SCHUBERT POLYNOMIALS
FOR THE CLASSICAL GROUPS

Sara C. Billey

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Author addresses:

Dept. of Mathematics, UCSD, La Jolla, CA, 92093-0112

E-mail address: sara@macaulay.ucsd.edu

Contents

CHAPTER I

Introduction

Schubert polynomials are a fascinating family of polynomials indexed by elements in a Weyl group. In this thesis we will define and prove explicit formulas for computing families of Schubert polynomials for each of the classical groups. What we mean by the classical groups are the four infinite families of Lie groups, $SL(n, \mathbb{C}), SO(2n, \mathbb{C}), SO(2n+1, \mathbb{C})$ and $Sp(2n, \mathbb{C})$. Associated with each Lie group there is a root system and Weyl group. The formulas for Schubert polynomials are based only on the root system and its Weyl group. We will only need to discuss the Lie theory in order to give an abstract definition of Schubert polynomials.

1. History

The history of Schubert polynomials goes back to H. Schubert and his book on enumerative geometry published in 1874, in which he describes the Schubert calculus. The type of problem he was asking is the following: "Given 4 lines in 3-space, how many other lines can be drawn which intersect all four?" One can show that given different configurations of the first four lines there are either 0,1, 2 or an infinite number of other lines intersecting all four. The Schubert polynomials have the property that when you multiply two of these polynomials and expand again in the basis of Schubert polynomials, the coefficients are the multiplicities that H. Schubert was looking for.

Through the next century many people generalized the theory behind Schubert's original work, see [19] for a more complete history. In 1973, Bernstein, Gelfand, and

Gelfand [4], and Demazure [7] independently defined a map from the cohomomology of a flag manifold to classes of polynomials in a quotient space. Under this map Schubert varieties go to Schubert classes. The Schubert classes form a basis for the quotient space which is isomorphic to the cohomology ring. Furthermore, expansion of their products gives the intersection multiplicities of the corresponding varieties. These classes of polynomials are defined by recurrence relations involving divided difference operators. In fact, we will define the Schubert polynomials to be the unique solutions to the infinite number of divided difference equation taken in the inverse limit.

The study of Schubert polynomials of type A (indexed by permutations) was founded by Lascoux and Schützenberger in the early 1970's. In 1974 [28], they defined the Schubert polynomials to be explicit representatives of the Schubert classes. Their choice of representatives has the property that as the polynomials are stable under the inclusion of S_n into S_{n+1}. By defining these polynomials, Lascoux and Schützenberger brought the work of Bernstein, Gelfand, Gelfand and Demazure into the realm of combinatorics. Their contributions to the field have recently been summarized in "Notes on Schubert Polynomials" by I. G. Macdonald [34] along with many additional results on the subject. Our notation and presentation follows [34]. We will give a brief outline of the Schubert polynomials defined by Lascoux and Schützenberger. However, our approach to Schubert polynomials in this thesis is independent from their earlier work. More recent results related to Schubert polynomials not introduced in this thesis appear in [9,10,11,13,12,15,25,26,36].

2. Outline

In the remaining sections of Chapter I, we will give a complete outline of the 6 main theorems contained in this thesis. Along the way, we will present background information on permutations, Weyl groups, root systems and Schubert polynomials. In Section 4 we define permutations and Schubert polynomials indexed by permutations (type A) . The first two main theorems are formulas for computing Schubert polynomials (of type A). In analogy with permutations we will present the background on root systems and Weyl groups that we will use. We will define the

Schubert polynomials in complete generality as the solutions to divided difference equations and give the specific defining relations for each root system. Theorems 3 and 4 give formulas for computing Schubert polynomials of types B, C, and D. Theorem 5 says that each family of Schubert polynomials forms a basis for the space they span. Theorem 6 says that any family of polynomials satisfying all the divided difference equations simultaneously will be unique.

The main theorems are labeled as "Theorem". The proofs of the main theorems will appear in Chapters II through IV. Lemmas and propositions are intermediate results, propositions are considered to be more important. Propositions are also statements that have been proved by other people and for the most part we will just refer the reader to the appropriate article for the proof.

In Chapter II, we introduce a family of polynomials based on *rc-graphs*. These polynomials will turn out to be the Schubert polynomials of type A. The set of all rc-graphs for a permutation will be constructed by applying a sequence of transformations to particular starting graphs; we call these transformations "chutes" and "ladders". From these algorithms, a lot of insight on Schubert polynomials can be gained. Many identities that were known are obvious from the rc-graphs and a few new identities have been found. In particular, we will use two of these identities to prove these polynomials satisfy the divided difference equations. The content of this chapter appears in [3].

Chapter III is devoted to correspondences between reduced words and tableaux. In Section 1 we give a summary of the Edelman-Greene correspondence [8] between reduced words for permutations and tableaux. Using this correspondence, we prove the Stanley symmetric functions can be expanded in the basis of Schur functions with non-negative integer coefficients. Section 2 describes the Haiman correspondences of B_n [18] and D_n [6] reduced words and shifted tableaux. These two correspondences lead to B_n and D_n analogs of the Stanley functions which play an important role in the definitions of Schubert polynomials of type B, C and D. As with the A_n-Stanley functions, these analogs can be expanded in the bases of Schur P or Q functions with non-negative integer coefficients. We conclude this section with several special cases and general identities for Stanley functions. The content of Section 2, Chapter III appears in [6].

We will prove the formulas for the Schubert polynomials of all four types in Chapter IV. At this point we will only need to show the existence of solutions to the divided difference equations. The proofs for each type of root system involve carefully computing divided difference operators on Schubert polynomials at the monomial level. Each case requires a new trick. The proof of the formula for type A_n was originally given in [5]. The proof that we give here follows easily from the theory developed in Chapter II on *rc-graphs*. We prove that each family of Schubert polynomials forms a basis for the appropriate space and give simple formulas for special cases. These special cases allow us to compute Schubert polynomials on the computer by applying divided differences. The content of Section 2 appears in [6].

In Chapter V, we will outline one of the most important open problems in the field of Schubert polynomials, namely find a combinatorial proof that the coefficients in the expansion of products of Schubert polynomials are non-negative. We give two exciting conjectures for special cases.

In the appendix, we give tables of Schubert polynomials for the root systems A_3, B_3, C_3 and D_3.

3. Permutations

We will begin by studying the most familiar Weyl groups, the symmetric group or equivalently, the group of permutations. We denote a permutation w in one-line notation as $[w_1, w_2, \ldots, w_n]$. In S_n, the symmetric group on n elements, let σ_i denote the simple transposition $[1, 2, \ldots, i+1, i, \ldots, n]$ which interchanges the i^{th} and $i+1^{st}$ entries when multiplying on the right of a permutation, *i.e.* $[w_1, w_2, \ldots, w_n]\sigma_i = [w_1, \ldots, w_{i-1}, w_{i+1}, w_i, w_{i+2}, \ldots, w_n]$. It is well known that the elements $\{\sigma_i\colon 1 \leq i \leq n-1\}$ generate S_n and the following relations hold:

$$\begin{aligned} \sigma_i^2 &= 1 \\ \sigma_i\sigma_j &= \sigma_j\sigma_i \qquad \text{if } |i-j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}. \end{aligned} \tag{I.1}$$

We say that a permutation w has an *inversion* (i,j) if $i < j$ and $w_i > w_j$. For every $w \in S_n$, we denote the total number of inversions by $\ell(w)$, read the *length* of w. If the product $\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_p} = w$ and $p = \ell(w)$, we say the sequence $a_1a_2\cdots a_p$ is a *reduced word* for w. No word $\sigma_{a_1}\cdots\sigma_{a_k}$ equals w if $p < l(w)$, hence a reduced word $a_1a_2\cdots a_p$ corresponds to a minimal sequence of generators whose product is w. Let $R(w)$ denote the set of all reduced words for a permutation w. For example, let $w = [2,3,1,5,4]$, then

$$w = \sigma_1\sigma_2\sigma_4 = \sigma_1\sigma_4\sigma_2 = \sigma_4\sigma_1\sigma_2 \tag{I.2}$$

and w cannot be written as any other minimal length sequence of generators. Therefore, $R(w) = \{124, 142, 412\}$.

It is a fact that the graph of reduced words with edges given by the relations in (I.1) is connected. Hence we can get from any reduced word to any other simply by using the second two relations in all possible ways [33](2.5').

The permutation with the longest length in S_n, denoted w_0, is $[n, n-1, \ldots, 1]$. Its length is $\binom{n}{2}$. The permutation with the shortest length is of course the identity permutation $[1, 2, \ldots, n]$, whose length is 0.

Note that each permutation $w = [w_1, w_2, \ldots, w_n] \in S_n$ has the same set of reduced words as the permutation $v = [w_1, w_2, \ldots, w_n, n+1, \ldots, m] \in S_m$ for $m > n$. Throughout the rest of this thesis we will regard v and w as representing the same permutation in the group $S_\infty = \varinjlim S_n$. We will consider S_n to be the subgroup of S_∞ generated by $\{\sigma_i : i < n\}$ and denote its elements by $w = [w_1, w_2, \ldots, w_n]$.

Let $\mathbb{Z}[z_1, z_2, \ldots, z_n]$ denote the ring of polynomials in n variables with coefficients in $\mathbb{Z}$. We define the action of $w = [w_1, w_2, \ldots, w_n] \in S_n$ on $f \in \mathbb{Z}[z_1, \ldots, z_n]$ as follows: $wf(z_1, z_2, \ldots, z_n) = f(z_{w_1}, z_{w_2}, \ldots, z_{w_n})$. From this we can define the *divided difference operators*

$$\partial_i f(z_1, z_2, \ldots, z_n) = \frac{f(z_1, \ldots, z_n) - \sigma_i f(z_1, \ldots, z_n)}{z_i - z_{i+1}} \tag{I.3}$$

for $1 \le i \le n-1$. For example,

$$\begin{aligned} \partial_2\left(z_1^3 z_2 + z_3^2\right) &= \frac{z_1^3 z_2 + z_3^2 - \sigma_2(z_1^3 z_2 - z_3^2)}{z_2 - z_3} \\ &= \frac{z_1^3 z_2 + z_3^2 - z_1^3 z_3 - z_2^2}{z_2 - z_3} \\ &= z_1^3 - z_3 - z_2. \end{aligned} \tag{I.4}$$

The result of applying a divided difference operator to a polynomial is again a polynomial of degree one less than the original. The result will also be symmetric in z_i and z_{i+1}.

It is easy to check that the following relations hold for divided difference operators:

$$\begin{aligned} \partial_i^2 &= 0 \\ \partial_i\partial_j &= \partial_j\partial_i \qquad \text{if } |i-j| > 1 \\ \partial_i\partial_{i+1}\partial_i &= \partial_{i+1}\partial_i\partial_{i+1}. \end{aligned} \tag{I.5}$$

The easiest way to check the last relations is to expand both sides on a polynomial and compare the result. Note the similarity between the relations in (I.5) and (I.1).

Let $\partial_w = \partial_{a_1}\partial_{a_2}\cdots\partial_{a_p}$ for some $a_1 a_2 \cdots a_p \in R(w)$. From the relations in (I.5), one may deduce that ∂_w does not depend on the choice of reduced word $a_1 a_2 \cdots a_p \in R(w)$. If, however, $a_1 a_2 \cdots a_p$ is not reduced then one can show $\partial_{a_1}\partial_{a_2}\cdots\partial_{a_p} = 0$ [33](2.6).

We discuss divided difference operators in more generality in Section 7.

4. Schubert polynomials of type A

We now have all the pieces to define the Schubert polynomials of type A. We give the formula for Schubert polynomials $\tilde{\mathfrak{S}}_w$ as defined by Lascoux and Schützenberger. Then we define a family of polynomials, $\mathfrak{S}_w$, which are also Schubert polynomials.

Definition. For every $w \in S_\infty$, the *Schubert polynomial* $S_w \in \mathbb{Z}[z_1, z_2, \ldots]$ satisfies the equation

$$\partial_i S_w = \begin{cases} S_{w\sigma_i} & \text{if } l(w\sigma_i) < l(w) \\ 0 & \text{if } l(w\sigma_i) > l(w) \end{cases} \tag{I.6}$$

for all $i \geq 1$, together with the condition that the constant term of S_w is 1 if $w = [1, 2, \ldots]$ and 0 otherwise.

We will show in Section 8 that if a family of polynomials exist that satisfy (I.6) they are unique. Lascoux and Schützenberger first gave explicit formulas for computing such polynomials.

PROPOSITION 4.1. [28] *For each permutation $w \in S_n$, the Schubert polynomial indexed by w is given by*

$$\tilde{\mathfrak{S}}_w = \partial_{w^{-1}w_0^{(n)}} \left(z_1^{n-1} z_2^{n-2} \cdots z_{n-1}^1 z_n^0 \right), \tag{I.7}$$

where $\partial_{w^{-1}w_0^{(n)}} = \partial_{a_1} \cdots \partial_{a_p}$ for any $a_1 a_2 \cdots a_p \in R(w^{-1}w_0^{(n)})$ and $w_0^{(n)} = [n, n-1, \ldots, 1]$ is the longest element of S_n.

It is not obvious that (I.7) is independent of n when we consider a permutation w as an element of S_∞. The stability of Schubert polynomials under the inclusion of S_n into S_{n+1} has been studied carefully. Stability was considered to be the difficult part in finding Schubert polynomials for arbitrary root systems. In fact, we will define Schubert polynomials in a new way so that they are automatically stable. The fact that the polynomials $\tilde{\mathfrak{S}}_w$ are stable is proved in [33]. The stability of Schubert polynomials of type A will be clear from Theorem 2, which gives a formula in terms of reduced words.

Here are some examples of Schubert polynomials which are not hard to find using Proposition 4.1

- $\tilde{\mathfrak{S}}_{w_0} = z_1^{n-1} z_2^{n-2} \cdots z_{n-1}^1$
- $\tilde{\mathfrak{S}}_{\sigma_i} = z_1 + z_2 + \ldots + z_i$
- $\tilde{\mathfrak{S}}_{id} = 1$

Next we develop a second family of polynomials which will also satisfy the defining equations for Schubert polynomials. It will follow from the proof of uniqueness these polynomials are equal to the $\tilde{\mathfrak{S}}_w$'s.

DEFINITION. If $\mathbf{a} = a_1 a_2 \ldots a_p$ is a reduced word for $w \in S_n$, we say the sequence $j_1 j_2 \cdots j_p$ of positive integers is $\mathbf{a} -$ *compatible* if

(1) $j_1 \leq j_2 \leq \cdots \leq j_p$
(2) $j_i = j_{i+1}$ implies $a_i > a_{i+1}$
(3) $j_i \leq a_i$ for all i.

Let $C(\mathbf{a})$ be the set of all **a**-compatible sequences. We will be using compatible sequences to make admissible monomials for the reduced word **a**. [1] We will say a monomial $z_{j_1} \cdots z_{j_p}$ is **a**-admissible if $j_1 \cdots j_p$ is **a**-compatible. Let $\mathcal{A}_z(\mathbf{a})$ be the set of all **a**-admissible monomials in the variables $z_1, z_2, \ldots$. We can collapse notation by saying $z_{j_1} \cdots z_{j_p} = z_1^{\alpha_1} \cdots z_n^{\alpha_n} = z^\alpha$ if there are α_1, j_i's equal to 1, α_2, j_i's equal to 2, etc.

[2]

Definition. For all $w \in S_\infty$, let

$$\mathfrak{S}_w(z_1, z_2, \ldots) = \sum_{\mathbf{a} \in R(w)} \sum_{z^\alpha \in \mathcal{A}_z(\mathbf{a})} z^\alpha. \tag{I.8}$$

We compute $\mathfrak{S}_w$ for $w = [2,3,1,5,4]$. As we noted before, $R(w) = \{124, 142, 412\}$. We compute all compatible sequences for each reduced word as follows:

(I.9)

<u>124</u>	<u>142</u>	<u>412</u>
123	122	112
124		

Therefore, $\mathfrak{S}_w = z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_2^2 + z_1^2 z_2$

It is easy to see that $\mathfrak{S}_{\sigma_i} = z_1 + z_2 + \ldots + z_i$ because i is the only reduced word for σ_i.

Next, we introduce the machinery of rc-graphs which is used to prove the polynomials $\mathfrak{S}_w$ are Schubert polynomials.

Definition. Given any reduced word $\mathbf{a} = a_1 a_2 \ldots a_p$ and an **a**-compatible sequence $\mathbf{j} = j_1 j_2 \ldots j_p$ the *reduced word compatible sequence graph* or *rc-graph* of the

[1] We have introduced both notations because they are used in the literature and there are conceptual benefits to both.

[2] We can reverse this process to reconstruct the compatible sequence from the admissible monomial. Namely, write a monomial as $z_{j_1} \cdots z_{j_p}$, then the compatible sequence is $j_1 \cdots j_p$ in increasing order.

pair $(\mathbf{a}, \boldsymbol{j})$ is $D(\mathbf{a}, \boldsymbol{j}) = \{(j_k, a_k - j_k + 1)\}$. Let $\mathcal{R}C(w) = \{D(\mathbf{a}, \boldsymbol{j}) : \mathbf{a} \in R(w), \boldsymbol{j} \in C(\mathbf{a})\}$.

In Chapter II, we will give two algorithms for computing the set of all reduced word-compatible sequence graphs for w or $\mathcal{R}C(w)$. One algorithm applies *chute* moves to canonical top rc-graphs which we call $D_{top}(w)$. The second algorithm is in a sense dual to the first; it applies *ladder* moves to the canonical bottom rc-graph, $D_{bot}(w)$. Every distinct diagram that appears as the result of a sequence of chute moves (or ladder moves) contributes an admissible monomial to the sum. We will just state the theorem here and refer the reader to Chapter II for complete details. The content of this chapter is joint work with Nantel Bergeron, [3].

Theorem 1. *Given any* $w \in S_\infty$,

$$\mathfrak{S}_w = \sum_{D \in \mathcal{R}C(w)} z_D = \sum_{D \in \mathcal{C}(D_{top}(w))} z_D = \sum_{D \in \mathcal{L}(D_{bot}(w))} z_D. \tag{I.10}$$

The theory of Schubert polynomials is intertwined with the study of reduced words. The key to understanding this relationship is stated in Theorem 2. This alternative definition of Schubert polynomials, originally conjectured by Richard Stanley, was first proved in [5] by William Jockusch, Stanley and myself and subsequently in [11] by Fomin and Stanley. We will give a third proof in Chapter IV which shows these polynomials satisfy the divided difference equations and hence must be the unique solutions as well. This new proof does not depend on any of the previous theory developed to prove Proposition 4.1.

Theorem 2. *Given any* $w \in S_\infty$, *the polynomials* $\mathfrak{S}_w$ *defined by*

$$\mathfrak{S}_w(z_1, z_2, \ldots) = \sum_{\mathbf{a} \in R(w)} \sum_{z^\alpha \in \mathcal{A}_z(\mathbf{a})} z^\alpha \tag{I.11}$$

are Schubert polynomials.

Note that the complexity for computing Schubert polynomials using this rule is bounded below by the complexity of computing the set of all reduced words for permutations. At this time the best algorithms are $O(l(w)^2)$. Any improved algorithm for computing reduced words would make computations with Schubert polynomials easier.

Using rc-graphs we can avoid computing all reduced words. However, the algorithms given in Chapter II are still not optimal. There exist multiple paths to some rc-graphs from the starting graph. It is an open problem to find a way to compute all rc-graphs without ever finding a repeat.

5. Stanley polynomials

The Stanley polynomials (also called stable Schubert polynomials) are a beautiful example of how mathematical theory is developed. They form a bridge between the original definition of Schubert polynomials (I.7) and our alternative definition given by (I.8). The connection between the two families of polynomials is given by Proposition 5.1. Furthermore, the analogs of the Stanley polynomials play a very important role in the formulas for Schubert polynomials of type B, C, and D.

Stanley [43] originally defined polynomials

$$G_w = \sum_{\mathbf{a}\in R(w)} \sum_{\substack{i_1\le i_2\le\cdots\le i_{l(w)}\\ i_k<i_{k+1} \text{ if } a_k<a_{k+1}}} z_{i_1} z_{i_2} \cdots z_{i_{l(w)}} \tag{I.12}$$

Compare the limits on the summation with conditions (1) and (2) of the definition of compatible sequences (see page 7). Then the following proposition is proved in [33](7.18).

PROPOSITION 5.1. *For* $w = [w_1, \ldots, w_n] \in S_n$*, let* $1_k \times w$ *be the permutation* $[1, 2, \ldots, k, w_1, \ldots, w_n]$*. Then*

$$\lim_{k\to\infty} \mathfrak{S}_{1_k\times w} = G_w. \tag{I.13}$$

This proposition is a hint for finding a formula for the Schubert polynomials. Note the similarity between (I.12) and (I.8). The only difference is that in (I.12) the i_j's are not bounded above by the reduced words.

These polynomials are symmetric functions in an infinite number of variables. Their expansion in the basis of Schur functions can be expressed in terms non-negative integers given by the Edelman-Greene correspondence of reduced words of the permutation w.

