Enumeration of Parabolic Double Cosets for Coxeter Groups

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Based on joint work with:
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Outline

Symmetric Groups

Notation.

- S_n is the group of permutations.
- $t_{i,j} = (i \leftrightarrow j) =$ transposition for $i < j$,
- $s_i = (i \leftrightarrow i+1) =$ simple transposition for $1 \leq i < n$.

Example. $w = [3,4,1,2,5] \in S_5$,

$$ws_4 = [3,4,1,5,2] \quad \text{and} \quad s_4 w = [3,5,1,2,4].$$

Symmetric Groups

Presentation.

S_n is generated by $s_1, s_2, \ldots, s_{n-1}$ with relations

$$
\begin{array}{l}
s_i s_i = 1 \\
(s_i s_j)^2 = 1 \text{ if } |i-j| > 1 \\
(s_i s_{i+1})^3 = 1
\end{array}
$$

This presentation of S_n by generators and relations is encoded an edge labeled chain, called a Coxeter graph.

$$S_7 \approx \quad \bullet_1 \overset{3}{\text{———}} \bullet_2 \overset{3}{\text{———}} \bullet_3 \overset{3}{\text{———}} \bullet_4 \overset{3}{\text{———}} \bullet_5 \overset{3}{\text{———}} \bullet_6$$

Symmetric Groups

Notation. Given any $w \in S_n$ write

$$w = s_{i_1} s_{i_2} \cdots s_{i_k}$$

in a minimal number of generators. Then

- k is the length of w denoted $\ell(w)$.
- $\ell(w) = \#\{(i < j) \mid w(i) > w(j)\}$ (inversions).
- $s_{i_1} s_{i_2} \cdots s_{i_k}$ is a reduced expression for w.

Example. $w = [2, 1, 4, 3, 7, 6, 5] \in S_7$ has 5 inversions, $\ell(w) = 5$.

$$w = [2, 1, 4, 3, 7, 6, 5] = s_1 s_3 s_6 s_5 s_6 = s_3 s_1 s_6 s_5 s_6 = s_3 s_1 s_5 s_6 s_5 = \dots$$

Symmetric Groups

Poincaré polynomials. Interesting q-analog of $n!$:

$$\sum_{w \in S_n} q^{\ell(w)} = (1+q)(1+q+q^2)\cdots(1+q+q^2+\ldots+q^{n-1}) = [n]_q!.$$

Examples.

$$[2]_q! = 1+q$$
$$[3]_q! = 1+2q+2q^2+q^3$$
$$[4]_q! = 1+3q+5q^2+6q^3+5q^4+3q^5+q^6$$

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Fact. The Poincaré polynomial $[n]_q!$ is the Hilbert series for $H^*(GL_n/B)$ and for the coinvariant algebra $\mathbb{Z}[x_1, \ldots, x_n]/\langle e_1, \ldots \rangle$.

Ascent Sets

Def. For $w \in S_n$, the (right) *ascent set* of w is

$$Ascents(w) = \{1 \leq i \leq n-1 \mid w(i) < w(i+1)\}$$

$$= \{1 \leq i \leq n-1 \mid \ell(w) < \ell(ws_i)\}.$$

Similarly, $Descents(w) = \{1 \leq i \leq n-1 \mid w(i) > w(i+1)\}$

Example. $Ascents([3,4,1,2,5]) = \{1,3,4\}$,
$Descents([3,4,1,2,5]) = \{2\}$

Ascent Sets

Eulerian polynomials. Another interesting q-analog of $n!$:

$$A_n(q) = \sum_{k=0}^{n-1} A_{n,k} q^k = \sum_{w \in S_n} q^{\#Ascents(w)}$$

where $Ascents(w) = \{i \mid w(i) < w(i+1)\}$.
See Petersen's book "Eulerian Numbers."

