shape $(n-1, n-2, \ldots, 1)$.

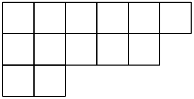
# Counting Standard Young Tableaux

**Defn.** A *partition* of a number $n$ is a weakly decreasing sequence of positive integers

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$$

such that $n = \sum \lambda_i = |\lambda|$.

Partitions can be visualized by their *Ferrers diagram*

$(6, 5, 2) \longrightarrow$ 

**Def.** A standard Young tableau $T$ of shape $\lambda$ is a bijective filling of the boxes by $1, 2, \ldots, n$ with rows and columns increasing.

**Example.** $T =$

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 6 | 8 |
| 4 | 5 | 9 | | |
| 7 | | | | |

The standard Young tableaux (SYT) index the bases of $S_n$-irreps.

# Counting Standard Young Tableaux

**Hook Length Formula.**(Frame-Robinson-Thrall, 1954) If $\lambda$ is a partition of $n$, then

$$\#SYT(\lambda) = \frac{n!}{\prod_{c \in \lambda} h_c}$$

where $h_c$ is the *hook length* of the cell $c$, i.e. the number of cells directly to the right of $c$ or below $c$, including $c$.

**Example.** Hook lengths of $\lambda = (5, 3, 1)$:

| | | | | |
|---|---|---|---|---|
| 7 | 5 | 4 | 2 | 1 |
| 4 | 2 | 1 | | |
| 1 | | | | |

So, $\#SYT(5, 3, 1) = \frac{9!}{7 \cdot 5 \cdot 4 \cdot 2 \cdot 4 \cdot 2} = 162$.

**Remark.** Notable other proofs by Greene-Nijenhuis-Wilf '79 (probabalistic), Krattenthaler '95 (bijective), Novelli-Pak-Stoyanovskii '97 (bijective), Bandlow '08.

# Counting Reduced Words

**Theorem.**(Edelman-Greene, 1987) For all $w \in S_n$,

$$|R(w)| = \sum a_{\lambda,w} \#SYT(\lambda)$$

for some nonnegative integer coefficients $a_{\lambda,w}$ with $\lambda \vdash \ell(w)$ in a given interval in dominance order.

Proof via an insertion algorithm like the RSK:
$\mathbf{a} = a_1 a_2 \ldots a_p \longleftrightarrow (P(\mathbf{a}), Q(\mathbf{a}))$.
$P(\mathbf{a})$ is strictly increasing in rows and columns has reading word equal to a reduced word for $w$.
$Q(\mathbf{a})$ can be any standard tableau of the same shape as $P(\mathbf{a})$.

**Corollary.** Every reduced word for $w_0$ inserts to the same $P$ tableau of staircase shape $\delta$, so $|R(w_0)| = \#SYT(\delta)$.

# Random Reduced Word

The formula $|R(w)| = \sum a_{\lambda,w} \# SYT(\lambda)$ gives rise to an easy way to choose a random reduced word for $w$ using the Hook Walk Algorithm (Greene-Nijenhuis-Wilf) for random STY of shape $\lambda$.

**Algorithm.** Input: $w \in S_n$, Output: $a_1 a_2 \dots a_p \in R(w)$ chosen uniformly at random.

1. Choose a $P$-tableau for $w$ in proportion to $\# SYT(sh(P))$.
2. Set $\lambda = sh(P)$.
3. Loop for $k$ from $n$ down to 1. Choose one of the $k$ empty cells $c$ in $\lambda$ with equal probability. Apply hook walk from $c$.
4. Hook walk: If $c$ is in an outer corner of $\lambda$, place $k$ in that cell. Otherwise, choose a new cell in the hook of $c$ uniformly. Repeat step until $c$ is an outer corner.

# Random Reduced Word

**Def.** For $\mathbf{a} = a_1 a_2 \ldots a_p \in R(w)$, let $B(\mathbf{a})$ be the random variable counting the number of *braids* in $\mathbf{a}$, i.e. consecutive letters $i, i+1, i$ or $i+1, i, i+1$.

**Examples.** $B(321323) = 1$ and $B(232123) = 2$

**Question.** What is the expected value of $B$ on $R(w)$?

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**Examples.** $B(321323) = 1$ and $B(232123) = 2$

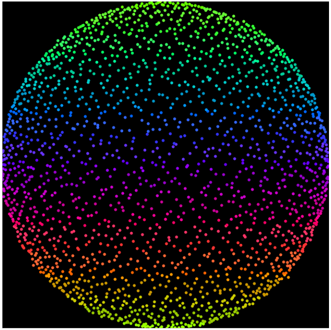
**Question.** What is the expected value of $B$ on $R(w)$?

