Coxeter-Knuth Graphs and a signed Little Bijection

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AMS-MAA Joint Meetings
January 17, 2014

Outline

Based on joint work with *Zach Hamaker*, *Austin Roberts*, and *Ben Young*.

1. Transition Equations for Schubert classes of classical types
2. (signed) Little Bijection on Reduced words
3. Kraśkiewicz insertion and Coxeter-Knuth moves on Reduced Words
4. Shifted Dual Equivalence graphs
5. Open problems

Background

- $\boldsymbol{G}$= Classical Complex Lie Group: $\boldsymbol{SL_n, SP_{2n}, SO_{2n+1}, SO_{2n}}$
- $\boldsymbol{B}$ = Borel subgroup = upper triangular matrices in $\boldsymbol{G}$
- $\boldsymbol{W}$ = Weyl groups $\subset \boldsymbol{B_n}$ = signed permutations
- $\boldsymbol{S}$ = simple reflections generating $\boldsymbol{W}$
- $\boldsymbol{T = \{wsw^{-1} : s \in S, w \in W\}}$ $\subset$ signed transpositions

Monk/Chevalley formula

Fact.: The Schubert classes $\mathfrak{S}_w,\ w \in W$ form a basis for $H^*(G/B, \mathbb{Q})$ and they satisfy Monk/Chevalley's formula

$$\mathfrak{S}_w \cdot \mathfrak{S}_s = \sum_{t \in T : l(wt) = l(w)+1} -(\mathfrak{S}_s, f_t) S_{wt},$$

for any simple reflection s where f_t is the linear polynomial negated by t.

Defn. *Transition Equations* are recurrences for Schubert classes derived from Monk/Chevalley's formula.

Transition Equations

Type A. (Lascoux-Schützenberger,1984) For all $w \neq id$, let $(r < s)$ be the largest pair of positions inverted in w in lexicographic order. If w has only one descent, then $\mathfrak{S}_w$ is the Schur polynomial $s_{\lambda(w)}(x_1, ..., x_r)$. Otherwise,

$$\mathfrak{S}_w = x_r \mathfrak{S}_v + \sum \mathfrak{S}_{w'}$$

where the sum is over all w' such that $l(w) = l(w')$ and $w' = wt_{rs}t_{ir}$ with $0 < i < r$. Call this set $T(w)$.

Types B, C, D. Transition equations are similar except some coefficients can be 2's (Billey,'95) and base cases are Schur P- or Q-functions (Pragacz,'91).

Stanley Symmetric Functions

Notation. For $w = [w_1, ..., w_n] \in S_n$, let $R(w)$ be the set of reduced words for w. Let $1 \times w = [1, (1 + w_1), ..., (1 + w_n)]$.

Defn. The *Stanley symmetric function* F_w is equivalently given by

1. (L-S) $F_w(x_1, x_2, \ldots) = \lim_{n\to\infty} \mathfrak{S}_{(1^n \times w)}$.

2. (L-S) If w has at most 1 descent, $F_w = s_{\lambda(w)}$. Otherwise,
$$F_w = \sum_{w' \in T(1\times w)} F_{w'}.$$

3. (Stanley, 1984) If $a = (a_1, ..., a_p) \in R(w)$, let $I(a)$ be the set of weakly increasing positive integer sequences $(i_1, ..., i_p)$ such that $i_j < i_{j+1}$ whenever $a_j < a_{j+1}$. Then
$$F_w = \sum_{a\in R(w)} \sum_{(i_1,\ldots,i_p)\in I(a)} x_{i_1} x_{i_2} \cdots x_{i_p}.$$

Combinatorics from Schubert Calculus

Cor. If w has at most 1 descent then $|R(w)| = f^{\lambda(w)}$ where f^{λ} is the number of standard tableaux of shape λ. Otherwise, $|R(w)| = \sum_{w' \in T(1 \times w)} |R(w')|$.

Cor. $|R(w)| = \sum_{\lambda} a_{\lambda,w} f^{\lambda}$.

Thm.(Edelman-Greene, 1987) The coefficient $a_{\lambda,w}$ counts the number of distinct P tableaux that arise when inserting all reduced words for w via EG-insertion (variation on RSK). Furthermore, for each such P and standard tableau Q of the same shape, there exists a unique $a \in R(w)$ which inserts to P and has recording tableau Q.