Examples.

$$A_2(q) = 1 + q$$
$$A_3(q) = 1 + 4q + q^2$$
$$A_4(q) = 1 + 11q + 11q^2 + q^3$$

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Theorem. (Holte 1997, Diaconis-Fulman 2009) When adding together n large randomly chosen numbers in any base, the probability of carrying a k for $0 \le k < n$ is approximately $A_{n,k}/n!$.

Parabolic Subgroups and Cosets

Defn. For any subset $I \in \{1, 2, \ldots, n-1\} = [n-1]$, let W_I be the parabolic subgroup of S_n generated by $\langle s_i \mid i \in I \rangle$.

Defn. Sets of permutations of the form wW_I (or $W_I w$) are left (or right) parabolic cosets for W_I for any $w \in S_n$.

Example. Take $I = \{1, 3, 4\}$ and $w = [3, 4, 1, 2, 5]$. Then the left coset wW_I includes the 12 permutations

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Facts.

- Every parabolic coset has a unique minimal and a unique maximal length element.
- Every parabolic coset for W_I has size $|W_I|$.
- S_n is the disjoint union of the $n!/|W_I|$ left parabolic cosets S_n/W_I.

Parabolic Double Cosets

Defn. Let $I, J \in [n-1]$ and $w \in S_n$, then the sets of permutations the form $W_I \cdot w \cdot W_J$ are parabolic double cosets.

Example. Take $I = \{2\},\ J = \{1,3,4\}$ and $w = [3,4,1,2,5]$. Then the parabolic double coset $W_I w W_J$ includes

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Example. $W_I\ [4,5,1,2,3]\ W_J$ has 12 elements.

Parabolic Double Cosets

Facts.

- Parabolic double coset for W_I, W_J can have different sizes.
- S_n is the disjoint union of the parabolic double cosets

$$W_I \backslash S_n / W_J = \{W_I w W_J \mid w \in S_n\}.$$

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Thm.(Kobayashi 2011) Every parabolic double coset is an interval in Bruhat order. The corresponding Poincaré polynomials are palindromic

$$P_{I,w,J}(q) = \sum_{v \in W_I w W_J} q^{\ell(v)}.$$

Connection to Richardson Varieties

Thm. The Richardson variety in $GL_n(\mathbb{C})/B$ indexed by $u < v$ is smooth if and only if the following polynomial is palindromic

$$\sum_{u \leq v \leq w} q^{\ell(v)}.$$

References on smooth Richardson varieties: See book by Billey-Lakshmibai, and papers by Carrell-Kuttler, Billey-Coskun, Lam-Knutson-Speyer, Kreiman-Lakshmibai, Knutson-Woo-Yong, Lenagan-Yakimov and many more.

Connections to Algebra

- **Solomon** (1976) gives a formula for the structure constants in his descent algebra basis elements in terms of parabolic double cosets.
- **Garsia-Stanton** (1984) use parabolic double cosets in their construction of basic sets for the Stanley-Reisner Rings of Coxeter complexes.
- **Stembridge** (2005) uses parabolic double cosets to characterize tight quotients and embeddings of Bruhat order into $\mathbb{R}^d$.

Counting Parabolic Double Cosets

Question 1. For a fixed I, J, how many distinct parabolic double cosets are there in $W_I \backslash S_n / W_J$?

Question 2. Is there a nice formula for $f(n) = \sum_{I,J} |W_I \backslash S_n / W_J|$?

Question 3. How many distinct parabolic double cosets are there in S_n in total?

Counting Double Cosets

- G= finite group
- H, K = subgroups of G
- $H\backslash G/K$ = *double cosets* of G with respect to H, K
  $= \{HgK : \; g \in G\}$

Generlization of Question 1. What is the size of $H\backslash G/K$?

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Generlization of Question 1. What is the size of $H\backslash G/K$?

One Answer..
The size of $H\backslash G/K$ is given by the inner product of the characters of the two trivial representations on H and K respectively induced up to G.

Reference: Stanley's "Enumerative Combinatorics" Ex 7.77a. Parabolic subgroups are Young subgroups for S_n so this translates into a symmetric function computation.

