Schubert Varieties Under a Microscope

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Famous Quotations

Arnold Ross (and PROMYS).

"Think deeply of simple things."

Angela Gibney. Why do algebraic geometers love moduli spaces?

"It is just like with people, if you want to get to know someone, go to their family reunion."

Goal. Focus our microscope on a particular family of varieties which are indexed by combinatorial data where lots is known about their structure and yet lots is still open.

Schubert Varieties

A *Schubert variety* is a member of a family of projective varieties which is defined as the closure of some orbit under a group action in a homogeneous space G/H.

Typical properties:

- They are all Cohen-Macaulay, some are "mildly" singular.
- They have a nice torus action with isolated fixed points.
- This family of varieties and their fixed points are indexed by combinatorial objects; e.g. partitions, permutations, or Weyl group elements.

Schubert Varieties

"Honey, Where are my Schubert varieties?"

Typical contexts:

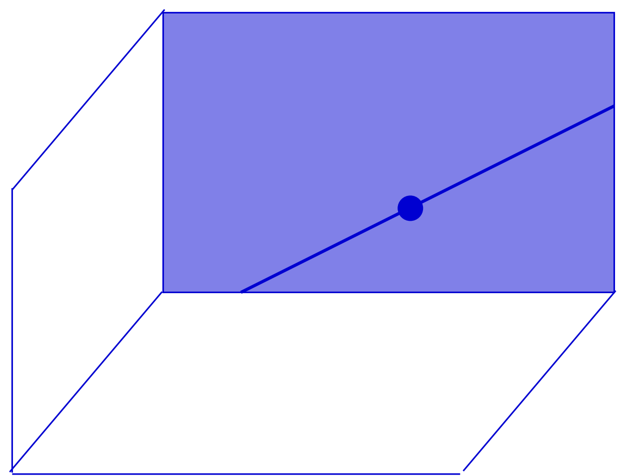
- The Grassmannian Manifold, $G(n, d) = GL_n/P$.
- *The Flag Manifold: Gl_n/B.*
- Symplectic and Orthogonal Homogeneous spaces: Sp_{2n}/B, O_n/P
- Homogeneous spaces for semisimple Lie Groups: G/P.
- Homogeneous spaces for Kac-Moody Groups: G/P.
- Goresky-MacPherson-Kotwitz spaces.

The Flag Manifold

Defn. A *complete flag* $F_\bullet = (F_1, \ldots, F_n)$ in $\mathbb{C}^n$ is a nested sequence of vector spaces such that $\dim(F_i) = i$ for $1 \leq i \leq n$. $F_\bullet$ is determined by an ordered basis $\langle f_1, f_2, \ldots f_n \rangle$ where $F_i = \mathrm{span}\langle f_1, \ldots, f_i \rangle$.

Example.

$$F_\bullet = \langle 6e_1 + 3e_2,\quad 4e_1 + 2e_3,\quad 9e_1 + e_3 + e_4,\quad e_2 \rangle$$



The Flag Manifold

Canonical Form.

$$F_\bullet = \langle 6e_1 + 3e_2, \quad 4e_1 + 2e_3, \quad 9e_1 + e_3 + e_4, \quad e_2 \rangle$$

$$\approx \begin{bmatrix} 6 & 3 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 9 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\approx \langle 2e_1 + e_2, \quad 2e_1 + e_3, \quad 7e_1 + e_4, \quad e_1 \rangle$$

$\mathcal{F}l_n(\mathbb{C}) :=$ *flag manifold* over $\mathbb{C}^n \subset \prod_{k=1}^n G(n,k)$

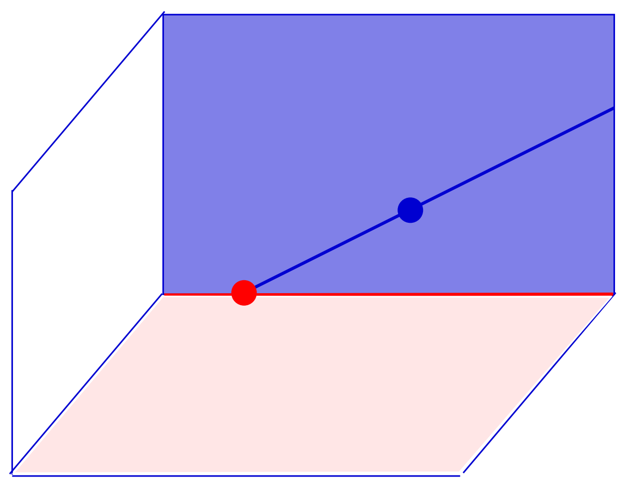
$= \{$complete flags $F_\bullet\}$

$= B \setminus GL_n(\mathbb{C}), \quad B =$ lower triangular mats.

Flags and Permutations

Example. $F_\bullet = \langle 2e_1 + e_2,\ \ 2e_1 + e_3,\ \ 7e_1 + e_4,\ \ e_1 \rangle \approx \begin{bmatrix} 2 & \textcircled{1} & 0 & 0 \\ 2 & 0 & \textcircled{1} & 0 \\ 7 & 0 & 0 & \textcircled{1} \\ \textcircled{1} & 0 & 0 & 0 \end{bmatrix}$

Note. If a flag is written in canonical form, the positions of the leading 1's form a permutation matrix. There are 0's to the right and below each leading 1. This permutation determines the *position* of the flag $F_\bullet$ with respect to the reference flag $E_\bullet = \langle e_1,\ e_2,\ e_3,\ e_4 \ \rangle$.

