Boolean Product Polynomials and the Resonance Arrangement

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Outline

Symmetric Polynomials

Schur Positivity via GL_n representation theory and vector bundles

Corollaries and Generalizations (Work in Progress)

Motivation

Symmetric Polynomials

Notation.

- Fix an alphabet of variables $X = \{x_1, x_2, \dots, x_n\}$.
- The symmetric group S_n acts on $\mathbb{C}[x_1, x_2, \dots, x_n]$ by permuting the variables: $w.x_i = x_{w(i)}$.
- A polynomial $f \in \mathbb{C}[x_1, x_2, \dots, x_n]$ is *symmetric* if $w.f = f$ for all $w \in S_n$.
- Let Λ_n denote the *ring of symmetric polynomials* in $\mathbb{C}[x_1, x_2, \dots, x_n]$.

Symmetric Polynomials

Examples. Let $[n] = \{1, 2, \ldots, n\}$.

Elementary: $e_k = \sum_{\substack{A \subset [n] \\ |A| = k}} \prod_{i \in A} x_i$

Homogeneous: $h_k = \sum_{\substack{\text{multisets } A \subset [n] \\ |A| = k}} \prod_{i \in A} x_i$

Power sum: $p_k = \sum_{i=1}^{n} x_i^k$

$$e_2(x_1, x_2, x_3, x_4) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$$

$$p_2(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

$$h_2 = e_2 + p_2.$$

Symmetric Polynomials

Fact. $\Lambda_n = \mathbb{C}[e_1, \ldots, e_n] = \mathbb{C}[h_1, \ldots, h_n] = \mathbb{C}[p_1, \ldots, p_n]$

Symmetric Polynomials

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Question. What other symmetric polynomials are "natural"?

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Question. What other symmetric polynomials are "natural"?

Monomials: $m_\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} +$ other monomials in S_n-orbit

Stanley's chromatic symmetric functions on a graph $G = (V, E)$:

$$X_G(x_1, \ldots, x_n) = \sum_{\substack{c: V \to [n] \\ \text{proper coloring}}} \prod_{v \in V} x_{c(v)}.$$

Observe. These examples are all sums of products.

Schur Polynomials

Defn. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, the *Schur polynomial*

$$s_\lambda(x_1, \ldots, x_n) = \sum_{T \in SSYT(\lambda,n)} \prod_{i \in T} x_i$$

where $SSYT(\lambda, n)$ are the semistandard fillings of λ with positive integers in $[n]$. Semistandard implies strictly increasing in columns and leniently increasing in rows.

Example. For $\lambda = (2, 1)$ and $n = 2$, $SSYT(\lambda, n)$ has two fillings

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

so $s_{(2,1)}(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2$.

Boolean Product Polynomials

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Defn. For $X = \{x_1, \ldots, x_n\}$, define

- *(n,k)-Boolean Product Polynomial*: For $1 \leq k \leq n$,

$$B_{n,k}(X) := \prod_{\substack{A \subseteq [n] \\ |A| = k}} \sum_{i \in A} x_i$$

- *n-th Total Boolean Product Polynomial*:

$$B_n(X) := \prod_{k=1}^{n} B_{n,k}(X) = \prod_{\substack{A \subseteq [n] \\ A \neq \emptyset}} \sum_{i \in A} x_i$$

Example. $B_2 = (x_1)(x_2)(x_1 + x_2) = x_1^2 x_2 + x_1 x_2^2 = s_{(2,1)}(x_1, x_2)$

Boolean Product Polynomials

Examples.

$$B_{3,1} = (x_1)(x_2)(x_3) = e_3(x_1, x_2, x_3) = s_{(1,1,1)}(x_1, x_2, x_3)$$

$$B_{3,2} = (x_1 + x_2)(x_1 + x_3)(x_2 + x_3) = s_{(2,1)}(x_1, x_2, x_3)$$

$$B_{3,3} = (x_1 + x_2 + x_3) = e_1(x_1, x_2, x_3) = s_{(1)}(x_1, x_2, x_3)$$

$$B_3 = s_{(1,1,1)}s_{(2,1)}s_{(1)} = s_{(4,2,1)} + s_{(3,3,1)} + s_{(3,2,2)}.$$

Subset Alphabets

Defn. For $1 \leq k \leq n$, define a new alphabet of linear forms

$$X^{(k)} = \{x_A = \sum_{i \in A} x_i : A \subset [n], |A| = k\}.$$

Then

$$B_{n,k} = \prod_{\substack{A \subseteq [n] \\ |A| = k}} x_A = e_{\binom{n}{k}}(X^{(k)}).$$

