Cyclotomic Generating Functions

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arXiv:1809.07386 and more coming soon!

BIRS: Asymptotic Algebraic Combinatorics
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Outline

Motivating Example: $q$-enumeration of SYT's via major index

Cyclotomic Generating Functions

More Examples and Some Asymptotics

Open Problems
Standard Young Tableaux

**Defn.** A standard Young tableau of shape $\lambda$ is a bijective filling of $\lambda$ such that every row is increasing from left to right and every column is increasing from top to bottom.

\[
\begin{array}{cccccc}
1 & 3 & 6 & 7 & 9 \\
2 & 5 & 8 &   \\
4 &   &   &   \\
\end{array}
\]

**Important Fact.** The standard Young tableaux of shape $\lambda$, denoted $\text{SYT}(\lambda)$, index a basis of the irreducible $S_n$ representation indexed by $\lambda$. 
Counting Standard Young Tableaux

**Hook Length Formula.** (Frame-Robinson-Thrall, 1954)

If $\lambda$ is a partition of $n$, then

$$\#SYT(\lambda) = \frac{n!}{\prod_{c \in \lambda} h_c}$$

where $h_c$ is the *hook length* of the cell $c$, i.e. the number of cells directly to the right of $c$ or below $c$, including $c$.

**Example.** Filling cells of $\lambda = (5, 3, 1) \vdash 9$ by hook lengths:

\[
\begin{array}{cccc}
7 & 5 & 4 & 2 & 1 \\
4 & 2 & 1 \\
1 & & & & \\
\end{array}
\]

So, $\#SYT(5, 3, 1) = \frac{9!}{7 \cdot 5 \cdot 4 \cdot 2 \cdot 4 \cdot 2} = 162$.

**Remark.** Notable other proofs by Greene-Nijenhuis-Wilf ’79 (probabilistic), Eriksson ’93 (bijective), Krattenthaler ’95 (bijective), Novelli -Pak -Stoyanovskii’97 (bijective), Bandlow’08,
**q-Counting Standard Young Tableaux**

**Def.** The *descent set* of a standard Young tableau $T$, denoted $D(T)$, is the set of positive integers $i$ such that $i+1$ lies in a row strictly below the cell containing $i$ in $T$.

The *major index* of $T$ is the sum of its descents:

$$\text{maj}(T) = \sum_{i \in D(T)} i.$$ 

**Example.** The descent set of $T$ is $D(T) = \{1, 3, 4, 7\}$ so $\text{maj}(T) = 15$ for $T = \begin{array}{c|c|c|c|c}
1 & 3 & 6 & 7 & 9 \\
2 & 4 & 8 \\
5 
\end{array}$.

**Def.** The *major index generating function* for $\lambda$ is

$$\text{SYT}(\lambda)^{\text{maj}}(q) := \sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)}.$$
Example. $\lambda = (5, 3, 1)$

$$\text{SYT}(\lambda)^{\text{maj}}(q) := \sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)} =$$

$$q^{23} + 2q^{22} + 4q^{21} + 5q^{20} + 8q^{19} + 10q^{18} + 13q^{17} + 14q^{16} + 16q^{15} + 16q^{14} + 16q^{13} + 14q^{12} + 13q^{11} + 10q^{10} + 8q^9 + 5q^8 + 4q^7 + 2q^6 + q^5$$

Note, at $q = 1$, we get back 162.
Computation of \( \text{SYT}(\lambda)^{\text{maj}}(q) \)

**Thm.** (Stanley’s \( q \)-analog of the Hook Length Formula for \( \lambda \vdash n \))

\[
\text{SYT}(\lambda)^{\text{maj}}(q) = \frac{q^{b(\lambda)} [n]_q !}{\prod_{c \in \lambda} [h_c]_q}
\]

where

- \( b(\lambda) := \sum (i - 1) \lambda_i \)
- \( h_c \) is the hook length of the cell \( c \)
- \( [n]_q := 1 + q + \cdots + q^{n-1} = \frac{q^n - 1}{q - 1} \)
- \( [n]_q ! := [n]_q [n-1]_q \cdots [1]_q \)
Corollaries of Stanley’s formula

**Thm.** (Stanley’s $q$-analog of the Hook Length Formula for $\lambda \vdash n$)

$$\text{SYT}(\lambda)^{\text{maj}}(q) = \frac{q^{b(\lambda)} [n]_q !}{\prod_{c \in \lambda} [h_c]_q}$$

**Corollaries.**

1. $\text{SYT}(\lambda)^{\text{maj}}(q) = \text{SYT}(\lambda')^{\text{maj}}(q)$.
2. The coefficients of $\text{SYT}(\lambda)^{\text{maj}}(q)$ are symmetric.
3. There is a unique min-maj and max-maj tableau of shape $\lambda$. 