Edelman and Greene [8] define an analog of the Robinson-Schensted-Knuth correspondence between reduced words and standard tableaux. We give a summary of their results in Section 1 of Chapter III. (See [37] or [32] for background information on tableaux and symmetric functions.) Under this correspondence, each reduced word $\mathbf{a}$ is mapped to a standard tableau $Q(\mathbf{a})$. Each standard tableaux of shape μ is the image of the same number of reduced words for w. In other words, for each standard tableau S of shape μ, let

$$g_w^\mu = |\{\mathbf{a} \in R(w) \mid Q(\mathbf{a}) = S\}|. \tag{I.14}$$

Then g_w^μ depends only on w and on the shape μ of S, and not on the choice of standard tableau.

Proposition 5.2. [43] *For any* $w \in S_\infty$,

$$G_w(z_1, z_2, \ldots) = \sum_\mu g_w^\mu S_\mu(z_1, z_2, \ldots) \tag{I.15}$$

where $S_\mu(z_1, z_2, \ldots)$ *is the Schur function corresponding to the shape* μ.

6. Background on Root Systems and Weyl Groups

We will give the general definitions of root systems and Weyl groups first and then state explicitly the information we need for the classical groups. As one reads this section, keep in mind the case A_{n-1} is the case of the symmetric group. We will define analogs of the following concepts we have already studied: 1) the symmetric group which will be the Weyl groups, 2) the action of the Weyl group on polynomials, 3) divided difference operators, 4) Schubert polynomials, 5) Stanley symmetric functions. Our exposition of Weyl groups and root systems is far from complete, for more information see [20,34].

Let V be any vector space over $\mathbb{Q}$ with a positive definite symmetric bilinear form (α, β). Each vector $\alpha \in V$ determines a reflection σ_α found by fixing the hyperplane perpendicular to α and sending α to $-\alpha$.

Definition. [34](A subset R of V is called a *root system* if

(1) R is finite and $0 \notin R$.

(2) If $\alpha \in R, \sigma_\alpha$ leaves R invariant.

(3) If $\alpha, \beta \in R$, then $(\alpha, \beta) \in \mathbb{Z}$.

The root systems of types A, B, C, and D are in addition reduced and irreducible (all at the same time). Hence, we will assume as part of the definition, if $\alpha \in R$ then $-\alpha$ is the only other scalar multiple of α in R. Also, R cannot be written as a union of two smaller root systems.

Definition. [34] The *Weyl group* W associated with R is the group generated by the reflections σ_α where $\alpha \in R$.

Let $\{e_i\}$ be the usual basis of $\mathbb{Q}^n$ where each e_i has a 1 in the i^{th} coordinate and 0's elsewhere. It is not hard to see that S_n is the group generated by the reflections $\sigma_i = \sigma_{e_{i+1}-e_i}$, which interchange e_i and e_{i+1}. Therefore, S_n is the Weyl group corresponding to the root system A_{n-1}.

Definition. [34] A *basis* of R is a subset B of R such that

(1) B is linearly independent.

(2) For each $\alpha \in R$, $\alpha = \sum_{\beta \in B} m_\beta \beta$ with coefficients $m_\beta \in \mathbb{Z}$ and either all $m_\beta \geq 0$ or $m_\beta \leq 0$.

This twist of the usual definition of a basis leads to many interesting differences between a root system and vector spaces. As in the case of vector spaces, one can show that every root system has a basis [34].

Proposition 6.1. [20, page 11] *If B is a basis of R and W is the associated Weyl group then W is generated by $(\sigma_\beta : \beta \in B)$. This is a minimal set of generators for W.*

Definition. Given any Weyl group W fix an order on the basis elements in B, *i.e.* $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Assume $w \in W$ can be written as $\sigma_{\alpha_{i_1}} \cdots \sigma_{\alpha_{i_p}}$. If p is the minimal number of generators needed to write w, then we say $p = l(w)$, $l(w)$ is the *length* of w and $i_1 \cdots i_p$ is a *reduced word* for w. Let $R(w)$ be the set of reduced words for w.

Reduced irreducible root systems can be completely classified to be one of one of 9 types; four infinite families A_{n-1}, B_n, C_n, D_n for any positive integer n, and five exceptional root systems E_6, E_7, E_8, F_4, G_2, [34]. We will only discuss the infinite families of root systems in this thesis. Lists of bases and Weyl group generators can be found in [20] for the exceptional root systems.

Next we give a basis for each of the different root systems that we will refer to this thesis. (The subscripts are the number of elements in the basis.)

(I.16)
$$\begin{aligned} A_{n-1} : B =& \{e_{i+1} - e_i : 1 \le i \le n-1\} \\ B_n : B =& \{e_1\} \cup \{e_{i+1} - e_i : 1 \le i \le n-1\} \\ C_n : B =& \{2e_1\} \cup \{e_{i+1} - e_i : 1 \le i \le n-1\} \\ D_n : B =& \{e_1 + e_2\} \cup \{e_{i+1} - e_i : 1 \le i \le n-1\} \end{aligned}$$

These bases are slightly different than those stated in [20] or [34]. We have changed the bases so that the root system $R_i \subset R_{i+1}$ for each case. Again we can consider $W_i \subset W_{i+1}$ by considering W_i as the group generated by reflections of all but the last basis element in R_{i+1}.

From Proposition 6.1, we can compute generators for the corresponding Weyl groups of each type. The linear transformation which sends $e_{i+1} - e_i$ to its negative and fixes all other basis elements is σ_i, the adjacent transposition on coordinates which switches the elements in positions i and $i+1$ when acting on the right. Hence, $W_{A_{n-1}} = S_n$ the symmetric group generated by $\{\sigma_i : 1 \le i \le n\}$.

Each of the other types of root systems B_n, C_n, D_n contain the basis for A_{n-1}, hence the σ_i's are generators for their Weyl groups as well. In addition, B_n has the basis element e_1. The linear transformation which sends e_1 to $-e_1$ can be thought of as an operator which acts on a permutation w by sending $w_1 \mapsto -w_1$. We will call this transformation σ_0, *i.e.* $[w_1, w_2, \ldots, w_n]\sigma_0 = [-w_1, w_2, \ldots, w_n]$. The group generated by $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}$ is the *hyperoctahedral group on n elements* or the group of signed permutations. One can think of this group also as the group of $n \times n$ matrices with exactly one non-zero entry in each row and each column and that entry is allowed to be ± 1. For example,

$$\begin{bmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}. \tag{I.17}$$

We represent an element in the hyperoctahedral group as a permutation with bars over some elements. The bars are just a nice notation for a sign, hence applying bar twice cancels the operation, *i.e.* $(\bar{\bar{k}}) = k$. For example, the signed permutation matrix in (I.17) is the element $w = [\bar{4}, 5, 1, 2, \bar{3}] \in W_{B_5}$. The longest element in W_{B_n} is the signed permutation $[\bar{1}, \bar{2}, \ldots, \bar{n}]$. Its length is $n^2 = (2n-1)+(2n-3)+\ldots+1$.

For C_n, the only difference from the basis for B_n is the first vector e_1 or $2e_1$. Hence, the same linear transformation, namely σ_0, sends them both to their negative. Therefore, $W_{C_n} = W_{B_n}$.

Finally, D_n has the extra basis element $e_1 + e_2$. The reflection sending $e_1 + e_2$ to $-e_1 - e_2$ either sends $e_1 \mapsto -e_1$ and $e_2 \mapsto -e_2$ *or* $e_1 \mapsto -e_2$ and $e_2 \mapsto -e_1$. One can check the second transformation is correct because it fixes the perpendicular hyperplane. Let $\sigma_{\hat{1}}$ multiply on the right of a permutation by sending the first two positions of a signed permutation to their negative (or bar using that notation) and switching the order, $[w_1, w_2, \ldots, w_n]\sigma_{\hat{1}} = [\bar{w}_2, \bar{w}_1, \ldots, w_n]$. Note, $\sigma_{\hat{1}} = \sigma_0\sigma_1\sigma_0$. Hence, W_{D_n} is a subgroup of the hyperoctahedral group whose elements all have an even number of sign changes. In terms of matrices, W_{D_n} is the subgroup of signed permutation matrices with an even number of -1's. The longest element in W_{D_n} is the signed permutation $[\pm 1, \bar{2}, \ldots, \bar{n}]$ where 1 is made positive or negative depending on whether n is odd or even. Its length is $n(n-1) = (2n-2) + (2n-4) + \ldots + 2$.

From this point on, we will consider every signed permutation $w = [w_1, w_2, \ldots, w_n]$ as a signed permutation on an infinite number of elements where only a finite number are not fixed. We use the notation $B_\infty = \varinjlim W_{B_n} = \varinjlim W_{C_n}$

and $D_\infty = \varinjlim W_{D_n}$. Below we summarize the generators of the infinite groups:

$$
\begin{aligned}
A:&\quad S_\infty = (\sigma_i : i \geq 1)\\
B:&\quad B_\infty = (\sigma_0) \cup (\sigma_i : i \geq 1)\\
C:&\quad B_\infty\\
D:&\quad D_\infty = (\sigma_{\hat{1}}) \cup (\sigma_i : i \geq 1)
\end{aligned}
\tag{I.18}
$$

The generators of any of the Weyl groups act on power series in $\mathbb{Q}[[z_1, z_2, \ldots]]$ as with permutations.

Definition. An element $\sigma \in W$ *acts* on a power series $f \in \mathbb{Q}[[z_1, z_2, \ldots]]$ by

$$\sigma f(\mathbf{z}) = f(\mathbf{z}\sigma). \tag{I.19}$$

One can think of $\mathbf{z}$ as the vector $(z_1, z_2, \ldots)$ and σ as a signed permutation matrix (with a finite number of non-fixed points). Then $\mathbf{z}\sigma$ is just the product of a vector and a matrix. In particular,

$$\sigma_0 f(z_1, \ldots, z_n) = f(-z_1, z_2, \ldots, z_n) \tag{I.20}$$

$$\sigma_{\hat{1}} f(z_1, \ldots, z_n) = f(-z_2, -z_1, z_3, \ldots, z_n). \tag{I.21}$$

For example, $\sigma_0\sigma_1 f(z_1, z_2, z_3) = \sigma_0 f(z_2, z_1, z_3) = f(-z_2, z_1, z_3)$. So if $f(z_1, z_2, z_3) = z_1 z_2^2 z_3$ then $\sigma_0\sigma_1 f = -z_2 z_1^2 z_3$.

7. Geometry and Schubert polynomials

We sketch the geometrical construction of Schubert varieties and the flag manifold in this section. The Schubert varieties have been studied extensively in the literature, see [13,12,15,19,21] and all of their references. The geometry is the motivation for studying Schubert polynomials. However, we show that we can abstract the notion of Schubert polynomials away from algebraic geometry and define them in combinatorial terms once we have proved that the geometry implies the existence of polynomials with certain properties.

To each of the 4 types of root systems we can assign a classical Lie group G as follows:

root system	Weyl group	Lie group
A_{n-1}	S_n	$Sl(n,\mathbb{C})$
B_n	W_{B_n}	$SO(2n+1,\mathbb{C})$
C_n	W_{B_n}	$Sp(2n,\mathbb{C})$
D_n	W_{D_n}	$SO(2n,\mathbb{C})$

For type A_{n-1}, $G_n = SL(n,\mathbb{C})$ acts naturally on $\mathbb{C}^n$. For types B_n ($G_n = SO(2n+1,\mathbb{C})$), C_n ($G_n = Sp(2n,\mathbb{C})$), and D_n ($G_n = SO(2n,\mathbb{C})$), G_n is by definition the subgroup of automorphisms preserving a non-degenerate bilinear form $(-,-)$ on $V = \mathbb{C}^{2n}$ or $\mathbb{C}^{2n+1}$. For SO, this will be a symmetric form; for Sp, a skew form $(x,y) = -(y,x)$. To be definite, let J_l be the $l\times l$ 'reverse identity matrix' with entries 1 on the anti-diagonal and 0 elsewhere, and let $\tilde{J}_{2n} = \left[\begin{smallmatrix} 0 & J_n \\ -J_n & 0\end{smallmatrix}\right]$. Then $SO(n,\mathbb{C})$ is the subset of $Sl(n,\mathbb{C})$ such that each matrix P satisfies $P^tJ_nP = J_n$ and $Sp(2n,\mathbb{C})$ is the subset of matrices P such that $P^t\tilde{J}_{2n}P = \tilde{J}_{2n}$.

Fix a Lie group G, root system R and Weyl group W from the list above all indexed by n. In each case let B be the Borel subgroup consisting of the upper triangular matrices in G. Abstractly, the *flag manifold* is the space of cosets $X_n = X = G/B = \{gB \mid g \in G\}$, which is a smooth complex projective variety. The flag manifold can also be described as the set of flags or isotropic flags on a vector space V of dimension n. However, we will only present the algebraic version of the space X, namely we give explicit matrix representatives for the cosets of G/B.

From the Bruhat decomposition, G is the disjoint union of double cosets BwB for $w \in W$. Hence, we can examine the subsets corresponding to BwB in G/B.

Definition. For each $w \in W$, define the *Schubert cells* in G/B to be the image of the double cosets

$$X_w = \frac{BwB}{B}. \tag{I.22}$$

The *Schubert variety* , $\overline{X_w}$, is defined to be the closure of the Schubert cell, X_w.

Proposition 7.1. [45] *The Schubert variety $\overline{X_w} = \bigcup X_v$ where the union is over all $v \leq w$ in the strong Bruhat order, i.e. if $a_1a_2\cdots a_p$ is a reduced word for w, then* *there exists a subsequence* $b_1 \cdots b_q$ *such that* $\sigma_{b_1} \cdots \sigma_{b_q} = v$.

It is not difficult to compute a set of matrix representatives for the cosets in G/B [33](A.2). These representatives depend on the Schubert cell X_w and the diagram of the permutation.

DEFINITION. Let $D(w) = \{(i,j) \in [1,n]^2 : i < w_j^{-1} \text{ and } j < w_i\}$ be the *diagram of the permutation* for $w \in S_n$.

In other words, $D(w)$ is obtained from $[1,n]^2$ by removing the points (i,j) which are east or south of (k, w_k) for any $k \in [1,n]$ (including the point (k, w_k)). It can be shown that the number of points in $D(w)$ is equal to the length of w, $l(w)$. See [33] for details on the diagram of a permutation.

For type A_{n-1}, X_w can be represented by the set of matrices

(I.23)
$$M_w = \{A \in Sl(n, \mathbb{C}) : A_{k,w_k} = 1 \text{ and } A_{i,j} = 0 \text{ if } (i,j) \notin D(w) \cup \{(i, w_i)\}\},$$

i.e. each right coset bwB in the double coset BwB contains exactly one matrix from M_w. For example, for $w = [2,1,5,4,3]$, M_w is the set of matrices of the general form

(I.24)
$$\begin{bmatrix} * & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 1 \\ 0 & 0 & * & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Note that X_w is isomorphic to $\mathbb{C}^{l(w)}$ since the diagram of the permutation is a set of size $l(w)$.

For the other classical groups we will again be able to use the diagram of a permutation by embedding the Weyl group into S_{2n} or S_{2n+1}. For $w \in W_{B_n}$, embed w into S_{2n+1} by sending w to $i(w)$ where

(I.25)
$$i(w)_k = \begin{cases} n+1-w_{n+1-k} & \text{if } i \leq n \\ n+1 & \text{if } i = n+1 \\ n+1+w_{k-n-1} & \text{if } i > n+1 \end{cases}$$

For $w \in W_{C_n}$ or W_{D_n}, embed w into S_{2n} by sending w to $i(w)$ where

$$i(w)_k = \begin{cases} n - w_{n+1-k} + \chi(w_{n+1-k} > 0) & \text{if } i \leq n \\ n + w_{k-n} + \chi(w_{k-n} < 0) & \text{if } i > n \end{cases} \tag{I.26}$$

where $\chi(arg)$ is 1 if arg is true and 0 if arg is false.

In general, for $w \in W$ the Schubert cell X_w can be represented by matrices in $M_{i(w)} \cap G$, where $i(w)$ is the permutation given by the embedding and G is the corresponding Lie group. Fulton [16] points out that $M_{i(w)} \cap G$ can be obtained by adding conditions to the diagram of the permutation which force the rows to be orthogonal under the bilinear form. The set $M_{i(w)} \cap G$, is isomorphic to $\mathbb{C}^{l(w)}$ where $l(w)$ is the length of w considered as an element of W. Note $\mathbb{C}^n$ has dimension $2n$.

If X is a flag manifold then the cohomology ring $H^*(X)$ is the same as the Chow ring [14][Cor. 19.2(b), Example 19.1.11]. Therefore, each closed subvariety V of X determines an element $[V] \in H^*(X)$. In particular, each Schubert variety corresponds to a cohomology class $[\overline{X_w}]$. Cup product in $H^*(X)$ corresponds to intersection of subvarieties (if defined in general position). The decomposition of X into Schubert cells X_w, which are of even real dimension and whose boundaries are unions of smaller Schubert cells, implies that the cohomology ring $H^*(X, \mathbb{Z})$ is concentrated in even dimensions (hence commutative), and induces a corresponding $\mathbb{Z}$-basis for $H^*(X, \mathbb{Z})$ of *Schubert class* C_w. [3]

Next, we introduce the theory developed by Bernstein, Gelfand, and Gelfand in [4] and independently by Demazure [7]. We associate to each root $\alpha \in R$ the equation of the perpendicular hyperplane, say $\gamma(\alpha)$. In general, $\gamma(\alpha)$ is obtained from α by replacing e_i with z_i. For example, $\gamma(e_{i+1} - e_i)$ is $z_{i+1} - z_i$.

Definition. Assume we are given a root system R with basis B. For each root

[3]To be more correct, one would typically call these elements Schubert cycles. Then Schubert classes are classes of polynomials in a quotient space which is isomorphic to $H^*(X_n)$ and they are the image of the Schubert cycles under this isomorphism. However, for our purposes the Schubert classes have the same properties as the Schubert cycles and we will usually be discussing classes of polynomials.

$\alpha \in B$, define the *divided difference operator* ∂_α on $f \in \mathbb{C}[[z_1, z_2, \ldots]]$ by

$$\partial_\alpha f = \frac{f - \sigma_\alpha f}{-\gamma(\alpha)} \tag{I.27}$$

PROPOSITION 7.2. *$f - \sigma_\alpha f$ is divisible by $\gamma(\alpha)$. Hence, $\partial_\alpha f$ is a polynomial if f is a polynomial.*

PROOF. Every point in the hyperplane perpendicular to α is fixed by σ_α. Therefore, $f - \sigma_\alpha f$ is 0 whenever $\gamma(\alpha) = 0$. By commutative algebra, this implies the ideal generated by $f - \sigma_\alpha f$ is contained in the ideal generated by $\gamma(\alpha)$. Hence, $f - \sigma_\alpha f = g \cdot \gamma(\alpha)$ for some polynomial g. $\square$

The following proposition is the fundamental step between Schubert varieties and Schubert polynomials, due to Bernstein, Gelfand and Gelfand.

PROPOSITION 7.3. [4] *The Schubert classes C_w satisfy the equations*

$$\partial_\alpha C_w = \begin{cases} C_{w\sigma_\alpha} & \text{if } l(w\sigma_\alpha) < l(w) \\ 0 & \text{if } l(w\sigma_\alpha) > l(w). \end{cases} \tag{I.28}$$

These equations, together with the dimensions of the C_w and the fact that $C_1 = 1$, determine the classes C_w.

The surprising aspect of this theorem is that one can compute the cohomology of a Schubert variety from the cohomology of a point, C_{w_0}.

The cohomology ring $H^*(X)$ can be naturally identified with the quotient $\mathbb{Q}[P]/I$ of the symmetric algebra of the root space P by the ideal I generated by all non-constant homogeneous Weyl group invariant polynomials. This can be understood to imply that the cohomology ring is isomorphic to a quotient space of a polynomial ring, namely

$$H^*(X_n) \approx \frac{\mathbb{Q}[z_1, z_2, \ldots]}{I_n} = H_n, \tag{I.29}$$

where I_n is the ideal generated by the Weyl group invariant polynomials in n variables without constant terms and all variables beyond n. For our Weyl groups we have the following generators for the ideal I, a discussion of these generators can be found in [20]:

$$\begin{aligned} S_n: &\quad I_n = < p_1, p_2, \dots ; z_{n+1}, z_{n+2}, \dots > \\ W_{B_n}: &\quad I_n = < p_2, p_4, p_6 \dots ; z_{n+1}, z_{n+2}, \dots > \\ W_{D_n}: &\quad I_n = < z_1 z_2 \cdots z_n ; p_2, p_4, \dots ; z_{n+1}, z_{n+2}, \dots > . \end{aligned}$$

Here $p_k = z_1^k + z_2^k + \dots$ is the k^{th} power sum. We have written our quotient spaces in a slightly different form than is found in the literature in order to make clear that we can define surjective ring homomorphisms $\rho_n H^*(X_{n+1}) \to H^*(X_n)$ for each n. In each case,

$$H_n = \frac{H_{n+1}}{< z_{n+1} >}. \tag{I.30}$$

so ρ_n is defined by

$$\rho_n(z_i) = \begin{cases} z_i & \text{if } i \leq n \\ 0 & \text{if } i > n. \end{cases} \tag{I.31}$$

The divided difference operators ∂_α corresponding to simple roots (from any root system in the same family) act on each cohomology ring H_n. Recall that the Schubert classes are uniquely determined by the divided difference equations. Therefore, for $w \in W_n$ the Schubert class C_w in $H^*(X_{n+1})$ maps to the corresponding Schubert class in $H^*(X_n)$ under ρ_n. This property holds in the geometry as well. From our matrix representatives of the Schubert cells, one can check each Schubert cell X_w in X_n embeds as the Schubert cell X_w in X_{n+1}. By Proposition 7.1, the same result holds for Schubert varieties.