Counting Parabolic Double Cosets

Question 2. Is there a nice formula for $f(n) = \sum_{I,J} |W_I \backslash S_n / W_J|$?

Data. $1, 5, 33, 281, 2961, 37277, 546193, 9132865, 171634161$ (A120733 in OEIS)

This counts the number of "two-way contingency tables" (see Diaconis-Gangoli 1994), the dimensions of the graded components of the Hopf algebra MQSym (see Duchamp-Hivert-Thibon 2002), and the number of cells in a two-sided analogue of the Coxeter complex (Petersen 2016).

Counting Parabolic Double Cosets

Question 3. How many distinct parabolic double cosets are there in S_n in total?

Data.: $p(n) = |\{W_I v W_J \mid v \in S_n, I, J \subset [n-1]\}|$,

$1, 3, 19, 167, 1791, 22715, 334031, 5597524, 105351108, 2200768698$

Not formerly in the OEIS! Now, see A260700.

Counting Parabolic Double Cosets

Question 3. How many distinct parabolic double cosets are there in S_n in total?

Defn. For $w \in S_n$, let c_w be the number of distinct parabolic double cosets with w minimal.

One Answer. $p(n) = \sum_{w \in S_n} c_w$.

Representing Parabolic Double Cosets

Lemma. w is minimal in $W_I w W_J$ if and only if $\ell(s_i w) > \ell(w)$ for all $i \in I$ and $\ell(w s_j) > \ell(w)$ for all $j \in J$. So

$$c_w = \#\{W_I w W_J \mid I \subset Ascent(w^{-1}), J \subset Ascent(w)\}.$$

Observation. Frequently $W_I w W_J = W_{I'} w W_{J'}$ even if $I, I' \subset Ascent(w^{-1})$ and $J, J' \subset Ascent(w)$.

Dilemma. Which representation is best for enumeration?

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Defn. A small ascent for w is an ascent j such that $ws_j = s_i w$.
Every other ascent is large.

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Enumeration Principle. To count distinct parabolic double cosets $W_I w W_J$ with w minimal, J can contain any subset of large ascents for w, I can contain any subset of large ascents for w^{-1}, count the small ascents very carefully!

Counting Parabolic Double Cosets

Theorem. (Billey-Konvalinka-Petersen-Slofstra-Tenner)

1. There is a finite family of 81 integer sequences $\{b_m^{\mathcal{I}} \mid m \geq 0\}$, such that for any permutation w, the total number of parabolic double cosets with minimal element w is equal to

$$c_w = 2^{|\operatorname{Floats}(w)|} \sum_{T \subseteq \operatorname{Tethers}(w)} \left(\prod_{R \in \operatorname{Rafts}(w)} b_{|R|}^{\mathcal{I}(R,T)} \right).$$

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2. The sequences $b_m^{\mathcal{I}}$ satisfy a linear homogeneous constant coefficient recurrence, and thus can be easily computed in time linear in m.
3. The expected number of tethers for $w \in S_n$ approaches $\frac{1}{n}$.

The Marine Model

Main Formula. For $w \in S_n$,

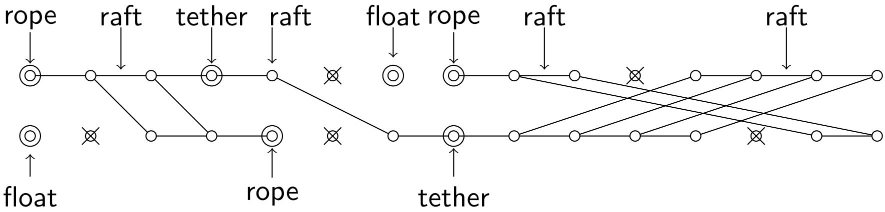
$$c_w = 2^{|\text{Floats}(w)|} \sum_{T \subseteq \text{Tethers}(w)} \left(\prod_{R \in \text{Rafts}(w)} b_{|R|}^{\mathcal{I}(R,T)} \right).$$

The w-Ocean.