Motivation for $q$-Counting Standard Young Tableaux

**Thm.** (Lusztig-Stanley 1979) Given a partition $\lambda \vdash n$, say

$$\text{SYT}(\lambda)^{\text{maj}}(q) := \sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)} = \sum_{k \geq 0} b_{\lambda,k} q^k.$$ 

Then $b_{\lambda,k} := \#\{T \in \text{SYT}(\lambda) : \text{maj}(T) = k\}$ is the number of times the irreducible $S_n$ module indexed by $\lambda$ appears in the decomposition of the coinvariant algebra $\mathbb{Z}[x_1, x_2, \ldots, x_n]/I_+$ in the homogeneous component of degree $k$. 
Key Questions for SYT(\(\lambda\))^{maj}(q)

Recall \(\text{SYT}(\lambda)^{maj}(q) = \sum_{T \in \text{SYT}(\lambda)} q^{maj(T)} = \sum b_{\lambda,k} q^k.\)

Existence Question. For which \(\lambda, k\) does \(b_{\lambda,k} = 0\) ?

Distribution Question. What patterns do the coefficients in the list \((b_{\lambda,0}, b_{\lambda,1}, \ldots)\) exhibit?

Unimodality Question. For which \(\lambda\), are the coefficients of \(\text{SYT}(\lambda)^{maj}(q)\) unimodal, meaning

\[b_{\lambda,0} \leq b_{\lambda,1} \leq \ldots \leq b_{\lambda,m} \geq b_{\lambda,m+1} \geq \ldots?\]
Example. \( \lambda = (5, 3, 1) \)

\[
\begin{array}{ccccccc}
\text{Notation: } & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 5 & 8 & 10 & 13 & 14 & 16 & 16 & 16 & 14 & 13 & 10 & 8 & 5 & 4 & 2 & 1
\end{array}
\]
$q$-Counting Standard Young Tableaux

Examples. 

$(2,2) \vdash 4$: 

$(0\ 0\ 1\ 0\ 1)$

$(5,3,1)$: 

$(00000\ 1\ 2\ 4\ 5\ 8\ 10\ 13\ 14\ 16\ 16\ 14\ 13\ 10\ 8\ 5\ 4\ 2\ 1)$

$(6,4) \vdash 10$: 

$(0\ 0\ 0\ 0\ 1\ 1\ 2\ 2\ 4\ 4\ 6\ 6\ 8\ 7\ 8\ 7\ 8\ 6\ 6\ 4\ 4\ 2\ 2\ 1\ 1)$

$(6,6) \vdash 12$: 

$(0\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 2\ 2\ 4\ 3\ 5\ 5\ 7\ 6\ 9\ 7\ 9\ 8\ 9\ 7\ 9\ 6\ 7\ 5\ 5\ 3\ 4\ 2\ 2\ 1\ 1\ 0\ 1)$

$(11,5,3,1) \vdash 20$: 

$(1\ 3\ 8\ 16\ 32\ 57\ 99\ 160\ 254\ 386\ 576\ 832\ 1184\ 1645\ 2255\ 3031\ 4027\ 5265\ 6811\ 8689\ 10979\ 13706\ 16959\ 20758\ 25200\ 30296\ 36143\ 42734\ 50163\ 58399\ 67523\ 77470\ 88305\ 99925\ 112370\ 125492\ 139307\ 153624\ 168431\ 183493\ 198778\ 214017\ 229161\ 243913\ 258222\ 271780\ 284542\ 296200\ 306733\ 315853\ 323571\ 329629\ 334085\ 336727\ 337662\ 336727\ 334085\ 329629\ 323571\ 315853\ 306733\ 296200\ 284542\ 271780\ 258222\ 243913\ 229161\ 214017\ 198778\ 183493\ 168431\ 153624\ 139307\ 125492\ 112370\ 99925\ 88305\ 77470\ 67523\ 58399\ 50163\ 42734\ 36143\ 30296\ 25200\ 20758\ 16959\ 13706\ 10979\ 8689\ 6811\ 5265\ 4027\ 3021\ 2255\ 1645\ 1123\ 662\ 332\ 156\ 78\ 39\ 19\ 9\ 5\ 3\ 2\ 1)$
Visualizing the coefficients of \( SYT(5, 3, 1)^{maj}(q) \): 

