Recent advances in symmetric functions and tableaux combinatorics

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UW: Current Problems Seminar
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Outline

1. Tale of Two Rings: Symmetric Functions and Quasisymmetric Functions
2. Schur functions, LLT polynomials and Macdonald polynomials
3. k-Schur functions
4. Affine dual equivalence graphs

New results based on joint work with Sami Assaf (preprint on arXiv).

Tale of Two Rings

Power Series Ring.: $\mathbb{Z}[[\boldsymbol{X}]]$ over a a finite or countably infinite alphabet $\boldsymbol{X = \{x_1, x_2, \ldots, x_n\}}$ or $\boldsymbol{X = \{x_1, x_2, \ldots\}}$.

Two subrings. of $\mathbb{Z}[[\boldsymbol{X}]]$:

- Symmetric Functions (SYM)
- Quasisymmetric Functions (QSYM)

Ring of Symmetric Functions

Defn. $f(x_1, x_2, \dots) \in \mathbb{Z}[[X]]$ is a *symmetric function* if for all i

$$f(\dots, x_i, x_{i+1}, \dots) = f(\dots, x_{i+1}, x_i, \dots).$$

Example. $x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + \dots$

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Defn. $f(x_1, x_2, \dots) \in \mathbb{Z}[[X]]$ is a *quasisymmetric function* if

$$\text{coef}(f; x_1^{\alpha_1}x_2^{\alpha_2}\dots x_k^{\alpha_k}) = \text{coef}(f; x_a^{\alpha_1}x_b^{\alpha_2}\dots x_c^{\alpha_k})$$

for all $1 < a < b < \dots < c$.

Example. $f(X) = x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + \dots$

Why study SYM and QSYM?

- Symmetric Functions (SYM): Used in representation theory, combinatorics, algebraic geometry over past 200+ years.

- Quasisymmetric Functions (QSYM): 0-Hecke algebra representation theory, Hopf dual of NSYM=non-commutative symmetric functions, Schubert calculus.

Take Math: 583A to find out more about the applications.

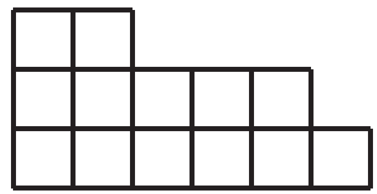
Monomial Basis of SYM

Defn. A *partition* of a number n is a weakly decreasing sequence of positive integers

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$$

such that $n = \sum \lambda_i = |\lambda|$.

Partitions can be visualized by their *Ferrers diagram*

$(6, 5, 2) \longrightarrow$ 

Defn/Thm. The *monomial symmetric functions* indexed by partitions of n

$$m_\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k} + x_2^{\lambda_1} x_1^{\lambda_2} \cdots x_k^{\lambda_k} + \textbf{all other perms of vars}$$

form a basis for SYM_n = homogeneous symmetric functions of degree n.

Fact. $\dim SYM_n = p(n) =$ number of partitions of n.

Monomial Basis of QSYM

Defn. A *composition* of a number n is a sequence of positive integers

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$$

such that $n = \sum \alpha_i = |\alpha|$.

Defn/Thm. The *monomial quasisymmetric functions* indexed by compositions of n

$$M_\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} + x_2^{\alpha_1} x_3^{\alpha_2} \cdots x_{k+1}^{\alpha_k} + \textbf{all other shifts}$$

form a basis for $QSYM_n$ = homogeneous quasisymmetric functions of deg n.

Fact. $\dim QSYM_n$ = number of compositions of $n = 2^{n-1}$.

Monomial Basis of QSYM

Fact. $\dim QSYM_n$ = number of compositions of $n = 2^{n-1}$.

Bijection:

$$
\begin{aligned}
(\alpha_1, \alpha_2, \ldots, \alpha_k) \longrightarrow \{ & \alpha_1, \\
& \alpha_1 + \alpha_2, \\
& \alpha_1 + \alpha_2 + \alpha_3, \\
& \cdots \\
& \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1} \}
\end{aligned}
$$

Counting Partitions

Asymptotic Formula:. (Hardy-Ramanujan)

$$p(n) \approx \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$$

Schur basis for SYM

Let $X = \{x_1, x_2, \ldots, x_m\}$ be a finite alphabet.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ and $\lambda_p = 0$ for $p > k$.