**Thm.**(Reiner, 2005) For all $n \geq 1$, the expected value of $B$ on $R(w_0)$ is exactly 1.

# Random Reduced Word

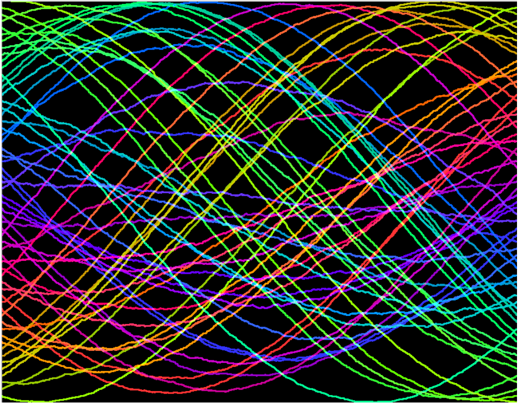
Angel-Holroyd-Romik-Virag: "Random Sorting Networks" (2007)

**Conjecture.** Assume $a_1 a_2 \ldots a_p \in R(w_0)$ is chosen uniformly at random. The distribution of 1's in the permutation matrix for $w = s_{a_1} s_{a_2} \cdots s_{a_{p/2}}$ converges as $n$ approaches infinity to the projected surface measure of the 2-sphere.



# Random Reduced Word

Alexander Holroyd's picture of a uniformly random 2000-element sorting network (selected trajectories shown):



# Macdonald's Formula

**Thm.** (Macdonald, 1991) For $w_0 \in S_n$,

$$\sum_{\mathbf{a} \in R(w_0)} a_1 \cdot a_2 \cdots a_{\binom{n}{2}} =$$

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**Question.**(Holroyd) Is there an efficient algorithm to choose a reduced word randomly with $P(a_1 a_2 \ldots a_{\binom{n}{2}})$ proportional to $a_1 \cdot a_2 \cdots a_{\binom{n}{2}}$?

# Consequences of Macdonald's Formula

**Thm.**(Young, 2014) There exists a Markov growth process using Little's bumping algorithm adding one crossing in a wiring diagram at a time to obtain a random reduced word for $w_0 \in S_n$ in $\binom{n}{2}$ steps.

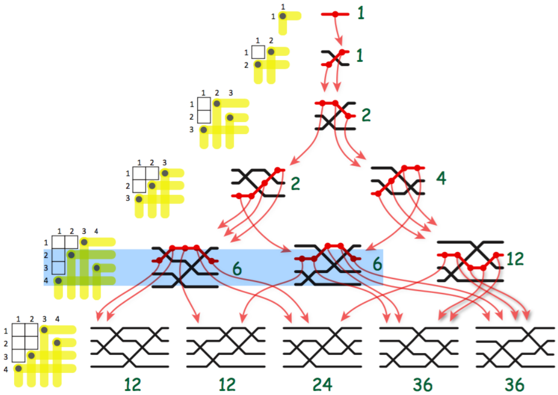
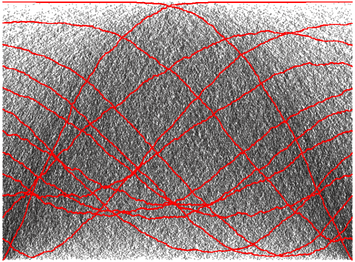



Image credit: Kristin Potter.

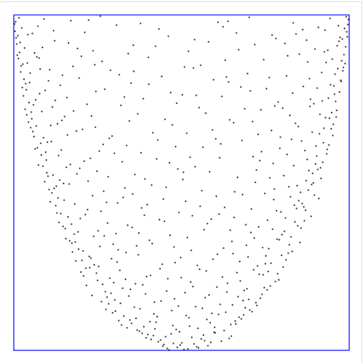
# Consequences of Macdonald's Formula

The wiring diagram for a random reduced word for $w_0 \in S_{600}$ chosen with Young's growth process.



# Macdonald's Formula

The permutation matrix for the product of the first half of a random reduced word for $w_0 \in S_{600}$ chosen with Young's growth process.