\( (1, 2, 4, 5, 8, 10, 13, 14, 16, 16, 16, 14, 13, 10, 8, 5, 4, 2, 1) \)
Visualizing Major Index Generating Functions

Visualizing the coefficients of $\text{SYT}(11, 5, 3, 1)^{\text{maj}}(q)$.

**Question.** What type of curve is that?
Visualizing the coefficients of SYT(10, 6, 1)^{maj}(q) along with the Normal distribution with $\mu = 34$ and $\sigma^2 = 98$. 
Visualizing Major Index Generating Functions

Visualizing the coefficients of $\text{SYT}(8, 8, 7, 6, 5, 5, 5, 2, 2)^\text{maj}(q)$
Cyclotomic Polynomials

**Def.** The irreducible factors of $q^n - 1$ over the integers are called *cyclotomic polynomials*. There is one for each positive integer $d$, given by

\[
\Phi_d(q) = \prod_{d|n} (q^d - 1)^{\mu(n/d)} = \frac{q^n - 1}{\prod_{c|n, c<n} \Phi_c(q)},
\]

where $\mu(n/d)$ is the Möbius function given by

\[
\mu(k) = \begin{cases} 
1 & k = 1 \\
0 & k > 1 \text{ has repeated prime factors} \\
(-1)^l & k > 1 \text{ is product of } l \text{ distinct prime factors.}
\end{cases}
\]

**Fact.** Each $q$-integer $[n]_q = (q^n - 1)/(q - 1)$ factors into a product of distinct cyclotomic polynomials

\[
[n]_q = 1 + q + \cdots + q^{n-1} = \prod_{1<d|n} \Phi_d(q).
\]
Cyclotomic Polynomials

**Examples.**

\[
\begin{align*}
\Phi_1(q) &= q - 1 \\
\Phi_2(q) &= q + 1 \\
\Phi_3(q) &= q^2 + q + 1 \\
\Phi_4(q) &= q^2 + 1 \\
\Phi_5(q) &= q^4 + q^3 + q^2 + q + 1 \\
\Phi_6(q) &= q^2 - q + 1 \\
\Phi_7(q) &= q^6 + q^5 + q^4 + q^3 + q^2 + q + 1 \\
\Phi_8(q) &= q^4 + 1 \\
\Phi_9(q) &= q^6 + q^3 + 1 \\
\Phi_{10}(q) &= q^4 - q^3 + q^2 - q + 1
\end{align*}
\]
Cyclotomic Polynomials

**Bigger Example.**

\[ \Phi_{105}(q) = q^{48} + q^{47} + q^{46} + q^{43} - q^{42} - 2q^{41} - q^{40} - q^{39} + q^{36} + q^{35} + q^{34} + q^{33} + q^{32} + q^{31} - q^{28} - q^{26} - q^{24} - q^{22} - q^{20} + q^{17} + q^{16} + q^{15} + q^{14} + q^{13} + q^{12} - q^{9} - 2q^{8} - q^{7} - q^{6} - q^{5} + q^{2} + q^{1} + 1 \]
“Fast” Computation of \( \text{SYT}(\lambda)^{\text{maj}}(q) \)

**Thm.** (Stanley’s \( q \)-analog of the Hook Length Formula for \( \lambda \vdash n \))

\[
\text{SYT}(\lambda)^{\text{maj}}(q) = \frac{q^{b(\lambda)} [n]_q !}{\prod_{c \in \lambda} [h_c]_q}
\]

**Trick for conjectures.** Cancel all of the cyclotomic factors of the denominator from the numerator, and then expand the remaining product.
Existence Question

Recall $\text{SYT}(\lambda)^\text{maj}(q) = \sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)} = \sum b_{\lambda,k} q^k$.