In summary, the Schubert classes C_w have three main properties we will use:

(1) The Schubert classes are stable under the surjection ρ_n for large n, *i.e.* if $w \times 1 = [w_1, \dots, w_n, n+1]$ then $\rho_n : C_{w\times 1} = C_w$.

(2) For each $w \in W$ and $\alpha \in B$ the basis of the root system,

$$\partial_\alpha C_w = \begin{cases} C_w & \text{if } l(w\sigma_\alpha) < l(w) \\ 0 & \text{if } l(w\sigma_\alpha) > l(w). \end{cases} \tag{I.32}$$

(3) Each C_w belongs to $H^{2l(w)}(X)$

DEFINITION. Given a sequence of rings A_n with homomorphisms $\gamma_n : A_{n+1} \to A_n$ we can construct the *inverse limit*, denoted $\varprojlim A_n$, to be the set of coherent sequences $(a_1, a_2, \ldots)$ such that $a_i \in A_i$ and $\gamma_i a_{i+1} = a_i$.

Now we can give a general definition of Schubert polynomials.

DEFINITION. For any of the classical families A, B, C, or D, let the n-th group be G_n, with Weyl group W_n and flag manifold X_n. Let $W_\infty = \varinjlim W_n$ be the direct limit of the Weyl groups. Recall $H_n \approx H^*(X_n)$ from (I.29). For $w \in W_\infty$ the *Schubert polynomial* S_w is the element $\varprojlim C_w$ in the inverse limit $\varprojlim H_n$ of the system

$$\cdots \leftarrow H_n \leftarrow H_{n+1} \leftarrow \cdots \tag{I.33}$$

THEOREM 7.4. *The Schubert polynomial S_w can be represented by a polynomial in $\mathbb{Z}[z_1, z_2, \ldots]$ for type A, or in $\mathbb{Q}[z_1, z_2, \ldots; p_1, p_3, p_5, \ldots]$ for types B,C, and D.*

PROOF. For each $w \in S_\infty$, there exists an N such that $\partial_i C_w = 0$ for all $i > N$. It suffices to take N larger than any letter in a reduced word for w. Therefore, C_w is symmetric in the variables $z_{N+1}, z_{N+2}, \ldots$ all of which are zero except for a finite number in any particular H_n. Hence, C_w only involves the first N variables, *i.e.*

$$C_w \in \frac{\mathbb{Z}[z_1, \ldots, z_N]}{I_n \cap \mathbb{Z}[z_1, \ldots, z_N]}. \tag{I.34}$$

This subring is constant in the inverse limit and equal to

$$\frac{\mathbb{Z}[z_1, \ldots, z_N]}{< p_1, \ldots, p_n >}. \tag{I.35}$$

Therefore, we can write $\varprojlim C_w$ as a polynomial in the basis of monomials $z_1^{\epsilon_1} \ldots z_N^{\epsilon_n}$ where $\epsilon_k < N - k$.

Almost the same proof works for B, C, and D. First, note that in the inverse limit of H_n the special invariant $z_1 \cdots z_n$ for D_n will be 0 since $z_1 \cdots z_n$ is 0 in H_{n-1} for each n. For each $w \in B_\infty$ or D_∞ there exists an N such that $\partial_i C_w = 0$ for all $i > N$. So by the same reasoning,

$$\varprojlim C_w \in \frac{\mathbb{Q}[z_1, \ldots, z_N]}{< p_2, p_4, \ldots >}. \tag{I.36}$$

Since C_w is homogeneous of degree $l(w)$ any representative in H_n cannot involve p_i for $i > l(w)$. Hence, $\varprojlim C_w$ can be represented by a polynomial in $\mathbb{Q}[z_1, \ldots, z_n; p_1, p_3, \ldots, p_{odd[l(w)]}]$, where $odd[k]$ is k or $k-1$ whichever is odd. □

8. Schubert polynomials of types B, C, and D

In this section we interpret the abstract definition of the Schubert polynomials given in Section 7 for the root systems of type B,C and D. We state the formulas for computing Schubert polynomials of types B, C and D in Theorems 3 and 4. We motivate the formulas by outlining a general formula for polynomials which automatically satisfy almost all of the divided difference equations. Finally, we prove that Schubert polynomials are unique as we have defined them. The content of this section and all of the theorems are a product of joint work with Mark Haiman, [6].

First, we need to interpret the equations for the divided differences operators in (I.27) for each case. Recall, that for $f \in \mathbb{C}[[z_1, z_2, \ldots]]$ and each basis element α, we have

$$\partial_\alpha f = \frac{f - \sigma_\alpha f}{-\gamma(\alpha)}. \tag{I.37}$$

Below we have a table of divided difference operators for each of the 4 root systems computed using the basis elements in (I.16).

$$\begin{aligned} A, B, C, D:\ \partial_i f &= \frac{f - \sigma_i f}{z_i - z_{i+1}} \qquad \forall i \geq 1 \\ B:\ \partial_0^B f &= \frac{f - \sigma_0 f}{-z_1} \\ C:\ \partial_0^C f &= \frac{f - \sigma_0 f}{-2z_1} \\ D:\ \partial_{\hat{1}} f &= \frac{f - \sigma_{\hat{1}} f}{-z_1 - z_2} r \end{aligned} \tag{I.38}$$

Definition. For every $w \in B_\infty$, the *Schubert polynomial* $\mathfrak{B}_w \in \mathbb{Q}[z_1, z_2, \ldots; p_1, p_3, \ldots]$ satisfies the equation

$$\partial_i \mathfrak{B}_w = \begin{cases} \mathfrak{B}_{w\sigma_i} & \text{if } l(w\sigma_i) < l(w) \\ 0 & \text{if } l(w\sigma_i) > l(w) \end{cases} \tag{I.39}$$

for all $i \geq 1$ and ∂_0^B, together with the condition that the constant term of S_w is 1 if $w = [1, 2, \ldots]$ and 0 otherwise.

Definition. For every $w \in B_\infty$, the *Schubert polynomial* $\mathfrak{C}_w \in \mathbb{Q}[z_1, z_2, \ldots; p_1, p_3, \ldots]$ satisfies the equation

$$\partial_i \mathfrak{C}_w = \begin{cases} \mathfrak{C}_{w\sigma_i} & \text{if } l(w\sigma_i) < l(w) \\ 0 & \text{if } l(w\sigma_i) > l(w) \end{cases} \tag{I.40}$$

for all $i > 1$ and ∂_0^C, together with the condition that the constant term of S_w is 1 if $w = [1, 2, \ldots]$ and 0 otherwise.

Definition. For every $w \in D_\infty$, the *Schubert polynomial* $\mathfrak{D}_w \in \mathbb{Q}[z_1, z_2, \ldots; p_1, p_3, \ldots]$ satisfies the equation

$$\partial_i \mathfrak{D}_w = \begin{cases} \mathfrak{D}_{w\sigma_i} & \text{if } l(w\sigma_i) < l(w) \\ 0 & \text{if } l(w\sigma_i) > l(w) \end{cases} \tag{I.41}$$

for all $i > 1$ and $i = \hat{1}$, together with the condition that the constant term of S_w is 1 if $w = [1, 2, \ldots]$ and 0 otherwise.

Our approach to finding a general formula for arbitrary Schubert polynomials is to use the Schubert polynomials of type A from Section 4. We can get a solution to the divided difference equations for $i \in \{1, 2, \ldots\}$ simply by assuming the Schubert polynomials S_w for a fixed root system have the form

$$S_w(Z) = \sum_{\substack{uv=w \\ l(u)+l(v)=l(w) \\ v \in S_\infty}} F_u(Z)\mathfrak{S}_v(Z). \tag{I.42}$$

Here F_u is a symmetric function in the variables $Z = \{z_1, z_2, \ldots\}$ *i.e.* $\partial_i F_u(Z) = F_u(Z)\partial_i$ and $\mathfrak{S}_w$ is the Schubert polynomials of type A. Then for $i = 1, 2, 3, \ldots$, we have

$$\partial_i S_w(Z) = \sum F_u(Z)\partial_i \mathfrak{S}_w(Z) \tag{I.43}$$

$$= \sum_{\substack{uv\sigma_i = w\sigma_i \\ v\sigma_i \in S_\infty \\ l(v\sigma_i)<l(v)}} F_u \mathfrak{S}_{v\sigma_i} \tag{I.44}$$

$$= \begin{cases} S_{w\sigma_i} \text{ if } l(w\sigma_i) < l(w) \\ 0 \text{ if } l(w\sigma_i) > l(w) \end{cases} \tag{I.45}$$

Now we only need to check $\partial_0^B, \partial_0^C$, or $\partial_{\hat{1}}$ act correctly in cases B, C, or D respectively.

In order to present the Schubert polynomials in their most natural (combinatorial) state, we would like to replace the ring $\mathbb{Q}[z_1, z_2, \ldots; p_1, p_3, \ldots]$ with the isomorphic ring $\mathbb{Q}[z_1, z_2, \ldots; p_1(X), p_3(X), \ldots]$, where $p_k(X) = x_1^k + x_2^k + \cdots$ are power sums in new variables, and we identify $p_k(X)$ with $-p_k(Z)/2$. Warning, this plethistic substitution needs to be handled very carefully. We can only make the substitution for the p_k's. We cannot take this to mean $z_i/2 = x_i$ for any i. We will use this in the proof of Theorem 3 of Chapter IV.

We will define F_u in terms of the X variables. Of course, if F_u is symmetric in the x's then it can be written as a polynomial in the $p_i(X)$'s, hence substituting $p_i(Z)$ for $p_i(X)$ we have F_u is a symmetric function in the z's. We will show in Chapter III, there is in fact a second way of defining each F_w and E_w in terms of Schur Q-functions. These functions are the B_∞ and D_∞ analogs of the Stanley functions in (I.12).

Definition. Given a reduced word $\mathbf{a} \in R(w)$. Let $P(\mathbf{a}) = \{i : a_{i-1} < a_i > a_{i+1}\}$ be the *peak set* for $\mathbf{a}$. Then $A_x(\mathbf{a})$ is the set of monomials $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m} = x_{i_1} x_{i_2} \cdots x_{i_p}$ such that

(1) $i_1 \geq i_2 \geq \ldots \geq i_l$
(2) $i_{k-1} = i_k = i_{k+1} \implies k \notin P(a)$

For example, $z_6^5 z_4^2 = z_6 z_6 z_6 z_6 z_6 z_4 z_4 \in A_x(4312573)$. Note there is no upper bound on any i_k as in (I.8). Hence, $A_x(\mathbf{a})$ is an infinite set of monomials.

DEFINITION. Let $i(\alpha)$ be the number of distinct variables with non-zero exponent in x^α and $o(\alpha)$ be the number of 1's and $\hat{1}$'s in a. Then define the Stanley symmetric functions for B_∞ and D_∞ respectively as

$$F_w(x_1, x_2, \dots) = \sum_{\substack{a \in R(w) \\ x^\alpha \in A_x(a)}} 2^{i(\alpha)} x^\alpha \tag{I.46}$$

$$E_w(x_1, x_2, \dots) = \sum_{\substack{a \in R(w) \\ x^\alpha \in A_x(a)}} 2^{i(\alpha)-o(\alpha)} x^\alpha. \tag{I.47}$$

F_w and E_w can also be written as polynomials in the Schur-P and -Q functions.

THEOREM 3. *The Schubert polynomials* $\mathfrak{C}_w$ *are given by*

$$\mathfrak{C}_w = \sum_{\substack{uv=w \\ l(u)+l(v)=l(w) \\ v \in S_\infty}} F_u(X) \mathfrak{S}_v(Z) \tag{I.48}$$

$$= \sum_{\substack{uv=w \\ v \in S_\infty}} \sum_{\substack{a \in R(u) \\ x^\alpha \in A_x(a)}} \sum_{\substack{b \in R(v) \\ z^\beta \in A_z(b)}} 2^{i(\alpha)} x^\alpha z^\beta \tag{I.49}$$

where $F_u(X)$ *is a certain non-negative integral linear combination of Schur Q-functions computed from* $u \in B_\infty$ *via the Haiman correspondence (see Chapter* III*). Given a partition* $\mu = (\mu_1 > \mu_2 > \dots > \mu_l)$ *with distinct parts, let*

$$w = \overline{\mu_1}\, \overline{\mu_2} \dots \overline{\mu_l} 12 \dots . \tag{I.50}$$

where (I.50) *the bars denote minus signs, and the ellipsis at the end stands for the remaining positive integers, omitting the* μ_i*'s, in increasing order. Then we have*

$$\mathfrak{C}_w = Q_\mu(X), \quad \mathfrak{B}_w = P_\mu(X). \tag{I.51}$$

THEOREM 4. *The Schubert polynomials* $\mathfrak{D}_w$ *are given by*

$$\mathfrak{D}_w = \sum_{\substack{uv=w \\ l(u)+l(v)=l(w) \\ v \in S_\infty}} E_u(X) \mathfrak{S}_v(Z) \tag{I.52}$$

$$= \sum_{\substack{uv=w \\ v \in S_\infty}} \sum_{\substack{a \in R(u) \\ x^\alpha \in A_x(a)}} \sum_{\substack{b \in R(v) \\ z^\beta \in A_z(b)}} 2^{i(\alpha)-o(\alpha)} x^\alpha z^\beta \tag{I.53}$$

where $E_u(X)$ is a certain non-negative integral linear combination of Schur P-functions computed from $u \in D_\infty$ via the D_n Haiman correspondence (see Chapter III*). Given a partition $\mu = (\mu_1 > \mu_2 > \cdots > \mu_l)$ with distinct parts, let $\nu_i = 1 + \mu_i$, taking $\mu_l = 0$ if necessary to make the number of parts even. Then for*

$$w = \overline{\nu_1}\,\overline{\nu_2}\ldots\overline{\nu_l}12\ldots, \tag{I.54}$$

we have

$$\mathfrak{D}_w = P_\mu(X). \tag{I.55}$$

THEOREM 5. *The Schubert polynomials $\mathfrak{C}_w$ of type C are a $\mathbb{Z}$-basis for the ring $\mathbb{Z}[z_1, z_2, \ldots; Q_\mu]$. The polynomials $\mathfrak{B}_w$ and $\mathfrak{D}_w$ are both $\mathbb{Z}$-bases for the ring $\mathbb{Z}[z_1, z_2, \ldots; P_\mu]$. Hence, the Schubert polynomials of type B, C, and D each are bases for the space $\mathbb{Z}[z_1, z_2, \ldots; p_1, p_2, \ldots]$.*

THEOREM 6. *Solutions of the defining equations for each type of Schubert polynomials are unique.*

PROOF. Let $\{S_w\}$ be a family of polynomials satisfying the defining recurrence relations, together with the constant term conditions. Suppose $\{S'_w\}$ is another solution. For each i,

$$\partial_i(S_w - S'_w) = \begin{cases} S_{w\sigma_i} - S'_{w\sigma_i} & \text{if } l(w\sigma_i) < l(w) \\ 0 & \text{if } l(w\sigma_i) > l(w). \end{cases} \tag{I.56}$$

By induction on the length of w, we may assume $S_{w\sigma_i} - S'_{w\sigma_i} = 0$. Then $\partial_i(S_w - S'_w) = 0$ for each appropriate i, so $S_w - S'_w$ is invariant for the relevant group S_∞, B_∞ or D_∞. The only S_∞ invariants in $\mathbb{Q}[z_1, z_2, \ldots]$ are constants, as are the B_∞ or D_∞ invariants in $\mathbb{Q}[z_1, z_2, \ldots; p_1, p_3, \ldots]$, because the even power sums are missing. Hence $S_w - S'_w$ is constant, so $S_w = S'_w$ by the constant term conditions. □

THEOREM 7. *In the product expansions*

$$\mathfrak{S}_u\mathfrak{S}_v = \sum_w c^w_{uv}\mathfrak{S}_w \tag{I.57}$$

and like expansions for types B, C, and D, the coefficients c^w_{uv} are non-negative.

PROOF. From intersection theory we know products of Schubert classes expand into Schubert classes with non-negative integer coefficients [14]. Since Schubert polynomials are representatives of these polynomial classes the same result holds. $\square$

CHAPTER II

RC-graphs

Our approach to computing Schubert polynomials is an algorithmic one. The idea is related to a conjecture originally due to Axel Kohnert [23]. Kohnert conjectured that the Schubert polynomials could be constructed by applying a recursive algorithm on the diagram of a permutation $D(w) = \{(i,j) : j = w_{i'} < w_i \text{ for } i' > i\}$. Each diagram that appears in the recursion contributes a term to the Schubert polynomial. At this time, Kohnert's conjecture has not been proved except in the special case that w is a vexillary permutation (or 2143-avoiding). We have verified the conjecture for every permutation in S_7.

Bergeron published an algorithm similar to Kohnert's for computing Schubert polynomials [2]. This algorithm again starts with $D(w)$ but it is computationally more complex. Some identities about Schubert polynomials cannot readily be established using this method of computation. Furthermore, the algorithm given in [33] is wrong. The permutation $w = [2, 1, 6, 4, 5, 3]$ is a counterexample. Therefore, we were driven to find yet another constructive method of computing Schubert polynomials from some other set of diagrams.

Fomin and Kirillov introduced in [9] a new set of diagrams which encode the Schubert polynomials. We call this object an rc-graph (reduced word/compatible sequence graph). In the spirit of Kohnert's conjecture, we are interested in constructing Schubert polynomials by doing "moves" on rc-graphs. We will define and prove two algorithms for constructing the set of all rc-graphs for a given permutation in Section 1. These two algorithms have been much more efficient in time and space than previously known algorithms. The algorithm can be extended to generate the double Schubert polynomials as well. Using these two constructions, many of the identities known for Schubert polynomials become more apparent and new identities have emerged.

Computers have facilitated our work immensely. We have gained an invaluable amount of intuition about Schubert polynomials by looking at data, we have been able to rule out false conjectures quickly and we have found two very interesting conjectures. In Chapter V, we conjecture two analogs of Pieri's rule for multiplying Schubert polynomials. We explain how we used our computers to find these conjectures.

1. Constructing RC-Graphs

In this section we define the rc-graphs and an algorithm for computing the polynomials, $\mathfrak{S}_w$. The goal of our algorithm is to start with a particular rc-graph and apply a sequence of transformations; thereby obtaining all rc-graphs for a permutation. The transformations will be of two types, namely *chute moves* and *ladder moves*. After proving several lemmas, we will state and prove Theorem 2, our main theorem, which states that this algorithm constructs the Schubert polynomials. Theorem 3 gives the second algorithm for computing the polynomials. The proof follows easily from Theorem 2 after defining an involution sending the rc-graphs for w into rc-graphs for w^{-1}. We conclude this section with four corollaries which follow easily from Theorems 2 and Theorem 3.

Recall the following definitions from Chapter I, Section 4. If $\mathbf{a} = a_1 a_2 \dots a_p$ is a reduced word for $w \in S_n$, we say the sequence $j_1 j_2 \cdots j_p$ is $\mathbf{a} -$ *compatible* if

(1) $j_1 \leq j_2 \leq \cdots \leq j_p$
(2) $j_i = j_{i+1}$ implies $a_i > a_{i+1}$
(3) $j_i \leq a_i$ for all i.

Definition. For all $w \in S_\infty$, let

$$\mathfrak{S}_w(z_1, z_2, \dots) = \sum_{\mathbf{a} \in R(w)} \sum_{z^\alpha \in \mathcal{A}_z(\mathbf{a})} z^\alpha. \tag{II.1}$$

Definition. Given any reduced word $\mathbf{a} = a_1 a_2 \ldots a_p$ and an $\mathbf{a}$-compatible sequence $\boldsymbol{\alpha} = \alpha_1\alpha_2\ldots\alpha_p$ the *reduced word compatible sequence graph* or *rc-graph* of the pair $(\mathbf{a},\boldsymbol{\alpha})$ is $D(\mathbf{a},\boldsymbol{\alpha}) = \{(\alpha_k, a_k - \alpha_k + 1)\}$. Let $\mathcal{RC}(w) = \{D(\mathbf{a},\boldsymbol{\alpha}) : \mathbf{a} \in R(w), \boldsymbol{\alpha} \in C(\mathbf{a})\}$.

We realize $D(\mathbf{a},\boldsymbol{\alpha})$ geometrically as the graph of $\{(\alpha_k, a_k - \alpha_k + 1)\}$. For example, let $\mathbf{a} = 521345$ and $\boldsymbol{\alpha} = 111235$ then $D(\mathbf{a},\boldsymbol{\alpha}) \in \mathcal{RC}[3,1,4,6,5,2]$ is

(II.2)

	1	2	3	4	5	6
1	+	+	·	·	+	·
2	·	+	·	·	·	
3	·	+	·	·		
4	·	·	·			
5	+	·				
6	·					

where a + represents an occupied position and a $\cdot$ represents an unoccupied position in the graph. In some cases, it is convenient to use $\overline{D}(\mathbf{a},\boldsymbol{\alpha}) = \{(\alpha_k, a_k)\}$. However, it will become clear why $D(\mathbf{a},\boldsymbol{\alpha})$ is more natural.

