Rank Varieties

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Outline

1. Background and history of Grassmannians
2. Rank Varieties and connections to q-Stirling numbers
3. Relating Rank Varieties to Richardson Varieties (Motivation)

Based on joint work with Izzet Coskun (arXiv:1008.2785).

The Grassmannian Manifolds

Definition. Fix a vector space V over $\mathbb{C}$ (or $\mathbb{R}$, $\mathbb{Z}_2$,...) with basis $B = \{e_1, \ldots, e_n\}$. The *Grassmannian manifold/variety*

$$G(k, n) = \{k\text{-dimensional subspaces of } V\}.$$

Question.

How can we impose the structure of a variety or a manifold on this set?

The Grassmannian Manifolds

Answer. Relate $G(k, n)$ to the set of $k \times n$ matrices.

$$U = \text{span}\langle 6e_1 + 3e_2, \quad 4e_1 + 2e_3, \quad 9e_1 + e_3 + e_4\rangle \in G(3, 4)$$

$$M_U = \begin{bmatrix} 6 & 3 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 9 & 0 & 1 & 1 \end{bmatrix}$$

- $U \in G(k, n) \iff$ rows of M_U are independent vectors in $V \iff$ some $k \times k$ minor of M_U is NOT zero.

The Grassmannian Manifolds

Canonical Form. Every subspace in $G(k, n)$ can be represented by a unique matrix in row echelon form.

Example.

$$U = \text{span}\langle 6e_1 + 3e_2, \quad 4e_1 + 2e_3, \quad 9e_1 + e_3 + e_4\rangle \in G(3, 4)$$

$$\approx \begin{bmatrix} 6 & 3 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 9 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \end{bmatrix}$$

$$\approx \langle 2e_1 + e_2, \quad 2e_1 + e_3, \quad 7e_1 + e_4\rangle$$

Subspaces and Subsets

Example.

$$U = \mathrm{RowSpan}\begin{bmatrix} 5 & 9 & \textcircled{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 8 & 0 & 9 & 7 & 9 & \textcircled{1} & 0 & 0 & 0 \\ 4 & 6 & 0 & 2 & 6 & 4 & 0 & 3 & \textcircled{1} & 0 \end{bmatrix} \in G(3,10).$$

$$\mathrm{position}(U) = \{3,7,9\}$$

Definition.

If $U \in G(k,n)$ and M_U is the corresponding matrix in canonical form then the columns of the leading 1's of the rows of M_U determine a subset of size k in $\{1,2,\ldots,n\} := [n]$. There are 0's to the right of each leading 1 and 0's above and below each leading 1. This k-subset determines the *position* of U with respect to the fixed basis.

The Schubert Cell C_{j} in $G(k, n)$

Defn. Let $\mathrm{j} = \{j_1 < j_2 < \cdots < j_k\} \in [n]$. A *Schubert cell* is

$$C_{\mathrm{j}} = \{U \in G(k,n) \mid \text{position}(U) = \{j_1, \ldots, j_k\}\}$$

Example. $C_{\{3,7,9\}} = \left\{ \begin{bmatrix} * & * & \textcircled{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & * & \textcircled{1} & 0 & 0 & 0 \\ * & * & 0 & * & * & * & 0 & * & \textcircled{1} & 0 \end{bmatrix} \right\} \subset G(3, 10).$

Observations.

- $\dim(C_{\{3,7,9\}}) = 2 + 5 + 6 = 13.$
- In general, $\dim(C_{\mathrm{j}}) = \sum j_i - i.$
- $G(k,n) = \bigcup C_{\mathrm{j}}$ over all k-subsets of $[n]$.
- Summing $q^{\dim(C_{\mathrm{j}})}$ over all Schubert cells equals the q-analog of $\binom{n}{k}$.