# Macdonald's Formula

**Thm.**(Macdonald, 1991) For any $w \in S_n$ with $\ell(w) = p$,

$$\sum_{\mathbf{a} \in R(w)} a_1 \cdot a_2 \cdots a_p = p! \mathfrak{S}_w(1, 1, 1, \ldots)$$

where $\mathfrak{S}_w(1, 1, 1, \ldots)$ is the number of monomials in the corresponding Schubert polynomial.

**Question.**(Young, Fomin, Kirillov,Stanley, Macdonald, ca 1990) Is there a bijective proof of this formula?

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**Question.**(Young, Fomin, Kirillov,Stanley, Macdonald, ca 1990) Is there a bijective proof of this formula?

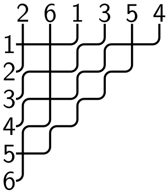
**Answer.** Yes! Based on joint work with Holroyd and Young, and builds on Young's growth process.

# Schubert polynomials

**History.** Schubert polynomials were originally defined by Lascoux-Schützenberger early 1980's. Via work of Billey-Jockusch-Stanley, Fomin-Stanley, Fomin-Kirillov, Billey-Bergeron in the early 1990's we know the following equivalent definition.

**Def.** For $w \in S_n$, $\mathfrak{S}_w(x_1, x_2, \ldots x_n) = \sum_{D \in RP(w)} x^D$ where $RP(w)$ are the *reduced pipe dreams* for $w$, aka *rc-graphs*.

**Example.** A reduced pipe dream $D$ for $w = [2, 6, 1, 3, 5, 4]^{-1}$ where $x^D = x_1^3 x_2 x_3 x_5$.



# Bijective Proof of Macdonald's Formula

To show:

$$\sum_{\mathbf{a} \in R(w)} a_1 \cdot a_2 \cdots a_p = p! \cdot \#RP(w)$$

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**Def.** $b_1 b_2 \ldots b_p$ is a *bounded word* for $a_1 a_2 \ldots a_p$ provided $1 \leq b_i \leq a_i$ for each $i$.

**Def.** The pair $(\mathbf{a}, \mathbf{b}) = ((a_1 a_2 \ldots a_p), (b_1 b_2 \ldots b_p))$ is a *bounded pair* for $w$ provided $\mathbf{a} \in R(w)$ and $\mathbf{b}$ is a bounded word for $\mathbf{a}$.

**Def.** A word $\mathbf{c} = c_1 c_2 \ldots c_p$ is a *sub-staircase word* provided $1 \leq c_i \leq i$ for each $i$.

# Bijective Proof of Macdonald's Formula

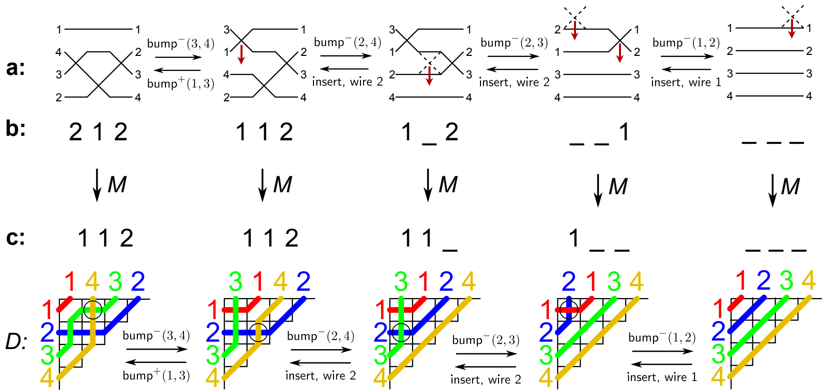
To show:

$$\sum_{\mathbf{a} \in R(w)} a_1 \cdot a_2 \cdots a_p = p! \cdot \#RP(w)$$

Want: A bijection $BP(w) \longrightarrow cD(w)$ where

- $BP(w)$ := bounded pairs for $w$,
- $cD(w)$ := $cD$-pairs for $w$ of the form $(\mathbf{c}, D)$ where $D$ is a reduced pipe dream for $w$ and $\mathbf{c}$ is a sub-staircase word of the same length as $w$.

# Bijective Proof of Macdonald's Formula



# Transition Equations

**Thm.**(Lascoux-Schützenberger,1984) For all $w \neq id$, let $(r < s)$ be the largest pair of positions inverted in $w$ in lexicographic order. Then,

$$\mathfrak{S}_w = x_r \mathfrak{S}_{wt_{rs}} + \sum \mathfrak{S}_{w'}$$

where the sum is over all $w'$ such that $\ell(w) = \ell(w')$ and $w' = wt_{rs}t_{ir}$ with $0 < i < r$. Call this set $T(w)$.