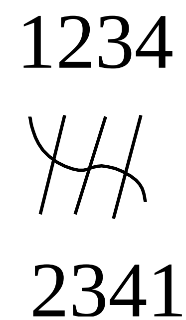


Many ways to represent a permutation

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix} = 2341 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

matrix notation — two-line notation — one-line notation — rank table

$$\begin{array}{cccc} * & . & . & . \\ * & . & . & . \\ * & . & . & . \\ . & . & . & . \end{array} = \qquad =$$



$$= (1, 2, 3)$$

diagram of a permutation — rc-graph — string diagram — reduced word

The Schubert Cell $C_w(E_\bullet)$ in $\mathcal{F}l_n(\mathbb{C})$

Defn. $C_w(E_\bullet) =$ All flags $F_\bullet$ with $\mathbf{position}(E_\bullet, F_\bullet) = w$

$$= \{F_\bullet \in \mathcal{F}l_n \mid \dim(E_i \cap F_j) = \mathrm{rk}(w[i,j])\}$$

Example. $F_\bullet = \begin{bmatrix} 2 & ① & 0 & 0 \\ 2 & 0 & ① & 0 \\ 7 & 0 & 0 & ① \\ ① & 0 & 0 & 0 \end{bmatrix} \in C_{2341} = \left\{ \begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \\ * & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} : * \in \mathbb{C} \right\}$

Easy Observations.

- $\mathtt{dim}_{\mathbb{C}}(C_w) = l(w) = \#$ inversions of w.
- $C_w = w \cdot B$ is a B-orbit using the right B action, e.g.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_{1,1} & 0 & 0 & 0 \\ b_{2,1} & b_{2,2} & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & 0 \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} \end{bmatrix} = \begin{bmatrix} b_{2,1} & b_{2,2} & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & 0 \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} \\ b_{1,1} & 0 & 0 & 0 \end{bmatrix}$$

The Schubert Variety $X_w(E_\bullet)$ in $\mathcal{F}l_n(\mathbb{C})$

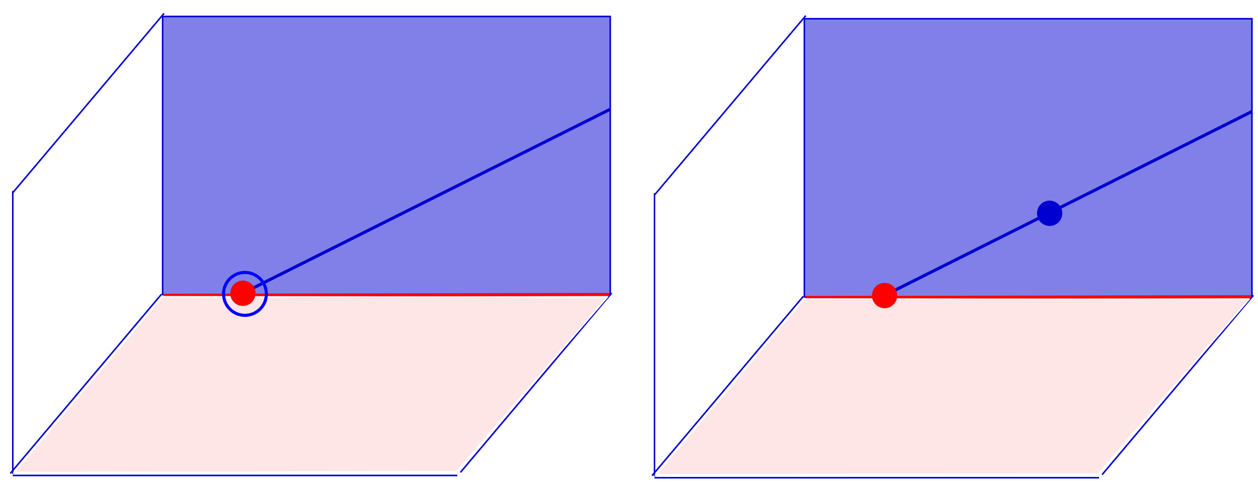
Defn. $X_w(E_\bullet)$ = Closure of $C_w(E_\bullet)$ under the Zariski topology

$$= \{F_\bullet \in \mathcal{F}l_n \mid \dim(E_i \cap F_j) \geq \mathrm{rk}(w[i,j])\}$$

where $E_\bullet = \langle e_1,\ e_2,\ e_3,\ e_4\ \rangle$.

Example. $$\begin{bmatrix} ① & 0 & 0 & 0 \\ 0 & * & ① & 0 \\ 0 & * & 0 & ① \\ 0 & ① & 0 & 0 \end{bmatrix} \in X_{2341}(E_\bullet) = \overline{\left\{ \begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \\ * & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right\}}$$

Why?.



The Main Combinatorial Tool

Bruhat Order. The closure relation on Schubert varieties defines a nice partial order.

$$X_w = \bigcup_{v \leq w} C_v \qquad = \bigcup_{v \leq w} X_v$$

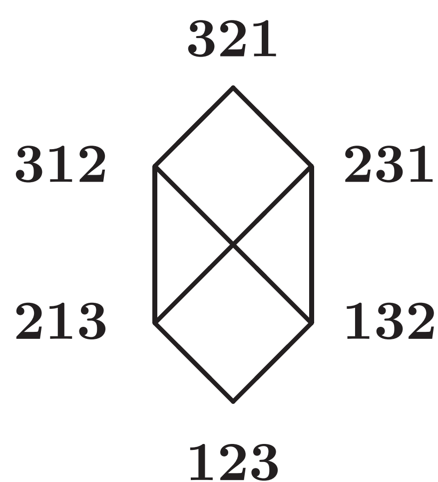
Bruhat order (Ehresmann 1934, Chevalley 1958) is the transitive closure of