Question. Is there a bijection from $R(w)$ to $\cup_{w' \in T(w)} R(w')$ which preserves the descent set and the Q tableau?

Little's Bijection

Question. Is there a bijection from $R(w)$ to $\cup_{w' \in T(w)} R(w')$ which preserves the descent set and the Q tableau?

Answer. Yes! It's called Little's bijection named for David Little (Little, 2003) + (Hamaker-Young, 2013).

Thomas Lam's Conjecture.(proved by Hamaker-Young, 2013) Every reduced word for any permutation with the same Q tableau is connected via Little bumps. Every communication class under Little bumps contains a unique reduced word for a unique fixed point free Grassmannian permutation.

Thus, the Little bumps are analogous to jeu de taquin!

Stanley Symmetric Functions

Notation. If $w = [w_1, ..., w_n]$ is a signed permutation, then w is *increasing* if $w_1 < w_2 < \ldots < w_n$.

Defn. The type C Stanley symmetric function F_w for $w \in B_n$ are given by the following equivalent conditions:

1. (Billey-Haiman, 1994)(T.K.Lam, 1994)(Fomin-Kirilov, 1996)
   Let $R(w)$ be the set of reduced words for $w \in B_n$. If $a = (a_1, ..., a_p) \in R(w)$, let $I(a)$ be the set of weakly increasing positive integer sequences $(i_1, ..., i_p)$ such that $i_{j-1} = i_j = i_{j+1}$ only occurs if a_j is not bigger than both a_{j-1} and a_{j+1}. Then

$$F_w^C = \sum_{a \in R(w)} \sum_{i=(i_1,\ldots,i_p) \in I(a)} 2^{|i|} x_{i_1} x_{i_2} \cdots x_{i_p}.$$

2. (Billey, 1996) If w is increasing, $F_w = Q_{\mu(w)}$. Otherwise,

$$F_w^C = \sum_{w' \in T(w)} F_{w'}^C.$$

Combinatorics from Schubert Calculus

Cor. If w is not increasing, $|R(w)| = \sum_{w' \in T(w)} |R(w')|$

Cor. $|R(w)| = \sum_{\lambda} c_{\lambda,w} g^{\lambda}$ where g^{λ} is the number of *shifted* standard tableaux of shape λ and $c_{\lambda,w}$ is a nonnegative integer.

Thm.(Kraśkiewicz, 1995) The coefficient $b_{\lambda,w}$ counts the number of distinct P tableaux that arise when inserting all reduced words for w via Kraśkiewicz-insertion (variation on EG). Furthermore, for each such P and standard tableau Q of the same shape, there exists a unique $a \in R(w)$ which inserts to P and has recording tableau Q. (See also Haiman evacuation procedure.)

Question. Is there a bijection from $R(w)$ to $\cup_{w' \in T(w)} R(w')$ which preserves the *peak* set and the Q tableau for signed permutations?

Signed Little Bijection

Question. Is there a bijection from $R(w)$ to $\cup_{w' \in T(w)} R(w')$ which preserves the peak set and the Q tableau for signed permutations?

Answer. Yes!

Thm. (Billey-Hamaker-Roberts-Young, 2014)

- The signed Little bijection preserves peak sets, shifted recording tableaux under Kraśkiewicz insertion and realizes the bijection given by the transition equation.
- Every reduced word for any signed permutation with the same Q tableau is connected via signed Little bumps.
- Every communication class under signed Little bumps contains a unique reduced word for a unique increasing signed permutation.

Again, the signed Little bumps are analogous to jeu de taquin on shifted shapes!

