Coxeter-Knuth Graphs
and a signed Little Bijection

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Outline

Based on joint work with Zach Hamaker, Austin Roberts, and Ben Young.

1. Transition Equations for Schubert classes of classical types
2. (signed) Little Bijection on Reduced words
3. Kraśkiewicz insertion and Coxeter-Knuth moves on Reduced Words
4. Shifted Dual Equivalence graphs
5. Open problems
Background

- \( G = \text{Classical Complex Lie Group: } SL_n, SP_{2n}, SO_{2n+1}, SO_{2n} \)
- \( B = \text{Borel subgroup} = \text{upper triangular matrices in } G \)
- \( W = \text{Weyl groups } \subset B_n = \text{signed permutations} \)
- \( S = \text{simple reflections generating } W \)
- \( T = \{ wsw^{-1} : s \in S, w \in W \} \subset \text{signed transpositions} \)
Monk/Chevalley formula

**Fact.** The Schubert classes $S_w$, $w \in W$ form a basis for $H^*(G/B, \mathbb{Q})$ and they satisfy Monk/Chevalley’s formula

$$S_w \cdot S_s = \sum_{t \in T: l(wt) = l(w) + 1} - (S_s, f_t) S_{wt},$$

for any simple reflection $s$ where $f_t$ is the linear polynomial negated by $t$.

**Defn.** *Transition Equations* are recurrences for Schubert classes derived from Monk/Chevalley’s formula.
Transition Equations

**Type A.** (Lascoux-Schützenberger, 1984) For all \( w \neq id \), let \((r < s)\) be the largest pair of positions inverted in \( w \) in lexicographic order. If \( w \) has only one descent, then \( \mathcal{S}_w \) is the Schur polynomial \( s_{\lambda(w)}(x_1, \ldots, x_r) \). Otherwise,

\[
\mathcal{S}_w = x_r \mathcal{S}_v + \sum \mathcal{S}_{w'}
\]

where the sum is over all \( w' \) such that \( l(w) = l(w') \) and \( w' = wt_{rs}t_{ir} \) with \( 0 < i < r \). Call this set \( T(w) \).

**Types B, C, D.** Transition equations are similar except some coefficients can be 2’s (Billey,’95) and base cases are Schur P- or Q-functions (Pragacz,’91).
Stanley Symmetric Functions

**Notation.** For \( w = [w_1, \ldots, w_n] \in S_n \), let \( R(w) \) be the set of reduced words for \( w \). Let \( 1 \times w = [1, (1 + w_1), \ldots, (1 + w_n)] \).

**Defn.** The *Stanley symmetric function* \( F_w \) is equivalently given by

1. (L-S) \( F_w(x_1, x_2, \ldots) = \lim_{n \to \infty} \Theta(1^n \times w) \).

2. (L-S) If \( w \) has at most 1 descent, \( F_w = s_{\lambda(w)} \). Otherwise,

\[
F_w = \sum_{w' \in T(1 \times w)} F_{w'}.
\]

3. (Stanley, 1984) If \( a = (a_1, \ldots, a_p) \in R(w) \), let \( I(a) \) be the set of weakly increasing positive integer sequences \((i_1, \ldots, i_p)\) such that \( i_j < i_{j+1} \) whenever \( a_j < a_{j+1} \). Then

\[
F_w = \sum_{a \in R(w)} \sum_{(i_1, \ldots, i_p) \in I(a)} x_{i_1} x_{i_2} \cdots x_{i_p}.
\]
Combinatorics from Schubert Calculus

Cor. If $w$ has at most 1 descent then $|R(w)| = f^\lambda(w)$ where $f^\lambda$ is the number of standard tableaux of shape $\lambda$. Otherwise, $|R(w)| = \sum_{w' \in T(1 \times w)} |R(w')|$. 

Cor. $|R(w)| = \sum_\lambda a_{\lambda, w} f^\lambda$. 

Thm. (Edelman-Greene, 1987) The coefficient $a_{\lambda, w}$ counts the number of distinct $P$ tableaux that arise when inserting all reduced words for $w$ via EG-insertion (variation on RSK). Furthermore, for each such $P$ and standard tableau $Q$ of the same shape, there exists a unique $a \in R(w)$ which inserts to $P$ and has recording tableau $Q$.

Question. Is there a bijection from $R(w)$ to $\bigcup_{w' \in T(w)} R(w')$ which preserves the descent set and the $Q$ tableau?
Little’s Bijection

**Question.** Is there a bijection from $R(w)$ to $\bigcup_{w' \in T(w)} R(w')$ which preserves the descent set and the $Q$ tableau?

**Answer.** Yes! It’s called Little’s bijection named for David Little (Little, 2003) + (Hamaker-Young, 2013).

**Thomas Lam’s Conjecture.** (proved by Hamaker-Young, 2013) Every reduced word for any permutation with the same $Q$ tableau is connected via Little bumps. Every communication class under Little bumps contains a unique reduced word for a unique fixed point free Grassmannian permutation.

Thus, the Little bumps are analogous to jeu de taquin!
Notation. If \( w = [w_1, ..., w_n] \) is a signed permutation, then \( w \) is increasing if \( w_1 < w_2 < ... < w_n \).