Schubert Varieties in $G(k, n)$

Defn. Given $\mathbf{j} = \{j_1 < j_2 < \cdots < j_k\} \in [n]$, the *Schubert variety* is

$$X_{\mathbf{j}} = \text{Closure of } C_{\mathbf{j}} \text{ under Zariski topology.}$$

Question. In $G(3, 10)$, which minors vanish on $C_{\{3,7,9\}}$?

$$C_{\{3,7,9\}} = \left\{ \begin{bmatrix} * & * & \textcircled{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & * & \textcircled{1} & 0 & 0 & 0 \\ * & * & 0 & * & * & * & 0 & * & \textcircled{1} & 0 \end{bmatrix} \right\}$$

Answer. All minors f_{j_1, j_2, j_3} with $\left\{ \begin{array}{c} 4 \leq j_1 \leq 8 \\ \text{or } j_1 = 3 \text{ and } 8 \leq j_2 \leq 9 \\ \text{or } j_1 = 3, j_2 = 7 \text{ and } j_3 = 10 \end{array} \right\}$

In other words, the canonical form for any subspace in $X_{\mathbf{j}}\overline{C_{\mathbf{j}}}$ has 0's to the right of column j_i in each row i.

Rank Varieties in $G(k, n)$

Recall we have fixed a basis $e_1, e_2, \ldots, e_n$ for $\mathbb{C}^n$.

Let W be the span of a non-empty collection of consecutive basis vectors: $W = \mathrm{span}(e_i, \ldots, e_j)$. Say $\ell(W) = i$ and $r(W) = j$.

Defn. In $G(k, n)$, a *rank set* $M = \{W_1, \ldots, W_k\}$ is a collection of k vector spaces in $\mathbb{C}^n$ such that each W_i is the span of consecutive basis elements and $\ell(W_i) \neq \ell(W_j)$ and $r(W_i) \neq r(W_j)$ for all $i \neq j$.

Defn. A *rank variety* $X(M)$ in $G(k, n)$ is the closure of the set of all $U \in G(k, n)$ such that U has a basis $u_1, \ldots, u_k$ where each $u_i \in W_i \in M$.

Rank Varieties in $G(k, n)$

Example. In $G(3,6)$, $M = \{\langle e_1, e_2, e_3\rangle, \langle e_2, e_3, e_4, e_5, e_6\rangle, \langle e_3, e_4\rangle\}$ is a rank set. $X(M)$ is the closure of the set of 3-planes specified by rank 3 matrices of the form

$$\left\{ \begin{bmatrix} * & * & * & 0 & 0 & 0 \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & 0 & 0 \end{bmatrix} \right\}$$

Example. $G(3,6)$ is a rank variety itself associated to

$$\left\{ \begin{bmatrix} * & * & * & * & 0 & 0 \\ 0 & * & * & * & * & 0 \\ 0 & 0 & * & * & *3 & * \end{bmatrix} \right\}$$

Rank Varieties in $G(k,n)$

Other examples of rank varieties .

- Every $G(k,n)$ is a rank variety.
- Every Schubert variety in $G(k,n)$ is a rank variety.
- Every Richardson variety in $G(k,n)$ is a rank variety.

There are many more rank varieties than Schubert varieties in $G(k,n)$ in general. For example, in $G(2,4)$ there are $\binom{4}{2} = 6$ Schubert varieties and 25 rank varieties.

Dimensions of Rank Varieties

Lemma. Let $M = \{W_1, \ldots, W_k\}$ be a rank set. Then

$$\dim X(M) = \sum_{i=1}^{k} \dim(W_i) - \sum_{i=1}^{k} \#\{W_j \in M : W_j \subset W_i\}.$$

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Example. In $G(3,6)$, $M = \{\langle e_1, e_2, e_3\rangle, \langle e_2, e_3, e_4, e_5, e_6\rangle, \langle e_3, e_4\rangle\}$, $X(M)$ is the closure of the set of 3-planes specified by rank 3 matrices of the form

$$\left\{ \begin{bmatrix} * & * & * & 0 & 0 & 0 \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & 0 & 0 \end{bmatrix} \right\}$$

$$\dim(X(M)) = 3 + 5 + 2 - 1 - 2 - 1 = 6$$

Dimensions of Rank Varieties

Defn. Consider the sum over all rank sets for $G(k, n)$

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Example. $g[2, 4] = 6 + 8q + 7q^2 + 3q^3 + q^4$.

