Cyclotomic Generating Functions

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Based on joint work with:
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arXiv:1809.07386 and more coming soon!

BIRS: Asymptotic Algebraic Combinatorics
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Outline

Standard Young Tableaux

Defn. A *standard Young tableau* of shape λ is a bijective filling of λ such that every row is increasing from left to right and every column is increasing from top to bottom.

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 6 & 7 & 9 \\ \hline 2 & 5 & 8 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array}$$

Important Fact. The standard Young tableaux of shape λ, denoted $\mathrm{SYT}(\lambda)$, index a basis of the irreducible S_n representation indexed by λ.

Counting Standard Young Tableaux

Hook Length Formula.(Frame-Robinson-Thrall, 1954)
If λ is a partition of n, then

$$\#SYT(\lambda) = \frac{n!}{\prod_{c \in \lambda} h_c}$$

where h_c is the *hook length* of the cell c, i.e. the number of cells directly to the right of c or below c, including c.

Example. Filling cells of $\lambda = (5, 3, 1) \vdash 9$ by hook lengths:

7	5	4	2	1
4	2	1		
1				

So, $\#SYT(5, 3, 1) = \frac{9!}{7 \cdot 5 \cdot 4 \cdot 2 \cdot 4 \cdot 2} = 162.$

Remark. Notable other proofs by Greene-Nijenhuis-Wilf '79 (probabilistic), Eriksson '93 (bijective), Krattenthaler '95 (bijective), Novelli -Pak -Stoyanovskii'97 (bijective), Bandlow'08,

q-Counting Standard Young Tableaux

Def. The *descent set* of a standard Young tableau T, denoted $D(T)$, is the set of positive integers i such that $i+1$ lies in a row strictly below the cell containing i in T.

The *major index* of T is the sum of its descents:

$$\mathrm{maj}(T) = \sum_{i \in D(T)} i.$$

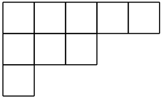
Example. The descent set of T is $D(T) = \{1, 3, 4, 7\}$ so $\mathrm{maj}(T) = 15$ for $T = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 6 & 7 & 9 \\ \hline 2 & 4 & 8 \\ \cline{1-3} 5 \\ \cline{1-1} \end{array}$.

Def. The *major index generating function* for λ is

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) := \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)}$$

q-Counting Standard Young Tableaux

Example. $\lambda = (5, 3, 1)$



$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) := \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} =$

$$q^{23} + 2q^{22} + 4q^{21} + 5q^{20} + 8q^{19} + 10q^{18} + 13q^{17} + 14q^{16} + 16q^{15}$$

$$+16q^{14} + 16q^{13} + 14q^{12} + 13q^{11} + 10q^{10} + 8q^{9} + 5q^{8} + 4q^{7} + 2q^{6} + q^{5}$$

Note, at $q = 1$, we get back 162.

Computation of $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$

Thm.(Stanley's q-analog of the Hook Length Formula for $\lambda \vdash n$)

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \frac{q^{b(\lambda)}[n]_q!}{\prod_{c\in\lambda}[h_c]_q}$$

where

- $b(\lambda) := \sum (i-1)\lambda_i$
- h_c is the hook length of the cell c
- $[n]_q := 1 + q + \cdots + q^{n-1} = \frac{q^n - 1}{q-1}$
- $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$

Corollaries of Stanley's formula

Thm.(Stanley's q-analog of the Hook Length Formula for $\lambda \vdash n$)

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \frac{q^{b(\lambda)}[n]_q!}{\prod_{c\in\lambda}[h_c]_q}$$

Corollaries.

1. $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \mathrm{SYT}(\lambda')^{\mathrm{maj}}(q)$.
2. The coefficients of $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ are symmetric.
3. There is a unique min-maj and max-maj tableau of shape λ.

Motivation for q-Counting Standard Young Tableaux

Thm.(Lusztig-Stanley 1979) Given a partition $\lambda \vdash n$, say

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) := \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} = \sum_{k \geq 0} b_{\lambda,k} q^k.$$

Then $b_{\lambda,k} := \#\{T \in \mathrm{SYT}(\lambda) : \mathrm{maj}(T) = k\}$ is the number of times the irreducible S_n module indexed by λ appears in the decomposition of the coinvariant algebra $\mathbb{Z}[x_1, x_2, \ldots, x_n]/I_+$ in the homogeneous component of degree k.

Key Questions for $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$

Recall $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} = \sum b_{\lambda,k} q^k$.

Existence Question. For which λ, k does $b_{\lambda,k} = 0$?