1. Take 2 parallel copies of the Coxeter graph G of S_n.
2. Connect vertex $i \in \mathit{Ascent}(w^{-1})$ and vertex $j \in \mathit{Ascent}(w)$ by a new edge called a plank whenever $ws_j = s_i w$.
3. Remove all edges not incident to a small ascent.

The Marine Model

Example. Rafts, tethers, floats and ropes of the w-ocean $w = [1, 3, 4, 5, 7, 8, 2, 6, 14, 15, 16, 9, 10, 11, 12, 13]$.

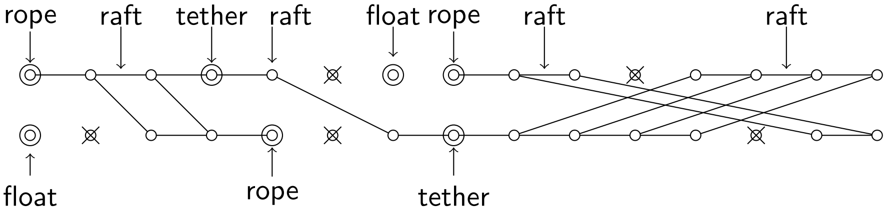


The Marine Model Terminology.

1. Raft – a maximal connected component of adjacent planks.
2. Float – a large ascent not adjacent to any rafts.
3. Rope – a large ascent adjacent to exactly one raft.
4. Tether – a large ascent connected to two rafts.

The Marine Model

Example. $w = (1, 3, 4, 5, 7, 8, 2, 6, 14, 15, 16, 9, 10, 11, 12, 13)$.



Formula. $c_w = 2^{|\text{Floats}(w)|} \sum_{T \subseteq \text{Tethers}(w)} \left(\prod_{R \in \text{Rafts}(w)} b_{|R|}^{\mathcal{I}(R,T)} \right).$

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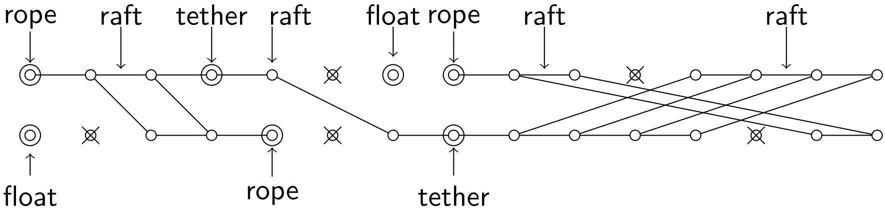

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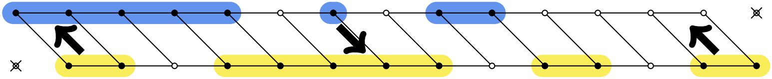
$$= 2^2\big(b_2^{(4,8)} \cdot b_1^{(4,8)} \cdot b_2^{(4,8)} \cdot b_4^{(4,8)} + b_2^{(4)} \cdot b_1^{(4)} \cdot b_2^{(4)} \cdot b_4^{(4)}$$
$$+ b_2^{(8)} \cdot b_1^{(8)} \cdot b_2^{(8)} \cdot b_4^{(8)} + b_2^{()} \cdot b_1^{()} \cdot b_2^{()} \cdot b_4^{()}\big)$$
$$= 2^2(71280 + 136620 + 144180 + 245640) = 2,390,880$$

Proof Sketch

Defn. A presentation (I, w, J) is lex minimal for $D = W_I w W_J$ provided $|I| < |I'|$ or $|I| = |I'|$ and $|J| < |J'|$ for all other presentations (I', w, J') for D.