Defn. The type \( C \) Stanley symmetric function \( F_w \) for \( w \in B_n \) are given by the following equivalent conditions:

   Let \( R(w) \) be the set of reduced words for \( w \in B_n \). If \( a = (a_1, ..., a_p) \in R(w) \), let \( I(a) \) be the set of weakly increasing positive integer sequences \((i_1, ..., i_p)\) such that \( i_{j-1} = i_j = i_{j+1} \) only occurs if \( a_j \) is not bigger than both \( a_{j-1} \) and \( a_{j+1} \). Then
   \[
   F^C_w = \sum_{a \in R(w)} \sum_{i=(i_1, ..., i_p) \in I(a)} 2^{\|i\|} x_{i_1} x_{i_2} \cdots x_{i_p}.
   \]

2. (Billey, 1996) If \( w \) is increasing, \( F_w = Q_{\mu(w)} \). Otherwise,
   \[
   F^C_w = \sum_{w' \in T(w)} F^C_{w'}.
   \]
Combinatorics from Schubert Calculus

Cor. If \( w \) is not increasing, \(|R(w)| = \sum_{w' \in T(w)} |R(w')|\)

Cor. \(|R(w)| = \sum_{\lambda} c_{\lambda,w} g_{\lambda} \) where \( g_{\lambda} \) is the number of shifted standard tableaux of shape \( \lambda \) and \( c_{\lambda,w} \) is a nonnegative integer.

Thm. (Kraśkiewicz, 1995) The coefficient \( b_{\lambda,w} \) counts the number of distinct \( P \) tableaux that arise when inserting all reduced words for \( w \) via Kraśkiewicz-insertion (variation on EG). Furthermore, for each such \( P \) and standard tableau \( Q \) of the same shape, there exists a unique \( a \in R(w) \) which inserts to \( P \) and has recording tableau \( Q \). (See also Haiman evacuation procedure.)

Question. Is there a bijection from \( R(w) \) to \( \bigcup_{w' \in T(w)} R(w') \) which preserves the peak set and the \( Q \) tableau for signed permutations?
Signed Little Bijection

**Question.** Is there a bijection from $R(w)$ to $\bigcup_{w' \in \mathcal{T}(w)} R(w')$ which preserves the peak set and the $Q$ tableau for signed permutations?

**Answer.** Yes!

**Thm.** (Billey-Hamaker-Roberts-Young, 2014)
- The signed Little bijection preserves peak sets, shifted recording tableaux under Kraskiewicz insertion and realizes the bijection given by the transition equation.
- Every reduced word for any signed permutation with the same $Q$ tableau is connected via signed Little bumps.
- Every communication class under signed Little bumps contains a unique reduced word for a unique increasing signed permutation.

Again, the signed Little bumps are analogous to jeu de taquin on shifted shapes!
The Algorithm

Given a reduced word, there is an associated reduced wiring diagram. If removing a crossing leaves another reduced wiring diagram, that crossing is a candidate to initiate a Little bump either pushing up or down. Pushing down (up) means reduce (increase) the corresponding letter in the word by 1.

- Check if the resulting word is reduced. If so, stop and return the new word.
- Otherwise, find the other point where the same two wires cross, and push them in the same direction.

Signed Little Bijection. Initiate a Little bump at the crossing \((r, s)\) corresponding to the lex largest inversion. This maps \(R(w)\) to \(\bigcup_{w' \in T(w)} R(w')\) bijectively.
The Algorithm in Pictures

0-12
The Algorithm in Pictures

0-13
The Algorithm in Pictures
Coxeter-Knuth Relations

The proof that (signed) Little bumps preserves $Q$ tableaux uses a key lemma about how the $Q$ changes with elementary Coxeter relations.

**Coxeter relations.** (commutation relations) $ij \equiv ji$ when $|i - j| > 1$, (braid relations) $0101 \equiv 1010$ and $i(i + 1)i \equiv (i + 1)i(i + 1)$ when $i > 0$.

**Defn.** *Coxeter-Knuth* moves on reduced words are minimally witnessed Coxeter relations that preserve $P$ tableau.

**Type A Coxeter-Knuth Moves.** *(EG)* Braids and witnessed commutations: $bac \equiv bca$, and $acb \equiv cab$ for $a < b < c$.

**Type B/C Coxeter-Knuth relations.** *(K)* 9 rules on windows of length 4 including $0101 \equiv 1010$ and $a(b + 1)b(b + 1) \equiv ab(b + 1)b$ for $a < b$, but no other braids.
Coxeter-Knuth Graphs and Dual Equivalence

**Defn.** The Coxeter-Knuth graph for $w$ has $V = R(w)$ and two reduced words are connected by an edge labeled $i$ if they agree in all positions except for a single Coxeter-Knuth relation starting in position $i$.

**Defn.** (Assaf, 2008) Dual equivalence graphs are graphs with labeled edges whose connected components are isomorphic to the graph on standard tableaux of a fixed partition shape with an edge labeled $i$ connecting any two vertices which differ by a transposition $(i, i+1)$ or $(i+1, i+2)$ with the third number on a diagonal in between the transposing pair.
Coxeter-Knuth Graphs and Dual Equivalence

**Thm.** The Coxeter-Knuth graphs in type $A$ are dual equivalence graphs and the isomorphism is given by the $Q$ tableaux in Edelman-Greene insertion.

In type $A$, this is a nice corollary of (Roberts, 2013) + (Hamaker-Young, 2013).

**Thm.** The Coxeter-Knuth graphs in type $B$ are shifted dual equivalence graphs and the isomorphism is given by the $Q$ tableaux in Kraśkiewicz insertion.

**Thm.** Shifted dual equivalence graphs can be axiomitized using a local rule and a commutation rule.

Could turn this talk around and start with the axioms for shifted dual equivalence and prove the transition rule hold for type B Stanley symmetric functions.
Open Problems

1. What does the (signed) Little bijection mean geometrically?

2. Is there a Little bump algorithm for all Weyl/Coxeter groups? 
   See (Lam-Shimozono, 2005) for affine type $A$.

3. How can Coxeter-Knuth relations be defined independent of Lie type?

4. How can the Little bump algorithm be useful for Schubert calculus in 
   analogy with jeu de taquin for Schur functions?