Defn. The following are equivalent definitions for the Schur functions $S_\lambda(X)$:

1. $S_\lambda = \frac{\det(x_i^{\lambda_j + n - j})}{\det(x_i^{n-j})}$ with indices $1 \leq i, j \leq m$.
2. $S_\lambda = \sum x^T$ summed over all *column strict tableaux* T *of shape* λ.

Defn. T is *column strict* if entries strictly increase along columns and weakly increase along rows.

Example. A column strict tableau of shape $(5, 3, 1)$

$$T = \begin{array}{|c|c|c|c|c|} \hline 7 \\ \cline{1-1} 4 & 7 & 7 \\ \cline{1-3} 2 & 2 & 3 & 4 & 8 \\ \hline \end{array} \qquad x^T = x_2^2 x_3 x_4^2 x_7^3 x_8$$

Multiplying Schur Functions

Littlewood-Richardson Coefficients.

$$S_\lambda(X) \cdot S_\mu(X) = \sum_{|\nu|=|\lambda|+|\mu|} c^\nu_{\lambda,\mu} S_\nu(X)$$

$c^\nu_{\lambda,\mu} = \#$ skew tableaux of shape ν/λ such that $x^T = x^\mu$ and the reverse reading word is a lattice word.

Example. If $\nu = (4,3,2)$, $\lambda = (2,1)$, $\lambda = (3,2,1)$ then

2	3		
	1	2	
		1	1

$\textbf{readingword} = 231211$

Fundamental basis for QSYM

Defn. Let $A \subset [p-1] = \{1, 2, \dots, p-1\}$. The fundamental quasisymmetric function

$$F_A(X) = \sum x_{i_1} \cdots x_{i_p}$$

summed over all $1 \leq i_1 \leq \dots \leq i_p$ such that $i_j < i_{j+1}$ whenever $j \notin A$.

Example. $F_{++-+} = x_1 x_1 x_1 x_2 x_2 + x_1 x_2 x_2 x_3 x_3 + x_1 x_2 x_3 x_4 x_5 + \dots$

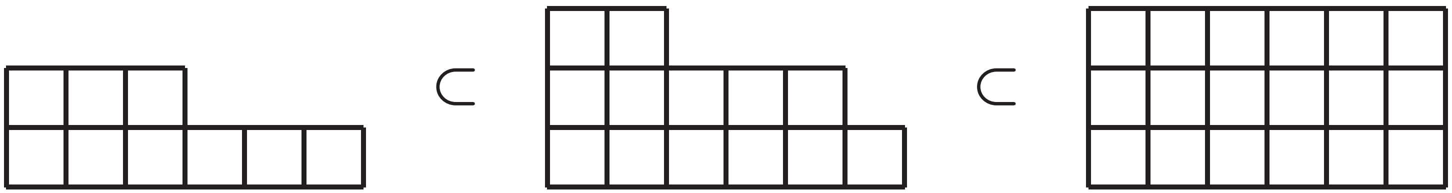
Here $++-+ = \{1, 2, 4\} \subset \{1, 2, 3, 4\}$.

Other bases of QSYM: quasi Schur basis (Haglund-Luoto-Mason-vanWilligenburg), matroid friendly basis (Luoto)

A Poset on Partitions

Defn. A *partial order* or a *poset* is a reflexive, anti-symmetric, and transitive relation on a set.

Defn. *Young's Lattice* on all partitions is the poset defined by the relation $\boldsymbol{\lambda} \subset \boldsymbol{\mu}$ if the Ferrers diagram for $\boldsymbol{\lambda}$ fits inside the Ferrers diagram for $\boldsymbol{\mu}$.



Defns. A standard tableau $\boldsymbol{T}$ of shape $\boldsymbol{\lambda}$ is a saturated chain in Young's lattice from $\emptyset$ to $\boldsymbol{\lambda}$.

Example. $\boldsymbol{T} =$

7				
4	5	9		
1	2	3	6	8

Schur functions

Thm. (Gessel,1984) For all partitions $\boldsymbol{\lambda}$,

$$S_{\lambda}(X) = \sum F_{D(T)}(X)$$

summed over all standard tableaux $\boldsymbol{T}$ of shape $\boldsymbol{\lambda}$.