**Existence Question.** For which $\lambda, k$ does $b_{\lambda,k} = 0$?
Recall \( \text{SYT}(\lambda)^{\text{maj}}(q) = \sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)} = \sum b_{\lambda,k} q^k. \)

**Existence Question.** For which \( \lambda, k \) does \( b_{\lambda,k} = 0 \) ?

**Cor of Stanley’s formula.** For every \( \lambda \vdash n \geq 1 \) there is a unique tableau with minimal major index \( b(\lambda) \) and a unique tableau with maximal major index \( \binom{n}{2} - b(\lambda') \). These two agree for shapes consisting of one row or one column, and otherwise they are distinct.
Example. The min-maj and max-maj tableaux for $(6, 4, 3, 3, 1)$.

\[
\begin{array}{cccccc}
1 & 3 & 4 & 11 & 16 & 17 \\
2 & 6 & 7 & 15 \\
5 & 9 & 10 \\
8 & 13 & 14 \\
12
\end{array}
&
\begin{array}{cccccc}
1 & 2 & 3 & 5 & 9 & 13 \\
4 & 6 & 10 & 14 \\
7 & 11 & 15 \\
8 & 12 & 16 \\
17
\end{array}
\]

\[b(\lambda) = \sum (i - 1) \lambda_i = 23\]

\[
\binom{17}{2} - b(\lambda') = 109
\]
Existence Question

Recall $\text{SYT}(\lambda)^{\text{maj}}(q) = \sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)} = \sum b_{\lambda,k} q^k$.

**Existence Question.** For which $\lambda, k$ does $b_{\lambda,k} = 0$?

**Cor of Stanley’s formula.** The coefficient of $q^{b(\lambda)+1}$ in $\text{SYT}(\lambda)^{\text{maj}}(q) = 0$ if and only if $\lambda$ is a rectangle. If $\lambda$ is a rectangle with more than one row and column, then coefficient of $q^{b(\lambda)+2}$ is 1.

**Question.** Are there other internal zeros?
Classifying All Nonzero Fake Degrees

**Thm.** (Billey-Konvalinka-Swanson, 2018) For any partition \( \lambda \) which is not a rectangle, \( \text{SYT}(\lambda)^{\text{maj}}(q) := \sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)} \) has no internal zeros. If \( \lambda \) is a rectangle with at least two rows and columns, \( \text{SYT}(\lambda)^{\text{maj}}(q) \) has exactly two internal zeros, one at degree \( b(\lambda) + 1 \) and the other at degree \( \text{maxmaj}(\lambda) - 1 \).
Classifying All Nonzero Fake Degrees

**Thm.** (Billey-Konvalinka-Swanson, 2018)  
For any partition $\lambda$ which is not a rectangle,

$$\text{SYT}(\lambda)^{\text{maj}}(q) := \sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)}$$

has no internal zeros. If $\lambda$ is a rectangle with at least two rows and columns, $\text{SYT}(\lambda)^{\text{maj}}(q)$ has exactly two internal zeros, one at degree $b(\lambda) + 1$ and the other at degree $\maxmaj(\lambda) - 1$.

**Proof Outline.** We identify block and rotation rules on tableaux giving rise to two posets on $\text{SYT}(\lambda)$—exceptional cases for rectangles which is ranked according to maj.
Strong and Weak Poset on SYT(3, 2, 1)
**Cor.** The irreducible $S_n$-module indexed by $\lambda$ appears in the decomposition of the degree $k$ component of the coinvariant algebra if and only if $b_{\lambda,k} > 0$ as characterized above.

Similar results hold for all Shepard-Todd groups $G(m, d, n)$.

Converting \( q \)-Enumeration to Discrete Probability

**Distribution Question.** What is the limiting distribution(s) for the coefficients in \( \text{SYT}(\lambda)^{\text{maj}}(q) \)?