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**Example.** If $w = [7325614]$, then $r = 5$, $s = 7$

$$\mathfrak{S}_w = x_5 \mathfrak{S}_{[7325416]} + \mathfrak{S}_{[7425316]} + \mathfrak{S}_{[7345216]}$$

So, $T(w) = \{[7425316], [7345216]\}$.

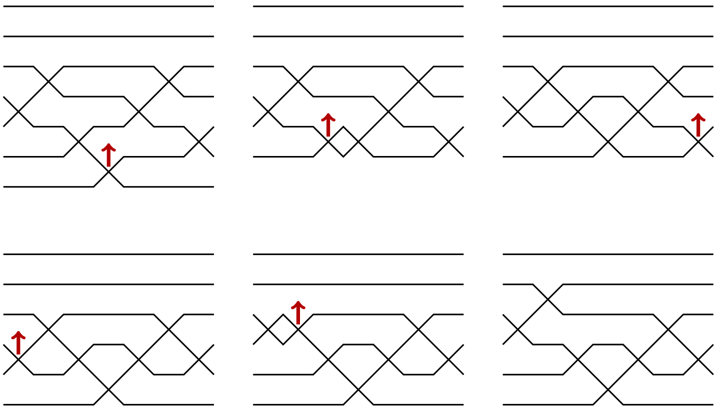
# Little's Bijection

**Theorem.**(David Little, 2003)
There exists a bijection from $R(w)$ to $\cup_{w' \in T(w)} R(w')$ which preserves the ascent set provided $T(w)$ is nonempty.

**Theorem.** (Hamaker-Young, 2013) Little's bijection also preserves the Coxeter-Knuth classes and the $Q$-tableaux under the Edelman-Greene correspondence. Furthermore, every reduced word for any permutation with the same $Q$ tableau is connected via Little bumps.

# Little Bumps

**Example.** The Little bump applied to $\mathbf{a} = 4356435$ in col 4.



# Push and Delete operators

Let $\mathbf{a} = a_1 \dots a_k$ be a word. Define the *decrement-push*, *increment-push*, and *deletion* of $\mathbf{a}$ at column $t$, respectively, to be

$$\mathcal{P}_t^- \mathbf{a} = (a_1, \dots, a_{t-1}, a_t - 1, a_{t+1}, \dots, a_k);$$
$$\mathcal{P}_t^+ \mathbf{a} = (a_1, \dots, a_{t-1}, a_t + 1, a_{t+1}, \dots, a_k);$$
$$\mathcal{D}_t \mathbf{a} = (a_1, \dots, a_{t-1}, a_{t+1}, \dots, a_k);$$

# Bounded Bumping Algorithm

**Input**: $(\mathbf{a}, \mathbf{b}, t_0, d)$, where **a** is a word that is nearly reduced at $t_0$, and **b** is a bounded word for **a**, and $d \in \{-, +\}$.

**Output**: $\mathrm{Bump}_{t_0}^{d}(\mathbf{a}, \mathbf{b}) = (\mathbf{a}', \mathbf{b}', i, j, \text{outcome})$.

1. Initialize $\mathbf{a}' \leftarrow \mathbf{a}$, $\mathbf{b}' \leftarrow \mathbf{b}$, $t \leftarrow t_0$.
2. Push in direction $d$ at column $t$, i.e. set $\mathbf{a}' \leftarrow \mathcal{P}_t^d \mathbf{a}'$ and $\mathbf{b}' \leftarrow \mathcal{P}_t^d \mathbf{b}'$.
3. If $b'_t = 0$, return $(\mathcal{D}_t \mathbf{a}', \mathcal{D}_t \mathbf{b}', \mathbf{a}'_t, t, \textit{deleted})$ and **stop**.
4. If $\mathbf{a}'$ is reduced, return $(\mathbf{a}', \mathbf{b}', \mathbf{a}'_t, t, \textit{bumped})$ and **stop**.
5. Set $t \leftarrow \mathrm{Defect}_t(\mathbf{a}')$ and **return to step 2.**

# Generalizing the Transition Equation

1. We use the bounded bumping algorithm applied to the $(r, s)$ crossing in a reduced pipe dream for $w$ to bijectively prove