Defn. The descent set of $\boldsymbol{T}$, denoted $D(T)$, is the set of indices $\boldsymbol{i}$ such that $i+1$ appears northwest of $\boldsymbol{i}$.

Example. Expand $S_{(3,2)}$ in the fundamental basis

$$\begin{array}{|c|c|c} \hline 4 & 5 & \multicolumn{1}{|c}{} \\ \hline 1 & 2 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \begin{array}{|c|c|c} \hline 3 & 5 & \multicolumn{1}{|c}{} \\ \hline 1 & 2 & \multicolumn{1}{c|}{4} \\ \hline \end{array} \quad \begin{array}{|c|c|c} \hline 3 & 4 & \multicolumn{1}{|c}{} \\ \hline 1 & 2 & \multicolumn{1}{c|}{5} \\ \hline \end{array} \quad \begin{array}{|c|c|c} \hline 2 & 5 & \multicolumn{1}{|c}{} \\ \hline 1 & 3 & \multicolumn{1}{c|}{4} \\ \hline \end{array} \quad \begin{array}{|c|c|c} \hline 2 & 4 & \multicolumn{1}{|c}{} \\ \hline 1 & 3 & \multicolumn{1}{c|}{5} \\ \hline \end{array}$$

$$S_{(3,2)}(X) = F_{++-+}(X) + F_{+-+-}(X) + F_{+-++}(X) + F_{-++-}(X) + F_{-+-+}$$

Macdonald Polynomials

Defn/Thm. (Macdonald 1988, Haiman-Haglund-Loehr 2005)

$$\widetilde{H}_{\mu}(X;q,t) = \sum_{w \in S_n} q^{inv_{\mu}(w)} t^{maj_{\mu}(w)} F_{D(w^{-1})}$$

where $D(w)$ is the descent set of w in one-line notation.

Thm. (Haiman 2001) Expanding $\widetilde{H}_{\mu}(X;q,t)$ into Schur functions

$$\widetilde{H}_{\mu}(X;q,t) = \sum_{i} \sum_{j} \sum_{|\lambda|=|\mu|} c_{i,j,\lambda} q^i t^j S_{\lambda},$$

the coefficients $c_{i,j,\lambda}$ are all non-negative integers.

$\Longrightarrow$ Macdonald polynomials are *Schur positive*,

Open I. Find a "nice" combinatorial algorithm to compute $c_{i,j,\lambda}$ showing these are non-negative integers.

Lascoux-Leclerc-Thibon Polynomials

Defn. Let $\bar{\mu} = (\mu^{(1)}, \mu^{(1)}, \ldots, \mu^{(k)})$ be a list of partitions.

$$LLT_{\bar{\mu}}(X;q) = \sum q^{inv_{\mu}(T)} F_{D(w^{-1})}$$

summed over all bijective fillings w of $\bar{\mu}$ where each $\mu^{(i)}$ filled with rows and columns increasing. Each w is recorded as the permutation given by the content reading word of the filling.

Thm. For all $\bar{\mu} = (\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(k)})$

1. $LLT_{\bar{\mu}}(X;q)$ is symmetric. (Lascoux-Leclerc-Thibon)

2. $LLT_{\bar{\mu}}(X;q)$ is Schur positive. (Assaf**)

** Proof still in revision/verification stage.

Lascoux-Leclerc-Thibon Polynomials

Open II. Find a "nice" combinatorial algorithm to compute the expansion coefficients for LLT's to Schurs.

Known. Each $\widetilde{H}_\mu(X;q,t)$ expands as a positive sum of LLT's so Open II implies Open I. (Haiman-Haglund-Loehr)

k-Schur Functions

Defn. (Lam-Lapointe-Morse-Shimozono + Lascoux, 2003-2010)

$$S_\lambda^{(k)}(X;q) = \sum_{S^* \in SST(\mu,k)} q^{\mathrm{spin}(S^*)} F_{D(S^*)}.$$

Nice Properties.: Consider $\{S_\lambda^{(k)}(X;q=1)\}$

1. These are a Schubert basis for the homology ring of the affine Grassmannian of type A_k. (Lam)

2. Structure constants are related to Gromov-Witten invariants of flag manifolds (Lapointe-Morse,Peterson, Lam-Shimozono).