**From Combinatorics to Probability.**
If \( f(q) = a_0 + a_1 q + a_2 q^2 + \cdots + a_n q^n \) where \( a_i \) are nonnegative integers, then construct the random variable \( X_f \) with discrete probability distribution

\[
\mathbb{P}(X_f = k) = \frac{a_k}{\sum_j a_j} = \frac{a_k}{f(1)}.
\]

If \( f \) is part of a family of \( q \)-analog of an integer sequence, we can study the limiting distributions.
Converting $q$-Enumeration to Discrete Probability

**Example.** For $\text{SYT}(\lambda)^{\text{maj}}(q) = \sum b_{\lambda,k} q^k$, define the integer random variable $X_{\lambda}[\text{maj}]$ with discrete probability distribution

$$P(X_{\lambda}[\text{maj}] = k) = \frac{b_{\lambda,k}}{|\text{SYT}(\lambda)|}.$$ 

We claim the distribution of $X_{\lambda}[\text{maj}]$ “usually” is approximately normal for most shapes $\lambda$. Let’s make that precise!
**Thm.** (Adin-Roichman, 2001)
For any partition $\lambda$, the mean and variance of $X_\lambda[\text{maj}]$ are

$$
\mu_\lambda = \frac{(|\lambda|) - b(\lambda') + b(\lambda)}{2} = b(\lambda) + \frac{1}{2} \left[ \sum_{j=1}^{\frac{|\lambda|}{2}} j - \sum_{c \in \lambda} h_c \right],
$$

and

$$
\sigma^2_\lambda = \frac{1}{12} \left[ \sum_{j=1}^{\frac{|\lambda|}{2}} j^2 - \sum_{c \in \lambda} h_c^2 \right].
$$

**Def.** The *standardization* of $X_\lambda[\text{maj}]$ is

$$
X_\lambda^*[\text{maj}] = \frac{X_\lambda[\text{maj}] - \mu_\lambda}{\sigma_\lambda}.
$$

So $X_\lambda^*[\text{maj}]$ has mean 0 and variance 1 for any $\lambda$. 
Asymptotic Normality

**Def.** Let $X_1, X_2, \ldots$ be a sequence of real-valued random variables with standardized cumulative distribution functions $F_1(t), F_2(t), \ldots$. The sequence is *asymptotically normal* if

$$\forall t \in \mathbb{R}, \quad \lim_{n \to \infty} F_n(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2} = \mathbb{P}(N < t)$$

where $N$ is a Normal random variable with mean 0 and variance 1.
Asymptotic Normality

**Def.** Let $X_1, X_2, \ldots$ be a sequence of real-valued random variables with standardized cumulative distribution functions $F_1(t), F_2(t), \ldots$. The sequence is *asymptotically normal* if

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where $N$ is a Normal random variable with mean 0 and variance 1.

**Question.** In what way can a sequence of partitions approach infinity?
The Aft Statistic

**Def.** Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$, let

$$aft(\lambda) := n - \max\{\lambda_1, k\}.$$  

**Example.** $\lambda = (5, 3, 1)$ then $aft(\lambda) = 4$.  

Look it up: Aft is now on FindStat as St001214
Theorem (Billey-Konvalinka-Swanson, 2019)

Suppose $\lambda^{(1)}, \lambda^{(2)}, \ldots$ is a sequence of partitions, and let $X_N := X_{\lambda^{(N)}}[\text{maj}]$ be the corresponding random variables for the maj statistic. Then, the sequence $X_1, X_2, \ldots$ is asymptotically normal if and only if $\text{aft}(\lambda^{(N)}) \to \infty$ as $N \to \infty$. 

Question. What happens if $\text{aft}(\lambda^{(N)})$ does not go to infinity as $N \to \infty$?
Thm. (Billey-Konvalinka-Swanson, 2019)

Suppose $\lambda^{(1)}, \lambda^{(2)}, \ldots$ is a sequence of partitions, and let $X_N := X_{\lambda^{(N)}}[\text{maj}]$ be the corresponding random variables for the maj statistic. Then, the sequence $X_1, X_2, \ldots$ is asymptotically normal if and only if $\text{aft}(\lambda^{(N)}) \to \infty$ as $N \to \infty$.