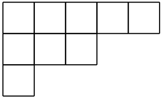
Distribution Question. What patterns do the coefficients in the list $(b_{\lambda,0}, b_{\lambda,1}, \ldots)$ exhibit?

Unimodality Question. For which λ, are the coefficients of $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ *unimodal*, meaning

$$b_{\lambda,0} \leq b_{\lambda,1} \leq \ldots \leq b_{\lambda,m} \geq b_{\lambda,m+1} \geq \ldots ?$$

q-Counting Standard Young Tableaux

Example. $\lambda = (5, 3, 1)$



$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) := \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} = \sum b_{\lambda,k} q^k =$

$$q^{23} + 2q^{22} + 4q^{21} + 5q^{20} + 8q^{19} + 10q^{18} + 13q^{17} + 14q^{16} + 16q^{15}$$

$$+16q^{14} + 16q^{13} + 14q^{12} + 13q^{11} + 10q^{10} + 8q^{9} + 5q^{8} + 4q^{7} + 2q^{6} + q^{5}$$

Notation: (00000 1 2 4 5 8 10 13 14 16 16 16 14 13 10 8 5 4 2 1)

q-Counting Standard Young Tableaux

Examples. $(2,2) \vdash 4$: (0 0 1 0 1)

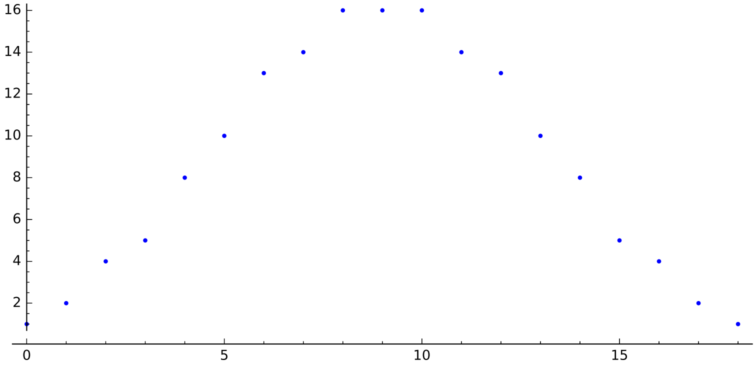
$(5,3,1)$: (00000 1 2 4 5 8 10 13 14 16 16 16 14 13 10 8 5 4 2 1)

$(6,4) \vdash 10$: (0 0 0 0 1 1 2 2 4 4 6 6 8 7 8 7 8 6 6 4 4 2 2 1 1)

$(6,6) \vdash 12$: (0 0 0 0 0 0 1 0 1 1 2 2 4 3 5 5 7 6 9 7 9 8 9 7 9 6 7 5 5 3 4 2 2 1 1 0 1)

$(11,5,3,1) \vdash 20$: (1 3 8 16 32 57 99 160 254 386 576 832 1184 1645 2255 3031 4027 5265 6811 8689 10979 13706 16959 20758 25200 30296 36143 42734 50163 58399 67523 77470 88305 99925 112370 125492 139307 153624 168431 183493 198778 214017 229161 243913 258222 271780 284542 296200 306733 315853 323571 329629 334085 336727 337662 336727 334085 329629 323571 315853 306733 296200 284542 271780 258222 243913 229161 214017 198778 183493 168431 153624 139307 125492 112370 99925 88305 77470 67523 58399 50163 42734 36143 30296 25200 20758 16959 13706 10979 8689 6811 5265 4027

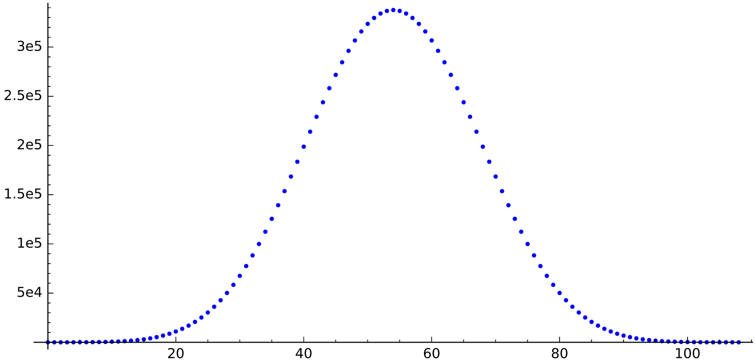
Visualizing Major Index Generating Functions