Given any rc-graph one can find the reduced word by reading out the numbers $j + i - 1$ of the occupied positions (i, j) going right to left, top to bottom in each row. The compatible sequence is found by reading the row numbers of the occupied positions in the same order. If $a_1 a_2 \ldots a_p$ is the reduced word read from the rc-graph D then let

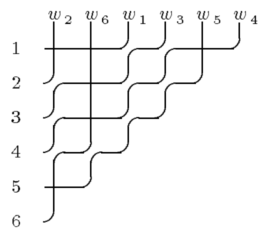
$$perm(D) = s_{a_1} s_{a_2} \ldots s_{a_p} \tag{II.3}$$

be the permutation such that $D \in \mathcal{RC}(perm(D))$. These graphs can be defined more generally to include words which are not reduced, but we will not be using this property.

It follows from the definition of a compatible sequence that all rc-graphs lie in $\mathbb{P} \times \mathbb{P}$. Moreover, if $perm(D) \in S_n$ then the elements (i, j) of D are such that $i + j < n$. Conversely, any graph in $\mathbb{P} \times \mathbb{P}$ that lies in the area $i + j < n$ and gives a reduced word $\mathbf{a}$ by using the above reading, is an rc-graph of a permutation of S_n, and the corresponding sequence of row numbers will be $\mathbf{a}$-compatible.

Fomin and Kirillov [9] originally introduced the rc-graphs above with more structure. The idea is to consider the rc-graph as a *planar history* of the inversions of $w = perm(D)$. To this end, we draw strings which cross at the positions $(i, j) \in D$ and avoid each other at the positions $(i, j) \notin D$. Below, we give an example for the permutation $w = [3, 1, 4, 6, 5, 2]$ and the rc-graph in (II.2).

(II.4)



We label the strings by the number $1, 2, 3, \ldots$ from top to bottom on the left end of the graph D. In our examples, we will eliminate the *sea* of strings labeled i if $w_i = i$ for all $i > n$. For $D \in \mathcal{RC}(w)$, it is easy to see that the strings will be permuted, through the rc-graph D, according to the permutation w. More precisely, the string labeled i will end up in column w_i on the top row of D. Clearly, no two strings cross each other more than once since the underlying picture is an rc-graph, and hence has the minimum number of crossings. The set $\mathcal{RC}(w)$ is the set of all such strings configurations with exactly $\ell(w)$ crossings. For the rest of this paper, we will consider the set $\mathcal{RC}(w)$ to be the set of rc-graphs with labeled strings as described above. We will draw the strings only when needed.

Lemma 1.1. *The transpose of an rc-graph $D \in \mathcal{RC}(w)$ is an rc-graph $D^t \in \mathcal{RC}(w^{-1})$. Hence, the map $\rho : \mathcal{RC}(w) \to \mathcal{RC}(w^{-1})$ given by $\rho(D) = D^t$ is an involution.*

V. Reiner suggested the same involution using only reduced words and compatible sequences.

Proof. If $D \in \mathcal{RC}(w)$, the strings in D^t trace out the permutation w^{-1}. Furthermore, $\ell(w) = \ell(w^{-1})$. Therefore, there number of crossings is minimal. $\square$

If we use the notation

$$x_D = \prod_{(i,j)\in D} x_i, \tag{II.5}$$

then the following corollary is a simple consequence of Theorem 1. This was also noted in [9].

COROLLARY 1.2. *Given any permutation $w \in S_\infty$,*

$$\mathfrak{S}_w = \sum_{D(\mathbf{a},\boldsymbol{\alpha})\in \mathcal{R}C(w)} x_{D(\mathbf{a},\boldsymbol{\alpha})}. \tag{II.6}$$

There are two particular rc-graphs for each permutation that are special in our situation. We define these below.

DEFINITION. For $w \in S_\infty$, let $D_{bot}(w) = \{(i,c) : c \leq m_i\} \in \mathcal{R}C(w)$ where $m_i = \#\{j : j > i \textit{ and } w_j < w_i\}$. This corresponds with graphing the largest reduced word in reverse lexicographic order and the largest compatible sequence for this word in ordinary lexicographic order.

DEFINITION. For $w \in S_\infty$, let $D_{top}(w) = \{(c,j) : c \leq n_j\} = D^t_{bot}(w^{-1})$ where $n_j = \#\{i : i < w_j^{-1} \text{ and } w_i > j\}$. This corresponds corresponds with graphing the smallest reduced word in reverse lexicographic order and the smallest compatible sequence for this word in ordinary lexicographic order.

Continuing the example in (II.2), the following rc-graphs are D_{top} and D_{bot} for $w = [3,1,4,6,5,2]$:

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & + & + & \cdot & \cdot & + \\ 2 & \cdot & + & \cdot & \cdot & \\ 3 & \cdot & + & \cdot & & \\ 4 & \cdot & + & & & \\ 5 & \cdot & & & & \end{array} \qquad \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & + & + & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & \cdot & \cdot & \\ 3 & + & \cdot & \cdot & & \\ 4 & + & + & & & \\ 5 & + & & & & \end{array} \tag{II.7}$$

For these new objects, we tried to find *moves* that would be analogous to the *moves* in Kohnert's conjecture [23]. This naturally lead us to the following definitions. Fix a permutation $w \in S_\infty$.

DEFINITION. Given any $D(\mathbf{a},\boldsymbol{\alpha}) \in \mathcal{RC}(w)$ we transform $D(\mathbf{a},\boldsymbol{\alpha})$ into $D(\mathbf{b},\beta)$ by a *ladder move*, $\mathcal{L}_{(i,j)}$, provided $D(\mathbf{a},\boldsymbol{\alpha})$ and $D(\mathbf{b},\beta)$ have the following configurations on two adjacent columns:

$$
\begin{array}{ccc} & j & j{+}1 \\ i{-}m & \cdot & \cdot \\ & + & + \\ & + & + \\ & + & + \\ i & + & \cdot \end{array}
\quad \mapsto \quad
\begin{array}{ccc} & j & j{+}1 \\ i{-}m & \cdot & + \\ & + & + \\ & + & + \\ & + & + \\ i & \cdot & \cdot \end{array}
\tag{II.8}
$$

We have only drawn two columns because no other points in the rc-graph will affect the possibility of doing a move. Formally, $\mathcal{L}_{(i,j)}(D) = D \cup \{(i-m, j+1)\} \setminus \{(i,j)\}$ provided

- $(i,j) \in D,\ (i,j+1) \notin D$.
- There exist $0 < m < i$ such that $(i-m,j), (i-m,j+1) \notin D$.
- For each $1 \leq k < m,\ \ (i-k,j), (i-k,j+1) \in D$.

Let $\mathcal{L}(D) = \{D' : D' = \mathcal{L}_{(i_1,j_1)} \cdots \mathcal{L}_{(i_k,j_k)}(D)$ for some sequence $(i_1,j_1), \ldots, (i_k,j_k)\}$.

DEFINITION. We transform $D(\mathbf{a},\boldsymbol{\alpha})$ into $D(\mathbf{b},\beta)$ by a *chute move*, $\mathcal{C}_{(i,j)}$, provided $D(\mathbf{a},\boldsymbol{\alpha})$ and $D(\mathbf{b},\beta)$ have the following configurations:

$$
\begin{array}{cccccc} & j{-}m & & & & j \\ i & \cdot & + & + & + & + \\ i{+}1 & \cdot & + & + & + & \cdot \end{array}
\quad \mapsto \quad
\begin{array}{cccccc} & j{-}m & & & & j \\ i & \cdot & + & + & + & \cdot \\ i{+}1 & + & + & + & + & \cdot \end{array}
\tag{II.9}
$$

Formally, $\mathcal{C}_{(i,j)}(D) = D \cup \{(i+1, j-m)\} \setminus \{(i,j)\}$ provided:

- $(i,j) \in D,\ (i+1,j) \notin D$.
- There exits $0 < m < j$ such that $(i,j-m), (i+1,j-m) \notin D$.
- For each $1 \leq k < m,\ \ (i,j-k), (i+1,j-k) \in D$.

Let $\mathcal{C}(D) = \{D' : D' = \mathcal{C}_{(i_1,j_1)} \cdot \mathcal{C}_{(i_k,j_k)}(D)$ for some sequence $(i_1,j_1), \ldots, (i_k,j_k)\}$.

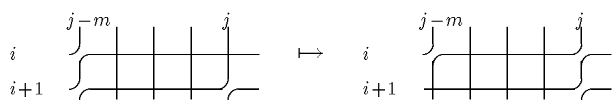
Comparing the two definitions above and using the Lemma 1.1, we have proven the following lemma.

Lemma 1.3. *Given any rc-graph D, $\rho(\mathcal{L}_{(i,j)}(D)) = \mathcal{C}_{(j,i)}(\rho(D))$, that is $\mathcal{L}_{(i,j)} = \rho \circ \mathcal{C}_{(j,i)} \circ \rho$.*

Lemma 1.4. *If $D \in \mathcal{RC}(w)$ then both $\mathcal{C}_{(i,j)}(D)$ and $\mathcal{L}_{(i,j)}(D)$ are in $\mathcal{RC}(w)$, if they exist.*

Proof. It is clear from (II.10) below that $\mathcal{C}_{(i,j)}(D)$ permutes the strings $1, 2, 3, \ldots$ exactly as D does. Hence $perm(\mathcal{C}_{(i,j)}(D)) = perm(D) = w$.

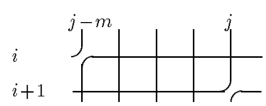
(II.10)



Transposing (II.10) shows that $\mathcal{L}_{(i,j)}(D) \in \mathcal{RC}(w)$. □

Lemma 1.5. *The following configuration cannot appear in $D(\mathbf{a}, \boldsymbol{\alpha})$ if $\mathbf{a}$ is a reduced word:*

(II.11)



Proof. This configuration cannot happen in an rc-graph since no two strings are allowed to cross twice. □

Lemma 1.6. *If $D \in \mathcal{RC}(w)$ and somewhere in D we have*

$$(i, j) \notin D \quad \text{and} \quad (i + 1, j) \in D \tag{II.12}$$

then it is possible to perform an inverse chute move somewhere on D.

Proof. Note that both chutes and ladders have well defined inverse operations. Starting with $(i + 1, j)$ look right along row $i + 1$ for the smallest $k > j$ such that $(i+1, k) \notin D$. There must be some unoccupied position in row $i+1$ since D contains only a finite number of points. The position (i, k) cannot be in D or there would be a contradiction of Lemma 1.5.

(II.13)
$$\begin{array}{cccccccccccccc} & j & & & & & & & m & & & & & k \\ i & \cdot & + & + & \cdot & + & \cdot & \cdot & \cdot & + & + & + & + & \cdot \\ i{+}1 & + & + & + & + & + & + & + & + & + & + & + & + & \cdot \end{array}$$

Now look to the left of (i,k) for the largest $m < k$ such that $(i,m) \notin D$. Then $(i+1,m)$ is able to move to (i,k) by an inverse chute move. □

THEOREM 2. *Given any* $w \in S_\infty$,

(II.14)
$$\mathfrak{S}_w = \sum_{D \in \mathcal{C}(D_{top}(w))} x_D.$$

PROOF. Starting with $D_{top}(w)$ and applying all possible sequences of chute moves we get a set of rc-graphs contained in $\mathcal{RC}(w)$ by Lemma 1.4.

From Lemma 1.6, the only possible rc-graphs that have no inverse chute moves are ones where there is no (i,j) satisfying (II.12). We can easily count the number of possible rc-graphs for permutations in S_n which do not contain this configuration. We have n choices of placing the first empty point in the first column, after this point there cannot be any occupied positions. Then $n-1$ choices for placing the first empty point in the second column, etc. Therefore, there are exactly $n!$ rc-graphs not satisfying (II.12) anywhere for permutations in S_n. Each permutation in S_n has at least one rc-graph not satisfying (II.12), namely $D_{top}(w)$. Hence, $D_{top}(w)$ is the only rc-graph in $RC(w)$ that cannot be transformed by some inverse chute move.

Each inverse chute move pushes a point of an rc-graph up. Therefore, every $D \in \mathcal{RC}(w)$ can be transformed into $D_{top}(w)$ by a finite sequence of inverse chute moves. Reversing this sequence and applying chutes to $D_{top}(w)$ we arrive at D. Hence $\mathcal{C}(D_{top}(w)) = \mathcal{RC}(w)$. The theorem follows from Corollary 1.2. □

Using the involution ρ of Lemma 1.1, we can also generate the rc-graphs by ladder moves from $D_{bot}(w)$.

THEOREM 3. *Given any* $w \in S_\infty$,

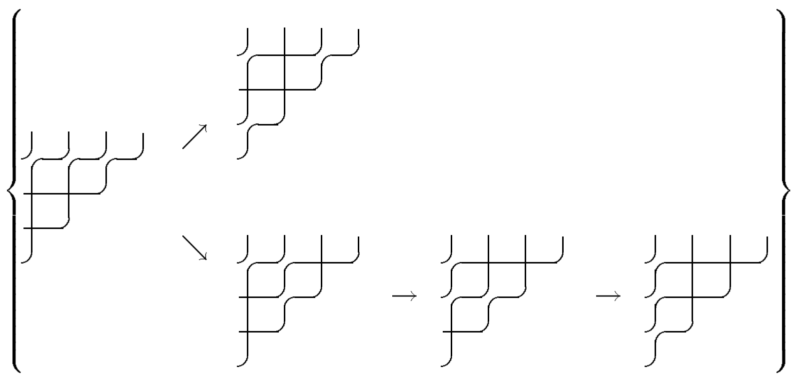
(II.15)
$$\mathfrak{S}_w = \sum_{D \in \mathcal{L}(D_{bot}(w))} x_D.$$

Proof. Note that $D_{top}(w^{-1}) = \rho(D_{bot}(w))$. From Theorem 2, Lemma 1.1 and Lemma 1.3, we have $\mathcal{RC}(w) = \rho(\mathcal{RC}(w^{-1})) = \rho(\mathcal{C}(D_{top}(w^{-1}))) = \rho(\mathcal{C}(\rho(D_{bot}(w)))) = \mathcal{L}(D_{bot}(w))$. $\square$

We compute $\mathfrak{S}_{[1432]}$ using ladder moves as follows:

(II.16)

$\mathcal{RC}[1,4,3,2] =$



Hence $\mathfrak{S}_{[1432]} = x_2^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_1^2x_3 + x_1^2x_2$.

Remark 1.7. *Chute moves and ladder moves define two posets on $\mathcal{RC}(w)$ with the covering relations $\mathcal{L}_{(i,j)}(D) \mapsto D$ and $\mathcal{C}_{(i,j)}(D) \mapsto D$ respectively. These are dual posets.*

The following corollary restricts the relations among the elements in the poset. However, there are still multiple paths to some of the rc-graphs.

Corollary 1.8. *We can generate $\mathcal{RC}(w)$ by only chute moves $\mathcal{C}_{(i,j)}$ such that i is the largest in column j, i.e. $(k,j) \notin D$ for all $k > j$. Similarly, we can generate $\mathcal{RC}(w)$ using only* rightmost *ladder moves $\mathcal{L}_{(i,j)}$, i.e. $(i,k) \notin D$ for all $k > i$.*

Proof. Given any $D \in \mathcal{RC}(w)$ different from $D_{top}(w)$, there exists at least one possible inverse chute move. Choose the inverse chute move $\mathcal{C}^{-1}_{(i,j)}$ such that i is as large as possible. The point (i,j) must be the lowest point in column j of $\mathcal{C}^{-1}_{(i,j)}(D)$, otherwise there exists a point (k,j) satisfying (II.12), hence, another possible inverse chute move $\mathcal{C}^{-1}_{(k,j)}$ with $k > i$. Next choose the lowest inverse chute move possible on $\mathcal{C}^{-1}_{(i,j)}(D)$. Continue applying the lowest move until there are no inverse chute moves possible. Reversing this sequence gives a sequence of chute moves $\mathcal{C}_{(i,j)}$ such that (i,j) is the lowest point in column i which transforms $D_{top}(w)$ to D. Transposing this proof, we get the result for ladder moves. □

Algebraic Proofs of Corollaries 1.9, 1.10, and 1.12 appear in [33]. Corollary 1.11 first appeared in [5].

Corollary 1.9. *The Schubert polynomials, indexed by permutations in S_∞, are an integral basis for $\mathbb{Z}[x_1, x_2, \ldots]$.*

Proof. The leading term of each $\mathfrak{S}_w$, in reverse lexicographic order, is given by $D_{bot}(w)$. Each D_{bot} is unique which implies each leading term is unique. Furthermore, given any monomial $x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ there exits a permutation w such that $x_{D_{bot}(w)} = x_1^{\alpha_1} \cdots x_m^{\alpha_m}$, simply put α_1 plusses left justified on row 1, α_2 plusses on row 2, etc. □

Corollary 1.10. *Given permutations $u \in S_m$ and $v \in S_n$, let $u \times v = [u_1, \ldots, u_m, v_1 + m, \ldots, v_n + m]$ and let $1_m \times v = [1, \ldots, m, v_1 + m, \ldots, v_n + m]$. We have*

$$\mathfrak{S}_u \mathfrak{S}_{1_m \times v} = \mathfrak{S}_{u \times v}. \tag{II.17}$$

Proof. Every rc-graph in $\mathcal{RC}(u)$ will be contained in $\mathbb{P} \times \mathbb{P} \cap \{(i,j) : i+j < m\}$, and every rc-graph in $\mathcal{RC}(1_m \times v)$ will contain no points in $\mathbb{P} \times \mathbb{P} \cap \{(i,j) : i+j \leq m\}$. No rc-graph in $\mathcal{RC}(u \times v)$ will contain a point on the line $i + j = m$. Therefore, there is a bijection between $\mathcal{RC}(u) \times \mathcal{RC}(1_m \times v)$ and $\mathcal{RC}(u \times v)$ given by sending $(D_1, D_2) \mapsto D_1 \cup D_2$. □

Given any permutation $v \in S_n$ let $(1 \times v)$ be the permutation $[1, v_1 + 1, v_2 + 1, \ldots v_n + 1]$. We define the inverse operation to be $\downarrow$ so $\downarrow (1 \times v) = v$. Note that $\downarrow v$ is well-defined only if $v_1 = 1$.

Corollary 1.11. *Given any $w \in S_\infty$,*

$$\mathfrak{S}_w(x_1, x_2, \ldots) = \sum x_1^{\ell(v)} \mathfrak{S}_{\downarrow vw}(x_2, x_3, \ldots)$$

where the sum is over all permutations $v \in S_\infty$ *such that* $\ell(w) = \ell(vw) + \ell(v)$, $v = s_{i_1}s_{i_2}\cdots s_{i_p}$ *with* $i_1 < i_2 < \ldots i_p$, *and* $(vw)_1 = 1$.

Proof. There is a bijection from $\mathcal{RC}(w) \mapsto \cup(v, \mathcal{RC}(\downarrow vw))$ where the union is over all permutations $v \in S_n$ such that $\ell(w) = \ell(vw) + \ell(v)$, $v = s_{i_1}s_{i_2}\cdots s_{i_p}$ with $i_1 < i_2 < \ldots i_p$, and $(vw)_1 = 1$. The bijection is given by sending $D \in \mathcal{RC}(w)$ to (v, D') if $v = s_{i_1}s_{i_2}\cdots s_{i_p}$ where the first row of D are points in columns $i_1, i_2, \ldots, i_p$, and D' is the rc-graph obtained by removing the first row of D. $\square$

Corollary 1.12 is a generalization of Corollary 1.11.

Corollary 1.12. *For any fixed positive integer* m *and any* $w \in S_n$, *we have the decomposition*

$$\mathfrak{S}_w(x_1, \ldots, x_n) = \sum d^w_{uv}\mathfrak{S}_u(x_1, \ldots, x_m)\mathfrak{S}_v(x_{m+1}, \ldots, x_n) \tag{II.18}$$

where the d^w_{uv} *are non-negative integers.*

Proof. Given a polynomial $f(x_1, x_2, \ldots, x_m)$, let $\rho_m f = f(x_1, \ldots, x_m, 0, 0, \ldots)$. By an abuse of notation, we also let $\rho_m(\mathcal{RC}(w)) = \{D \in \mathcal{RC}(w) : \rho_m(x_D) = x_D\}$. For each $w \in S_\infty$ and each m, there exists a bijection $\phi : \mathcal{RC}(w) \to \cup\, \rho_m(\mathcal{RC}(u)) \times \mathcal{RC}(v)$ where the union is over all permutations u, v such that $l(u) + l(v) = l(w)$ and $1_m \times v = u^{-1}w$. We define $\phi(D) = \{(i,j) \in D : j \leq m\} \times \{(i, j-m) : (i,j) \in D \text{ and } j > m\}$. Therefore,

$$\mathfrak{S}_w(x_1, \ldots, x_n) = \sum_{\substack{l(u)+l(v)=l(w)\\ 1_m \times v = u^{-1}w}} [\rho_m\mathfrak{S}_u(x_1, \ldots, x_n)]\mathfrak{S}_v(x_{m+1}, \ldots, x_n). \tag{II.19}$$

Finally, we can expand $\rho_m\mathfrak{S}_u$ in a positive sum of Schubert polynomials by the transition equation (4.16) of [33]. $\square$

Given permutations $v = v_1, \cdots, v_j \in \mathcal{S}_j$ and $w = w_1, \cdots, w_k \in \mathcal{S}_k$, we let $v * w$ and $v \times w$ denote the permutations $v_1 + k, \ldots, v_j + k, w_1, \ldots, w_k \in \mathcal{S}_{j+k}$ and $v_1, \ldots, v_j, w_1 + j, \ldots, w_k + j \in \mathcal{S}_{j+k}$, respectively. We then have the following result.