Question. What happens if $\text{aft}(\lambda^{(N)})$ does not go to infinity as $N \to \infty$?
Thm. (Billey-Konvalinka-Swanson, 2019)
Let \( \lambda^{(1)}, \lambda^{(2)}, \ldots \) be a sequence of partitions. Then \((X_{\lambda(N)}[\text{maj}])^*)\) converges in distribution if and only if

(i) \( \text{aft}(\lambda^{(N)}) \to \infty \); or
(ii) \( |\lambda^{(N)}| \to \infty \) and \( \text{aft}(\lambda^{(N)}) \) is eventually constant; or
(iii) the distribution of \( X_{\lambda^{(N)}}^*[\text{maj}] \) is eventually constant.

The limit law is \( \mathcal{N}(0,1) \) in case (i), \( \Sigma^*_M \) in case (ii), and discrete in case (iii).

Here \( \Sigma_M \) denotes the sum of \( M \) independent identically distributed uniform \([0,1]\) random variables, known as the Irwin–Hall distribution or the uniform sum distribution.
Example. \( \lambda = (100, 2) \) looks like the distribution of the sum of two independent uniform random variables on \([0, 1]\):
Example. $\lambda = (100, 2, 1)$ looks like the distribution of the sum of three independent uniform random variables on $[0, 1]$. 

![Graph](graph.png)
**Example.** $\lambda = (100, 3, 2)$ looks like the normal distribution, but not quite!
Proof ideas: Characterize the Moments and Cumulants

Definitions.

- For \( d \in \mathbb{Z}_{\geq 0} \), the \( d \)th moment

\[
\mu_d := \mathbb{E}[X^d]
\]

- The moment-generating function of \( X \) is

\[
M_X(t) := \mathbb{E}[e^{tX}] = \sum_{d=0}^{\infty} \mu_d \frac{t^d}{d!},
\]

- The cumulants \( \kappa_1, \kappa_2, \ldots \) of \( X \) are defined to be the coefficients of the exponential generating function

\[
K_X(t) := \sum_{d=1}^{\infty} \kappa_d \frac{t^d}{d!} := \log M_X(t) = \log \mathbb{E}[e^{tX}].
\]
Nice Properties of Cumulants

1. *(Familiar Values)* The first two cumulants are $\kappa_1 = \mu$, and $\kappa_2 = \sigma^2$.

2. *(Shift Invariance)* The second and higher cumulants of $X$ agree with those for $X - c$ for any $c \in \mathbb{R}$.

3. *(Homogeneity)* The $d$th cumulant of $cX$ is $c^d\kappa_d$ for $c \in \mathbb{R}$.

4. *(Additivity)* The cumulants of the sum of *independent* random variables are the sums of the cumulants.

5. *(Polynomial Equivalence)* The cumulants and moments are determined by polynomials in the other sequence.
Examples of Cumulants and Moments

**Example.** Let $X = \mathcal{N}(\mu, \sigma^2)$ be the normal random variable with mean $\mu$ and variance $\sigma^2$. Then the cumulants are

$$\kappa_d = \begin{cases} 
\mu & d = 1, \\
\sigma^2 & d = 2, \\
0 & d \geq 3.
\end{cases}$$

and for $d > 1$,

$$\mu_d = \begin{cases} 
0 & \text{if } d \text{ is odd}, \\
\sigma^d (d-1)!! & \text{if } d \text{ is even}.
\end{cases}$$

**Example.** For a Poisson random variable $X$ with mean $\mu$, the cumulants are all $\kappa_d = \mu$, while the moments are $\mu_d = \sum_{i=1}^d \mu^i S_{i,d}$. 
Cumulants for Major Index Generating Functions

**Thm.** (Billey-Konvalinka-Swanson, 2019)

Let $\lambda \vdash n$ and $d \in \mathbb{Z}_{>1}$. If $\kappa^\lambda_d$ is the $d$th cumulant of $X^\lambda_{\text{maj}}$, then

$$\kappa^\lambda_d = \frac{B_d}{d} \left[ \sum_{j=1}^{n} j^d - \sum_{c \in \lambda} h^d_c \right]$$

(1)

where $B_0, B_1, B_2, \ldots = 1, \frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, \ldots$ are the Bernoulli numbers (OEIS A164555 / OEIS A027642).