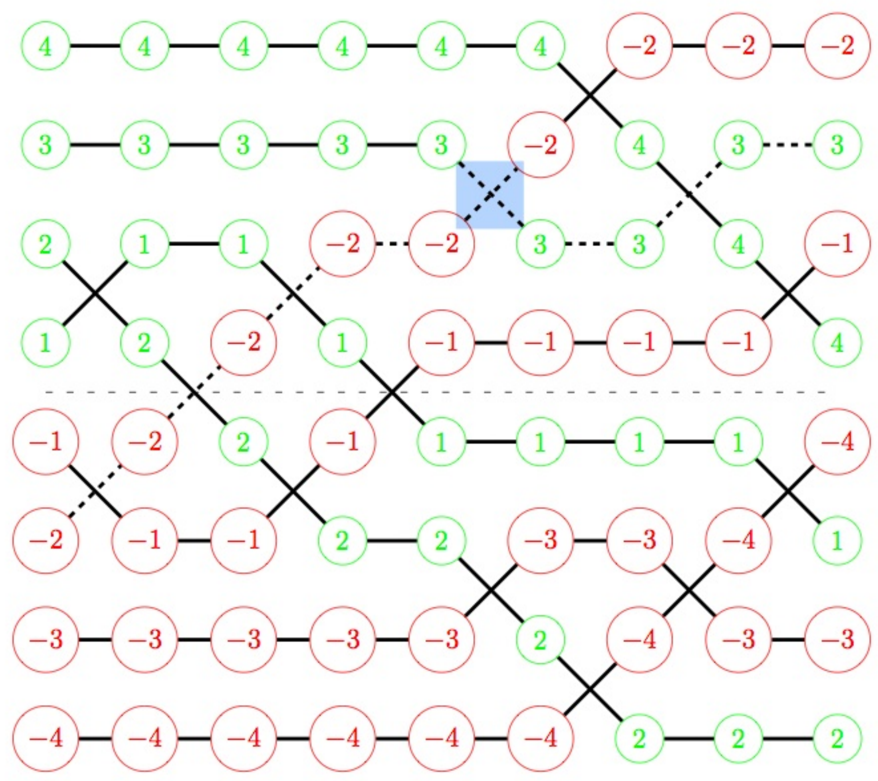
The Algorithm

Given a reduced word, there is an associated reduced wiring diagram. If removing a crossing leaves another reduced wiring diagram, that crossing is a candidate to initiate a Little bump either pushing up or down. Pushing down (up) means reduce (increase) the corresponding letter in the word by 1.

- Check if the resulting word is reduced. If so, stop and return the new word.
- Otherwise, find the other point where the same two wires cross, and push them in the same direction.

Signed Little Bijection. Initiate a Little bump at the crossing (r, s) corresponding to the lex largest inversion. This maps $R(w)$ to $\cup_{w' \in T(w)} R(w')$ bijectively.