Corollary 1.13 (Block Decomposition Formula).

$$\mathfrak{S}_{v*w} = (x_1 \ldots x_j)^k \mathfrak{S}_v \uparrow^j \mathfrak{S}_w. \tag{II.20}$$

Corollary 1.14. *Assume $u(1) > u(2)$ and $\rho_2 \mathfrak{S}_u \neq 0$, then*

$$\rho_2 \mathfrak{S}_u = z_1^t z_2^s h_\lambda(z_1, z_2) \tag{II.21}$$

$$\rho_2 \mathfrak{S}_{us_1} = z_1^s z_2^s h_\lambda(z_1, z_2) h_{t-s-1}(z_1, z_2) \tag{II.22}$$

where $t \geq s$, $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots$ is a partition, $h_k(z_1, z_2) = \sum_{i=0}^{k} z_1^i z_2^{k-i}$, and $h_\lambda(z_1, z_2) = \prod h_{\lambda_i}$ is the homogeneous symmetric function.

Proof. If $\rho_2 \mathfrak{S}_u \neq 0$, then $D_{top}(u)$ is contained in the first two rows. We can construct all rc-graphs for u which are contained in the first two rows by applying chute moves to $D_{top}(u)$, each point can make at most one move. Each column containing 2 occupied positions will contribute a factor $z_1 z_2$ to every monomial in $\rho_2 \mathfrak{S}_u$. Furthermore, these columns will not effect any other possible moves from the first row to the second row. Let v be the permutation obtained from u by removing every column of $D_{top}(u)$ which has two points, say we removed s columns. For example,

$$D_{top}(u) = \begin{array}{c|cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & + & + & + & \cdot & + & + & + & + \\ 2 & \cdot & + & \cdot & \cdot & \cdot & + & + & \end{array} \tag{II.23}$$

$$D_{top}(v) = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ 1 & + & + & \cdot & + & + \\ 2 & \cdot & \cdot & \cdot & \cdot & \end{array} \tag{II.24}$$

Then,

$$\rho_2 \mathfrak{S}_u = z_1^s z_2^s \rho_2 \mathfrak{S}_v. \tag{II.25}$$

Now, $D_{top}(v)$ is contained in the first row. Let $t+1$ be the be the first unoccupied column in $D_{top}(v)$. Then the first t points cannot move down. Hence $\rho_2 \mathfrak{S}_v = z_1^t \rho_2 \mathfrak{S}_w$, where $D_{top}(w)$ is obtained from $D_{top}(w)$ by removing the first t columns. In our example, this implies

(II.26)
$$D_{top}(w) = \begin{array}{c|ccc} & 1 & 2 & 3 \\ 1 & \cdot & + & + \\ 2 & \cdot & \cdot & \end{array}$$

At this point $D_{top}(w)$ is contained in the first row and the point $(1,1)$ is empty. Note, if $D_{top}(w) = \{i : j < i \leq k+j\}$ then $\rho_2\mathfrak{S}_w = \sum_{i=0}^{k} z_1^i z_2^{k-i} = h_k(z_1, z_2)$. Hence each consecutive sequence of k occupied positions will contribute a factor of $h_k(z_1, z_2)$. Therefore, in general if $D_{top}(w)$ has l consecutive sequences, $\rho_2\mathfrak{S}_w = h_{\lambda_1}h_{\lambda_2}\cdots h_{\lambda_l}$. $\square$

2. Double Schubert Polynomials

The double Schubert polynomials generalize the normal Schubert polynomials in two alphabets. The original definition, given by Lascoux and Schützenberger, was written in terms of divided difference operators. Our definition follows from Eq(6.3) of [33]. Here we show that the double Schubert polynomials can also be represented graphically and these graphs can constructed by ladder moves on an initial graph.

Definition. Let $X = \{x_1, x_2, \ldots\}$ and $Y = \{y_1, y_2, \ldots\}$ be two alphabets. Given $w \in S_\infty$ we define the *double Schubert polynomial*, $\mathfrak{S}_w(X, Y)$, to be

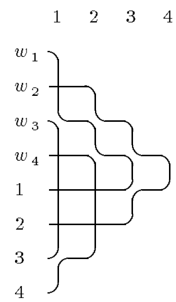
(II.27)
$$\mathfrak{S}_w(X,Y) = \sum_{\substack{v^{-1}u=w \\ l(u)+l(v)=l(w)}} (-1)^{\ell(v)}\mathfrak{S}_u(X)\mathfrak{S}_v(Y).$$

Definition. A *double rc-graph* E for a permutation w is a collection of points (i,j) such that $i \neq 0, j > 0$, $\{(i,j) \in E : i > 0\}$ is an rc-graph for a permutation u, $\{(i,j) \in E : i < 0\}$ is an upside down rc-graph for a permutation v, $v^{-1}u = w$, and $l(w) = l(u) + l(v^{-1})$. Denote the set of all double rc-graphs for w by $\widetilde{\mathcal{RC}}(w)$. Given a double rc-graph E we define the associated monomial

(II.28)
$$(xy)_E = \prod_{\substack{(i,j)\in E \\ i>0}} x_i \prod_{\substack{(i,j)\in E \\ i<0}} (-y_i)$$

For example, take $w = [4,3,2,1]$ then

(II.29)
$$E = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ -4 & \cdot & & & \\ -3 & + & \cdot & & \\ -2 & \cdot & \cdot & \cdot & \\ -1 & + & \cdot & \cdot & \cdot \\ 1 & + & + & \cdot & \cdot \\ 2 & + & + & \cdot & \\ 3 & \cdot & \cdot & & \\ 4 & \cdot & & & \end{array}$$



is a double rc-graph (without and with strings) for w. If we separate E into its upper and lower half, we have

(II.30)
$$E_{i>0} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ 1 & + & + & \cdot \\ 2 & + & + & \\ 3 & \cdot & & \end{array} \quad \text{and} \quad E_{i<0} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ 1 & + & \cdot & \cdot \\ 2 & \cdot & \cdot & \\ 3 & + & & \end{array}$$

where $E_{i>0} \in \mathcal{RC}[3,4,1,2]$ and $E_{i<0} \in \mathcal{RC}[2,1,4,3]$. Notice that the natural involution ρ from $\widetilde{\mathcal{RC}}(w)$ to $\widetilde{\mathcal{RC}}(w^{-1})$ is now the reflection across $i = 0$.

We can extend the definition of a ladder or chute move for double rc-graphs to include moves which go above the line $i = 0$. We will call such a move a *d-ladder* or a *d-chute* move. We will study d-ladder moves only and leave the d-chute moves to the reader. To describe a d-ladder move $\widetilde{\mathcal{L}}_{(i,j)}E = E \cup \{(i', j')\} \setminus \{(i,j)\}$, there are three cases to consider: (a) $i > i' > 0$, (b) $0 > i > i'$ and (c) $i > 0 > i'$. In case (a), we allow only usual ladder moves $\widetilde{\mathcal{L}}_{(i,j)}E = \mathcal{L}_{(i,j)}E = E \cup \{(i-m, j+1)\} \setminus \{(i,j)\}$ provided $m < i$ satisfies the conditions of Section 1. In case (b), since we are in an upside down rc-graph of an inverse permutation, we allow only upside down inverse ladder moves $\widetilde{\mathcal{L}}_{(i,j)}E = \mathcal{L}^{-1}_{(i+m,j+1)}E = E \cup \{(i+m, j+1)\} \setminus \{(i,j)\}$ provided $m < |i|$ satisfies the condition of Section 1, *upside down*. In case (c), it is enough to allow only moves of the form $\widetilde{\mathcal{L}}_{(i,j)}E = E \cup \{(-1,j)\} \setminus \{(i,j)\}$ provided:

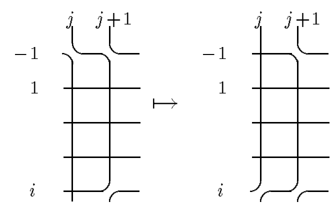
- $(i,j) \in E,\ (i, j+1) \notin E.$
- $(-1,j), (-1,j+1) \notin E.$
- For each $1 \leq k < i,\ \ (k,j), (k,j+1) \in E.$

Let $\widetilde{\mathcal{L}}(E)$ be the set of all possible combinations of d-ladder moves on E.

LEMMA 2.1. *If* $E \in \widetilde{\mathcal{RC}}(w)$ *then* $\widetilde{\mathcal{L}}_{(i,j)}E$ *is in* $\widetilde{\mathcal{RC}}(w)$.

PROOF. The proof in cases (a) and (b) is given in Lemma 1.4. For case (c), simply compare the permutation of the strings in (II.31)

(II.31)



□

THEOREM 4. *Let $E_{bot}(w)$ be the double rc-graph for w which consists of the points in $D_{bot}(w)$. Given $w \in S_\infty$,*

$$\mathfrak{S}_w(X,Y) = \sum_{E \in \widetilde{\mathcal{L}}(E_{bot}(w))} (xy)_E. \tag{II.32}$$

R. Stanley first noted that double Schubert polynomials could be expressed in terms of generalized compatible sequences as in (II.1).

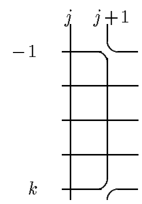
PROOF. By definition of a double Schubert polynomial,

$$\mathfrak{S}_w(X,Y) = \sum_{\substack{v^{-1}u=w \\ l(u)+l(v)=l(w)}} \sum_{C \in \mathcal{RC}(u)} \sum_{D \in \mathcal{RC}(v)} (-1)^{\ell(v)} x_C y_D \tag{II.33}$$

$$= \sum_{E \in \widetilde{\mathcal{RC}}} (xy)_E. \tag{II.34}$$

Therefore, we need to show $\widetilde{\mathcal{L}}(E_{bot}(w)) = \widetilde{\mathcal{RC}}(w)$. Lemma 2.1 shows that $\widetilde{\mathcal{L}}(E_{bot}(w)) \subset \widetilde{\mathcal{RC}}(w)$. For the other inclusion, we will proceed by induction on the cardinality of $E_{i<0}$. If $E_{i<0}$ is empty then we can apply Theorem 3 and there exists a sequence of inverse ladder moves which transform E into E_{bot}. Now suppose that $\mathrm{Card}(E_{i<0}) = c > 0$. Using Theorem 3 on $E_{i>0}$, we can assume that $E_{i>0} = D_{bot}(u)$ for some $u < w$. Using upside down ladder moves on $E_{i<0}$ we may also assume that the row $i = -1$ is not empty. Find $j > 0$ such that $(-1, j) \in E$ and $(-1, j+1) \notin E$. Let $i > 0$ be the smallest row such that $(i, j) \notin E$. That is for $0 < k < i$ we have $(k, j) \in E$. Note that since $E_{i>0} = D_{bot}(u)$, we must have $(i, j+1) \notin E$. We claim that $\widetilde{\mathcal{L}}^{-1}_{(i,j)}(E) = E \cup \{(i,j)\} \setminus \{(-1,j)\}$ is an inverse d-ladder move of type (c). For this, we need only show that for $0 < k < i$ we have $(k, j+1) \in E$. If we assume that for $0 < k < i$ we have $(k, j+1) \notin E$ then for the smallest such k two strings of E would cross twice, see (II.35),this would be a contradiction.

(II.35)



Hence, by inverse d-ladder moves we can transform E to E' where $\mathrm{Card}(E'_{i<0}) = c-1$. By the induction hypothesis we can now transform E' into E_{bot} with a sequence of inverse d-ladder moves. This shows that $\widetilde{\mathcal{RC}}(w) \subset \widetilde{\mathcal{L}}(E_{bot}(w))$. $\square$

From Theorem 4 one can check, $\mathfrak{S}_w(X, 0) = \mathfrak{S}_w(X)$.

The text of this chapter is a reprint of material as it appears in *RC-graphs and Schubert polynomials* to appear in Experimental Mathematics, co-authored with Nantel Bergeron. I made substantial contributions to the research and text as did my co-author.

CHAPTER III

Reduced Words and Tableaux

In this chapter we present correspondences between reduced words of an element in a Weyl group and tableaux. These correspondences play an important role in defining Schubert polynomials of type B, C, and D.

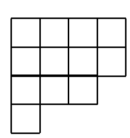
1. The original Edelman-Greene correspondence

Edelman and Greene have defined an analog of Schensted correspondence for reduced words [8]. As with Schensted's algorithm, there are several variations. Our main goal, in using this algorithm is to give the coefficients of the A_n–Stanley functions expressed in the basis of Schur functions. Therefore, we have chosen the algorithm from [8] which parallels the Haiman correspondence we will present in the next section and gives the coefficients we need.

We begin by reviewing the combinatorial definition of tableaux and Schur functions. Our treatment of these objects and the entire theory of symmetric functions is far from complete. For more information, one can read [37] and [32].

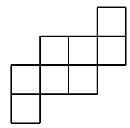
Recall that when $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_l)$ is a partition with weakly decreasing parts, the corresponding *shape* is called the *Ferrers diagram* of μ, found by placing μ_i squares in row i, and left justifying the picture. Here we have the Ferrers diagram for $\mu = (4, 4, 3, 1)$.

(III.1)



We say μ is a partition of n if $|\mu| = \sum \mu_i = n$. In the example above $|\mu| = 11$. Let λ and μ be partitions such that $\lambda \subset \mu$. A *skew shape*, μ/λ is the Ferrers diagram obtained by removing the squares in λ from μ. For example, the diagram for $(4,4,3,1)/(3,1)$ is

(III.2)



If λ is the empty shape, then μ/λ is a normal shape.

A *tableau* of shape μ is a function assigning to each cell in the shape an entry from some totally ordered alphabet, so that the entries are non–decreasing along each row and column. If the alphabet is the set of numbers $\{1, \ldots, n\}$, where $n = |\mu|$, and the assignment of numbers to cells is bijective, the tableau is a *standard tableau*. If the entries of a tableau T are strictly increasing in the rows and weakly increasing in the columns, we say T is *semi–standard*. If T is any tableau, its *weight* is the monomial x^T in the variables $x_1, x_2, \ldots$ formed by taking the product over all entries in T of the variable x_i for each entry labeled i.

Definition. The *Schur function* $S_\mu(X)$ is the sum $\sum_T x^T$, taken over all semi–standard tableaux of shape μ.

We say that a standard tableau S has a descent at i if $i+1$ appears to the right of i. Let $D(S)$ be the set of positions of descents in S. A sequence $\mathbf{i} = i_1 \cdots i_p$ is *admissible* for a standard tableau S of size p if $i_1 \leq \ldots \leq p$ and $i_k = i_{k+1} \implies k \in D(S)$. Let S be a standard tableau, then define

$$\Theta(S) = \sum_{\substack{i_1 \leq \ldots \leq p \\ i_k = i_{k+1} \implies k \in D(S)}} z_{i_1} \cdots z_{i_p}. \tag{III.3}$$

Proposition 1.1. *Given any Schur function, S_μ, we have*

$$S_\mu = \sum \Theta(S) \tag{III.4}$$

where the sum is over all standard tableaux S of shape μ.

Proof. We will prove this by giving a bijection between column–strict tableaux and pairs $(S, \mathbf{i})$ where S is a standard tableau and $\mathbf{i}$ is admissible for S. Clearly given a standard tableau S and a sequence $\mathbf{i}$, we can form a column strict tableau by T replacing each label k in S by i_k. T will be column strict since $i_1 \leq \cdots \leq i_p$ and equality is only allowed to happen at positions of descents. Conversely, given a column strict tableau T, we can find a standard tableau S by first numbering all the 1's in T in left to right order, then numbering all the 2's from left to right *etc.*. This process guarantees that squares with the same label will be descending in the standard tableau. See the example of standardization below. We obtain an admissible sequence from T just by ordering the labels in T in increasing order. One can check these operations are inverses of each other. $\square$

For example, we standardize the following column strict tableau:

(III.5)
$$\begin{array}{|c|c|c|}\hline 1&3&3\\\hline 3&4\\\cline{1-2} 5&5\\\cline{1-2}\end{array} \rightarrow \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2} 6&7\\\cline{1-2}\end{array}.$$

Given a standard tableau, specify an empty square s along the right edge of the tableau. We perform a *jeu–de–taquin* slide by moving the larger of the two entries above and to the left of s into s. Say we obtained the new label from square t. Then remove the label on t, and do a *jeu–de–taquin* slide by labeling t by the larger of the two entries above and to the left of t and leave that new square empty. Continue this process until the empty square is in the upper left hand corner. If we had started with a skew shape, we will finish the algorithm whenever the remaining diagram is a skew shape.

For example,

(III.6)
$$\begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4&\\\hline\end{array} \xrightarrow[p]{} \begin{array}{|c|c|}\hline 1&2\\\hline 3&\\\hline 4&5\\\hline\end{array} \xrightarrow[p]{} \begin{array}{|c|c|}\hline 1&2\\\hline &3\\\hline 4&5\\\hline\end{array} \xrightarrow[p]{} \begin{array}{cc}\cline{2-2} &\multicolumn{1}{|c|}{2}\\\hline \multicolumn{1}{|c|}{1}&3\\\hline \multicolumn{1}{|c|}{4}&5\\\hline\end{array}$$

DEFINITION. Define the *promotion operator* p on a standard tableau T by the following algorithm: delete the largest entry of T and perform a *jeu-de-taquin* slide into its cell, leaving a skew shape. Let $\boldsymbol{\alpha}_n$ denote the 'staircase' shape $(n-1, n-2, \ldots, 1)$ of size $\binom{n}{2}$. Let its corners be labeled $1, \ldots, n-1$ from the bottom row to the top. If T is a standard tableau of shape $\boldsymbol{\alpha}_n$, its *promotion sequence* $\hat{p}(T)$ is the sequence $a_1 \ldots a_{\binom{n}{2}}$ in which a_i is the label of the corner occupied by the largest entry of $p^{\binom{n}{2}-i}(T)$.

For example, taking $n = 4$, let T be the first tableau pictured below. Its promotions $p(T)$, $p^2(T)$, $p^3(T)$ $p^4(T)$, and $p^5(T)$ are shown to its right. In each step we show only the result of the *jeu-de-taquin* slide.

$$\text{(III.7)} \quad \begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&6& \\ \cline{1-2} 5& & \\ \cline{1-1}\end{array} \xrightarrow[p]{} \begin{array}{|c|c|c|}\hline \ &1&4\\ \hline 2&3& \\ \cline{1-2} 5& & \\ \cline{1-1}\end{array} \xrightarrow[p]{} \begin{array}{|c|c|c|}\hline \ &1&4\\ \hline \ &3& \\ \cline{1-2} 2& & \\ \cline{1-1}\end{array} \xrightarrow[p]{} \begin{array}{|c|c|c|}\hline \ & \ &1\\ \hline \ &3& \\ \cline{1-2} 2& & \\ \cline{1-1}\end{array} \xrightarrow[p]{} \begin{array}{|c|c|c|}\hline \ & \ &1\\ \hline \ & \ & \\ \cline{1-2} 2& & \\ \cline{1-1}\end{array} \xrightarrow[p]{} \begin{array}{|c|c|c|}\hline \ & \ &1\\ \hline \ & \ & \\ \cline{1-2} \ & & \\ \cline{1-1}\end{array}$$

Each a_i is 1, 2, or 3 according to which corner is occupied by the largest entry of $p^{6-i}(T)$. Removing the 6 from the second row implies $a_6 = 2$, removing the 5 from the bottom row implies $a_5 = 1$, etc. In general, note that the largest entry of $p^{6-i}(T)$ is i itself, so a_i records the corner ultimately reached by entry i in the promotion process. Here the sequence $\mathbf{a} = \hat{p}(T)$ is 312312.

Proposition 1.2. (Edelman-Greene correspondence) *The map $T \mapsto \hat{p}(T)$ is a bijection from standard tableaux of shape $\boldsymbol{\alpha}_n$ to reduced words for the longest element $w_0 = [n, n-1, \ldots, 1]$ of S_n. The initial segment $\mathbf{a} = a_1 \ldots a_k$ of the reduced word $\hat{p}(T)$ determines the initial segment $T|_k$ containing entries 1 through k of T. Let $Q(\mathbf{a}) = T|_k$, then the number*

$$g_w^\mu = |\{\mathbf{a} \in R(w) \mid Q(\mathbf{a}) = S\}| \tag{III.8}$$

depends only on w and on the shape μ of S. Finally, we have $D(\mathbf{a}) = D(Q(\mathbf{a}))$ for the descent sets.