**Remark.** We use this theorem to prove that as $n$ approaches infinity the standardized cumulants for $d \geq 3$ all go to 0 proving the Asymptotic Normality Theorem.
Cumulants for Major Index Generating Functions

**Thm.** (Billey-Konvalinka-Swanson, 2019)
Let $\lambda \vdash n$ and $d \in \mathbb{Z}_{>1}$. If $\kappa_d^\lambda$ is the $d$th cumulant of $X_\lambda[\text{maj}]$, then

$$\kappa_d^\lambda = \frac{B_d}{d} \left[ \sum_{j=1}^n j^d - \sum_{c \in \lambda} h_c^d \right]$$

where $B_0, B_1, B_2, \ldots = 1, \frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, \ldots$ are the Bernoulli numbers (OEIS A164555 / OEIS A027642).

**Remark.** We use this theorem to prove that as $n$ approaches infinity the standardized cumulants for $d \geq 3$ all go to 0 proving the Asymptotic Normality Theorem.

**Remark.** Note, $\kappa_2^\lambda$ is exactly the Adin-Roichman variance formula.
Cumulants of certain \( q \)-analogs

Suppose \( \{a_1, \ldots, a_m\} \) and \( \{b_1, \ldots, b_m\} \) are multisets of positive integers such that

\[
f(q) = \frac{\prod_{j=1}^{m}[a_j]_q}{\prod_{j=1}^{m}[b_j]_q} = \sum c_k q^k \in \mathbb{Z}_{\geq 0}[q]
\]

Let \( X \) be a discrete random variable with \( \mathbb{P}(X = k) = c_k / f(1) \).
Then the \( d \)th cumulant of \( X \) is

\[
\kappa_d = \frac{B_d}{d} \sum_{j=1}^{m} (a_j^d - b_j^d)
\]

where \( B_d \) is the \( d \)th Bernoulli number (with \( B_1 = \frac{1}{2} \)).

**Example.** This theorem applies to

\[
\text{SYT}(\lambda)_{\text{maj}}(q) := \sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)} = \frac{q^{b(\lambda)[n]_q}}{\prod_{c \in \lambda}[h_c]_q}
\]
Corollaries of the Distribution Theorem

1. Asymptotic normality also holds for block diagonal skew shapes with aft going to infinity.

2. New proof of asymptotic normality of 
\[
[n]_q! = \sum_{w \in S_n} q^{\text{maj}(w)} = \sum_{w \in S_n} q^{\text{inv}(w)}
\] due to Feller (1944).


Corollaries of the Distribution Theorem

1. Asymptotic normality also holds for block diagonal skew shapes with $a_l$ going to infinity.

2. New proof of asymptotic normality of $[n]_q! = \sum_{w \in S_n} q^{\text{maj}(w)} = \sum_{w \in S_n} q^{\text{inv}(w)}$ due to Feller (1944).


**Question.** Using Morales-Pak-Panova $q$-hook length formula, can we prove an asymptotic normality for most skew shapes?
Cyclotomic Generating Functions

**Def.** A polynomial \( f(q) \) with nonnegative integer coefficients is a *cyclotomic generating function* provided it satisfies one of the following equivalent conditions:

(i) (Rational form.) There are multisets \( \{a_1, \ldots, a_m\} \) and \( \{b_1, \ldots, b_m\} \) of positive integers and \( \alpha, \beta \in \mathbb{Z}_{\geq 0} \) such that

\[
f(q) = \alpha q^\beta \cdot \prod_{j=1}^{m} \frac{[a_j]_q}{[b_j]_q} = \alpha q^\beta \cdot \prod_{j=1}^{m} \frac{1 - q^{a_j}}{1 - q^{b_j}}.
\]  

(ii) (Cyclotomic form.) The polynomial \( f(q) \) can be written as a non-negative integer times a product of cyclotomic polynomials and factors of \( q \).

(iii) (Complex form.) The complex roots of \( f(q) \) are each either a root of unity or zero.
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Cyclotomic Generating Functions

More examples of cyclotomic generating functions:

1. Stanley: $s_{\lambda}(1, q, q^2, \ldots, q^m)$.

2. Björner-Wachs: $q$-hook length formula for forests.

3. Macaulay: Hilbert series of polynomial quotients $k[x_1, \ldots, x_n]/(\theta_1, \theta_2, \ldots, \theta_n)$ where $\deg(x_i) = b_i$, $\deg(\theta_i) = a_i$, and $(\theta_1, \theta_2, \ldots, \theta_n)$ is a homogeneous system of parameters $k[x_1, \ldots, x_n]/$.

