Consequences of the Lakshmibai-Sandhya Theorem

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Thanks!

The AWM has played a central role in improving the lives of women and men in mathematics. Keep up the good work!

Also, thanks to Rebecca Goldin and Julianna Tymoczko for organizing this session!

Outline

1. Schubert varieties
2. Pattern Avoidance in Permutations
3. The Lakshmibai-Sandhya Theorem and its Consequences
4. Open Problems

Enumerative Geometry

Approximately 150 years ago... Grassmann, Schubert, Pieri, Giambelli, Severi, and others began the study of *enumerative geometry*.

Early questions:

- What is the dimension of the intersection between two general lines in $\mathbb{R}^2$?
- How many lines intersect two given lines and a given point in $\mathbb{R}^3$?
- How many lines intersect four given lines in $\mathbb{R}^3$?

Modern questions:

- How many points are in the intersection of 2,3,4,... Schubert varieties in general position?

Why Study Schubert Varieties?

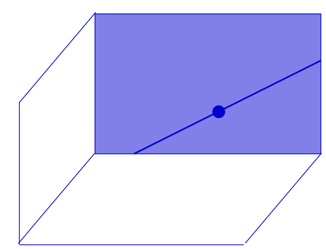
1. It can be useful to see points, lines, planes etc as families of Schubert varieties with certain properties.
2. Schubert varieties provide interesting examples for test cases and future research in algebraic geometry, combinatorics, representation theory, symplectic geometry, and theoretical physics.
3. Applications in discrete geometry, computer graphics, and computer vision.

The Flag Manifold

Defn. A *complete flag* $F_\bullet = (F_1, \dots, F_n)$ in $\mathbb{C}^n$ is a nested sequence of vector spaces such that $\dim(F_i) = i$ for $1 \le i \le n$. $F_\bullet$ is determined by an ordered basis $\langle f_1, f_2, \dots f_n \rangle$ where $F_i = \mathrm{span}\langle f_1, \dots, f_i \rangle$.

Example.

$$F_\bullet = \langle 6e_1 + 3e_2,\quad 4e_1 + 2e_3,\quad 9e_1 + e_3 + e_4,\quad e_2 \rangle$$



The Flag Manifold

Canonical Form. Every flag can be represented as a matrix in row echelon form.

$$F_\bullet = \langle 6e_1 + 3e_2,\quad 4e_1 + 2e_3,\quad 9e_1 + e_3 + e_4,\quad e_2 \rangle$$

$$\approx \begin{bmatrix} 6 & 3 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 9 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\approx \langle 2e_1 + e_2,\quad 2e_1 + e_3,\quad 7e_1 + e_4,\quad e_1 \rangle$$

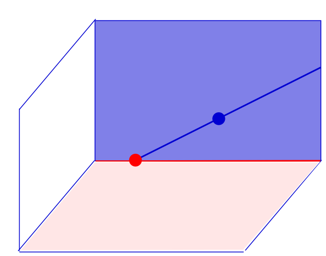
$\mathcal{F}l_n(\mathbb{C}) :=$ *flag manifold* over $\mathbb{C}^n = \{$complete flags $F_\bullet\}$

$= B \setminus GL_n(\mathbb{C}), \quad B =$ lower triangular mats.

Flags and Permutations

Example. $F_\bullet = \langle 2e_1 + e_2,\quad 2e_1 + e_3,\quad 7e_1 + e_4,\quad e_1 \rangle \approx \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

Note. If a flag is written in canonical form, the positions of the leading 1's form a permutation matrix. There are 0's to the right and below each leading 1. This permutation determines the *position* of the flag $F_\bullet$ with respect to the reference flag $E_\bullet = \langle e_1,\ e_2,\ e_3,\ e_4 \rangle$.