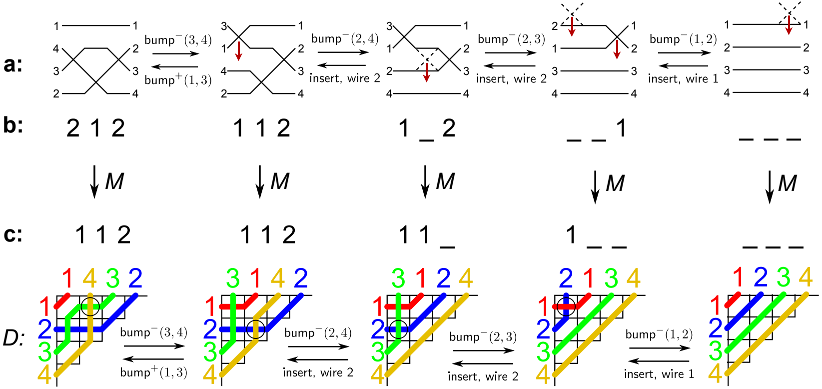
$$\mathfrak{S}_w = x_r \mathfrak{S}_{wt_{rs}} + \sum \mathfrak{S}_{w'}.$$

2. We use the bounded bumping algorithm applied to the $(r, s)$ crossing to give a bijection

$$BP(w) \longrightarrow BP(wt_{rs}) \times [1, p] \cup \bigcup_{w' \in T(w)} BP(w').$$

# Bijective Proof of Macdonald's Formula

$$\sum_{\mathbf{a} \in R(w)} a_1 \cdot a_2 \cdots a_p = p! \mathfrak{S}_w(1, 1, 1, \ldots)$$



# $q$-analog of Macdonald's Formula

**Def.** A $q$-analog of any integer sequence $f_1, f_2, \ldots$ is a family of polynomials $f_1(q), f_2(q), \ldots$ such that $f_i(1) = f_i$.

**Examples.**

- The standard $q$-analog of a positive integer $k$ is $[k] = [k]_q := 1 + q + q^2 + \cdots + q^{k-1}$.
- The standard $q$-analog of the factorial $k!$ is defined to be $[k]_q! := [k][k-1]\cdots[1]$.

Macdonald conjectured a $q$-analog of his formula using $[k]$, $[k]_q!$.

# $q$-analog of Macdonald's Formula

**Theorem.**(Fomin and Stanley, 1994)
Given a permutation $w \in S_n$ with $\ell(w) = p$,

$$\sum_{\mathbf{a} \in R(w)} [a_1] \cdot [a_2] \cdots [a_p] \; q^{\mathrm{comaj}(\mathbf{a})} = [p]_q! \, \mathfrak{S}_w(1, q, q^2, \ldots)$$

where

$$\mathrm{comaj}(\mathbf{a}) = \sum_{a_i < a_{i+1}} i.$$

**Remarks.** Our bijection respects the $q$-weight on each side so we get a bijective proof for this identity too. The key lemma is a generalization of Carlitz's proof that $\ell(w)$ and $\mathrm{comaj}(w)$ are equidistributed on $S_n$ and another generalization of the Transition Equation.

# Another generalization of Macdonald's formula

**Fomin-Kirillov, 1997.** We have the following identity of polynomials in $x$ for the permutation $w_0 \in S_n$:

$$\sum_{\mathbf{a} \in R(w_0)} (x + a_1) \cdots (x + a_{\binom{n}{2}}) = \binom{n}{2}! \prod_{1 \le i < j \le n} \frac{2x + i + j - 1}{i + j - 1}.$$

**Remarks.** Our bijective proof of Macdonald's formula plus a bijection due to Lenart, Serrano-Stump give a new proof of this identity answering a question posed by Fomin-Kirillov.

The right hand side is based on Proctor's formula for reverse plane partitions and Wach's characterization of Schubert polynomials for vexillary permutations.

# Open Problems

**Open.** Is there a common generalization for the Transition Equation for Schubert polynomials, bounded pairs, and its $q$-analog?

**Open.** Is there a nice formula for $|rpp^{\lambda}(x)|$ or $[rpp^{\lambda}(x)]_q$ for an arbitrary partition $\lambda$ as in the case of staircase shapes as noted in the Fomin-Kirillov Theorem?

**Open.** What is the analog of Macdonald's formula for Grothendieck polynomials and what is the corresponding bijection?