The following proposition was stated first by Stanley in [43] without an explicit algorithm for computing the coefficients g_w^μ. Edelman and Greene showed that their correspondence gave an explicit construction of the coefficients.

Proposition 1.3. *For $w \in S_n$,*

$$|R(w)| = \sum g_w^\mu f_\lambda \tag{III.9}$$

where f_λ is the number of standard tableaux of shape λ and the coefficients $m_\lambda(w)$ are non–negative integers.

Recall the definition of the A_n–Stanley function from (I.12),

$$G_w = \sum_{\mathbf{a} \in R(w)} \sum_{\substack{i_1 \leq i_2 \leq \ldots \leq i_{l(w)} \\ i_k = i_{k+1} \implies a_k > a_{k+1}}} z_{i_1} z_{i_2} \cdots z_{i_{l(w)}}. \tag{III.10}$$

THEOREM 1.4. *The A_n–Stanley symmetric functions, defined by* (I.12), *have the following expansion in the basis of Schur functions:*

$$G_w(z_1, z_2, \ldots, z_n) = \sum_{\lambda} g_w^{\mu} S_{\lambda}(x_1, x_2, \ldots, z_n) \tag{III.11}$$

PROOF. From Proposition 1.2, we know for each $\mathbf{a} \in R(w)$, $D(\mathbf{a}) = D(Q(\mathbf{a}))$. Hence,

$$\Theta(Q(\mathbf{a})) = \sum_{\substack{i_1 \leq \ldots \leq p \\ i_k = i_{k+1} \implies a_k > a_{k+1}}} z_{i_1} z_{i_2} \cdots z_{i_p} \tag{III.12}$$

and substituting (III.12) into (III.10) we have

$$G_w = \sum_{\mathbf{a} \in R(w)} \Theta(Q(\mathbf{a})) \tag{III.13}$$

$$= \sum_{\mu} g_w^{\mu} \sum \Theta(S). \tag{III.14}$$

by (III.8), where the second sum is over all standard tableaux of shape μ. By Proposition 1.1 the second sum is the Schur function of shape μ, which proves the proposition. □

2. The Haiman correspondence

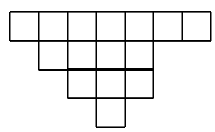
In this section we review the Haiman correspondences on shifted tableaux from [18] and use them to define symmetric functions associated with elements of the Weyl groups B_n and D_n. These symmetric functions are the natural B_n and D_n analogs of symmetric functions defined for elements of A_n by Stanley [43]. For this reason we call them *Stanley functions*. Just as the A_n Stanley functions are now understood to be 'stable' type A Schubert polynomials, the B_n and D_n Stanley functions turn out to be specializations of type B and D Schubert polynomials.

Our central results here are identities between the defining *tableau* forms of the Stanley functions and more explicit *monomial* forms given by Propositions 2.4 and 2.10. Expressed in tableau form, the Stanley functions are transparently non–negative integral combinations of Schur $Q-$ and $P-$ functions, respectively. Expressed in monomial form, they are amenable to detailed computations with divided difference operators. Both aspects are essential for the proofs of our main theorems in Chapter IV.

At the end of this section we evaluate the Stanley functions for various special elements of the Weyl groups. Most of these evaluations and some others not given here were also found by J. Stembridge, T.–K. Lam, or both, in work not yet published. They take the monomial forms as the definition, attributing this to Fomin. We give a self–contained treatment here, since our methods are new and the proofs simple. Note, however, that Propositions 2.14 and 2.13 were first proved by Stembridge, and Proposition 2.16 by Lam. They consider Proposition 2.17 to be well–known!

We begin by reviewing the combinatorial definition of Schur $Q-$ and $P-$functions. Recall that when $\mu = (\mu_1 > \mu_2 > \cdots > \mu_l)$ is a partition with distinct parts, the corresponding *shifted shape* is a sort of Ferrers diagram of μ, but with each row indented one space at the left from the preceding row, as shown here for $\mu = (7, 4, 3, 1)$.

(III.15)



We restate the relevant definitions from Section 1 in for shifted shapes. A *tableau* of shape μ is a function assigning to each cell in the shape an entry from some totally ordered alphabet, so that the entries are non–decreasing along each row and column. If the alphabet is the set of numbers $\{1, \ldots, n\}$, where $n = |\mu|$, and the assignment of numbers to cells is bijective, the tableau is a *standard tableau*. If the alphabet consists of natural numbers $1, 2, \ldots$ and circled natural numbers $1^\circ, 2^\circ, \ldots$, with the ordering $1^\circ < 1 < 2^\circ < 2 < \cdots$, the tableau is a *circled tableau* provided that no circled number is repeated in any row and no uncircled number is repeated in any column. For example,

(III.16)

1°	2°	3	3
	2°	4°	4
		4°	

is a circled tableau.

If T is a circled tableau, its *weight* is the monomial x^T in variables $x_1, x_2, \ldots$ formed by taking the product over all entries in T of the variable x_i for an entry i

or i°. In (III.16), the weight of the tableau is $x_1x_2^2x_3^2x_4^3$.

Definition. The *Schur Q-function* $Q_\mu(X)$ is the sum $\sum_T x^T$, taken over all circled tableaux of shifted shape μ. The *Schur P-function* $P_\mu(X)$ is defined to be $2^{-l(\mu)}Q_\mu(X)$, where $l(\mu)$ is the number of parts of μ.

Note that the rules defining circled tableaux always permit free choice of the circling for entries along the main diagonal. Consequently, P_μ can also be described as the sum $\sum_T x^T$, taken over circled tableaux with no circled entries on the diagonal.

The following well-known basic facts can be derived (albeit with some effort) from various theorems and exercises in [32].

Proposition 2.1. *The Schur P- and Q-functions are the specializations $P_\mu(X;-1)$ and $Q_\mu(X;-1)$ of the Hall-Littlewood polynomials $P_\mu(X;t)$ and $Q_\mu(X;t)$, for μ with distinct parts. Consequently, they are symmetric functions in the variables X and they depend only upon the power sums $p_k(X)$ for k odd. Moreover, the sets $\{P_\mu(X)\}$ and $\{Q_\mu(X)\}$ are $\mathbb{Q}$-bases for the algebra $\mathbb{Q}[p_1(X), p_3(X), \ldots]$ generated by odd power sums, and $\mathbb{Z}$-bases for the subrings $\mathbb{Z}[P_\mu]$ and $\mathbb{Z}[Q_\mu]$.*

For more detail on the combinatorial interpretation of P- and Q-functions, consult [47], [38].

Next we need a description of Q-functions in terms of standard tableaux. If T is a (shifted) standard tableau of size n, we say that $j \in \{1, \ldots, n-1\}$ is a *descent* of T if $j+1$ appears in a lower row than j in T. The set of descents is denoted $D(T)$. We shall say that $j \in \{2, \ldots, n-1\}$ is a *peak* of T if $j-1$ is an ascent and j is a descent. The set of peaks we denote $P(T)$.

Given a set $P \subseteq \{2, \ldots, n-1\}$ (to be thought of as a peak set), we say that a sequence $i_1 \leq i_2 \leq \cdots \leq i_n$ is *admissible* for P if we do not have $i_{j-1} = i_j = i_{j+1}$ for any $j \in P$. Letting $A(P)$ denote the set of P-admissible sequences, we define the *shifted quasi-symmetric function*

(III.17) $$\Theta_P^n(X) = \sum_{\substack{(i_1 \leq \cdots \leq i_n)\\ \in A(P)}} 2^{|\mathbf{i}|} x_{i_1} x_{i_2} \cdots x_{i_n},$$

where $|\mathbf{i}|$ denotes the number of *distinct* values i_j in the admissible sequence, *i.e.*, the number of distinct variables in the monomial.

PROPOSITION 2.2. *The Schur Q-function Q_μ is equal to the sum $\sum_T \Theta^{|\mu|}_{P(T)}(X)$, where T ranges over standard tableau of shifted shape μ.*

PROOF. Our argument is a routine one, involving subscripting the entries of each circled tableau to get a standard tableau, so we only sketch it. A similar proof of the analogous formula for Schur S-functions originated in unpublished work of I. Gessel.

Given a circled tableau T, all entries i° and i for any given i form a rim hook, not necessarily connected, with the i°'s occupying the vertical portions and the i's occupying the horizontals. To obtain an underlying standard tableau, we distinguish all occurrences of i° by subscripts $i^\circ_1, i^\circ_2, \ldots$, proceeding downward by rows. In a similar fashion we distinguish occurrences of i proceeding to the right by columns. By this subscripting we totally order all entries of T; replacing them by the numbers 1 through $n = |T|$ in the same order gives a standard tableau $S(T)$.

Given $S(T)$ and the weight monomial x^T, we immediately recover T, except for the circling. The entries of $S(T)$ corresponding to i° and i form a sequence which descends and then ascends, *i.e.*, a sequence with no peak. Henceforth we refer to such a sequence as a *vee*. We must have i° along the descending part of the vee and i along the ascending part. Only the circling at the 'valley' of the vee is undetermined. Thus there are $2^{|\mathbf{i}|}$ circled tableaux with this particular weight and underlying standard tableau, where $|\mathbf{i}|$ is the number of distinct indices in the weight monomial. Moreover, the combinations of standard tableau S and weight monomial $x_{i_1}x_{i_2}\cdots x_{i_n}$ that occur are exactly those where the sequence $i_1 \leq \cdots \leq i_n$ is admissible for the peak set $P(S)$. This proves the proposition. $\square$

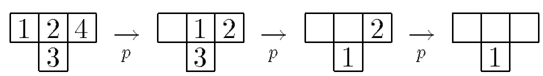
Having completed our review of Q- and P-functions, we turn to the Haiman correspondences and associated Stanley functions. We treat B_n first, everything we need is proven in [18]. For D_n we will have to add something.

DEFINITION. A *reduced word* for an element $w \in B_n$ is a sequence $\mathbf{a} = a_1a_2\ldots a_l$ of indices $0 \leq a_i \leq n-1$ such that w is the product of simple reflections $\sigma_{a_1}\cdots\sigma_{a_l}$

and $l = l(w)$ is minimal. We denote by $R(w)$ the set of reduced words for w. The *peak set* $P(\mathbf{a})$ is the set $\{i \in \{2, \ldots, l-1\} \mid a_{i-1} < a_i > a_{i+1}\}$.

Definition. Let $\boldsymbol{\beta}_n$ denote the shifted 'staircase' shape $(2n-1, 2n-3, \ldots, 1)$ of size n^2. Let its corners be labeled $0, 1, \ldots, n-1$ from the bottom row to the top. If T is a standard tableau of shape $\boldsymbol{\beta}_n$, its *promotion sequence* $\hat{p}(T)$ is the sequence $a_1 \ldots a_{n^2}$ in which a_i is the label of the corner occupied by the largest entry of $p^{n^2-i}(T)$. Here the *promotion operator* p is defined as follows: to compute $p(T)$, delete the largest entry of T, perform a (shifted) *jeu-de-taquin* slide into its cell, and fill the vacated upper-left corner with a new least entry.

Since this definition is a bit complicated, we illustrate with a simple example. Taking $n = 2$, let T be the first tableau pictured below. Its promotions $p(T)$, $p^2(T)$, $p^3(T)$ are shown to its right, except we have suppressed the new entries that should fill the upper left.

(III.18) 

Each a_i is 0 or 1, according to which corner is occupied by the largest entry of $p^{4-i}(T)$. Note that the largest entry of $p^{4-i}(T)$ is i itself, so a_i records the corner ultimately reached by entry i in the promotion process. Here the sequence $\mathbf{a} = \hat{p}(T)$ is 0101.

Proposition 2.3. (B_n Haiman correspondence) *The map $T \mapsto \hat{p}(T)$ is a bijection from standard tableaux of shape $\boldsymbol{\beta}_n$ to reduced words for the longest element $w_0 = \overline{1}\,\overline{2} \cdots \overline{n}$ of B_n. The initial segment $a_1 \ldots a_k$ of the reduced word $\hat{p}(T)$ determines the initial segment $T|_k$ containing entries 1 through k of T. Denoting $T|_k$ by $\Gamma(a_1 \ldots a_k)$, the number*

$$f_w^\mu = |\{\mathbf{a} \in R(w) \mid \Gamma(\mathbf{a}) = S\}| \tag{III.19}$$

depends only on w and on the shape μ of S. Finally, we have $P(\mathbf{a}) = P(\Gamma(\mathbf{a}))$ for the peak sets.

Proof. All but the part about peak sets is proved in Proposition 6.1 and Theorem 6.3 of [18]. For the peak set part it suffices to show $P(T) = P(\hat{p}(T))$ for T of shape $\boldsymbol{\beta}_n$. For a peak at position n^2-1, that is, involving the largest three entries of T, it is obvious that T has a peak if and only if $\hat{p}(T)$ does. For other positions, the result follows because shifted *jeu-de-taquin* preserves the peak set of a tableau. □

Using Proposition 2.3 we can now introduce well-defined symmetric functions associated with elements of B_n.

Definition. Let w be an element of B_n. The B_n *Stanley function* $F_w(X)$ is defined by

$$F_w(X) = \sum_{\mu} f_w^{\mu} Q_{\mu}(X). \tag{III.20}$$

The following crucial identity is an immediate consequence of Propositions 2.2 and 2.3.

Proposition 2.4.

$$\begin{aligned} F_w(X) &= \sum_{\mathbf{a} \in R(w)} \Theta_{P(\mathbf{a})}^{l(w)}(X) \\ &= \sum_{\mathbf{a} \in R(w)} \sum_{\substack{(i_1 \le \cdots \le i_l) \\ \in A(P(\mathbf{a}))}} 2^{|\mathbf{i}|} x_{i_1} x_{i_2} \cdots x_{i_l}. \end{aligned} \tag{III.21}$$

From (III.21) we obtain another important identity.

Corollary 2.5. *For all w, $F_w(X) = F_{w^{-1}}(X)$.*

Proof. Since F_w is a symmetric function, it is unaltered by reversing the indices of the variables. Therefore (III.21) is equal to

$$\sum_{\mathbf{a} \in R(w)} \sum_{\substack{(i_1 \ge \cdots \ge i_l) \\ \in A(P(\mathbf{a}))}} 2^{|\mathbf{i}|} x_{i_1} x_{i_2} \cdots x_{i_l}, \tag{III.22}$$

where the admissibility condition on a decreasing sequence is just as before: no $i_{j-1} = i_j = i_{j+1}$ when j is a peak. But then $(i_1 \ge \cdots \ge i_l)$ is admissible for $P(\mathbf{a})$ if and only if the reversed sequence $(i_l \le \cdots \le i_1)$ is admissible for $P(\mathbf{a}^r)$, where $\mathbf{a}^r$ is the reverse of $\mathbf{a}$, *i.e.*, a general element of $R(w^{-1})$. So (III.22) reduces to (III.21) for w^{-1}. □

The situation for D_n is analogous to that for B_n, but requires some new information about the relevant Haiman correspondence.

DEFINITION. A *reduced word* for $w \in D_n$ is a sequence $a_1 a_2 \dots a_l$ of the symbols $\hat{1}, 1, 2, \dots, n-1$ such that $\sigma_{a_1} \cdots \sigma_{a_l} = w$ and $l = l(w)$ is minimal. As before, $R(w)$ denotes the set of reduced words for w. A *flattened word* is a word obtained from a D_n reduced word by changing all the $\hat{1}$'s to 1's. The *peak set* $P(\mathbf{a})$ is defined to be the peak set (in the obvious sense) of the corresponding flattened word. A *winnowed word* is a word obtained from a B_n reduced word by deleting all the 0's.

DEFINITION. Let $\boldsymbol{\delta}_n$ denote the shifted 'staircase' shape $(2n-2, 2n-4, \dots, 2)$ of size $n(n-1)$. Let its corners be labeled $1, \dots, n-1$ from the bottom row to the top. If T is a standard tableau of shape $\boldsymbol{\delta}_n$, its *promotion sequence* $\hat{p}(T)$ is the sequence $a_1 \dots a_{n(n-1)}$ in which a_i is the label of the corner occupied by the largest entry of $p^{n(n-1)-i}(T)$.

In [18] it is shown that $T \to \hat{p}(T)$ defines a bijection from standard tableaux of shape $\boldsymbol{\delta}_n$ to winnowed words for the longest element of B_n and conjectured that initial segments of $\hat{p}(T)$ determine the corresponding initial segments of T. Here we extend these results by proving the conjecture just mentioned and relating the correspondence to D_n.

The first step is to identify both flattened words and winnowed words with words of a third kind. In what follows, flattened words and winnowed words are always for the longest element of D_n or B_n unless mention is made to the contrary. Recall that the longest element of D_n is $w_0^D = \overline{1}\,\overline{2} \dots \overline{n}$ if n is even, or $1\,\overline{2} \dots \overline{n}$ if n is odd.

DEFINITION. A *visiting word* $a_1 \dots a_{n(n-1)}$ is a sequence of symbols $1 \le a_i \le n$ such that

(1) the product $\sigma_{a_1} \cdots \sigma_{a_{n(n-1)}}$ is the identity in the symmetric group S_n, and
(2) for all $k \in \{1, 2, \dots, n\}$, there is a j such that $\sigma_{a_1} \cdots \sigma_{a_j}(1) = k$.

These conditions mean that as the adjacent transpositions $\sigma_{a_1}, \sigma_{a_2}, \dots$ are applied in succession, beginning with the identity permutation $1\,2 \dots n$, each of the numbers 1 through n visits the leftmost position at some point, and ultimately returns to its original position. Note that $n(n-1)$ is the minimum length for such a sequence, since each number has to switch places twice with every other.

PROPOSITION 2.6. *The sets of visiting words, flattened words, and winnowed words of order n are all the same.*

PROOF. Flattening or winnowing a reduced word gives its image under the natural homomorphism from D_n or B_n to S_n in which sign changes are ignored. In D_n and B_n, when the application of $\sigma_{\hat{1}}$ or σ_0 to a (signed) permutation changes the sign of a number, that number must occupy the leftmost position before or after the sign change. From this it is clear that every flattened word and every winnowed word is also a visiting word.

It is also clear that every visiting word is a winnowed word, since to un-winnow it is only necessary for each k to insert a 0 at some point during which k occupies the leftmost position.

The only difficulty is now to see that given a visiting word $\mathbf{a}$, there is always a way of changing some 1's to $\hat{1}$'s to make a reduced word for w_0^D. To $\mathbf{a}$ we associate a graph $G(\mathbf{a})$ with vertex set $\{1,\ldots,n\}$ by introducing for each a_j equal to 1 an edge connecting $v_j(1)$ and $v_j(2)$, where $v_j = \sigma_{a_1}\cdots\sigma_{a_j}$. In other words, applying the transpositions σ_{a_i} in succession, each time there is a change in the leftmost position we introduce an edge between the former occupant and its replacement. In general $G(\mathbf{a})$ can have multiple edges, but not loops.

Given a subset of the 1's in $\mathbf{a}$, there is a corresponding subset of the edges in $G(\mathbf{a})$, forming a subgraph H. If we change the 1's in the given subset to $\hat{1}$'s, we get a word describing an element $v \in D_n$ whose unsigned underlying permutation remains the identity. The sign of $v(k)$ is negative if and only if an odd number of edges in H are incident at vertex k, since these edges represent the transpositions $\sigma_{\hat{1}}$ involving k. To un-flatten $\mathbf{a}$, we need $v = w_0^D$; our word will automatically be reduced since its length is $n(n-1)$. Equivalently, we must find a function from the edges of G to $\mathbb{Z}_2$ such that its sum over all incident edges is 1 at every vertex, except possibly vertex 1. It is well-known and easy to prove that a suitable function exists if $G(\mathbf{a})$ is connected.

For each $i \in \{2, \ldots, n\}$, let $h(i)$ be the number which i replaces on its first visit to the leftmost position. Note that i and $h(i)$ are linked by an edge of $G(\mathbf{a})$. Moreover $h(i)$ makes its first visit to the leftmost position before i does, showing that the sequence $i, h(i), h(h(i)), \ldots$ never repeats and therefore ultimately reaches 1. This proves $G(\mathbf{a})$ is connected. □

From the above proof we can extract something more. The un-flattenings of a given flattened word correspond to solutions of a system of $n-1$ independent linear equations over $\mathbb{Z}_2$ in m variables, where m is the number of edges in $G(\mathbf{a})$. There are 2^{m-n+1} such solutions. More generally, the same reasoning applies to reduced words for an arbitrary $w \in D_n$, but with $G(\mathbf{a})$ only having vertices for numbers that actually reach the leftmost position. This gives the following result.

PROPOSITION 2.7. *If* $\mathbf{b}$ *is the flattened word of a reduced word for* $w \in D_n$*, then the number of reduced words* $\mathbf{a} \in R(w)$ *which flatten to* $\mathbf{b}$ *is* 2^{m-k+1}*, where* m *is the number of* 1*'s in* $\mathbf{b}$ *and* k *is the number of visitors to the leftmost position, i.e., the number of distinct values taken by* $\sigma_{b_1} \cdots \sigma_{b_j}(1)$ *as* j *varies from* 0 *to* $l(w)$*.*

Note that $m-k+1$ is the number of *repeat visits* occurring as the transpositions σ_{b_i} are successively applied, *i.e.*, the number of times an application of σ_1 moves a number into the leftmost position which has been there before. In what follows, we denote the number of repeat visits by $r(\mathbf{b})$ and the number $k-1 = m - r(\mathbf{b})$ of *first visits* by $f(\mathbf{b})$. Abusing notation, we also write $r(\mathbf{a})$ and $f(\mathbf{a})$ for these when $\mathbf{a}$ is a reduced word flattening to $\mathbf{b}$.