Example. $w = [2, 3, 4, \ldots, 15, 16, 1] \in S_{16}$



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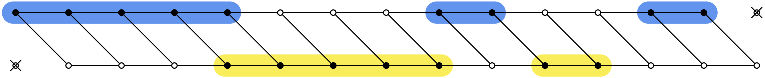

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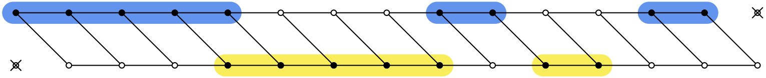
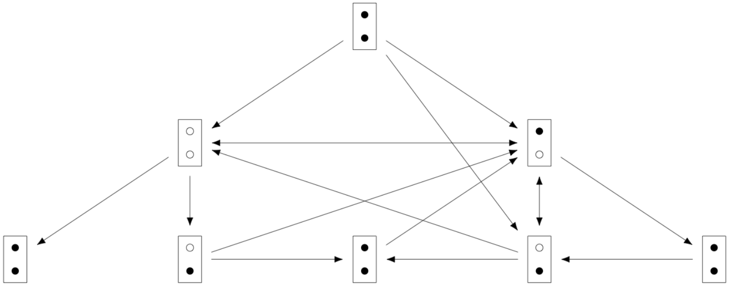


Note. Over the identity, lex-minimal presentations are two-level versions of the staircase diagrams in [Richmond-Slofstra 2016].

Key Steps

Lemma. Every parabolic double coset has a unique lex-minimal presentation.

Lemma. Lex minimal presentations along any one raft correspond with words in the finite automaton below (loops are omitted), hence they are enumerated by a rational generating function $P^{\mathcal{I}}(x)/Q(x)$ by the Transfer Matrix Method.



Review

Theorem. (Billey-Konvalinka-Petersen-Slofstra-Tenner)

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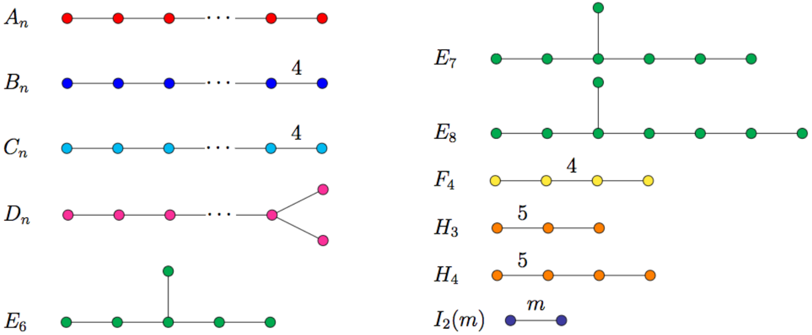
Coxeter Groups

- G = *Coxeter graph* with vertices $\{1, 2, \ldots, n\}$,
  edges labeled by $\mathbb{Z}_{\geq 3} \cup \infty$.

$$\bullet_1 \overset{4}{\text{---}} \bullet_2 \overset{3}{\text{---}} \bullet_3 \overset{3}{\text{---}} \bullet_4 \quad \approx \quad \bullet_1 \overset{4}{\text{---}} \bullet_2 \text{---} \bullet_3 \text{---} \bullet_4$$

- W = *Coxeter group* generated by $S = \{s_1, s_2, \ldots, s_n\}$ with relations
  1. $s_i^2 = 1$.
  2. $s_i s_j = s_j s_i$ if i, j not adjacent in G.
  3. $\underbrace{s_i s_j s_i \cdots}_{m(i,j) \text{ gens}} = \underbrace{s_j s_i s_j \cdots}_{m(i,j) \text{ gens}}$ if i, j connected by edge labeled $m(i, j)$.

Examples



Generalizing the notation from Symmetric Groups

- W = *Coxeter group* generated by $S = \{s_1, s_2, \ldots, s_n\}$ with special relations.
- $\ell(w)$ = *length* of w = length of a reduced expression for w.
- $W_I = \langle s_i \mid i \in I \rangle$ is a parabolic subgroup of W.
- $W_I w W_J$ is a parabolic double coset of W for any $I, J \subset [n]$, $w \in W$.
- c_w = number of distinct parabolic double cosets in W with minimal element w.