The Algorithm in Pictures



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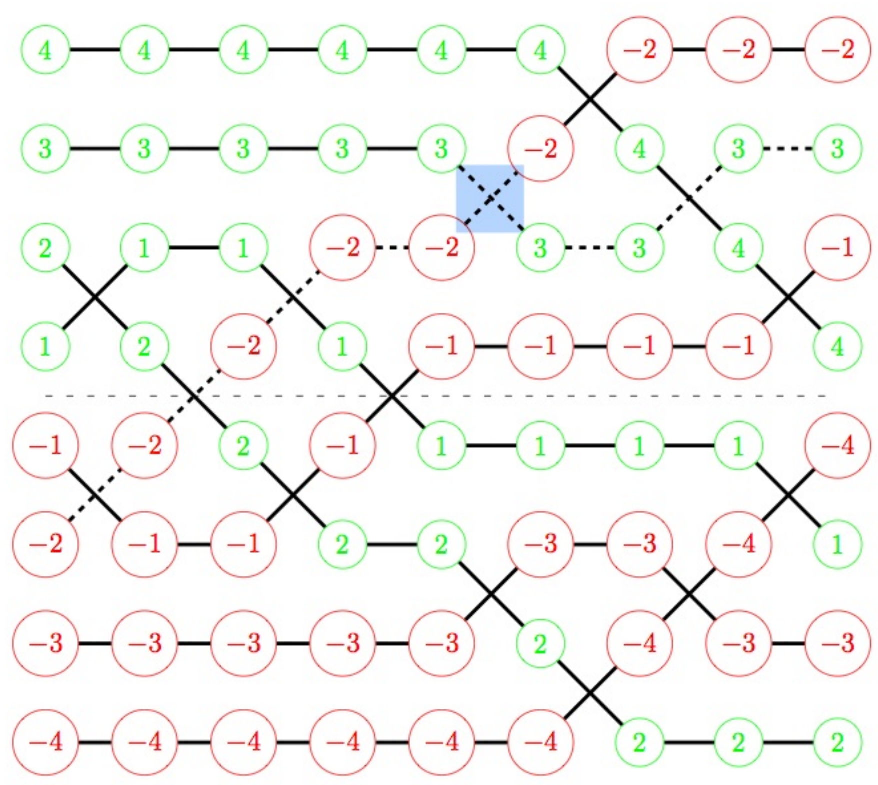 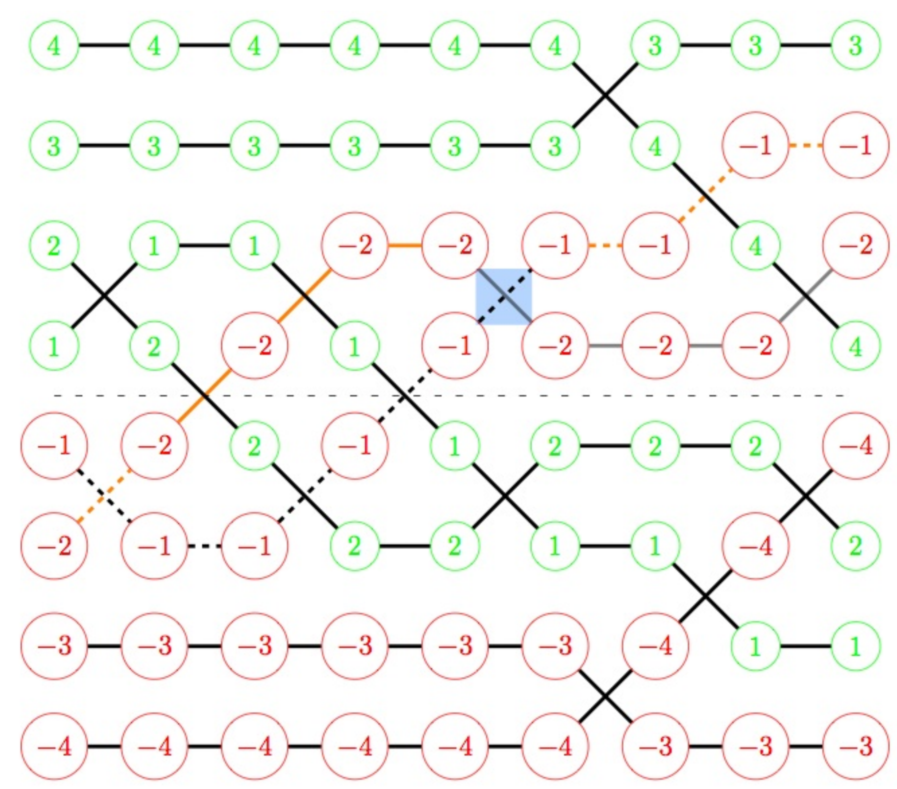