3. There exists a k-Schur analog the Murnaghan-Nakayma rule. (Bandlow-Schilling-Zabrocki)

k-Schur Functions

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Nice Conjectures.: Consider $\{S_\lambda^{(k)}(X;q)\}$ with q an indeterminate

1. Macdonald polynomials expand as a positive sum of k-Schurs. (LLLMS)
2. LLT's expand as a positive sum of k-Schurs (Assaf-Haiman)

Schur Positivity of k-Schurs

Theorem. (Lam-Lapointe-Morse-Shimozono, 2011) At $q = 1$, $\{S_\lambda^{(k)}(X;1)\}$ is Schur positive. In fact, each k-Schur expands as a positive sum of $k+1$-Schurs.

Conjecture. (see Assaf-Billey preprint) Using this definition the k-Schur function $S_\lambda^{(k)}(X;q)$ expands as a positive sum of Schur functions.

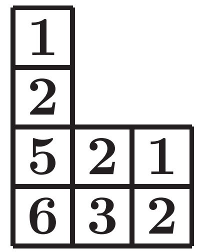
Benefits.

- Significantly reduces number of terms in the expansion so easier to store and manipulate.
- Simplifies computations of products in the homology ring.
- Each k-Schur can be associated to an S_n-module.

n-core poset

Defn. A partition λ is an *n-core* if it has no hooks of length n.

Example. (3,3,1,1) is a 4-core:



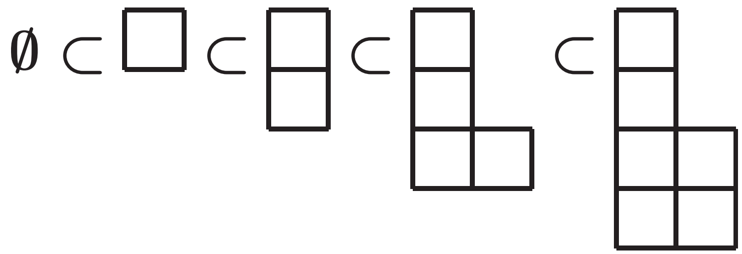
Defn. The *n-core poset* is the partial order on n-cores ordered by containment of Ferrers diagrams.

Thm. (Lascoux) The n-core poset is isomorphic to Bruhat order on $\widetilde{S}_n/S_n$.

n-core poset

Defn. A *strong tableau* is a saturated chain of inclusions in the n-core poset.

Example. $n = 3$:



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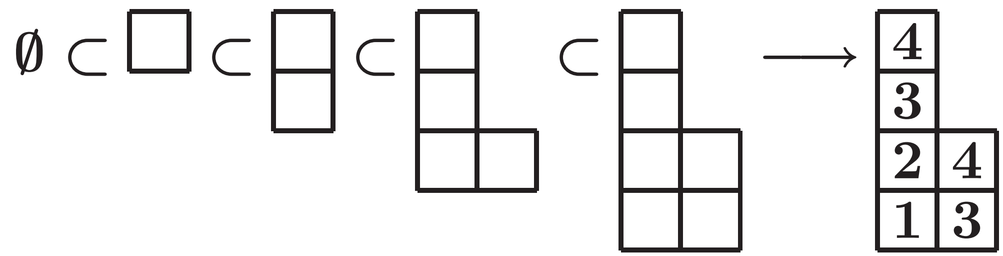