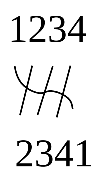


Many ways to represent a permutation

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix} = 2341 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

matrix notation — two-line notation — one-line notation — rank table

$$\begin{array}{cccc} * & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} =$$



$$= (1, 2, 3)$$

diagram of a permutation — string diagram — reduced word

The Schubert Cell $C_w(E_\bullet)$ in $\mathcal{F}l_n(\mathbb{C})$

Defn. $C_w(E_\bullet)$ = All flags $F_\bullet$ with $\mathbf{position}(E_\bullet, F_\bullet) = w$

$$= \{F_\bullet \in \mathcal{F}l_n \mid \dim(E_i \cap F_j) = \mathrm{rk}(w[i,j])\}$$

Example. $F_\bullet = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \in C_{2341} = \left\{ \begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \\ * & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} : * \in \mathbb{C} \right\}$

Easy Observations.

- $\mathtt{dim}_{\mathbb{C}}(C_w) = l(w) = \#$ inversions of w.
- $C_w = w \cdot B$ is a B-orbit using the right B action, e.g.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_{1,1} & 0 & 0 & 0 \\ b_{2,1} & b_{2,2} & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & 0 \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} \end{bmatrix} = \begin{bmatrix} b_{2,1} & b_{2,2} & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & 0 \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} \\ b_{1,1} & 0 & 0 & 0 \end{bmatrix}$$

The Schubert Variety $X_w(E_\bullet)$ in $\mathcal{F}l_n(\mathbb{C})$

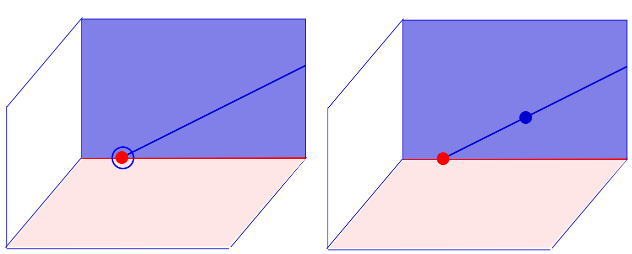
Defn. $X_w(E_\bullet)$ = Closure of $C_w(E_\bullet)$ under the Zariski topology

$$= \{F_\bullet \in \mathcal{F}l_n \mid \dim(E_i \cap F_j) \geq \mathrm{rk}(w[i,j])\}$$

where $E_\bullet = \langle e_1, e_2, e_3, e_4 \rangle$.

Example. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \\ 0 & * & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \in X_{2341}(E_\bullet) = \overline{\left\{ \begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \\ * & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right\}}$

Why?.



Combinatorics and Geometry

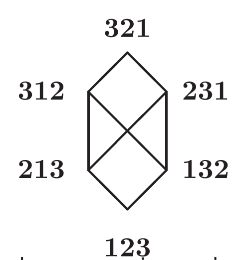
Fact. The closure relation on Schubert varieties defines a nice partial order.

$$X_w = \bigcup_{v \leq w} C_v \quad = \bigcup_{v \leq w} X_v$$

Bruhat order (Ehresmann 1934, Chevalley 1958) is the transitive closure of

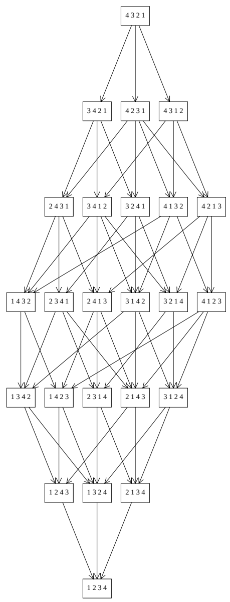
$$w < wt_{ij} \iff w(i) < w(j).$$

Example. Bruhat order on permutations in S_3.