$$w < wt_{ij} \iff w(i) < w(j).$$

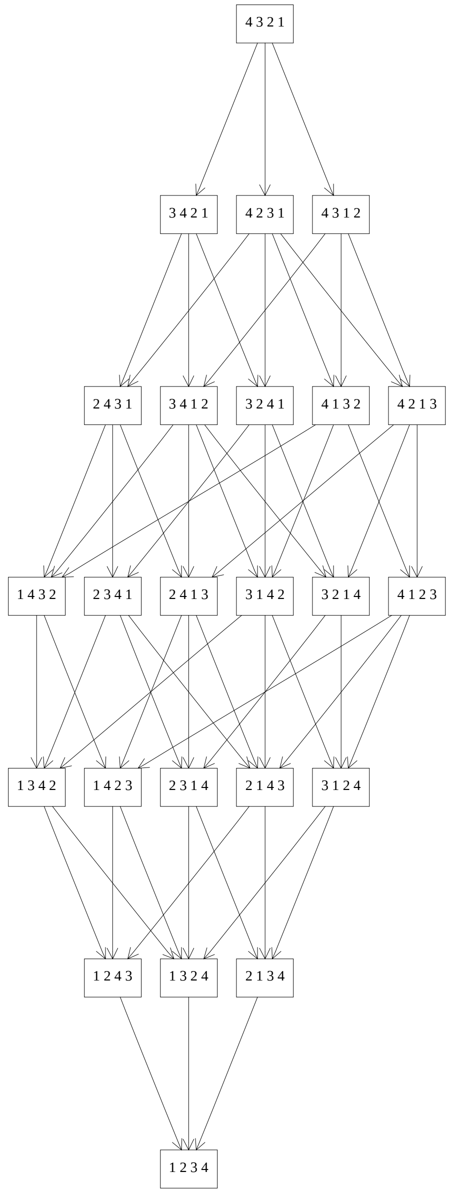
Equivalently,

$$t_{ij}w < w \iff w(i) < w(j).$$

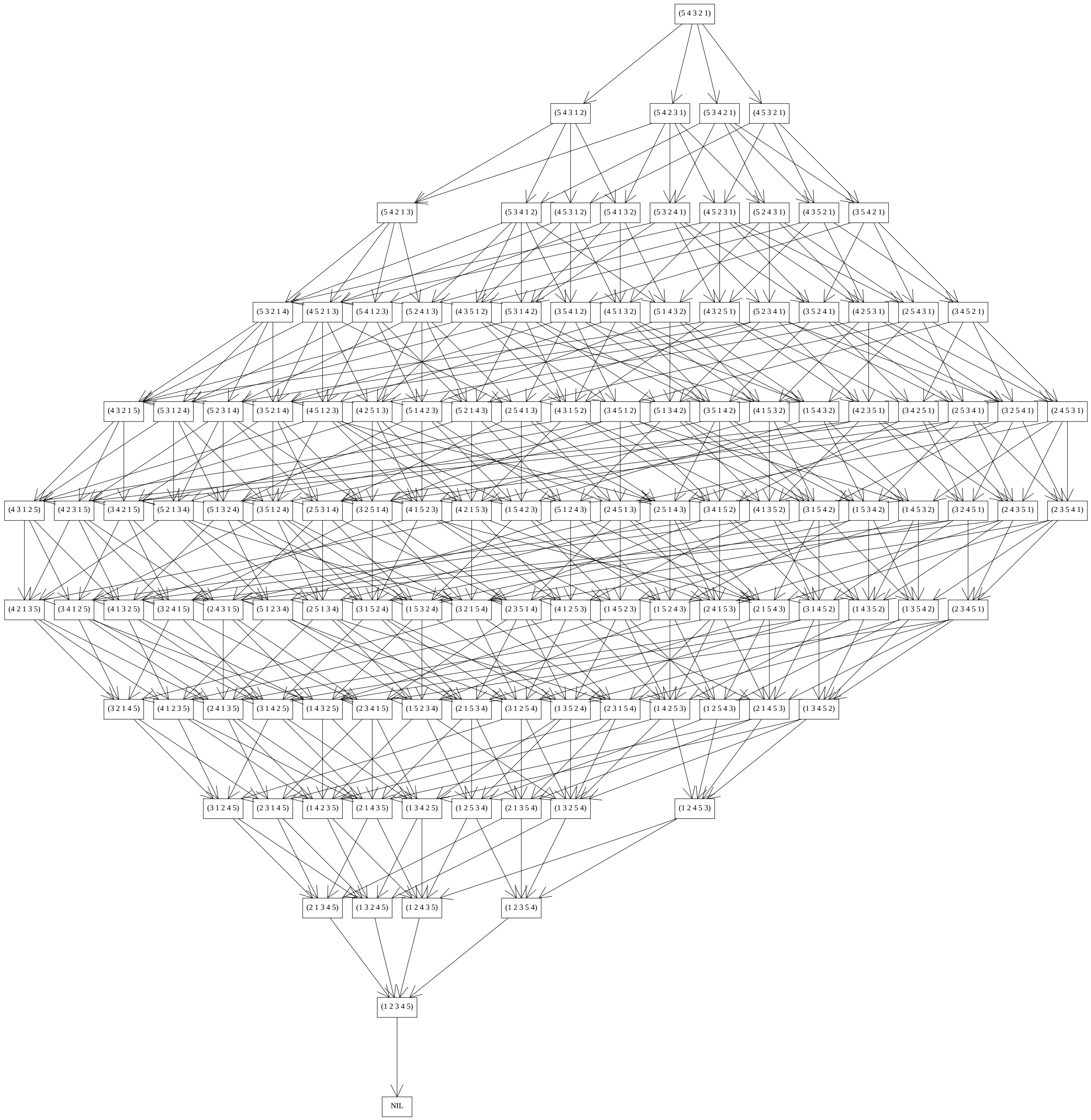
Example. Bruhat order on permutations in S_3.