Strong tableaux in the n-core poset

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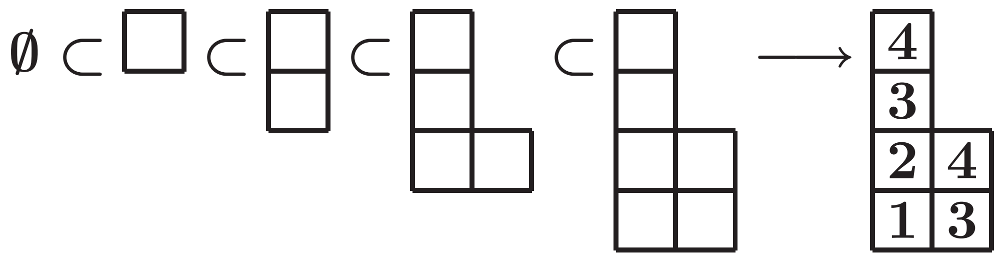
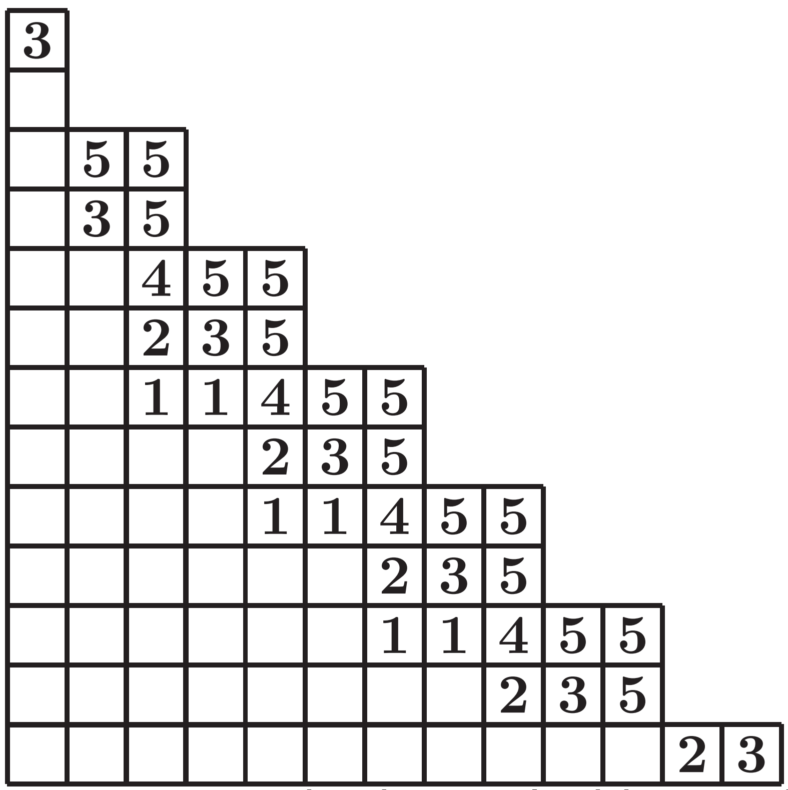


$n = 4$:

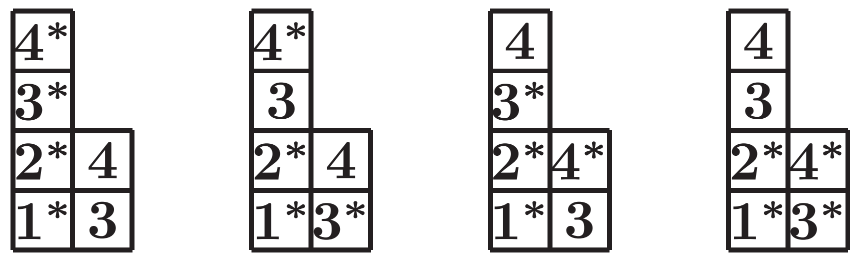


Note, i's live in connected, identical ribbons, shifted in diagonal with offset n.

Starred Strong tableaux

Defn. A *starred strong tableau* $S^* \in \mathbf{SST}(\mu, n)$ is a strong tableau S along with a choice of i-ribbon for each $i \in S$. Place $*$ in SE corner of the "starred" i-ribbon.

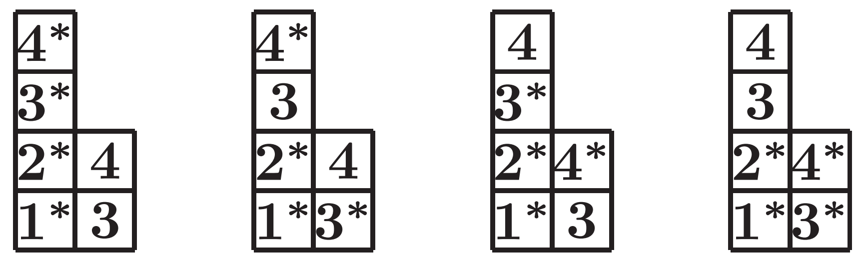
Example. All SST's for $n = 3$ and $\mu = (2, 2, 1, 1)$



Starred Strong tableaux

Defn. A *starred strong tableau* $S^* \in \mathbf{SST}(\lambda, n)$ is a strong tableau S of shape λ along with a choice of i-ribbon for each $i \in S$. Place $*$ in SE corner of the "starred" i-ribbon.