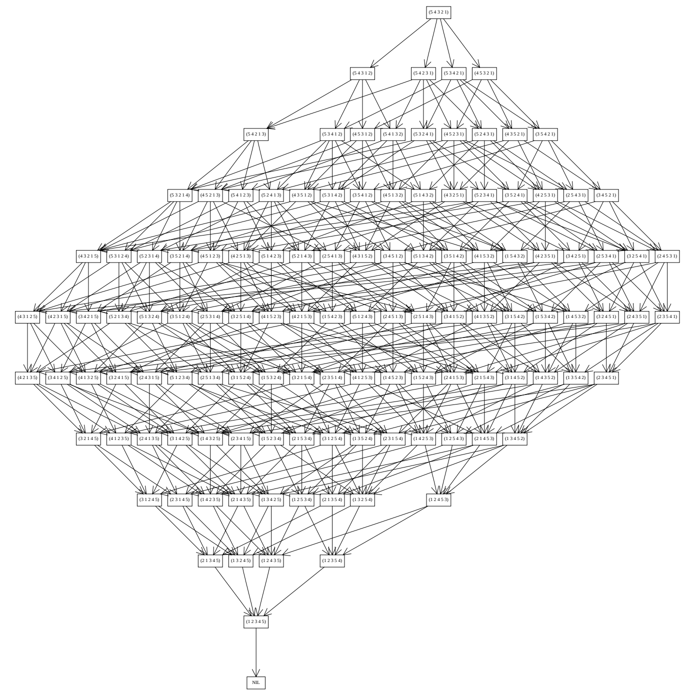


Observations. Self dual, rank symmetric, rank unimodal.

Bruhat order on S_4



Bruhat order on S_5



Poincaré polynomials

Fact. The *Poincaré polynomial* for $H^*(X_w)$ is $P_w(t) = \sum_{v \leq w} t^{l(v)}$.

Example. $w = 3412$

$$
\begin{array}{ll}
4: & (3412) \\
3: & (3142)(3214)(1432)(2413) \\
2: & (3124)(1342)(2143)(2314)(1423) \\
1: & (2134)(1243)(1324) \\
0: & (1234)
\end{array}
$$

Poincaré polynomial:. $P_{3412}(t) = 1 + 3t + 5t^2 + 4t^3 + t^4$.

10 Fantastic Facts on Bruhat Order

1. Bruhat Order Characterizes Inclusions of Schubert Varieties
2. Contains Young's Lattice in S_∞
3. Nicest Possible Möbius Function
4. Beautiful Rank Generating Functions
5. $[x, y]$ Determines the Composition Series for Verma Modules
6. Symmetric Interval $[\hat{0}, w] \iff X(w)$ rationally smooth
7. Order Complex of (u, v) is shellable
8. Rank Symmetric, Rank Unimodal and k-Sperner
9. Efficient Methods for Comparison
10. Amenable to Pattern Avoidance

Lakshmibai-Sandhya Theorem

Fact. There exists a simple criterion for characterizing smooth Schubert varieties using pattern avoidance.

Theorem: Lakshmibai-Sandhya 1990 (see also Haiman, Ryan, Wolper)
X_w is non-singular $\iff$ w has no subsequence with the same relative order as 3412 and 4231.

$w = 625431$ contains $6241 \sim 4231$ $\Longrightarrow$ X_{625431} is singular
Example: $w = 612543$ avoids 4231 $\&3412$ $\Longrightarrow$ X_{612543} is non-singul

21 Years Later . . .

Consequences of the Lakshmibai-Sandhya Theorem:

1. Testing for smoothness of Schubert varieties can be done in polynomial time, $O(n^4)$.
2. There is an explicit formula for counting the number v_n of smooth Schubert varieties for $w \in S_n$ due to Haiman (see also Bousquet-Mélou+Butler):

$$V(t) = \frac{1 - 5t + 3t^2 + t^2\sqrt{1-4t}}{1 - 6t + 8t^2 - 4t^3}$$
$$= t + 2t^2 + 6t^3 + 22t^4 + 88t^5 + 366t^6 + 1552t^7 + 6652t^8 + O$$

3. Many geometrical properties of Schubert varieties are now characterized by pattern avoidance or a variation on this theme.

Let me tell you about 10 of them!