To obtain a further corollary to the proof of Proposition 2.6, observe that the subgraph H can be chosen as a subgraph of any given spanning tree of $G(\mathbf{a})$. Indeed, H will then be unique, since it will be given by $k-1$ independent linear equations in $k-1$ variables. In particular, the last paragraph of the proof shows that edges of the form $(i, h(i))$ corresponding to first visits form a spanning tree, proving the following.

PROPOSITION 2.8. *If* $\mathbf{b}$ *is a flattened word for* w*, then there is a unique reduced word* $\mathbf{a}$ *for* w *with flattened word* $\mathbf{b}$*, such that all the* $\hat{1}$*'s in* $\mathbf{a}$ *correspond to* 1*'s representing first visits in* $\mathbf{b}$*.*

Now we come to the D_n analog of Proposition 2.3.

PROPOSITION 2.9. (D_n Haiman correspondence) *The map $T \mapsto \hat{p}(T)$ is a bijection from standard tableaux of shape $\boldsymbol{\delta}_n$ to flattened words for the longest element of D_n. The initial segment $b_1 \ldots b_k$ of $\hat{p}(T)$ determines the initial segment $T|_k$ containing entries 1 through k of T. Given a reduced word $\mathbf{a}$ with flattened word $\mathbf{b}$, denote $T|_k$ by $\Gamma(a_1 \ldots a_k)$. Then the number*

$$e_w^\mu = \sum_{\substack{\mathbf{a} \in R(w) \\ \Gamma(\mathbf{a}) = S}} 2^{-o(\mathbf{a})}, \tag{III.23}$$

where $o(\mathbf{a})$ denotes the total number of 1's and $\hat{1}$'s in $\mathbf{a}$, depends only on w and on the shape μ of S. Finally, we have $P(\mathbf{a}) = P(\Gamma(\mathbf{a}))$ for the peak sets.

PROOF. The bijection is Theorem 5.16 of [18], since we now know that flattened words and winnowed words are the same. The peak set statement follows exactly as in Proposition 2.3 above.

For the assertion about initial segments, we show that whenever $\mathbf{bc}$ and $\mathbf{bc}'$ are flattened words for the longest element, with common initial segment $\mathbf{b}$, then $\mathbf{c}$ and $\mathbf{c}'$ are connected by a chain of S_n Coxeter relations. This given, the proof of Proposition 6.1 in [18] applies, with one change. Namely, for the argument involving the Coxeter relation $121 \leftrightarrow 212$ to go through, when the two flattened words are $\mathbf{b}121$ and $\mathbf{b}212$, their corresponding tableaux must differ only in the largest three entries. But this is shown by the proof of Proposition 5.15 in [18].

Now consider two flattened words $\mathbf{bc}$ and $\mathbf{bc}'$. Treating them as winnowed words, note that a winnowed word can be canonically un-winnowed by inserting a 0 at the beginning, and after every 1 that represents a first visit. Since the presence of each 0 is controlled by the initial segment of the word up to that point, the words $\mathbf{bc}$ and $\mathbf{bc}'$ un-winnow to $\mathbf{ad}$ and $\mathbf{ad}'$ for some $\mathbf{a}$, $\mathbf{d}$, and $\mathbf{d}'$ whose winnowed words are $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{c}'$. Then $\mathbf{d}$ and $\mathbf{d}'$ are connected by a chain of B_n Coxeter relations, which after winnowing reduce to S_n Coxeter relations connecting $\mathbf{c}$ to $\mathbf{c}'$.

What remains is to show that the numbers e_w^μ don't depend upon the particular tableau S, only on its shape. Let S and S' be elementary dual equivalent tableaux of shape μ. Let $\mathbf{a}$ be a reduced word for w with $\Gamma(\mathbf{a}) = S$ and let $\mathbf{b}$ be the corresponding flattened word. Note that S is really a function of $\mathbf{b}$ and is the initial segment of any tableau corresponding to an extension of $\mathbf{b}$.

By Lemma 5.2 of [18], if we extend S to a tableau T of shape $\boldsymbol{\delta}_n$, and let T' be the corresponding extension of S', then $\hat{p}(T)$ and $\hat{p}(T')$ differ by a certain substitution in the positions corresponding to the segment involved in the elementary dual equivalence $S \approx S'$. The complete list of possible substitutions is given in Table 5 of [18].

All but two of these substitutions are special cases of S_n Coxeter relations other than $121 \leftrightarrow 212$. It is easy to see that whenever $\mathbf{b}$ is a flattened word for w and $\mathbf{b}'$ differs from $\mathbf{b}$ by any S_n Coxeter relation besides $121 \leftrightarrow 212$, then $\mathbf{b}'$ is also a flattened word for w and $f(\mathbf{b}') = f(\mathbf{b})$.

The two remaining substitutions are $1121 \leftrightarrow 1212$ and $1211 \leftrightarrow 2121$. For these pairs it is again easy to see that if a flattened word $\mathbf{b}$ for w contains one of the pair, substituting the other yields another flattened word $\mathbf{b}'$ for w. Furthermore, we have $f(\mathbf{b}) = f(\mathbf{b}')$, for the second of the consecutive 1's in 1121 or 1211 never represents a first visit, while the other two bring about visits by the same two numbers as do the two 1's in the substituted 1212 or 2121.

Summarizing, we have bijections between flattened words $\mathbf{b}$ for w with $\Gamma(\mathbf{b}) = S$ and words $\mathbf{b}'$ with $\Gamma(\mathbf{b}') = S'$, and these bijections preserve the number of first visits. Since $2^{r(\mathbf{b})}$ reduced words for w correspond to each flattened word $\mathbf{b}$ we find that the sum

$$\sum_{\substack{\mathbf{b}\in F(w)\\ \Gamma(\mathbf{b})=S}} t^{f(\mathbf{b})} = \sum_{\substack{\mathbf{a}\in R(w)\\ \Gamma(\mathbf{a})=S}} 2^{-r(\mathbf{a})} t^{f(\mathbf{a})} \tag{III.24}$$

is not changed by replacing S with S'. Here $F(w)$ denotes the set of flattened words for w. Since all tableaux of shape μ are connected by chains of elementary dual equivalences, (III.24) depends only on w and μ, and hence so does (III.23), by setting $t = 1/2$. $\square$

Now we have the D_n analog of (III.20).

Definition. Let w be an element of D_n. The D_n *Stanley function* $E_w(X)$ is defined by

$$E_w(X) = \sum_\mu e_w^\mu Q_\mu(X). \tag{III.25}$$

Just as for B_n, we immediately obtain an identity from Propositions 2.2 and 2.9, and the corresponding corollary, with the same proof as Corollary 2.5.

Proposition 2.10.

$$\begin{aligned} E_w(X) &= \sum_{\mathbf{a}\in R(w)} 2^{-o(\mathbf{a})}\Theta^{l(w)}_{P(\mathbf{a})}(X) \\ &= \sum_{\mathbf{a}\in R(w)} \sum_{\substack{(i_1\le\cdots\le i_l)\\ \in A(P(\mathbf{a}))}} 2^{|\mathbf{i}|-o(\mathbf{a})} x_{i_1}x_{i_2}\cdots x_{i_l}. \end{aligned} \tag{III.26}$$

Corollary 2.11. *For all w, $E_w(X) = E_{w^{-1}}(X)$.*

Although the coefficients e_w^μ need not be integers, it is nevertheless true that $E_w(X)$ is an integral linear combination of P-functions, as we show next.

For this purpose we must extract a concept which is implicit in the proof of Proposition 2.9. We define flattened words $\mathbf{b}$ and $\mathbf{b}'$ to be *dual equivalent* if they are connected by a chain of substitutions from Table 5 of [18]. The proof of Proposition 2.9 shows that $\mathbf{b}$ and $\mathbf{b}'$ are then flattened words for the same elements w, and that the map Γ is a bijection from each dual equivalence class to the set of all standard tableaux of some shape μ. Moreover, $f(\mathbf{b})$ is constant on dual equivalence classes.

Proposition 2.12. *The Stanley functions $E_w(X)$ are integral linear combinations of Schur P-functions.*

Proof. This amounts to saying that $2^{l(\mu)}e_w^\mu$ is an integer. Since e_w^μ is given by (III.24) with $t = 1/2$, and (III.24) is a polynomial with integer coefficients, it suffices to show that $l(\mu) \ge f(\mathbf{b})$ for every flattened word $\mathbf{b}$ such that $\Gamma(\mathbf{b})$ has shape μ.

Since both μ and $f(\mathbf{b})$ are constant on dual equivalence classes, we can assume that $\Gamma(\mathbf{b})$ is the tableau T_0 formed by numbering the cells of μ from left to right, one row at a time. The peaks of T_0 occur at the end of each row except the last, so

$|P(T_0)| = |P(\mathbf{b})| = l(\mu) - 1$. For any flattened word, we have $f(\mathbf{b}) \le |P(\mathbf{b})| + 1$, since each first visit is represented by a 1 in $\mathbf{b}$, no two of these 1's can be consecutive, and between every two non-consecutive 1's there is at least one peak. This shows $f(\mathbf{b}) \le l(\mu)$, as required. □

To close, we evaluate E_w and F_w for some special values of w.

PROPOSITION 2.13. *Let $\mu = (\mu_1 > \cdots > \mu_l)$, where μ_l is taken to be zero if necessary to make the number of parts even. Let $\nu_i = \mu_i + 1$ and let $w = \overline{\nu_1}\,\overline{\nu_2}\ldots\overline{\nu_l}12\ldots$. Then $E_w(X) = P_\mu(X)$.*

PROOF. Our method is to give an explicit description of the reduced words for w and compute E_w directly. In order to do this, we introduce a new bijection ϕ, different from Γ, from reduced words for w to standard tableaux of shape μ.

For any element v of D_n the *inversions* of v are (1) pairs $i < j$ for which $v(i) > v(j)$; (2) pairs $i < j$ for which the larger in absolute value of $v(i)$ and $v(j)$ is negative. (A pair can count twice, once in each category). The length $l(v)$ is the number of inversions. In particular, we have $l(w) = |\mu|$ since there are no inversions of type (1) and each i, $1 \le i \le l$, is involved in μ_i inversions of type (2). Let $m = |\mu| = l(w)$.

We now claim that $\mathbf{a} = a_1a_2\ldots a_m$ is a reduced word for w if and only if at every stage j, applying σ_{a_j} to the signed permutation $\sigma_{a_1}\cdots\sigma_{a_{j-1}}$ does one of two things:

(1) moves one of the numbers ν_k which is still positive at this stage to the left across a number which is not a positive ν_i, or

(2) if the smallest two currently positive ν_i's occupy positions 1 and 2, applies $\sigma_{\hat{1}}$ to exchange them and make them negative.

To justify the claim, we note first that such a sequence of operations clearly realizes w after m steps, hence $\mathbf{a}$ is a reduced word for w. To see that every reduced word for w has this form, it is only necessary to check that the form is preserved when $\mathbf{a}$ is modified by any D_n Coxeter relation. For this, note that Coxeter relations of the form $aba \leftrightarrow bab$ with a, b adjacent never apply, nor does $1\hat{1} \leftrightarrow \hat{1}1$. For all others, of the form $ac \leftrightarrow ca$ with a, c nonadjacent, the verification is trivial.

Now, given a reduced word **a** for w, let $v_j = \sigma_{a_1} \cdots \sigma_{a_j}$. Let k_j be the number of ν_i's which appear with positive sign and not in position 1 in the signed permutation v_j, and let λ_j be the partition whose parts are one less than the positions of these ν_i's, a partition with k_j distinct parts. Observe that in passing from v_j to v_{j+1} by move (1) or (2) above, exactly one part of λ_j is reduced by 1 to give λ_{j+1}, and the available choices for a move correspond one-to-one with the corners of the Ferrers diagram of λ_j . Also observe that $\lambda_0 = \mu$. Therefore the sequence of shapes $\emptyset = \lambda_m \subset \lambda_{m-1} \subset \cdots \subset \lambda_0 = \mu$ describes the initial segments of a unique standard tableau $\phi(\mathbf{a})$ of shape μ, every standard tableau occurs, and the tableau contains sufficient information to reconstruct the sequence of moves and thus **a**. This shows ϕ is a bijection from reduced words for w to standard tableaux of shape μ.

Note that $m-j$ is a descent of $\phi(\mathbf{a})$ if and only if the move made at stage j occurs to the left of the move made at stage $j+1$. This shows that the descent set $D(\phi(\mathbf{a}))$ is the same as that of the reversed reduced word $\mathbf{a}^r = a_m a_{m-1} \ldots a_1$. Hence their peak sets are also equal. Note also that each reduced word contains a total of l 1's and $\hat{1}$'s, all representing first visits, so there is one reduced word per flattened word, or in other words, the flattenings of the reduced words are all distinct.

Formula (III.26) for $E_{w^{-1}}$ thus reduces to

$$2^{-l} \sum_{\operatorname{sh} T=\mu} \Theta^{|\mu|}_{P(T)}(X), \tag{III.27}$$

which is $P_\mu(X)$ by Proposition 2.2. Since $E_w = E_{w^{-1}}$ by Corollary 2.11, the proof is complete. □

PROPOSITION 2.14. *Let* $\mu = (\mu_1 > \cdots > \mu_l)$ *and let* $w = \overline{\mu_1}\,\overline{\mu_2} \cdots \overline{\mu_l} 12 \ldots$. *Then* $F_w(X) = Q_\mu(X)$.

PROOF. Since the argument here is virtually identical to that used for preceding proposition, we only give a sketch.

Again we have $l(w) = |\mu|$ by straightforward considerations. (Inversions for B_n are the same as those for D_n, plus one for every negative $v(i)$.)

In this case the allowable "moves" associated with a reduced word are:

(1) move a currently positive μ_k left across anything except a positive μ_i, or

(2) if a positive μ_i occupies position 1, apply σ_0 to change its sign.

The tableau $\phi(\mathbf{a})$ is formed from a sequence of shapes λ_j exactly as before, except now the parts of λ_j are the positions of all the positive μ_i's (including in position 1, and without subtracting one). This ϕ is a bijection exactly as before, and again we have $P(\phi(\mathbf{a})) = P(\mathbf{a}^r)$. Hence using formula (III.21) for $F_{w^{-1}}$, Proposition 2.2, and Corollary 2.5, we find $F_w = Q_\mu$ as asserted. $\square$

For our remaining special case computations we require some facts about the unshifted Haiman correspondence.

DEFINITION. Let $\boldsymbol{\alpha}_n$ denote the straight (*i.e.*, not shifted) staircase shape $(n-1, n-2, \ldots, 1)$, of size $\binom{n}{2}$. Let its corners be labeled $1, 2, \ldots, n-1$ from bottom to top. If T is a standard tableau of shape $\boldsymbol{\alpha}_n$, its *promotion sequence* $\hat{p}(T)$ is the sequence $a_1 \ldots a_{\binom{n}{2}}$ in which a_i is the label of the corner occupied by the largest entry of $p^{\binom{n}{2}-i}(T)$.

PROPOSITION 2.15. (Haiman correspondence) *The map $T \to \hat{p}(T)$ is a bijection from standard tableaux of shape $\boldsymbol{\alpha}_n$ to reduced words for the longest element of S_n. The initial segment $a_1 \ldots a_k$ of $\hat{p}(T)$ determines the initial segment $T|_k$ containing entries 1 through k of T. Denoting $T|_k$ by $\Gamma(a_1 \ldots a_k)$, the number*

$$g_v^\lambda = |\{\mathbf{a} \in R(v) \mid \Gamma(\mathbf{a}) = S\}| \tag{III.28}$$

depends only on v and on the shape λ of S. We have $D(\mathbf{a}) = D(\Gamma(\mathbf{a}))$ for the descent sets.

DEFINITION. Let v be an element of S_n. The *Stanley function* $G_v(X)$ is defined by

$$G_v(X) = \sum_\lambda g_v^\lambda s_\lambda(X), \tag{III.29}$$

where s_λ denotes the usual Schur S-function.

Proposition 2.15 is proved in [18], where it is also shown that the above definition of S_n Stanley functions agrees with the original definition in [43]. (In [43], and also in Chapter 7 of [33], where G_v is shown to be a 'stable' Schubert polynomial of type A, G_v and g_v^λ are denoted F_v and $\alpha(\lambda, v)$.)

Now we can express the functions $F_{w_0^B v}$ and $E_{w_0^D v}$, for $v \in S_n$, in terms of the quantities just defined.

PROPOSITION 2.16. *Let* w_0^B, w_0^D, *and* v_0 *be the longest elements of* B_n, D_n, *and* S_n, *respectively. Let* δ_k *denote the partition* $(k, k-1, \ldots, 1)$. *Then we have for every* $v \in S_n$

$$F_{w_0^B v} = \sum_\lambda g^\lambda_{v_0 v} Q_{\delta_n + \lambda}, \tag{III.30}$$

$$E_{w_0^D v} = \sum_\lambda g^\lambda_{v_0 v} P_{\delta_{n-1} + \lambda}. \tag{III.31}$$

Equivalently, $F_{w_0^B v}$ *and* $E_{w_0^D v}$ *are the images of* $G_{v_0 v}$ *under linear transformations sending Schur functions* s_λ *to* $Q_{\delta_n+\lambda}$ *and* $P_{\delta_{n-1}+\lambda}$, *respectively.*

PROOF. Fix a reduced word $\mathbf{c}$ for v^{-1}. Then the reduced words for $v_0 v$ are exactly the initial parts $\mathbf{a}$ of those reduced words $\mathbf{ac}$ for v_0 which end in $\mathbf{c}$. Similar statements apply with w_0^B and w_0^D in place of v_0.

From this observation and Proposition 2.15 it follows that $g^\lambda_{v_0 v}$ is equal to the number of tableaux S of skew shape $\boldsymbol{\alpha}_n/\lambda$ for which $\hat{p}(S) = \mathbf{c}$. Similarly, $f^\mu_{w_0^B v}$ is the number of tableaux T of shape $\boldsymbol{\beta}_n/\mu$ for which $\hat{p}(T) = \mathbf{c}$. But there are no 0's in $\mathbf{c}$, and therefore $f^\mu_{w_0^B v}$ is non-zero only if the shape μ contains the corner with label 0, that is, if $\mu = \delta_n + \lambda$ for some λ. In this case, the rules for computing $\hat{p}(S)$ and $\hat{p}(T)$ are identical, showing that $f^{\delta_n+\lambda}_{w_0^B v} = g^\lambda_{v_0 v}$. This proves (III.30).

For (III.31), we need $2^{n-1} e^{\delta_{n-1}+\lambda}_{w_0^D v} = g^\lambda_{v_0 v}$ and $e^\mu_{w_0^D v} = 0$ if μ is not of the form $\delta_{n-1} + \lambda$. Since $(w_0^D v)^{-1}(k)$ is negative for all $k \in \{2, \ldots, n\}$, we have $f(\mathbf{a}) = n-1$ for all $\mathbf{a} \in R(w_0^D v)$. In the proof of Proposition 2.12 we showed that $f(\mathbf{b}) \leq l(\mu)$ whenever $\Gamma(\mathbf{b}))$ has shape μ. This shows $e^\mu_{w_0^D v}$ is non-zero only for μ of the form $\delta_{n-1}+\lambda$. Moreover, using the left-hand side of (III.24) with $t = 1/2$ to evaluate $e^\mu_{w_0^D v}$, we find that $2^{n-1} e^\mu_{w_0^D v}$ is the number of flattened words $\mathbf{b}$ for $w_0^D v$ with $\Gamma(\mathbf{b}) = S$, for any given tableau S of shape μ.

If $\mathbf{b}$ is a flattened word for $w_0^D v$, then $\mathbf{bc}$ is clearly a flattened word for w_0^D. Every element of S_n has a reduced word containing at most one 1, so we may choose $\mathbf{c}$ with this property. For such $\mathbf{c}$, we claim the converse holds: if $\mathbf{bc}$ is a flattened word for

w_0^D then $\mathbf{b}$ is a flattened word for $w_0^D v$. This amounts to saying that $\mathbf{bc}$ can be un-flattened without changing any 1's in $\mathbf{c}$ to $\hat{1}$'s. If there are no 1's in $\mathbf{c}$, this is trivial. If there is a single 1, then it is the last 1 in the visiting word $\mathbf{bc}$, corresponding to the transposition moving 1 into the leftmost position for the last time. As such, it represents a repeat visit, so Proposition 2.8 shows we can un-flatten $\mathbf{bc}$ without changing it to a $\hat{1}$.

In view of the claim just proven, Proposition 2.9 shows that $2^{n-1}e^{\mu}_{w_0^D v}$ is equal to the number of tableaux T of shape $\boldsymbol{\delta}_n/\mu$ for which $\hat{p}(T) = \mathbf{c}$. Exactly as in the argument above for B_n, this is the same as $g^{\lambda}_{v_0 v}$ for $\mu = \delta_{n-1} + \lambda$. □

We give one final special case evaluation for its inherent interest, even though we will not need it later.