Visualizing the coefficients of $\mathrm{SYT}(5,3,1)^{\mathrm{maj}}(q)$:

$$(1,2,4,5,8,10,13,14,16,16,16,14,13,10,8,5,4,2,1)$$

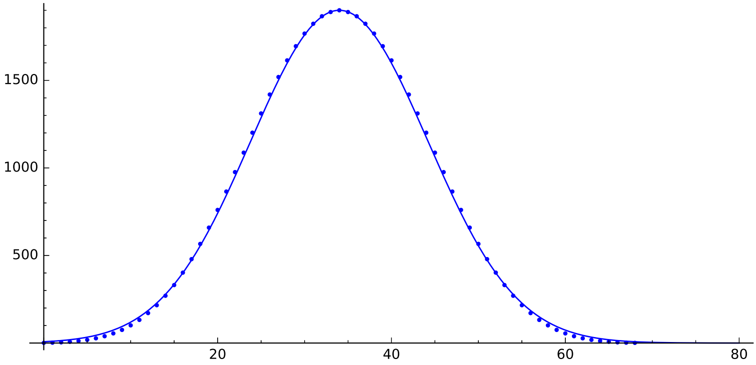
Visualizing Major Index Generating Functions



Visualizing the coefficients of $\mathrm{SYT}(11,5,3,1)^{\mathrm{maj}}(q)$.

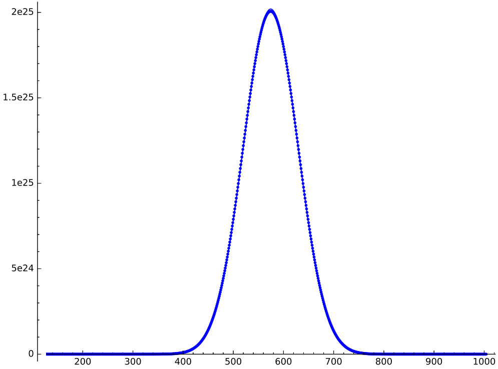
Question. What type of curve is that?

Visualizing Major Index Generating Functions



Visualizing the coefficients of $\mathrm{SYT}(10,6,1)^{\mathrm{maj}}(q)$ along with the Normal distribution with $\mu = 34$ and $\sigma^2 = 98$.

Visualizing Major Index Generating Functions



Visualizing the coefficients of $\mathrm{SYT}(8, 8, 7, 6, 5, 5, 5, 2, 2)^{\mathrm{maj}}(q)$

Cyclotomic Polynomials

Def. The irreducible factors of $q^n - 1$ over the integers are called *cyclotomic polynomials*. There is one for each positive integer d, given by

$$\Phi_d(q) = \prod_{d|n} (q^d - 1)^{\mu(n/d)} = \frac{q^n - 1}{\prod_{c|n, c<n} \Phi_c(q)},$$

where $\mu(n/d)$ is the Möbius function given by

$$\mu(k) = \begin{cases} 1 & k = 1 \\ 0 & k > 1 \text{ has repeated prime factors} \\ (-1)^\ell & k > 1 \text{ is product of } \ell \text{ distinct prime factors.} \end{cases}$$

Fact. Each q-integer $[n]_q = (q^n - 1)/(q - 1)$ factors into a product of distinct cyclotomic polynomials

$$[n]_q = 1 + q + \cdots + q^{n-1} = \prod_{1<d|n} \Phi_d(q).$$

Cyclotomic Polynomials

Examples.

$$
\begin{aligned}
\Phi_1(q) &= q - 1 \\
\Phi_2(q) &= q + 1 \\
\Phi_3(q) &= q^2 + q^1 + 1 \\
\Phi_4(q) &= q^2 + 1 \\
\Phi_5(q) &= q^4 + q^3 + q^2 + q^1 + 1 \\
\Phi_6(q) &= q^2 - q^1 + 1 \\
\Phi_7(q) &= q^6 + q^5 + q^4 + q^3 + q^2 + q^1 + 1 \\
\Phi_8(q) &= q^4 + 1 \\
\Phi_9(q) &= q^6 + q^3 + 1 \\
\Phi_{10}(q) &= q^4 - q^3 + q^2 - q^1 + 1
\end{aligned}
$$

Cyclotomic Polynomials

Bigger Example.

$$\Phi_{105}(q) = q^{48} + q^{47} + q^{46} + q^{43} - q^{42} - 2q^{41} - q^{40} - q^{39} + q^{36} + q^{35} + q^{34} + q^{33} + q^{32} + q^{31} - q^{28} - q^{26} - q^{24} - q^{22} - q^{20} + q^{17} + q^{16} + q^{15} + q^{14} + q^{13} + q^{12} - q^{9} + -1q^{8} - 2q^{7} - q^{6} - q^{5} + q^{2} + q^{1} + 1$$

"Fast" Computation of $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$

Thm.(Stanley's q-analog of the Hook Length Formula for $\lambda \vdash n$)

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \frac{q^{b(\lambda)}[n]_q!}{\prod_{c \in \lambda}[h_c]_q}$$

Trick for conjectures. Cancel all of the cyclotomic factors of the denominator from the numerator, and then expand the remaining product.