10 Pattern Properties

Prop 1. (Carrell-Peterson, Deodhar, Gasharov) (ca 1994)
The following are equivalent

1. X_w is smooth.
2. w avoids 3412 and 4231.
3. The Kazhdan-Lusztig polynomial $P_{id,w} = 1$.
4. The Bruhat graph for w is regular.
5. The Poincare polynomial for w, $P_w(t) = \sum_{v \leq w} t^{l(v)}$ is palindromic.
6. The Poincare polynomial for w factors nicely

$$P_w(t) = \prod_{i=1}^{k} (1 + t + t^2 + \cdots + t^{e_i})$$

Example. $P_{4321}(t) = (1+t)(1+t+t^2)(1+t+t^2+t^3)$

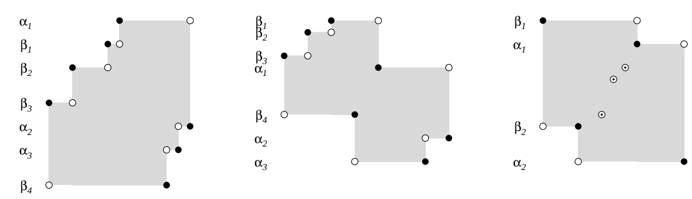
10 Pattern Properties

Prop 2. (Billey-Warrington, Manivel, Kassel-Lascoux-Reutenauer, Cortez) (2000)

X_v is an irreducible component of the singular locus of X_w $\iff$

$$v = w \cdot (\text{1-cycle permutation})$$

corresponding to a 4231 or 3412 or 45312 pattern of the following form



(1) (2) (3)

Here ○'s denote 1's in w, ●'s denote 1's in v.

10 Pattern Properties

Thm. (Zariski) X is a smooth variety $\iff$ the local ring at every point is regular.

Def. X is *factorial* at a point $\iff$ the local ring at that point is a unique factorization domain.

Prop 3. (Bousquet-Mélou+Butler, 2007, conj. by Woo-Yong)

X_w is factorial at every point $\iff$ w avoids $\mathbf{4231}$ and $\mathbf{3\underline{41}2}$.

10 Pattern Properties

Prop 4. There exists a simple criterion for characterizing Gorenstein Schubert varieties using modified pattern avoidance.

Def. X is *Gorenstein* if it is CM and its canonical sheaf is a line bundle.

Theorem: Woo-Yong (2004)

X_w is Gorenstein $\iff$

- w avoids $\mathbf{31542}$ and $\mathbf{24153}$ *with Bruhat restrictions* $\{t_{15}, t_{23}\}$ and $\{t_{15}, t_{34}\}$
- for each descent d in w, the associated partition $\lambda_d(w)$ has all of its inner corners on the same antidiagonal.

10 Pattern Properties

Prop 5. (Gasharov-Reiner, 2002)
X_w is defined by inclusions of the form $F_i \subset \mathbf{span}(e_1, \ldots, e_j)$ or $\mathbf{span}(e_1, \ldots, e_j) \subset F_i$ $\iff$ w avoids

$$\mathbf{4231, 35142, 42513, 351624.}$$

Gasharov-Reiner show that Schubert varieties defined by inclusions have a nice presentation for their cohomolog ring.

10 Pattern Properties

Prop 6. (Deodhar, Billey-Warrington, 1998)
The following are equivalent

1. The Bott-Samelson resolution of X_w is small.
2. w is 321-hexagon avoiding, i.e. avoids

$$\mathbf{321, 56781234, 56718234, 46781235, 46718235}$$

3. $\sum_{v \le w} t^{l(v)} P_{v,w}(t) = (1+t)^{l(w)}$.
4. For each $v \le w$, the Kazhdan-Lusztig polynomial $P_{v,w}(t) = \sum_{\sigma \in E(v,w)} t^{\mathbf{defec}}$

10 Pattern Properties

Prop 7. (Tenner, 2006)

The principle order ideal below w in Bruhat order is a Boolean lattice $\iff$ w is $\mathbf{321}$ and $\mathbf{3412}$ avoiding.