Bruhat order on S_4



Bruhat order on S_5



Bruhat Order and the Geometry of X_w

$$X_w = \bigcup_{v \le w} C_v \quad = \bigcup_{v \le w} X_v$$

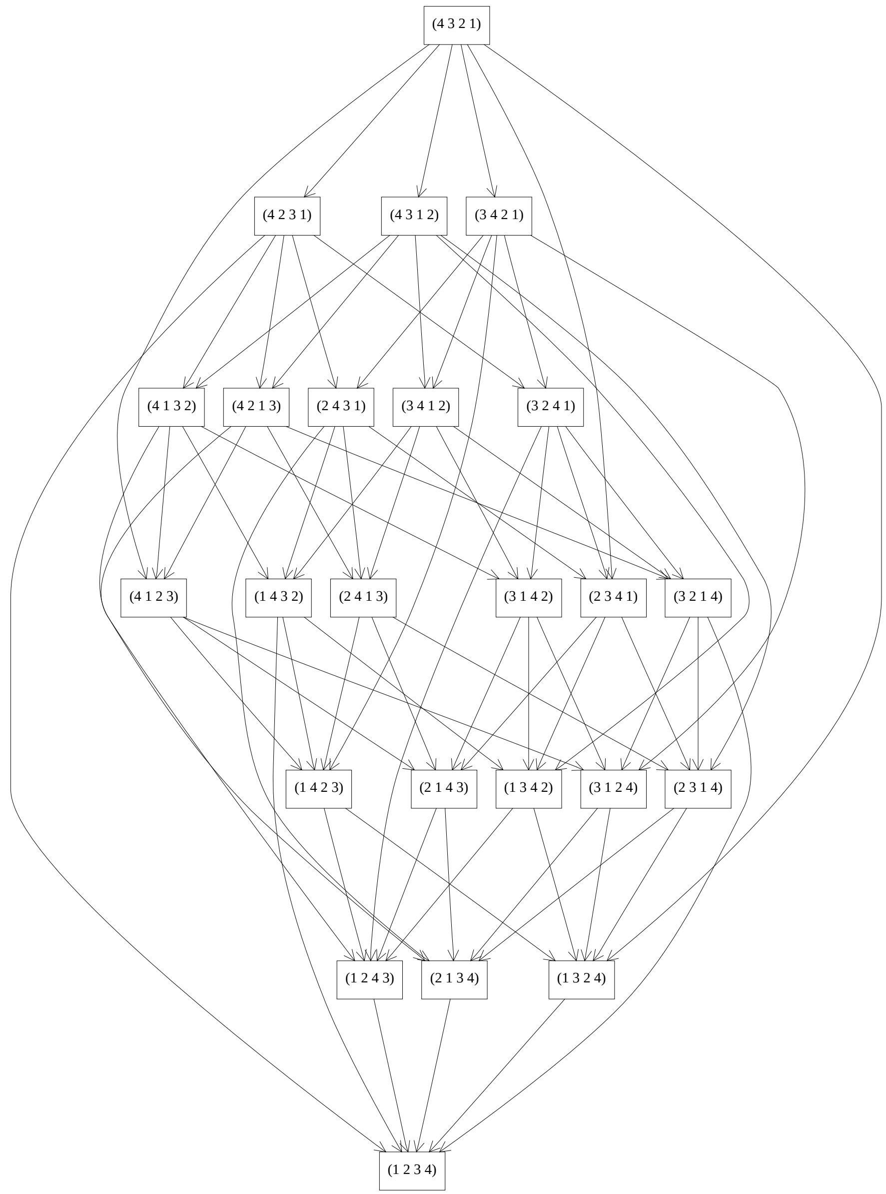
Consequences.

- The Schubert variety $X_{w_0} = \bigcup_{v \le w_0} C_v = \mathcal{F}l_n$ where $w_0 = n \dots 21$.
- The cohomology ring of $H^*(X_w)$ has linear basis $\{[X_v] \mid v \le w\}$. Therefore, the Poincare polynomial of $H^*(X_w)$ is

$$\sum \dim(H^{2k}(X_w))t^k = \sum_{v \le w} t^k.$$

- Let T = invertible diagonal matrices. The T-fixed points in X_w are the permutation matrices indexed by $v \le w$.
- If $u,\ ut_{ij}$ are T-fixed points in X_w they are connected by a T-stable curve. The set of all T-stable curves in X_w are represented by the *Bruhat graph* on $[id, w]$.

Bruhat Graph in S_4



Five Microscopic Focus Points

Focus Point 1. The tangent space to the Schubert variety X_w is completely determined by the Bruhat graph.

- $\mathfrak{g}$ = Lie algebra of $Gl_n = \bigoplus_{1 \leq i,j \leq n} x_{i,j}$ (Chevalley basis)
- $\mathfrak{b}$ = Lie algebra of $B = \bigoplus_{1 \leq j \leq i \leq n} x_{i,j}$.
- The tangent space to any point in GL_n/B looks like $\mathfrak{g}/\mathfrak{b} = \bigoplus_{1 \leq i < j \leq n} x_{i,j}$

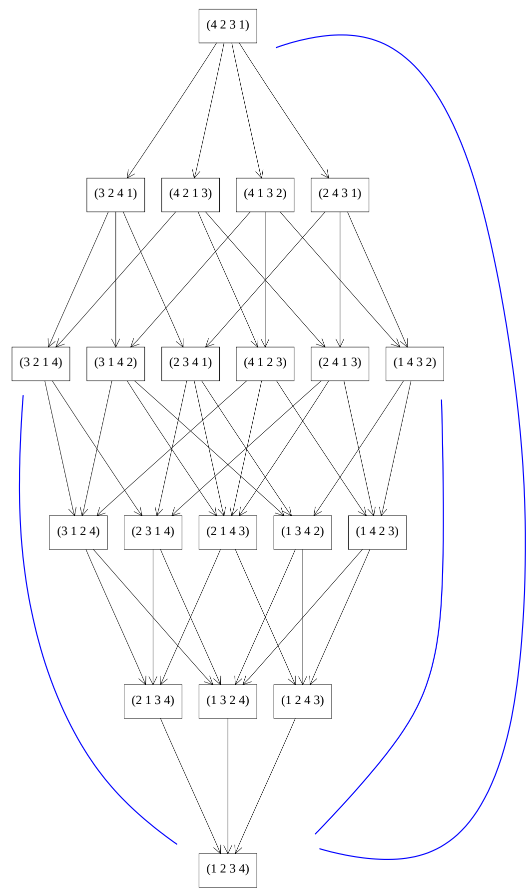
  where $\mathfrak{b}$ is the Lie algebra of B.
- The basis for the tangent space at a point $F_\bullet \in X_w \subset GL_n/B$ is a subset of the Chevalley basis for the whole space. The subset only depends on which Schubert cell C_v contains $F_\bullet$.