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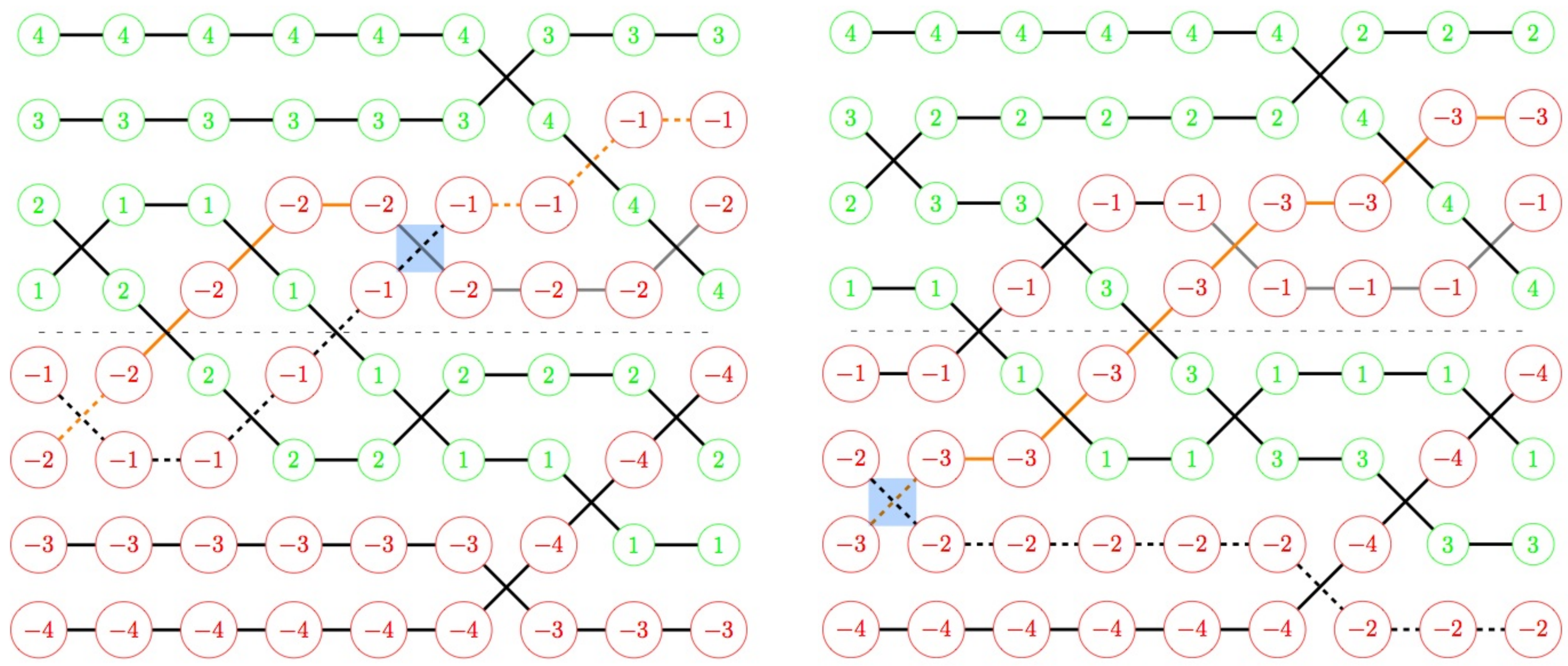

Coxeter-Knuth Relations

The proof that (signed) Little bumps preserves Q tableaux uses a key lemma about how the Q changes with elementary Coxeter relations.

Coxeter relations. (commutation relations) $ij \equiv ji$ when $|i - j| > 1$, (braid relations) $0101 \equiv 1010$ and $i(i+1)i \equiv (i+1)i(i+1)$ when $i > 0$.

Defn. *Coxeter-Knuth* moves on reduced words are minimally witnessed Coxeter relations that preserve P tableau.

Type A Coxeter-Knuth Moves. (EG) Braids and witnessed commutations: $bac \equiv bca$, and $acb \equiv cab$ for $a < b < c$.

Type B/C Coxeter-Knuth relations. (K) 9 rules on windows of length 4 including $0101 \equiv 1010$ and $a(b+1)b(b+1) \equiv ab(b+1)b$ for $a < b$, but no other braids.

Coxeter-Knuth Graphs and Dual Equivalence

Defn. The Coxeter-Knuth graph for w has $V = R(w)$ and two reduced words are connected by an edge labeled i if they agree in all positions except for a single Coxeter-Knuth relation starting in position i.

Defn. (Assaf, 2008) Dual equivalence graphs are graphs with labeled edges whose connected components are isomorphic to the graph on standard tableaux of a fixed partition shape with an edge labeled i connecting any two vertices which differ by a transposition (i,i+1) or (i+1,i+2) with the third number on a diagonal in between the transposing pair.

Coxeter-Knuth Graphs and Dual Equivalence

Thm. The Coxeter-Knuth graphs in type A are dual equivalence graphs and the isomorphism is given by the Q tableaux in Edelman-Greene insertion.

In type A, this is a nice corollary of (Roberts, 2013) + (Hamaker-Young, 2013).

Thm. The Coxeter-Knuth graphs in type B are shifted dual equivalence graphs and the isomorphism is given by the Q tableaux in Kraśkiewicz insertion.

Thm. Shifted dual equivalence graphs can be axiomitized using a local rule and a commutation rule.

Could turn this talk around and start with the axioms for shifted dual equivalence and prove the transition rule hold for type B Stanley symmetric functions.

Open Problems

1. What does the (signed) Little bijection mean geometrically?

2. Is there a Little bump algorithm for all Weyl/Coxeter groups?
   See (Lam-Shimozono, 2005) for affine type A.

3. How can Coxeter-Knuth relations be defined independent of Lie type?

4. How can the Little bump algorithm be useful for Schubert calculus in analogy with jeu de taquin for Schur functions?