Note: Boolean permutations are 321-hexagon avoiding.

10 Pattern Properties

Prop 8. (Woo, 2009)
The Kazhdan-Lusztig polynomial $P_{id,w}(1) = 2 \iff w$ avoids 653421, 632541, 463152, 526413, 546213, and 465132 and the singular locus of X_w has exactly 1 component.

Def. $KL_m = \{w \in S_\infty \mid P_{id,w}(1) \leq m\}$.

Example. KL_1 are the permutations indexing smooth Schubert varieties.

Extension (Billey-Weed): KL_2 is characterized by 66 permutation pattern on 5,6,7 or 8 entries.

Open: KL_m is closed under taking patterns. Can it always be described by a finite set of patterns?

10 Pattern Properties

Prop 9. (Billey-Postnikov, Billey-Braden, 2003) Pattern avoidance can be generalized to all Coxeter groups using inversion sets of positive roots.

In particular, for all semisimple simple, connected Lie groups G and Borel subgroups B, the smooth and rationally smooth Schubert varieties in G/B can be characterized by avoiding certain generalized patterns. Only requires checking patterns of types $A_3, B_2, B_3, C_2, C_3, D_4, G_2$.

10 Pattern Properties

Prop 10. Patterns are useful to describe smooth and rationally smooth Schubert varieties in the affine flag manifold.

Def. Let G be the Kac Moody group of type $\widetilde{A}_{n-1}$ and $I =$ Iwahori subgroup of G, then G/I is the *affine flag manifold*.

Equivalently, $G/I \approx \widetilde{G}/\widetilde{B}$ where

$$\widetilde{G} = SL_n(\mathbb{C}[t, t^{-1}]), \qquad \widetilde{B} = \{M \in \widetilde{G} \mid M|_{t=0} \in B\}.$$

10 Pattern Properties

In the affine flag manifold,

- Affine Schubert varieties are indexed by affine permutations $w \in \widetilde{S}_n$:

$$w : \mathbb{Z} \longrightarrow \mathbb{Z}$$

  s.t. $w(i+n) = w(i)+n \; \forall i$ and $w(1)+w(2)+\cdots+w(n) = \binom{n+1}{2}$.

- An affine permutation w contains a classical permutation v if there exists a subsequence of w with the same relative order as $v = v_1 \dots v_k$.

10 Pattern Properties

Thm. (Billey-Crites, 2011+)
An affine Schubert variety X_w is rationally smooth $\iff$ $w \in \widetilde{S}_n$ avoids 3412 and 4231 or w is a twisted spiral permutation of length $k(n-1)$ for some $k > 1$.

Thm. (Chen-Crites-Kuttler, manuscript)
An affine Schubert variety X_w is smooth $\iff$ $w \in \widetilde{S}_n$ avoids 3412 and 4231. Furthermore, the tangent space to X_w at the identity can be described in terms of reflection over real and imaginary roots.

Open Problems

1. Give a pattern based algorithm to produce the factorial and/or Gorenstein locus of a Schubert variety.
2. Describe the maximal singular locus of a Schubert variety for other semisimple Lie groups using generalized pattern avoidance.
3. Find a method to "learn" marked mesh patterns by computer.
4. Conjecture (Woo): The Schubert varieties with multiplicity ≤ 2 can be characterized by pattern avoidance.
5. (From Úlfarsson) Is there a nice generating function to count the number of factorial and/or Gorenstein permutations.
6. Find a geometric explanation why a finite number of patterns suffice in all cases above.
7. What is the right notion of patterns for GKM spaces?

Some Recommended Further Reading

1. Tenner's Database: http://math.depaul.edu/bridget/patterns.html
2. "A Unification Of Permutation Patterns Related To Schubert Varieties" by Henning Úlfarsson, arXiv:1002.4361v6.