Existence Question

Recall $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} = \sum b_{\lambda,k} q^k$.

Existence Question. For which λ, k does $b_{\lambda,k} = 0$?

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Existence Question. For which λ, k does $b_{\lambda,k} = 0$?

Cor of Stanley's formula. For every $\lambda \vdash n \geq 1$ there is a unique tableau with minimal major index $b(\lambda)$ and a unique tableau with maximal major index $\binom{n}{2} - b(\lambda')$. These two agree for shapes consisting of one row or one column, and otherwise they are distinct.

Patterns on Tableaux

Example. The min-maj and max-maj tableaux for $(6,4,3,3,1)$.

$$
\begin{array}{cccccc}
1 & 3 & 4 & 11 & \boxed{16} & \boxed{17} \\
2 & 6 & 7 & \boxed{15} & & \\
5 & 9 & 10 & & & \\
8 & \boxed{13} & \boxed{14} & & & \\
\boxed{12} & & & & &
\end{array}
\qquad\qquad
\begin{array}{cccccc}
1 & 2 & 3 & 5 & 9 & \boxed{13} \\
4 & 6 & 10 & \boxed{14} & & \\
7 & 11 & \boxed{15} & & & \\
8 & 12 & \boxed{16} & & & \\
\boxed{17} & & & & &
\end{array}
$$

$$b(\lambda) = \sum (i-1)\lambda_i = 23 \qquad\qquad \binom{17}{2} - b(\lambda') = 109$$

Existence Question

Recall $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} = \sum b_{\lambda,k} q^k$.

Existence Question. For which λ, k does $b_{\lambda,k} = 0$?

Cor of Stanley's formula. The coefficient of $q^{b(\lambda)+1}$ in $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = 0$ if and only if λ is a rectangle.
If λ is a rectangle with more than one row and column, then coefficient of $q^{b(\lambda)+2}$ is 1.

Question. Are there other internal zeros?

Classifying All Nonzero Fake Degrees

Thm.(Billey-Konvalinka-Swanson, 2018)
For any partition λ which is not a rectangle,

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) := \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)}$$

has no internal zeros. If λ is a rectangle with at least two rows and columns, $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ has exactly two internal zeros, one at degree $b(\lambda) + 1$ and the other at degree $maxmaj(\lambda) - 1$.

Classifying All Nonzero Fake Degrees

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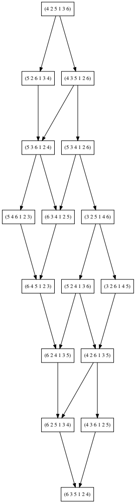
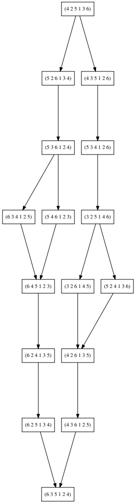
Proof Outline. We identify block and rotation rules on tableaux giving rise to two posets on $\mathrm{SYT}(\lambda)$– exceptional cases for rectangles which is ranked according to maj.

Strong and Weak Poset on SYT(3, 2, 1)



Strong



Weak

Classifying All Nonzero Fake Degrees

Cor. The irreducible S_n-module indexed by λ appears in the decomposition of the degree k component of the coinvariant algebra if and only if $b_{\lambda,k} > 0$ as characterized above.

Similar results hold for all Shepard-Todd groups $G(m, d, n)$.

See arXiv:1809.07386 for more details.

Converting q-Enumeration to Discrete Probability

Distribution Question. What is the limiting distribution(s) for the coefficients in $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$?

From Combinatorics to Probability.
If $f(q) = a_0 + a_1 q + a_2 q^2 + \cdots + a_n q^n$ where a_i are nonnegative integers, then construct the random variable X_f with discrete probability distribution

$$\mathbb{P}(X_f = k) = \frac{a_k}{\sum_j a_j} = \frac{a_k}{f(1)}.$$

If f is part of a family of q-analog of an integer sequence, we can study the limiting distributions.

Converting q-Enumeration to Discrete Probability

Example. For $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \sum b_{\lambda,k} q^k$, define the integer random variable $X_\lambda[\mathrm{maj}]$ with discrete probability distribution

$$\mathbb{P}(X_\lambda[\mathrm{maj}] = k) = \frac{b_{\lambda,k}}{|\mathrm{SYT}(\lambda)|}.$$

We claim the distribution of $X_\lambda[\mathrm{maj}]$ "usually" is approximately normal for most shapes λ. Let's make that precise!