Theorem. (Lakshmibai-Seshadri, 1984) For $v, w \in S_n$

$$T_v(X_w) \approx v \cdot \mathrm{span}\{x_{i,j} \mid t_{ij}v \leq w\}.$$

Tangent space of a Schubert Variety

Example. $T_{1234}(X_{4231}) = \text{span}\{x_{i,j} \mid t_{ij} \leq w\}$.



dim$X(4231)$=5 dim$T_{id}(4231) = 6 \implies X(4231)$ is singular!

Five Microscopic Focus Points

Focus Point 2. Simple criteria for characterizing singular Schubert varieties.

Theorem: (Carrell-Peterson,1994)

X_w is non-singular $\iff$ $P_w(t) = \sum_{v \le w} t^{l(v)}$ is palindromic.

Example: $P_{3412}(t) = 1 + 3t + 5t^2 + 4t^3 + t^4 \quad \Longrightarrow \quad X(3412)$ is singular.

Theorem: (Lakshmibai-Sandhya 1990 (see also Haiman, Ryan, Wolpert))

X_w is non-singular $\iff$ w has no subsequence with the same relative order as 3412 and 4231.

	$w = 625431$	contains	$6241 \sim 4231$	$\Longrightarrow$ X_{625431} is singular
Example:	$w = 612543$	avoids	4231 $\&3412$	$\Longrightarrow$ X_{612543} is non-singul

Five Microscopic Focus Points

Consequences.

- (Haiman ca 1990) Let v_n be the number of $w \in S_n$ for which $X(w)$ is non-singular. Then the generating function $V(t) = \sum_n v_n t^n$ is given by

$$V(t) = \frac{1 - 5t + 3t^2 + t^2\sqrt{1-4t}}{1 - 6t + 8t^2 - 4t^3}.$$

- (Billey-Warrington, Kassel-Lascoux-Reutenauer, Manivel 2003) The bad patterns in w can also be used to efficiently find the singular locus of X_w.

- (Billey-Postnikov 2005) Generalized pattern avoidance to all semisimple simply-connected Lie groups G and characterized smooth Schubert varieties X_w by avoiding these generalized patterns. Only requires checking patterns of types $A_3, B_2, B_3, C_2, C_3, D_4, G_2$.

Open Problems.

- Give a purely geometric reason why rank 4 patterns are enough.

- Identify the singular locus when G is an arbitrary semisimple Lie group.

Five Microscopic Focus Points

Focus Point 3. There exists a simple criterion for characterizing Gorenstein Schubert varieties using modified pattern avoidance.

Theorem: Woo-Yong (Sept. 2004)

X_w is Gorenstein $\iff$

- w avoids 35142 and 42513 *with Bruhat restrictions* $\{t_{15}, t_{23}\}$ and $\{t_{15}, t_{34}\}$

- for each descent d in w, the associated partition $\lambda_d(w)$ has all of its inner corners on the same antidiagonal.

Defn. X is Gorenstein if it is CM and its canonical sheaf is a line bundle. Their proof verifies a dependence relation among certain Schubert polynomials.

Five Microscopic Focus Points

Focus Point 4. Schubert varieties are useful for studying the cohomology ring of the flag manifold.

Theorem (Borel): $H^*(\mathcal{F}l_n) \cong \dfrac{\mathbb{Z}[x_1, \ldots, x_n]}{S_n - \text{invariants}}$

- $\{[X_w] \mid w \in S_n\}$ form a basis for $H^*(\mathcal{F}l_n)$ over $\mathbb{Z}$.

Question. What is the product of two basis elements?

$$[X_u] \cdot [X_v] = \sum [X_w] c_{uv}^{w}.$$

Cup Product in $H^*(\mathcal{F}l_n)$

Answer. Use Schubert polynomials! Due to Lascoux-Schützenberger, Bernstein-Gelfand-Gelfand, Demazure.

- BGG: Fix $w \in S_n$ and let $\mathfrak{S}_w \equiv [X_w] \mathrm{mod} \langle S_n - \text{invariants} \rangle$. For any i such that $w < wt_{i,i+1}$,

$$\partial_i \mathfrak{S}_w = \frac{\mathfrak{S}_w - t_{i,i+1}\mathfrak{S}_w}{x_i - x_{i+1}} \equiv [X_{ws_i}] \mathrm{mod} \langle S_n - \text{invariants} \rangle$$

  So by choosing a representative for $[X_{id}]$, we get reps for all $[X_w]$:

$$[X_{id}] \equiv x_1^{n-1} x_2^{n-2} \cdots x_{n-1} \equiv \prod_{i>j} (x_i - x_j) \equiv \ldots$$

- LS: Choosing $[X_{id}] \equiv x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$ works best because product expansion can be done without regard to the ideal!

Schubert polynomials for S_4

$$
\begin{aligned}
\mathfrak{S}_{w_0(1234)} &= 1\\
\mathfrak{S}_{w_0(2134)} &= x_1\\
\mathfrak{S}_{w_0(1324)} &= x_2 + x_1\\
\mathfrak{S}_{w_0(3124)} &= x_1^2\\
\mathfrak{S}_{w_0(2314)} &= x_1x_2\\
\mathfrak{S}_{w_0(3214)} &= x_1^2x_2\\
\mathfrak{S}_{w_0(1243)} &= x_3 + x_2 + x_1\\
\mathfrak{S}_{w_0(2143)} &= x_1x_3 + x_1x_2 + x_1^2\\
\mathfrak{S}_{w_0(1423)} &= x_2^2 + x_1x_2 + x_1^2\\
\mathfrak{S}_{w_0(4123)} &= x_1^3\\
\mathfrak{S}_{w_0(2413)} &= x_1x_2^2 + x_1^2x_2\\
\mathfrak{S}_{w_0(4213)} &= x_1^3x_2\\
\mathfrak{S}_{w_0(1342)} &= x_2x_3 + x_1x_3 + x_1x_2\\
\mathfrak{S}_{w_0(3142)} &= x_1^2x_3 + x_1^2x_2\\
\mathfrak{S}_{w_0(1432)} &= x_2^2x_3 + x_1x_2x_3 + x_1^2x_3 + x_1x_2^2 + x_1^2x_2\\
\mathfrak{S}_{w_0(4132)} &= x_1^3x_3 + x_1^3x_2\\
\mathfrak{S}_{w_0(3412)} &= x_1^2x_2^2\\
\mathfrak{S}_{w_0(4312)} &= x_1^3x_2^2\\
\mathfrak{S}_{w_0(2341)} &= x_1x_2x_3\\
\mathfrak{S}_{w_0(3241)} &= x_1^2x_2x_3\\
\mathfrak{S}_{w_0(2431)} &= x_1x_2^2x_3 + x_1^2x_2x_3
\end{aligned}
$$