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Furthermore, for $1 \leq p \leq \binom{n}{k}$ define the symmetric polynomials

$$e_p(X^{(k)}) = \sum_{\substack{S \subset k\text{-subsets of}[n] \\ |S| = p}} \prod_{A \in S} x_A.$$

Schur Positivity

Theorem. For all $1 \leq k \leq n$ and $1 \leq p \leq \binom{n}{k}$, the expansion

$$e_p(X^{(k)}) = \sum_{\lambda} c_\lambda s_\lambda(x_1, \ldots, x_n)$$

has nonnegative integer coefficients c_λ.

Corollary. Both $B_{n,k}$ and B_n are Schur positive.

Proof Setup

Notation. Fix a complex vector bundle $\mathcal{E}$ of rank n. The *total Chern class* $c(\mathcal{E})$ is the sum of the individual Chern classes

$$c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + \cdots + c_n(\mathcal{E}).$$

Via the Splitting Principle, we have $c(\mathcal{E}) = \prod_{i=1}^{n}(1 + x_i)$ where the x_i for $1 \leq i \leq n$ are the *Chern roots* of $\mathcal{E}$ associated to certain line bundles.

Prior Work

Thm.(Lascoux, 1978) The total Chern class of $\bigwedge^2\mathcal{E}$ and $\mathrm{Sym}^2\mathcal{E}$ is Schur-positive in terms of the Chern roots $x_1, \ldots, x_n$ of $\mathcal{E}$. Specifically, there exist integers $d_{\lambda,\mu} \geq 0$ for $\mu \subseteq \lambda$ such that

$$c(\textstyle\bigwedge^2\mathcal{E}) = \prod_{1\leq i<j\leq n}(1+x_i+x_j) = 2^{-\binom{n}{2}} \sum_{\mu\subseteq\delta_{n-1}} d_{\gamma_n,\mu}2^{|\mu|}s_\mu(X),$$

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Here $\gamma_n = (n-1, \ldots, 1, 0)$ and $\delta_n = (n, \ldots, 2, 1)$.

Binomial Determinants

Lascoux showed that for $\mu = (\mu_1, \ldots, \mu_n) \subseteq \lambda = (\lambda_1, \ldots, \lambda_n)$,

$$d_{\lambda,\mu} = \det\left(\binom{\lambda_i + n - i}{\mu_j + n - j}\right)_{1 \le i,j \le n} \ge 0.$$

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Thm.(Gessel-Viennot 1985) $d_{\lambda,\mu}$ counts the number of nonintersecting lattice paths from heights $\lambda + \delta_n$ along the y-axis to main diagonal points $\mu + \delta_n$ using east or south steps.

This highly influential theorem was inspired by Lascoux's theorem!

Vector Bundle Approach to Schur Positivity

Notation. Fix a complex vector bundle $\mathcal{E}$ of rank n over a smooth projective variety V. The *total Chern class*

$$c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + \cdots + c_n(\mathcal{E}) = \prod_{i=1}^{n}(1 + x_i)$$

where the x_i for $1 \le i \le n$ are the *Chern roots* of $\mathcal{E}$.

Construct another vector bundle $\mathbb{S}^{\lambda}(\mathcal{E})$ over V by applying the Schur functor from GL_n-representation theory on each fiber.

Thm.(Fulton) The Chern roots of $\mathbb{S}^{\lambda}(\mathcal{E})$ are indexed by semistandard tableaux:

$$\{x_T = \sum_{i \in T} x_i \text{ for } T \in SSYT(\lambda, n)\}.$$

Vector Bundle Approach to Schur Positivity

Notation. For any partitions λ and μ, consider the Schur function s_μ on the alphabet of Chern roots on $\mathbb{S}^\lambda(\mathcal{E})$, denoted $s_\mu(\mathbb{S}^\lambda(\mathcal{E}))$.