Standardization

Thm.(Adin-Roichman, 2001)
For any partition λ, the mean and variance of $X_\lambda[\mathrm{maj}]$ are

$$\mu_\lambda = \frac{\binom{|\lambda|}{2} - b(\lambda') + b(\lambda)}{2} = b(\lambda) + \frac{1}{2}\left[\sum_{j=1}^{|\lambda|} j - \sum_{c\in\lambda} h_c\right],$$

and

$$\sigma_\lambda^2 = \frac{1}{12}\left[\sum_{j=1}^{|\lambda|} j^2 - \sum_{c\in\lambda} h_c^2\right].$$

Def. The *standardization* of $X_\lambda[\mathrm{maj}]$ is

$$X_\lambda^*[\mathrm{maj}] = \frac{X_\lambda[\mathrm{maj}] - \mu_\lambda}{\sigma_\lambda}.$$

So $X_\lambda^*[\mathrm{maj}]$ has mean 0 and variance 1 for any λ.

Asymptotic Normality

Def. Let $X_1, X_2, \ldots$ be a sequence of real-valued random variables with standardized cumulative distribution functions $F_1(t), F_2(t), \ldots$. The sequence is *asymptotically normal* if

$$\forall\, t \in \mathbb{R}, \qquad \lim_{n\to\infty} F_n(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2} = \mathbb{P}(N < t)$$

where N is a Normal random variable with mean 0 and variance 1.

Asymptotic Normality

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where N is a Normal random variable with mean 0 and variance 1.

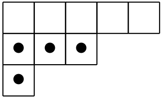
Question. In what way can a sequence of partitions approach infinity?

The Aft Statistic

Def. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$, let

$$\operatorname{aft}(\lambda) := n - \max\{\lambda_1, k\}.$$

Example. $\lambda = (5, 3, 1)$ then $\operatorname{aft}(\lambda) = 4$.



Look it up: Aft is now on FindStat as St001214

Distribution Question: From Combinatorics to Probability

Thm. (Billey-Konvalinka-Swanson, 2019)

Suppose $\lambda^{(1)}, \lambda^{(2)}, \ldots$ is a sequence of partitions, and let $X_N := X_{\lambda^{(N)}}[\mathrm{maj}]$ be the corresponding random variables for the maj statistic. Then, the sequence $X_1, X_2, \ldots$ is asymptotically normal if and only if $\mathrm{aft}(\lambda^{(N)}) \to \infty$ as $N \to \infty$.

Distribution Question: From Combinatorics to Probability

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Suppose $\lambda^{(1)}, \lambda^{(2)}, \dots$ is a sequence of partitions, and let $X_N := X_{\lambda^{(N)}}[\mathrm{maj}]$ be the corresponding random variables for the maj statistic. Then, the sequence $X_1, X_2, \dots$ is asymptotically normal if and only if $\mathrm{aft}(\lambda^{(N)}) \to \infty$ as $N \to \infty$.

Question. What happens if $\mathrm{aft}(\lambda^{(N)})$ does not go to infinity as $N \to \infty$?

Distribution Question: From Combinatorics to Probability

Thm.(Billey-Konvalinka-Swanson, 2019)
Let $\lambda^{(1)}, \lambda^{(2)}, \ldots$ be a sequence of partitions. Then $(X_{\lambda^{(N)}}[\mathrm{maj}]^*)$ converges in distribution if and only if

(i) $\mathrm{aft}(\lambda^{(N)}) \to \infty$; or

(ii) $|\lambda^{(N)}| \to \infty$ and $\mathrm{aft}(\lambda^{(N)})$ is eventually constant; or

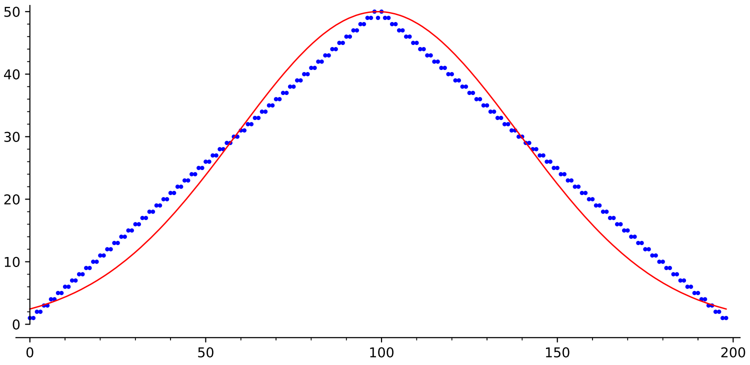
(iii) the distribution of $X^*_{\lambda^{(N)}}[\mathrm{maj}]$ is eventually constant.