Schubert polynomials

Theorem. (Billey-Jockush-Stanley, Fomin-Stanley 1993) Schubert polynomials have all nonnegative coefficients:

$$\mathfrak{S}_{w_0 w} = \sum_{D \in \mathcal{RC}-graphs(w)} x^D.$$

Example.

$$\mathfrak{S}_{w_0 1432} = x_2^2 x_3 + x_1 x_2 x_3 + x_1^2 x_3 + x_1 x_2^2 + x_1^2 x_2$$

Theorem. (Kogan-Miller, Knutson-Miller 2004) Matrix Schubert varieties degenerate to a union of toric varieties indexed by faces in the Gelfand-Tsetlin polytope. These faces are in bijection with rc-graphs.

Cup Product in $H^*(\mathcal{F}l_n)$

Key Feature. Schubert polynomials have distinct leading terms, therefore expanding any polynomial in the basis of Schubert polynomials can be done by linear algebra.

Buch: Fastest approach to multiplying Schubert polynomials uses Lascoux and Schützenberger's transition equations. Works up to about $n = 15$.

Draw Back. Schubert polynomials don't prove c_{uv}^{w}'s are nonnegative (except in special cases).

Cup Product in $H^*(\mathcal{F}l_n)$

Another Approach:.

- By intersection theory: $[X_u] \cdot [X_v] = [X_u(E_\bullet) \cap X_v(F_\bullet)]$

- Perfect pairing: $[X_u(E_\bullet)] \cdot [X_v(F_\bullet)] \cdot [X_{w_0 w}(G_\bullet)] = c_{uv}^w [X_{id}]$

$$||$$

$$[X_u(E_\bullet) \cap X_v(F_\bullet) \cap X_{w_0 w}(G_\bullet)]$$

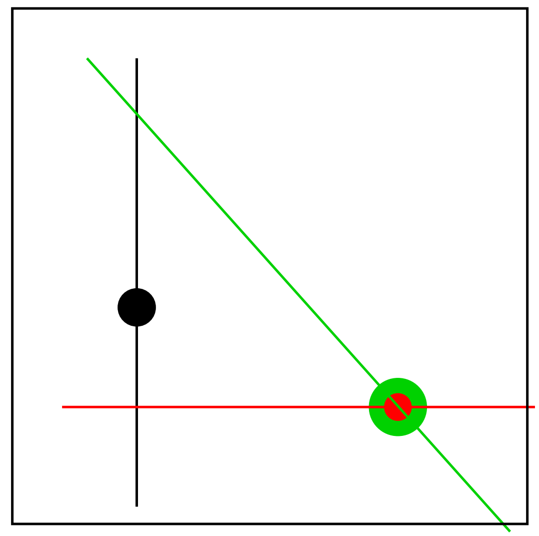
- The Schubert variety X_{id} is a single point in $\mathcal{F}l_n$.

Intersection Numbers: $c_{uv}^w = \# X_u(E_\bullet) \cap X_v(F_\bullet) \cap X_{w_0 w}(G_\bullet)$ assuming all flags $E_\bullet$, $F_\bullet$, $G_\bullet$ are in sufficiently general position. Hence all c_{uv}^w are nonnegative integers!

Open Problem. Find a combinatorial method to compute c_{uv}^w.

Intersecting Schubert Varieties

Example. Fix three flags $R_\bullet$, $G_\bullet$, and $B_\bullet$:



Find $X_u(R_\bullet) \cap X_v(G_\bullet) \cap X_w(B_\bullet)$ where u, v, w are the following permutations:

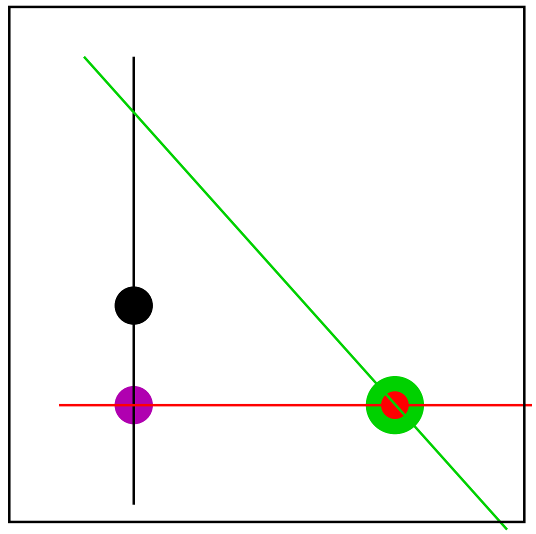
	R_1	R_2	R_3
P_1		1	
P_2			1
P_3	1		

	G_1	G_2	G_3
P_1			1
P_2		1	
P_3	1		

	B_1	B_2	B_3
P_1		1	
P_2	1		
P_3			1

Intersecting Schubert Varieties

Example. Fix three flags $R_\bullet$, $G_\bullet$, and $B_\bullet$:



Find $X_u(R_\bullet) \cap X_v(G_\bullet) \cap X_w(B_\bullet)$ where u, v, w are the following permutations:

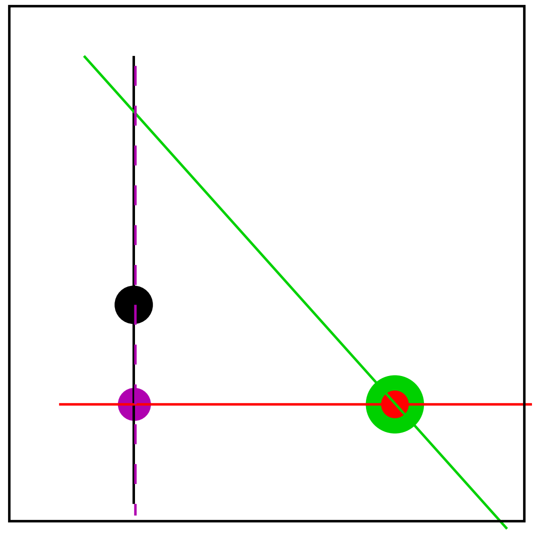
	R_1	R_2	R_3
P_1		1	
P_2			1
P_3	1		

	G_1	G_2	G_3
P_1			1
P_2		1	
P_3	1		

	B_1	B_2	B_3
P_1		1	
P_2	1		
P_3			1

Intersecting Schubert Varieties

Example. Fix three flags $R_\bullet$, $G_\bullet$, and $B_\bullet$:



Find $X_u(R_\bullet) \cap X_v(G_\bullet) \cap X_w(B_\bullet)$ where u, v, w are the following permutations:

	R_1	R_2	R_3
P_1		1	
P_2			1
P_3	1		

	G_1	G_2	G_3
P_1			1
P_2		1	
P_3	1		

	B_1	B_2	B_3
P_1		1	
P_2	1		
P_3			1

Intersecting Schubert Varieties

Schubert's Problem. How many points are there usually in the intersection of d Schubert varieties if the intersection is 0-dimensional?

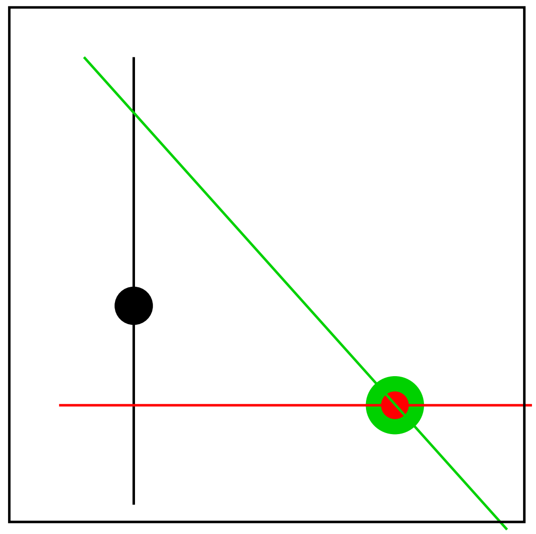
- Solving approx. n^d equations with $\binom{n}{2}$ variables is challenging!

Observation. We need more information on spans and intersections of flag components, e.g. $\dim(E^1_{x_1} \cap E^2_{x_2} \cap \cdots \cap E^d_{x_d})$.

Permutation Arrays

Theorem. (Eriksson-Linusson, 2000) For every set of d flags $E^1_\bullet, E^2_\bullet, \ldots, E^d_\bullet$, there exists a unique permutation array $P \subset [n]^d$ such that

$$\dim(E^1_{x_1} \cap E^2_{x_2} \cap \cdots \cap E^d_{x_d}) = \mathrm{rk} P[x].$$



	R_1	R_2	R_3
B_1			
B_2			
B_3	①	1	1

G_1

	R_1	R_2	R_3
B_1			
B_2			①
B_3	1	1	2

G_2

	R_1	R_2	R_3
B_1			①
B_2		①	2
B_3	1	2	3

G_3

Unique Permutation Array Theorem

Theorem. (Billey-Vakil 2005) If

$$X = X_{w^1}(E^1_\bullet) \cap \cdots \cap X_{w^d}(E^d_\bullet)$$

is nonempty 0-dimensional intersection of d Schubert varieties with respect to flags $E^1_\bullet, E^2_\bullet, \ldots, E^d_\bullet$ in general position, then there exists a unique permutation array $P \in [n]^{d+1}$ such that

$$X = \{F_\bullet \mid \dim(E^1_{x_1} \cap E^2_{x_2} \cap \cdots \cap E^d_{x_d} \cap F_{x_{d+1}}) = \mathrm{rk}P[x].\} \qquad (1)$$

Furthermore, we can recursively solve a family of equations for X using P.

Open Problem. Can one find a finite set of rules for moving dots in a 3-d permutation array which determines the c^w_{uv}'s analogous to Vakil's moves on checkerboards? (Vakil's rules only involve patterns in S_4.)

Generalizations of Schubert Calculus for G/B

1995-2005: A Highly Productive Decade.

$$\left\{\begin{array}{l} \text{A: } GL_n \\ \text{B: } SO_{2n+1} \\ \text{C: } SP_{2n} \\ \text{D: } SO_{2n} \\ \text{Semisimple Lie Groups} \\ \text{Kac-Moody Groups} \\ \text{GKM Spaces} \end{array}\right\} \times \left\{\begin{array}{l} \text{cohomolgy} \\ \text{quantum} \\ \text{equivariant} \\ \text{K-theory} \\ \text{eq. K-theory} \end{array}\right\}$$

Contributions from: Bergeron, Billey, Brion, Buch, Carrell, Ciocan-Fontainine, Coskun, Duan, Fomin, Fulton, Gelfand, Goldin, Graham, Griffeth, Guillemin, Haibao, Haiman, Holm, Kirillov, Knutson, Kogan, Kostant, Kresh, S. Kumar, A. Kumar, Lascoux, Lenart, Miller, Peterson, Pitti, Postnikov, Ram, Robinson, Sottile, Tamvakis, Vakil, Winkle, Yong, Zara...