Generalizing Main Theorem to Coxeter Groups

Theorem. (Billey-Konvalinka-Petersen-Slofstra-Tenner)

1. For every Coxeter group W and $w \in W$, we have

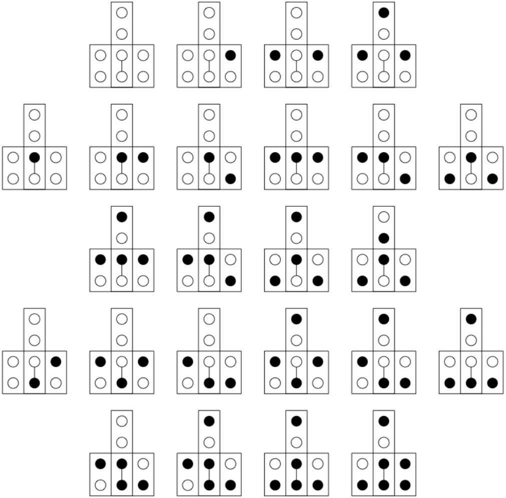
$$c_w = 2^{|\,\mathrm{Floats}(w)|} \sum_{\substack{T \subseteq \mathrm{Tethers}(w) \\ W \subseteq \mathrm{Wharfs}(w)}} \left(\prod_{R \in \mathrm{Rafts}(w)} b_{|R|}^{\mathcal{I}(R,T,W)} \right).$$

   - The w-ocean, floats, planks, rafts and tethers as before.
   - Wharf – a small ascent at a branch node of the Coxeter graph, along with decorations on the local neighborhood around branch node.

2. The sequences $b_m^{\mathcal{I}(R,T,W)}$ satisfy a similar constant coefficient, linear recurrence based on the same automaton for type A.

Examples

Consider the Coxeter group of type D_4 and $c_{id} = 72$. Up to symmetry of the 3 leaves around the central vertex, there are 24 distinct types of allowable lex-minimal presentations (I, id, J).



Examples

Types D_n and B_{n-1}. The number of parabolic double cosets with minimal element id gives rise to the sequence starting $20, 72, 234, 746, 2380, 7614, 24394, 78192$ for $n = 3, \ldots, 10$, and the generating function

$$\frac{t^3\left(20 - 28t + 14t^2\right)}{1 - 5t + 7t^2 - 4t^3}.$$

Type E_n. For $n = 6, \ldots, 10$, the analogous sequence starts with $750, 2376, 7566, 24198, 77532$, and the generating function is

$$\frac{t^4\left(66 - 96t + 42t^2\right)}{1 - 5t + 7t^2 - 4t^3}.$$

Type Affine A_n. For $n = 2, \ldots, 10$, the analogous sequence starts with $98, 332, 1080, 3474, 11146, 35738, 114566, 367248$, and the generating function is

$$\frac{2 - 8t + 22t^2 - 28t^3 + 20t^4 - 4t^5}{(1-t)\left(1-t+t^2\right)\left(1-5t+7t^2-4t^3\right)}.$$

Examples

Type F_4. The total number of distinct parabolic double cosets is 19,959. The number of parabolic double cosets with w minimal in F_4 is always in this set of 24:

$$1, 2, 4, 6, 8, 10, 12, 16, 20, 22, 24, 25, 26$$

$$30, 31, 32, 36, 38, 40, 44, 48, 52, 64, 66$$

Open Problems

1. Follow up to Question 3: Is there a simpler or more efficient formula for the total number of distinct parabolic double cosets are there in S_n than the one given here?

2. Follow up to Question 2: Is there a simpler or more efficient formula for $f(n) = \sum_{I,J} |W_I \backslash S_n / W_J|$?

3. What other families of double cosets for S_n and beyond have interesting enumeration formulas?

4. What further geometrical properties do Richardson varieties have when indexed by a parabolic double coset?