The limit law is $\mathcal{N}(0,1)$ in case (i), Σ^*_M in case (ii), and discrete in case (iii).

Here Σ_M denotes the sum of M independent identically distributed uniform $[0,1]$ random variables, known as the Irwin–Hall distribution or the *uniform sum distribution*.

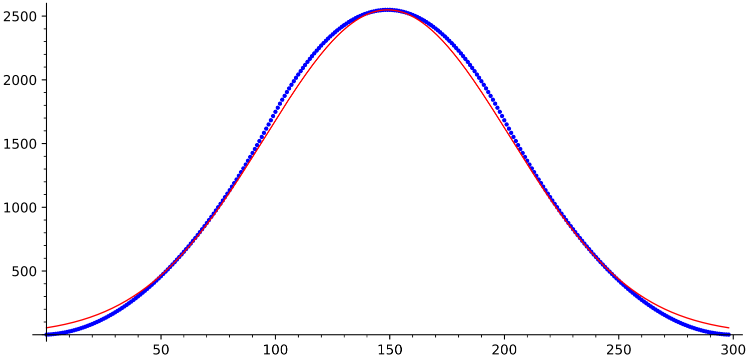
Distribution Question: From Combinatorics to Probability

Example. $\lambda = (100, 2)$ looks like the distribution of the sum of two independent uniform random variables on $[0, 1]$:



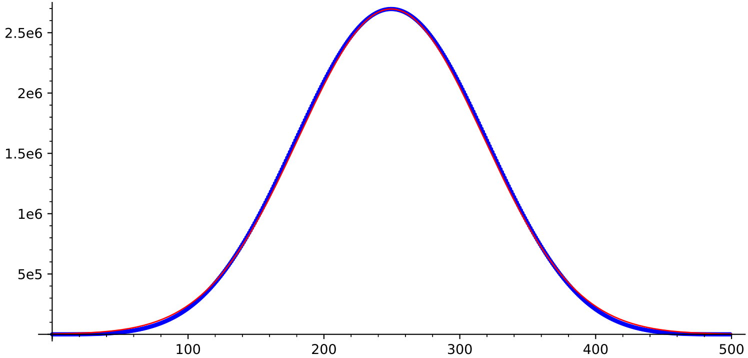
Distribution Question: From Combinatorics to Probability

Example. $\lambda = (100, 2, 1)$ looks like the distribution of the sum of three independent uniform random variables on $[0, 1]$:



Distribution Question: From Combinatorics to Probability

Example. $\lambda = (100, 3, 2)$ looks like the normal distribution, but not quite!



Proof ideas: Characterize the Moments and Cumulants

Definitions.

- For $d \in \mathbb{Z}_{\geq 0}$, the *dth moment*

$$\mu_d := \mathbb{E}[X^d]$$

- The *moment-generating function* of X is

$$M_X(t) := \mathbb{E}[e^{tX}] = \sum_{d=0}^{\infty} \mu_d \frac{t^d}{d!},$$

- The *cumulants* $\kappa_1, \kappa_2, \ldots$ of X are defined to be the coefficients of the exponential generating function

$$K_X(t) := \sum_{d=1}^{\infty} \kappa_d \frac{t^d}{d!} := \log M_X(t) = \log \mathbb{E}[e^{tX}].$$

Nice Properties of Cumulants

1. *(Familiar Values)* The first two cumulants are $\kappa_1 = \mu$, and $\kappa_2 = \sigma^2$.
2. *(Shift Invariance)* The second and higher cumulants of X agree with those for $X - c$ for any $c \in \mathbb{R}$.
3. *(Homogeneity)* The dth cumulant of cX is $c^d \kappa_d$ for $c \in \mathbb{R}$.
4. *(Additivity)* The cumulants of the sum of *independent* random variables are the sums of the cumulants.
5. *(Polynomial Equivalence)* The cumulants and moments are determined by polynomials in the other sequence.

Examples of Cumulants and Moments

Example. Let $X = \mathcal{N}(\mu, \sigma^2)$ be the normal random variable with mean μ and variance σ^2. Then the cumulants are

$$\kappa_d = \begin{cases} \mu & d = 1, \\ \sigma^2 & d = 2, \\ 0 & d \geq 3. \end{cases}$$

and for $d > 1$,

$$\mu_d = \begin{cases} 0 & \text{if } d \text{ is odd,} \\ \sigma^d (d-1)!! & \text{if } d \text{ is even.} \end{cases}$$

.

Example. For a Poisson random variable X with mean μ, the cumulants are all $\kappa_d = \mu$, while the moments are $\mu_d = \sum_{i=1}^{d} \mu^i S_{i,d}$.