See also A. Yong's slides on "Enumerative Formulas in Schubert Calculus"

http://math.berkeley.edu/ ayong/slides.html

Five Microscopic Focus Points

Focus Point 5. (Kazhdan-Lusztig, 1980) Poincare polynomials for the intersection cohomology sheaf of X_w at a point in C_v is determined by the Kazhdan-Lusztig polynomial $P_{v,w}(q)$

$$\sum \dim(IC_v^{2k}(X_w))q^k = P_{v,w}(q).$$

This proves that $P_{v,w}(q)$ has nonnegative integer coefficients!

Applications in many areas of mathematics:

1. $P_{v,w}(q) = 1 \iff X(w)$ is (rationally) smooth at v. (KL, 1980)
2. $\{C'_w = q^{l(w)/2} \sum P_{v,w} T_v\}$ in the Hecke algebra useful for computing irreducible representations of Hecke algebras. (KL, 1979)
3. $P_{v,w}(1) =$ multiplicity in decomposition series for Verma modules (Beilinson-Berstein, Brylinski-Kashiwara, 1981).
4. $P_{id,w}(1)$ used in Haiman's Immanent Conjectures. (Haiman, 1993)

Kazhdan-Lusztig Polynomials

Recursive Formula. (K-L) Base cases: $P_{w,w} = 1$, $P_{v,w} = 0$ if $v \not\leq w$. Otherwise for $s = t_{i,i+1}$ s.t. $sw < w$

$$P_{v,w}(q) = q^{1-c} P_{sv,sw} + q^{c} P_{v,sw} - \sum_{z < sw} \mu(z, sw) q^{\frac{l(w)-l(z)}{2}} P_{v,z}$$

$$\mu(z, y) = \text{coeff of } q^{\frac{l(y)-l(z)-1}{2}} \text{ in } P_{z,y}(q)$$

and

$$c = \begin{cases} 0 & v < sv \\ 1 & v > sv. \end{cases}$$

Observation. This formula only depends on Bruhat order.

Theorem.(Brenti) In fact, $P_{v,w}$ can be computed by only knowing the poset on $[id, w]$ after removing the labels on the vertices.

Kazhdan-Lusztig Polynomials

Below are all Kazhdan–Lusztig polynomials with $v = \mathbf{id}$ and $w \in S_5$ which are different from 1:

w	$P_{\mathrm{id},w}$
(14523) (15342) (24513) (25341) (34125) (34152) (35124) (35142) (35241) (35412) (41523) (42315) (42351) (42513) (42531) (43512) (45132) (45213) (51342) (52314) (52413) (52431) (53142) (53241) (53421) (54231)	$q+1$
(34512) (45123) (45231) (53412)	$2q+1$
(52341)	q^2+2q+1
(45312)	q^2+1

Kazhdan-Lusztig Polynomial are Mysterious!

Surprising Theorem. (Polo, 1999) Any polynomial $P(q) \in \mathbb{Z}_{\geq 0}[q]$ with constant term 1 is the Kazhdan-Lusztig polynomial for some pair of permutations.

Surprising Counterexamples. (McLarnan-Warrington 2003)
0-1 Conjecture: $\mu(v, w) \in \{0, 1\}$. True for all of S_9. False in S_{10}:

$$\mu(4321098765, 9467182350) = 4.$$

Conclusion. Our microscope is too small! We need better tools to understand them.

Open. Give a combinatorial formula for the coefficients of $P_{v,w}(q)$ and/or identify a "nice" basis for $IC_v(X_w)$.

Pattern avoidance and KL-Polynomials

- $P_{v,w} = 1$ for all $v \leq w$ iff w is $\mathbf{3412, 4231}$-avoiding.
- Say w is *321-hexagon avoiding* i.e. avoiding 5 patterns

$$\begin{array}{ccc} \mathbf{321} & \mathbf{56781234} & \mathbf{56718234} \\ & \mathbf{46781235} & \mathbf{46718235} \end{array}$$

Then following are equivalent (Billey-Warrington 2001):

1. w is 321-hexagon-avoiding.
2. $P_{x,w} = \sum_{\sigma \in E(x,w)} q^{d(\sigma)}$ for all $x \leq w$.
3. $\sum_i \dim(IC^{2i}(X_w))q^i = \sum_{v \leq w} P_{v,w}(q) = (1+q)^{l(w)}$.

- Let Σ be a group generated by a subset of transpositions, $\sigma \in S_n/\Sigma$, $v = \sigma v'$ and $w = \sigma w'$ for $v', w' \in \Sigma$. Then by (Billey-Braden, 2003)

$$P_{v,w}(1) \geq P_{v',w'}(1).$$

Connecting Schubert and KL polynomials

Question. Is there a master formula that relates Schubert polynomials and Kazhdan-Lusztig polynomials?

Evidence. Kumar's criteria (1996) for testing if $F_\bullet$ is a smooth point in X_w asks if a Kostant polynomial has a nice factorization:

$$\sum_{\alpha_1\alpha_2\ldots\alpha_k \in K(v,w)} \alpha_1\alpha_2\ldots\alpha_k = \prod_{\beta\in S(v,w)} \beta.$$

Kostant polynomials can also be used to expand $[X_u]\cdot[X_v]$ and determine the $c_{u,v}^w$'s. Therefore, singularities and cohomology ARE related!

Conclusions

"Combinatorics is the equivalent of nanotechnology in mathematics."

Key open problems:

1. Find a combinatorial formula for the structure constants in the cup product of $H^*(\mathcal{F}l_n)$.

2. Find a combinatorial formula for the coefficients of the Kazhdan-Lusztig polynomials.

3. Find a unified formula or connection between Schubert polynomials and Kazhdan-Lusztig polynomials.

4. Give a geometric explanation for the prevalence of properties characterized by pattern avoidance.

5. Give a formula for the multiplicity of a Schubert variety.