4. Chevalley: Length generating function restricted to minimum length coset representatives of a finite reflection group modulo a parabolic subgroup.
Remark. Corresponding with each cyclotomic generating function $f(q)$, there is a discrete random variable $X_f$ supported on $\mathbb{Z}_{\geq 0}$ with probability generating function $f(q)/f(1)$ and higher cumulants for $d \geq 2$,

$$\kappa_d^f = \frac{B_d}{d} \sum_{j=1}^{m} (a_j^d - b_j^d).$$

Therefore, we can study asymptotics for interesting sequences of cyclotomic generating functions much like SYT.
**Thm.** There exists statistics determining asymptotic normality and other limiting distributions in the following cases:

1. Stanley: \( s_\lambda(1, q, q^2, \ldots, q^m) \).

2. Björner-Wachs: \( q \)-hook length formula for forests.

3. Iwahori-Matsumoto, Stembridge-Waugh, Zabrocki: Coxeter length generating function restricted to coset representatives of the extended affine Weyl group of type \( A_{n-1} \) mod translations by coroots. The associated statistic is \( \text{baj} - \text{inv} \).
Beyond Cyclotomic Generating Functions

Another family of polynomials:

**Thm.** (Douvropoulos, N. Williams conjecture) There exists a cyclic sieving phenomena for factorizations of Coxeter elements along with the $q$-analog of $n^{n-2}$ given by $[n]_q^2 \cdots [n]_q^{n-1}$

The coefficients of $[n]_q^2 \cdots [n]_q^{n-1}$ also appear to be normal...
Local Limit Conjecture

**Conjecture.** Let $\lambda \vdash n > 25$. Uniformly for all $n$ and for all integers $k$, we have

$$|\mathbb{P}(X_{\lambda}[\text{maj}] = k) - N(k; \mu_{\lambda}, \sigma_{\lambda})| = O\left(\frac{1}{\sigma_{\lambda} \text{aft}(\lambda)}\right)$$

where $N(k; \mu_{\lambda}, \sigma_{\lambda})$ is the density function for the normal distribution with mean $\mu_{\lambda}$ and variance $\sigma_{\lambda}$.

The conjecture has been verified for $n \leq 50$ and $\text{aft}(\lambda) > 1$.

Up to $n = 50$, the constant $1/9$ works.

At $n = 50$, $1/10$ does not.
Unimodality Question

**Conjecture.** The polynomial $\text{SYT}^{\text{maj}}(q)$ is unimodal if $\lambda$ has at least 4 corners. If $\lambda$ has 3 corners or fewer, then $\text{SYT}^{\text{maj}}(q)$ is unimodal except when $\lambda$ or $\lambda'$ is among the following partitions:

1. Any partition of rectangle shape that has more than one row and column.
2. Any partition of the form $(k, 2)$ with $k \geq 4$ and $k$ even.
3. Any partition of the form $(k, 4)$ with $k \geq 6$ and $k$ even.
4. Any partition of the form $(k, 2, 1, 1)$ with $k \geq 2$ and $k$ even.
5. Any partition of the form $(k, 2, 2)$ with $k \geq 6$.
6. Any partition on the list of 40 special exceptions of size at most 28.
Unimodality Question

Special Exceptions.

(3, 3, 2), (4, 2, 2), (4, 4, 2), (4, 4, 1, 1),
(5, 3, 3), (7, 5), (6, 2, 1, 1, 1, 1),
(5, 5, 2), (5, 5, 1, 1), (5, 3, 2, 2), (4, 4, 3, 1),
(4, 4, 2, 2), (7, 3, 3), (8, 6), (6, 6, 2),
(6, 6, 1, 1), (5, 5, 2, 2), (5, 3, 3, 3), (4, 4, 4, 2),
(11, 5), (10, 6), (9, 7), (7, 7, 2),
(7, 7, 1, 1), (6, 6, 4), (6, 6, 1, 1, 1, 1), (6, 5, 5),
(5, 5, 3, 3), (12, 6), (11, 7), (10, 8),
(15, 5), (14, 6), (11, 9), (16, 6), (12, 10), (18, 6),
(14, 10), (20, 6), (22, 6).
Conclusion

Many Thanks!

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