Cumulants for Major Index Generating Functions

Thm.(Billey-Konvalinka-Swanson, 2019)
Let $\lambda \vdash n$ and $d \in \mathbb{Z}_{>1}$. If κ_d^λ is the dth cumulant of $X_\lambda[\mathrm{maj}]$, then

$$\kappa_d^\lambda = \frac{B_d}{d}\left[\sum_{j=1}^{n} j^d - \sum_{c \in \lambda} h_c^d\right] \tag{1}$$

where $B_0, B_1, B_2, \ldots = 1, \frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, \ldots$ are the Bernoulli numbers (OEIS A164555 / OEIS A027642).

Remark. We use this theorem to prove that as aft approaches infinity the standardized cumulants for $d \geq 3$ all go to 0 proving the Asymptotic Normality Theorem.

Cumulants for Major Index Generating Functions

Thm.(Billey-Konvalinka-Swanson, 2019)
Let $\lambda \vdash n$ and $d \in \mathbb{Z}_{>1}$. If κ_d^λ is the dth cumulant of $X_\lambda[\mathrm{maj}]$, then

$$\kappa_d^\lambda = \frac{B_d}{d}\left[\sum_{j=1}^{n} j^d - \sum_{c\in\lambda} h_c^d\right] \tag{1}$$

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Remark. We use this theorem to prove that as aft approaches infinity the standardized cumulants for $d \geq 3$ all go to 0 proving the Asymptotic Normality Theorem.

Remark. Note, κ_2^λ is exactly the Adin-Roichman variance formula.

Cumulants of certain q-analogs

Thm.(Chen–Wang–Wang-2008 and Hwang–Zacharovas-2015)
Suppose $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_m\}$ are multisets of positive integers such that

$$f(q) = \frac{\prod_{j=1}^m [a_j]_q}{\prod_{j=1}^m [b_j]_q} = \sum c_k q^k \in \mathbb{Z}_{\geq 0}[q]$$

.

Let X be a discrete random variable with $\mathbb{P}(X = k) = c_k / f(1)$.
Then the dth cumulant of X is

$$\kappa_d = \frac{B_d}{d} \sum_{j=1}^m (a_j^d - b_j^d)$$

where B_d is the dth Bernoulli number (with $B_1 = \frac{1}{2}$).

Example. This theorem applies to

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) := \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} = \frac{q^{b(\lambda)}[n]_q!}{\prod_{c \in \lambda}[h_c]_q}$$

Corollaries of the Distribution Theorem

1. Asymptotic normality also holds for block diagonal skew shapes with aft going to infinity.

2. New proof of asymptotic normality of $[n]_q! = \sum_{w \in S_n} q^{\mathrm{maj}(w)} = \sum_{w \in S_n} q^{\mathrm{inv}(w)}$ due to Feller (1944).

3. New proof of asymptotic normality of q-multinomial coefficients due to Diaconis (1988), Canfield-Jansen-Zeilberger (2011).

4. New proof of asymptotic normality of q-Catalan numbers due to Chen-Wang-Wang(2008).

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Question. Using Morales-Pak-Panova q-hook length formula, can we prove an asymptotic normality for most skew shapes?

Cyclotomic Generating Functions

Def. A polynomial $f(q)$ with nonnegative integer coefficients is a *cyclotomic generating function* provided it satisfies one of the following equivalent conditions:

(i) (Rational form.) There are multisets $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_m\}$ of positive integers and $\alpha, \beta \in \mathbb{Z}_{\geq 0}$ such that

$$f(q) = \alpha q^{\beta} \cdot \prod_{j=1}^{m} \frac{[a_j]_q}{[b_j]_q} = \alpha q^{\beta} \cdot \prod_{j=1}^{m} \frac{1 - q^{a_j}}{1 - q^{b_j}}. \tag{2}$$

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(iii) (Complex form.) The complex roots of $f(q)$ are each either a root of unity or zero.

Cyclotomic Generating Functions

More examples of cyclotomic generating functions:.

1. Stanley: $s_\lambda(1, q, q^2, \ldots, q^m)$.

2. Björner-Wachs: q-hook length formula for forests.

3. Macaulay: Hilbert series of polynomial quotients $k[x_1, \ldots, x_n]/(\theta_1, \theta_2, \ldots, \theta_n)$ where $deg(x_i) = b_i$, $deg(\theta_i) = a_i$, and $(\theta_1, \theta_2, \ldots, \theta_n)$ is a homogeneous system of parameters $k[x_1, \ldots, x_n]/$.