Example. All SST's for $n = 3$ and $\mu = (2, 2, 1, 1)$



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$D(S^*) := \{i \ : \ (i+1)^*$lies NW of i^*in$S^*\}$

k-Schur Functions

Definition.

$$S_\lambda^{(k)}(X;q) = \sum_{S^* \in SST(\lambda,k)} q^{\mathrm{spin}(S^*)} F_{D(S^*)}.$$

SST= Starred Strong Tableaux on the n-core poset. Here $n = k + 1$.

$D(S^*)$ = Descent Set of S^*

$\mathrm{spin}(S^*) = \sum_{i \in S} n(i)[h(i) - 1] + d(i)$

- $n(i)$ = number of connected i-ribbons in S
- $h(i)$ = height of i^*-ribbon
- $d(i)$ = number of i-ribbons NW of i^*-ribbon

Dual Equivalence Graphs

Theorem. (Haiman 1992) The graph on all standard tableaux on partitions of size n with edges given by *dual equivalence* has exactly one connected component for each partition of n.

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Theorem. (Assaf, preprint) The standard dual equivalence graphs can be characterized by 6 axioms.

Affine Dual Equivalence Graphs

Theorem. (Haiman 1992) The graph on all standard tableaux on partitions of size n with edges given by *dual equivalence* has exactly one connected component for each partition of n.

Theorem. (Assaf, preprint) The standard dual equivalence graphs can be characterized by 6 axioms.

Theorem. (Assaf-Billey) There exists an analogous graph structure on starred strong tableaux that satisfy the first 3 of Assaf's axioms and every vertex in a connected component of the graph has the same spin.

Attempted Proof for Schur Positivity

Assaf Machine.

Goal: Given any $G(V) = \sum_{T \in V} F_{D(T)}$, show $G(V)$ is Schur positive.

1. Impose a graph structure on V by finding a family of involutions ϕ_i for $1 < i < n$. Set $E_i = \{(x, \phi(x)) : x \in V, \phi(x) \neq x\}$. Each (V, E_i) is a matching.

2. Show graph satisfies Assaf's axioms including local Schur positivity on every connected component of $(V, E_{i-1} \cup E_i \cup E_{i+1})$.

Update: Computer verification of local Schur positivity for the graphs on k-Schur functions needs to find all possible graph isomorphism types for $n = 2, \ldots, 9$. So far $n = 2, 3, 4, 5$ finished. Case $n = 6$ running on 8 processors. There are 15,041 interval bottoms to check. Many take minutes, some have taken a week.

`http://www.math.washington.edu/~billey/kschur/d-graphs-11-2011.pdf`

Computer Assisted Proofs

Questions.

1. What is the value of a computer proof?
2. What data needs to be stored to convince reader that computer verification is complete?
3. How long is too long?
4. What are the standards for publishing a computer assisted proof?

Big Picture

Geometry	Combinatorics	Rep Theory
Grassmannians	Schur functions	S_n, GL_n irreducible reps
Affine Grassmannians	k-Schur	???? (Li-Chung Chen + Haiman)
Hilbert Schemes of points in plane	Macdonald polynomials	Garsia-Haiman module (n!-theorem)

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Grassmannians	Schur functions	S_n, GL_n irreducible reps
	$\uparrow$	
Affine Grassmannians	k-Schur	???? (Li-Chung Chen + Haiman)
	$\uparrow$	
Hilbert Schemes of points in plane	Macdonald polynomials	Garsia-Haiman module (n!-theorem)

Theorem.(A-B) k-Schur functions are Schur positive.

Conjecture.(Lapointe-Morse) Macdonald polynomials are k-Schur positive for the right k.

Open. Find a direct geometric connection from Hilbert Schemes to Affine Grassmannians.