dim	*ranksets*
0 :	(2, 1), (3, 1), (4, 1), (3, 2), (4, 2), (4, 3)
1 :	(23, 1), (34, 1), (3, 12), (4, 12), (2, 123), (34, 2), (4, 23), (3, 234)
2 :	(234, 1), (23, 12), (34, 12), (4, 123), (2, 1234), (3, 1234), (34, 23)
3 :	(234, 12), (34, 123), (23, 1234)
4 :	(234, 123)

Dimensions of Rank Varieties

Defn. Consider the sum over all rank sets for $G(k,n)$

$$g[k,n] = \sum_{M} q^{\dim(X(M))}.$$

Lemma. Let $[k] = 1 + q + \cdots + q^{k-1}$. Then

$$g[k,n] = g[k,n-1] + [n-k+1]g[k-1,n-1].$$

Proof: Partition the rank sets for $G(k,n)$ according to whether or not e_n appears as a right hand endpoint for some subspace in the set.

q-Stirling numbers

Defn. The *Stirling numbers* of the 2nd kind are

$$S(n,k) = \# \text{ set partitions of } \{1,\ldots,n\} \text{ into } k \text{ nonempty blocks.}$$

Define

$$S[n,k] = q^{k-1}S[n-1,k-1] + [k]S[n-1,k]$$

with boundary conditions $S[0,0] = 1, S[n,0] = 0$ for $n > 0$, $S[n,k] = 0$ for $k > n$. Note, $S[n,k]$ is divisible by $q^{\binom{k}{2}}$.

$S[n,k]$ q-counts sets partitions by crossing number in juggling patterns in work of Ehrenborg-Readdy. See also [Garsia-Remmel, Milne, Wachs-White].

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with boundary conditions $S[0,0] = 1, S[n,0] = 0$ for $n > 0$, $S[n,k] = 0$ for $k > n$. Note, $S[n,k]$ is divisible by $q^{\binom{k}{2}}$.

Cor. $g[k,n] = \sum_M q^{\dim(X(M))} = \dfrac{S[n+1, n-k+1]}{q^{\binom{n-k+1}{2}}}$.

Motivation: Projecting Richardson Varieties

Schubert varieties X_w can be defined in any *partial flag manifold*

$$FL(k_1 < \cdots < k_d; \mathbb{C}^n) = \{V_1 \subset V_2 \subset \cdots \subset V_d : \dim(V_i) = k_i\}.$$

Defn. A *Richardson variety* $R(u,v)$ is $X_u \cap gX_v$ where g generic.

We have a natural projection mapping a flag to its biggest subspace

$$\pi : FL(k_1 < \cdots < k_d; \mathbb{C}^n) \longrightarrow G(k_d, n).$$

Question. What is $\pi(R(u,v))$?

(Related question studied by Lusztig, Postnikov, Rietsch, Brown-Goodearl-Yakimov, Bergeron-Sottile, Lam-Knutson-Speyer)

Motivation: Projecting Richardson Varieties

Theorem. X is a projected Richardson variety in $G(k, n)$ under the "biggest subspace map" if and only if X is a rank variety.

Cor. Let X be a rank variety with rank set M. The following are equivalent.

1. X is smooth.
2. X is a Segre product of linearly embedded sub-Grassmannians.
3. M is a union of 1-dimensional subspaces and rank sets on disjoint intervals which correspond with sub-Grassmannians after quotienting out by the 1-dimensional subspaces.

Cor. X^{sing} is the set of all $x \in X$ such that either $\pi^{-1}|_{R(u,v)}(x) \in R(u, v)$ is singular or $\pi^{-1}|_{R(u,v)}(x)$ is positive dimensional.

Theorem (via Kleiman Transversality).

$$R(u, v)^{sing} = (X_u^{sing} \cap X^v) \cup (X_u \cap X^v_{sing}).$$

Open Problems on Rank Varieties

- Relate rank varieties to Lusztig's canonical bases.

- Give a nice expression for the cohomology class of a rank variety in terms of Schur functions.

- Is there a nice parameterization of an arbitrary Richardson variety similar to the rank sets?