4. Chevalley: Length generating function restricted to minimum length coset representatives of a finite reflection group modulo a parabolic subgroup.

Cyclotomic Generating Functions

Remark. Corresponding with each cyclotomic generating function $f(q)$, there is a discrete random variable X_f supported on $\mathbb{Z}_{\geq 0}$ with probability generating function $f(q)/f(1)$ and higher cumulants for $d \geq 2$,

$$\kappa_d^f = \frac{B_d}{d} \sum_{j=1}^{m} (a_j^d - b_j^d).$$

Therefore, we can study asymptotics for interesting sequences of of cyclotomic generating functions much like SYT.

Recent Progress

Thm. There exists statistics determining asymptotic normality and other limiting distributions in the following cases:

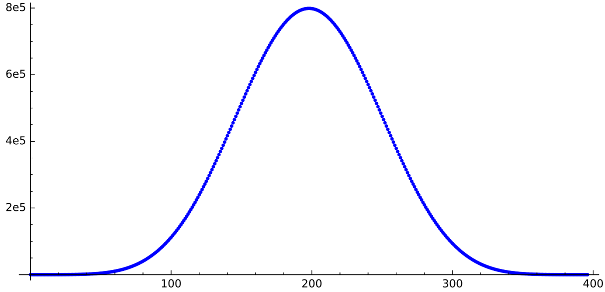
1. Stanley: $s_\lambda(1, q, q^2, \ldots, q^m)$.
2. Björner-Wachs: q-hook length formula for forests.
3. Iwahori-Matsumoto, Stembridge-Waugh, Zabrocki: Coxeter length generating function restricted to coset representatives of the extended affine Weyl group of type A_{n-1} mod translations by coroots. The associated statistic is $\mathrm{baj} - \mathrm{inv}$.

Beyond Cyclotomic Generating Functions

Another family of polynomials:

Thm.(Douvropoulos, N. Williams conjecture) There exists a cyclic sieving phenomena for factorizations of Coxeter elements along with the q-analog of n^{n-2} given by $[n]_{q^2} \cdots [n]_{q^{n-1}}$

The coefficients of $[n]_{q^2} \cdots [n]_{q^{n-1}}$ also appear to be normal...



Local Limit Conjecture

Conjecture. Let $\lambda \vdash n > 25$. Uniformly for all n and for all integers k, we have

$$|\mathbb{P}(X_\lambda[\mathrm{maj}] = k) - N(k; \mu_\lambda, \sigma_\lambda)| = O\left(\frac{1}{\sigma_\lambda \, \mathrm{aft}(\lambda)}\right)$$

where $N(k; \mu_\lambda, \sigma_\lambda)$ is the density function for the normal distribution with mean μ_λ and variance σ_λ.

The conjecture has been verified for $n \leq 50$ and $\mathrm{aft}(\lambda) > 1$.

Up to $n = 50$, the constant $1/9$ works.

At $n = 50$, $1/10$ does not.

Unimodality Question

Conjecture. The polynomial $\mathrm{SYT}^{\mathrm{maj}}(q)$ is unimodal if λ has at least 4 corners. If λ has 3 corners or fewer, then $\mathrm{SYT}^{\mathrm{maj}}(q)$ is unimodal except when λ or λ' is among the following partitions:

1. Any partition of rectangle shape that has more than one row and column.
2. Any partition of the form $(k, 2)$ with $k \geq 4$ and k even.
3. Any partition of the form $(k, 4)$ with $k \geq 6$ and k even.
4. Any partition of the form $(k, 2, 1, 1)$ with $k \geq 2$ and k even.
5. Any partition of the form $(k, 2, 2)$ with $k \geq 6$.
6. Any partition on the list of 40 special exceptions of size at most 28.

Unimodality Question

Special Exceptions.

$$\begin{gathered}(3,3,2),(4,2,2),(4,4,2),(4,4,1,1),\\(5,3,3),(7,5),(6,2,1,1,1,1),\\(5,5,2),(5,5,1,1),(5,3,2,2),(4,4,3,1),\\(4,4,2,2),(7,3,3),(8,6),(6,6,2),\\(6,6,1,1),(5,5,2,2),(5,3,3,3),(4,4,4,2),\\(11,5),(10,6),(9,7),(7,7,2),\\(7,7,1,1),(6,6,4),(6,6,1,1,1,1),(6,5,5),\\(5,5,3,3),(12,6),(11,7),(10,8),\\(15,5),(14,6),(11,9),(16,6),(12,10),(18,6),\\(14,10),(20,6),(22,6).\end{gathered}$$

Conclusion

Many Thanks!

To you all for listening, to the organizers of this workshop, and to BIRS for creating the mathematical atmosphere.