Example. Take $n = 3$, $\lambda = (1,1)$, then the Chern roots of $\mathbb{S}^\lambda(\mathcal{E})$ are the variables in the alphabet

$$X^{(2)} = \{x_1 + x_2, x_1 + x_3, x_2 + x_3\}.$$

For $\mu = (2,1)$, expand

$$\begin{aligned} s_\mu(\mathbb{S}^\lambda(\mathcal{E})) =& s_{(2,1)}(x_1 + x_2, x_1 + x_3, x_2 + x_3) \\ =& 2s_{(3)}(x_1, x_2, x_3) + 5s_{(2,1)}(x_1, x_2, x_3) + 4s_{(1,1,1)}(x_1, x_2, x_3). \end{aligned}$$

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Note. This operation is not plethysm $s_\mu[s_\lambda]$.

Key Ingredient

Thm.(Pragacz 1996) Let λ be a partition, and let

- $\mathcal{E}_1, \ldots, \mathcal{E}_k$ be vector bundles,
- $Y_1, \ldots, Y_k$ be the alphabets consisting of their Chern roots,
- $\mu^{(1)}, \ldots, \mu^{(k)}$ be partitions.

Then, there exists nonnegative integers $c_{(\nu^{(1)},\ldots,\nu^{(k)})}$ such that

$$s_\lambda(\mathbb{S}^{\mu^{(1)}}(\mathcal{E}_1)\otimes\cdots\otimes\mathbb{S}^{\mu^{(k)}}(\mathcal{E}_k)) = \sum_{\nu_1,\ldots,\nu_k} c_{(\nu^{(1)},\ldots,\nu^{(k)})} s_{\nu_1}(Y_1)\cdots s_{\nu_k}(Y_k).$$

Pragacz's proof uses work of Fulton-Lazarsfeld on numerical positivity for ample vector bundles. The Hard Lefschetz theorem is a key component.

Corollaries

Cor. For any partitions λ, μ, $s_\mu(\mathbb{S}^\lambda(\mathcal{E}))$ is Schur positive.

Cor. The expansion of $e_p(X^{(k)})$ is Schur positive since the Chern roots of $\mathbb{S}^{1^k}(\mathcal{E}) = X^{(k)}$ and $e_p = s_{1^p}$.

Cor. The analogue of Lascoux's theorem holds for all Schur functors $\mathbb{S}^\lambda(\mathcal{E})$), e.g.

$$c(\bigwedge^k \mathcal{E}) = \prod_{A \subseteq [n], |A| = k} \left(1 + \sum_{i \in A} x_i\right) = \sum_{p \geq 0} e_p(X^{(k)}).$$

Question. What are the Schur expansions?

Boolean Product Expansions for $B_{n,n-1}$

Thm. For $n \geq 2$,

$$B_{n,n-1} = \prod_{i=1}^{n} (x_1 + x_2 + \ldots + x_n - x_i) = \sum_{\lambda \vdash n} a_\lambda s_\lambda(X)$$

where a_λ is the number of $T \in SYT(\lambda)$ with smallest ascent given by an even number.

More generally, consider

$$B_{n,n-1}(X;q) := \prod_{i=1}^{n} (h_1(X) + qx_i)$$
$$= \sum_{j=0}^{n} q^j e_j(X) h_{(1^{n-j})}(X).$$

New: Brendon Rhoades has a new graded S_n-module with $B_{n,n-1}(X;q)$ as the graded Frobenius characteristic.

Motivation

Defn. Consider the real variety $V(B_n)$. Since each factor of B_n is linear, this variety is a hyperplane arrangement called the *Resonance Arrangement* or *All-Subsets Arrangement* $\mathcal{H}_n$. Each hyperplane is orthogonal to a nonzero 0-1-vector in $\mathbb{R}^n$.

Open. Find the characteristic polynomial for $\mathcal{H}_n$. Use it to count the number of regions and number of bounded regions by Zaslavsky's Theorem.

Thm.(Cavalieri-Johnson-Markwig, 2011) The regions of $\mathcal{H}_n$ are the domains of polynomiality of double Hurwitz numbers.

Motivation

Further Connections. See Lou Billera's talk slides
"On the real linear algebra of vectors of zeros and ones"

1. The chambers of the Resonance Arrangement $\mathcal{H}_n$ can be labeled by maximal unbalanced collections of 0-1 vectors. See Billera-Moore-Moraites-Wang-Williams, 2012.

2. Minimal balanced collections determine the minimum linear description of cooperative games possessing a nonempty core in Lloyd Shapley's economic game theory work from 1967. Finding a good formula for enumerating them is still open.

3. Hadamard's maximal determinant problem from 1893 can be rephrased in terms of finding maximal absolute value determinants of 0-1 matrices.

Many Thanks!