PROPOSITION 2.17. *Let ϕ be the homomorphism from the ring of symmetric functions onto the subring generated by odd power sums defined by*

$$\phi(p_k) = \begin{cases} 2p_k & \text{for } k \text{ odd,} \\ 0 & \text{for } k \text{ even.} \end{cases} \tag{III.32}$$

Then for $v \in S_n$, we have

$$F_v = \phi(G_v), \tag{III.33}$$

and if in addition $v(1) = 1$,

$$E_v = \phi(G_v). \tag{III.34}$$

PROOF. If $\mathbf{a}$ is a reduced word for $v \in S_n$, let us denote the corresponding tableaux $\Gamma(\mathbf{a})$ under the A_n, B_n, and D_n Haiman correspondences by $\Gamma_A(\mathbf{a})$, $\Gamma_B(\mathbf{a})$, and $\Gamma_D(\mathbf{a})$. It is easy to show that $\Gamma_B(\mathbf{a})$ and $\Gamma_D(\mathbf{a})$ are both identical to the tableau obtained by bringing $\Gamma_A(\mathbf{a})$ to normal shifted shape via shifted *jeu-de-taquin*. Hence for S of shape μ, $f_v^{\mu} = |\{\mathbf{a} \in R(v) \mid \Gamma_B(\mathbf{a}) = S\}| = \sum_{\lambda} k^{\mu}_{\lambda} g^{\lambda}_v$, where k^{μ}_{λ} is the number of standard tableaux of straight shape λ carried by shifted *jeu-de-taquin* to any given tableau of shifted shape μ.

In [47] it is shown that $\phi(s_\lambda) = \sum_\mu k_\lambda^\mu Q_\mu$. Equation (III.33) follows immediately. Equation (III.34) follows because when $v(1) = 1$, there are no 1's in any reduced word for v, and therefore $e_v^\mu = f_v^\mu$. □

The text of Section 2 of this chapter is a reprint of material as it appears in *Schubert polynomials for the classical groups* to appear in the Journal of the AMS, co-authored with Mark Haiman. Section 2 of Chapter 3 has been included for clarity of exposition and I was the secondary author.

CHAPTER IV

Formulas for Schubert Polynomials

In this chapter we prove Theorems 2, 3, 4, and 5. We conclude this chapter will tables of Schubert polynomials for of all four types and $n = 3$.

1. Proof of the formula for type A_n

In this section we will show, in Theorem 2, that the polynomials $\mathfrak{S}_w$ defined in (I.8) satisfy the recurrence relations

$$\text{(IV.1)} \qquad \partial_i \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{ws_i} & \text{if } l(w) > l(ws_i) \\ 0 & \text{if } l(w) < l(ws_i) \end{cases}$$

defined by the divided difference operators corresponding to the root system of type A. This will prove $\mathfrak{S}_w$ is the unique Schubert polynomial corresponding to $w \in S_\infty$.

We define a special product of permutations

$$\text{(IV.2)} \qquad u \cdot v = \begin{cases} uv & \text{if } l(uv) = l(u) + l(v) \\ 0 & \text{otherwise} \end{cases}$$

Note that if $w \in S_n$ and $u \cdot v = w$ then u and v must also be permutations in S_n. Let

$$\text{(IV.3)} \qquad 1_r \times v = [1, 2, \ldots, r, v(1), v(2), \ldots]$$

Note that $R(1_r \times v) = \{(a_1 + r)(a_2 + r) \cdots (a_p + r) : a_1 a_2 \cdots a_p \in R(w)\}$.

Let ρ_r be the operator on polynomials such that

$$\rho_r F(z_1, z_2, \dots) = F(z_1, \dots, z_r, 0, 0, \dots). \tag{IV.4}$$

In particular, $\rho_1 \mathfrak{S}_u = 0$ unless u has a strictly decreasing reduced word and in that case $\rho_1 \mathfrak{S}_u = z_1^{l(u)}$. A permutation with a strictly decreasing reduced word will be called a *decreasing* permutation. Note that any permutation has at most one strictly decreasing word.

We will define a restricted set of variables $Z_k = \{z_k, z_k + 1, \dots\}$. Then a polynomial $F(Z_k)$ is F with the first variable set to z_k, the second variable set to z_{k+1}, etc. In this notation, $F(Z_1)$ will be the usual representation of the polynomial.

We will need a technical lemma to complete the proof of our main theorem for this section.

LEMMA 1.1. *For any $u \in S_\infty$,*

$$\partial_1 \rho_2 \mathfrak{S}_u = \begin{cases} \rho_2 \mathfrak{S}_{us_1} & \text{if } l(u) > l(us_1) \\ 0 & \text{if } l(u) < l(us_1). \end{cases} \tag{IV.5}$$

PROOF. If every reduced word for u has at least three ascents then every admissible monomial for u has a factor different from z_1 and z_2. Hence, $\rho_2 \mathfrak{S}_u = 0$ unless u has a reduced word with at most 2 decreasing sequences.

Recall from Chapter II, Corollary 1.14 if $w(1) > w(2)$ and $\rho_2 \mathfrak{S}_u \neq 0$, then

$$\rho_2 \mathfrak{S}_u = z_1^t z_2^s h_\lambda(z_1, z_2) \tag{IV.6}$$

$$\rho_2 \mathfrak{S}_{us_1} = z_1^s z_2^s h_\lambda(z_1, z_2) h_{t-s-1}(z_1, z_2) \tag{IV.7}$$

where $t \geq s$, $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots$ is a partition, $h_k(z_1, z_2) = \sum_{i=0}^k z_1^i z_2^{k-i}$, and $h_\lambda(z_1, z_2) = \prod h_{\lambda_i}$ is the homogeneous symmetric function. Since ∂_1 commutes with any polynomial that is symmetric in z_1 and z_2, we have

$$\partial_1 \rho_2 \mathfrak{S}_u = z_1^s z_2^s h_\lambda(z_1, z_2) \left(\partial_1 z_1^{t-s}\right) \tag{IV.8}$$

$$= z_1^s z_2^s h_\lambda(z_1, z_2) \sum_{i=0}^{t-s-1} z_1^i z_2^{t-s-1-i} \tag{IV.9}$$

$$= \rho_2 \mathfrak{S}_{us_1}. \tag{IV.10}$$

If $w(1) < w(2)$, then $\rho_2\mathfrak{S}_u = z_1^s z_2^s h_\lambda(z_1, z_2)h_{t-s-1}(z_1, z_2)$, hence $\partial_1\rho_2\mathfrak{S}_u = 0$ since $\rho_2\mathfrak{S}_u$ is symmetric in z_1 and z_2. □

THEOREM 2. *For any positive integer i and any $w \in S_\infty$*

$$\partial_i\mathfrak{S}_w = \begin{cases} \mathfrak{S}_{ws_i} & \text{if } l(w) > l(ws_i) \\ 0 & \text{if } l(w) < l(ws_i) \end{cases} \tag{IV.11}$$

PROOF. The first case we consider is when $i > 1$. Assume the theorem is true for all $v \in S_{n-1}$, we will show by induction that it is true for $w \in S_n$. Recall, from Corollary 1.12, for any positive integer r,

$$\mathfrak{S}_w = \sum_{u\cdot(1_r\times v)=w} (\rho_r\mathfrak{S}_u(Z_1))\,\mathfrak{S}_v(Z_{r+1}) \tag{IV.12}$$

where $u \cdot v = w$ implies $l(u) + l(v) = l(w)$. Let $r = 1$ in (IV.12), then

$$\partial_i\mathfrak{S}_w = \partial_i \sum_{u\cdot(1\times v)=w} \rho_1\mathfrak{S}_u(Z_1)\mathfrak{S}_v(Z_2) \tag{IV.13}$$

$$= \sum_{\substack{u\cdot(1\times v)=w \\ u \text{ decreasing}}} z_1^{l(u)}\partial_i\mathfrak{S}_v(Z_2) \tag{IV.14}$$

since ∂_i commutes with z_1 if $i > 1$. If $w \in S_n$, then $u \cdot (1 \times v) = w$ implies $v \in S_{n-1}$, hence the recursion in (IV.11) holds for $\partial_i\mathfrak{S}_v(Z_1)$ or $\partial_{i+1}\mathfrak{S}_v(Z_2)$ if we take into account the relabeling of the variables. Therefore,

$$\partial_i\mathfrak{S}_w = \sum_{\substack{u\cdot(1\times v)=w \\ u \text{ decreasing} \\ l(v)>l(vs_{i+1})}} z_1^{l(u)}\mathfrak{S}_{vs_{i+1}}(Z_2). \tag{IV.15}$$

Note that $l(w) > l(ws_i)$ if and only there exists at least one pair u, v such that $u \cdot (1 \times v) = w$, u is decreasing, and $l(v) > l(vs_{i+1})$. If $l(w) < l(ws_i)$, the sum is over the empty set, hence $\partial_i\mathfrak{S}_w = 0$.

Assume $l(w) > l(ws_i)$, if $u\cdot(1\times v) = w$ and $l(v) > l(vs_{i+1})$ then $u\cdot(1\times vs_{i+1}) = u\cdot(1\times v)s_i = ws_i$. Conversely, if $u'\cdot(1\times v') = ws_i$ then $u'\cdot(1\times v')s_i = u\cdot(1\times v's_{i+1}) = w$ and $l(v's_i) > l(v')$ since $l(w) > l(ws_i)$. Furthermore, if u was decreasing in either case above then multiplying $(1\times v)$ by s_i on the right leaves u decreasing. Therefore,

$$\partial_i \mathfrak{S}_w = \begin{cases} \sum_{\substack{u\cdot(1\times v)=ws_i \\ u \text{ decreasing}}} z_1^{l(u)} \mathfrak{S}_v(Z_2) & \text{if } l(w) > l(ws_i) \\ 0 & \text{if } l(w) < l(ws_i) \end{cases} \tag{IV.16}$$

Using (IV.12) one more time we get (IV.11) for the case $i > 1$.

Next, assume $i = 1$ and let $r = 2$ in (IV.12). Then we have

$$\partial_1 \mathfrak{S}_w = \sum_{u\cdot(1_2\times v)=w} \partial_1 \rho_2(\mathfrak{S}_u(Z_1))\mathfrak{S}_v(Z_3). \tag{IV.17}$$

Applying Lemma 1.1,

$$\partial_1 \mathfrak{S}_w = \sum_{\substack{u\cdot(1_2\times v)=w \\ l(u)>l(us_1)}} \rho_2(\mathfrak{S}_{us_1}(Z_1))\mathfrak{S}_v(Z_3). \tag{IV.18}$$

Clearly, s_1 commutes with the permutation $(1_2 \times v)$ since every letter in any of the reduced words for $(1_2\times v) = [1, 2, v_1, \dots, v_{n-2}]$ is at least 3. Therefore, $u\cdot(1_2\times v) = w$ and $l(u) > l(us_1)$ if and only if $us_1 \cdot (1_2 \times v) = ws_1$ and $l(w) > l(ws_1)$. Hence,

$$\partial_1 \mathfrak{S}_w = \begin{cases} \sum_{u\cdot(1_2\times v)=ws_1} \rho_2(\mathfrak{S}_u(Z_1))\mathfrak{S}_v(Z_3) & \text{if } l(w) > l(ws_1) \\ 0 & \text{if } l(w) < l(ws_1) \end{cases} \tag{IV.19}$$

Applying (IV.12) on more time proves the theorem. □

2. Proof of the formulas for types B, C

Our central results for this section, are contained in Theorems 3 and 4. Below we have split each theorem into two separate statements, labeled A, and B. Theorems 3A and 4A are the promised formulas for Schubert polynomials. Parts B are the additional results that the Schubert polynomials reduce to Schur P- and Q-functions in the special cases corresponding to Schubert cycles for isotropic Grassmannians. In Theorem 5, we show each family of Schubert polynomials forms a $\mathbb{Z}$-basis for the relevant ring. We conclude the section with some auxiliary results useful for computing Schubert polynomials of type C and D.

Theorems 3A and 4A involve 'admissible monomial' forms of formulas (I.48) and (I.52), derived from formulas (III.21) and (III.26) and the admissible monomial formula for type A Schubert polynomials, (IV.22) below. In order to distinguish between two notions of admissibility we will make the following conventions. If $\mathbf{a} = a_1a_2\ldots a_l$ is a reduced word for an element $w \in B_\infty$ or D_∞, we let $\mathcal{A}_x(\mathbf{a})$ denote the set of admissible monomials $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_m^{\alpha_m} = x_{i_1}x_{i_2}\cdots x_{i_l}$, such that $(i_1 \leq i_2 \leq \cdots \leq i_l) \in A(P(\mathbf{a}^r))$, where $\mathbf{a}^r$ is the reversed sequence $a_l\cdots a_2a_1$. Equivalently, $\mathcal{A}_x(\mathbf{a})$ consists of monomials $x^\alpha = x_{j_1}\cdots x_{j_l}$ for which

(1) $j_1 \geq j_2 \geq \cdots \geq j_l$
(2) $j_{k-1} = j_k = j_{k+1}$ implies $k \notin P(\mathbf{a})$.

By Corollary 2.5 we have for $w \in B_n$,

(IV.20) $$F_w(X) = \sum_{\substack{\mathbf{a}\in R(w)\\ x^\alpha\in\mathcal{A}_x(\mathbf{a})}} 2^{i(\alpha)}x^\alpha,$$

where $i(\alpha)$ is the number of distinct variables with non-zero exponent in x^α. By Corollary 2.11, we have

(IV.21) $$E_w(X) = \sum_{\substack{\mathbf{a}\in R(w)\\ x^\alpha\in\mathcal{A}_x(\mathbf{a})}} 2^{i(\alpha)-o(\mathbf{a})}x^\alpha$$

for each $w \in D_n$.

If $\mathbf{a} = a_1a_2\ldots a_l$ is a reduced word for $w \in S_n$, we let $\mathcal{A}_z(\mathbf{a})$ denote the set of monomials $z^\alpha = z_{j_1}\cdots z_{j_l}$ satisfying the following admissibility constraints:

(1) $j_1 \leq j_2 \leq \cdots \leq j_l$
(2) $j_i = j_{i+1}$ implies $a_i > a_{i+1}$
(3) $j_i \leq a_i$ for all i.

Proposition 2.1. **[3]** *For all* $w \in S_\infty$,

(IV.22) $$\mathfrak{S}_w(z_1, z_2, \ldots) = \sum_{\substack{\mathbf{a}\in R(w)\\ z^\alpha\in\mathcal{A}_z(\mathbf{a})}} z^\alpha.$$

THEOREM 3A. *The Schubert polynomials $\mathfrak{C}_w$ are given by the two equivalent formulas:*

$$\mathfrak{C}_w = \sum_{\substack{uv=w \\ l(u)+l(v)=l(w) \\ v\in S_\infty}} F_u(X)\mathfrak{S}_v(Z) \tag{IV.23}$$

$$= \sum_{\substack{uv=w \\ l(u)+l(v)=l(w) \\ v\in S_\infty}} \sum_{\substack{\mathbf{a}\in R(u) \\ x^\alpha\in\mathcal{A}_x(\mathbf{a})}} \sum_{\substack{\mathbf{b}\in R(v) \\ z^\beta\in\mathcal{A}_z(\mathbf{b})}} 2^{i(\alpha)}x^\alpha z^\beta. \tag{IV.24}$$

THEOREM 4A. *The Schubert polynomials $\mathfrak{D}_w$ are given by the two equivalent formulas:*

$$\mathfrak{D}_w = \sum_{\substack{uv=w \\ l(u)+l(v)=l(w) \\ v\in S_\infty}} E_u(X)\mathfrak{S}_v(Z) \tag{IV.25}$$

$$= \sum_{\substack{uv=w \\ l(u)+l(v)=l(w) \\ v\in S_\infty}} \sum_{\substack{\mathbf{a}\in R(u) \\ x^\alpha\in\mathcal{A}_x(\mathbf{a})}} \sum_{\substack{\mathbf{b}\in R(v) \\ z_\beta\in\mathcal{A}_z(\mathbf{b})}} 2^{i(\alpha)-o(\mathbf{a})}x^\alpha z^\beta. \tag{IV.26}$$

We prove several lemmas before proving Theorems 3A and 4A.

LEMMA 2.2. *For any f and g, and any i, we have*

$$\partial_i(fg) = (\partial_i f)(\sigma_i g) + f\,\partial_i g. \tag{IV.27}$$

PROOF. Expand both sides and observe they are equal. □

LEMMA 2.3. *Let W denote any of the groups S_∞, B_∞, or D_∞. Let $G_u(Z)$ be arbitrary symmetric functions indexed by elements $u\in W$ and define*

$$H_w = \sum_{\substack{uv=w \\ l(u)+l(v)=l(w) \\ v\in S_\infty}} G_u(Z)\mathfrak{S}_v(Z). \tag{IV.28}$$

Then for all $i>0$ and $w\in W$,

$$\partial_i H_w = \begin{cases} H_{w\sigma_i} & \text{if } l(w\sigma_i) < l(w) \\ 0 & \text{if } l(w\sigma_i) > l(w). \end{cases} \tag{IV.29}$$

PROOF. For $i > 0$, the operator ∂_i commutes with multiplication by the symmetric function $G_u(Z)$. Hence,

$$\partial_i H_w = \sum_{\substack{uv=w \\ l(u)+l(v)=l(w) \\ v\in S_\infty}} G_u(Z)\partial_i \mathfrak{S}_v(Z) \tag{IV.30}$$

$$= \sum_{\substack{uv=w \\ l(u)+l(v)=l(w) \\ v\in S_\infty \\ l(v\sigma_i)<l(v)}} G_u(Z)\mathfrak{S}_{v\sigma_i}(Z) \tag{IV.31}$$

by the defining recurrence for the Schubert polynomials $\mathfrak{S}_w$.

If $l(w\sigma_i) > l(w)$ then the conditions $uv = w$, $l(u) + l(v) = l(w)$, $v \in S_\infty$ and $l(v\sigma_i) < l(v)$ are never satisfied, so (IV.31) is equal to zero.

On the other hand, if $l(w\sigma_i) < l(w)$ then the map $(u, v) \mapsto (u, v\sigma_i)$ is a bijection from all (u, v) such that $uv = w$, $l(u) + l(v) = l(w)$, $v \in S_\infty$, and $l(v\sigma_i) < l(v)$ to all (u', v') such that $u'v' = w\sigma_i$, $l(u') + l(v') = l(w\sigma_i)$, $v' \in S_\infty$. Therefore (IV.31) is $H_{w\sigma_i}$. $\square$

From here out, symmetric functions in X depend only on odd power sums and really represent symmetric functions in Z via the relation $p_k(X) = -p_k(Z)/2$.

LEMMA 2.4. *For any symmetric function $G(X)$ belonging to the ring generated by odd power sums, we have*

$$\sigma_0 G(X) = G(z_1 + X) \tag{IV.32}$$

$$\partial_0 G(X) = \frac{G(X) - G(z_1 + X)}{-2z_1}, \tag{IV.33}$$

where $G(z_1 + X) = G(z_1, x_1, x_2, \ldots)$.

PROOF. Because σ_0 is a ring homomorphism, it suffices to verify (IV.32) for

$G(X) = p_k(X)$, an odd power sum.

$$\sigma_0 p_k(X) = -\frac{1}{2}\sigma_0 p_k(Z) \tag{IV.34}$$
$$= -\frac{1}{2}p_k(-z_1, z_2, \dots)$$
$$= z_1^k - \frac{1}{2}p_k(z_1, z_2, \dots)$$
$$= z_1^k + p_k(X)$$
$$= p_k(z_1 + X).$$

Equation (IV.33) follows from (IV.32). □

We will not be using the next corollary for the proofs that follow. However, it is useful for computing tables of Schubert polynomials.

Corollary 2.5. *The action of ∂_0 on $Q_\mu(X)$ is given by*

$$\partial_0 Q_\mu(X) = \sum_{0<k\leq\mu_1} Q_{\mu/(k)}(X) z_1^{k-1}. \tag{IV.35}$$

Proof. Q_μ belongs to the ring generated by odd power sums, so Lemma 2.4 applies. We have $Q_\mu(z_1+X) = \sum_{\lambda\subseteq\mu} Q_\lambda(z_1)Q_{\mu/\lambda}(X)$, where $Q_{\mu/\lambda}$ for a skew shifted shape is given by the obvious extension of Proposition 2.2. The factor $Q_\lambda(z_1)$ is equal to zero unless $\lambda = (k)$ is a one row shape, in which case it is equal to 1 if $k = 0$ and $2z_1^k$ if $k > 0$. Therefore,

$$\partial_0 Q_\mu(X) = \frac{Q_\mu(X) - Q_\mu(z_1+X)}{-2z_1} \tag{IV.36}$$
$$= \frac{Q_\mu(X) - Q_\mu(X) - \sum_{0<k\leq\mu_1} Q_{\mu/(k)} 2z_1^k}{-2z_1},$$

which simplifies to (IV.35). □

Definition. A reduced word $\mathbf{a} = a_1a_2\dots a_l$ and monomial $x^\alpha \in \mathcal{A}_x(\mathbf{a})$ will be referred to as a *reduced word admissible monomial pair*, and denoted by $\binom{\mathbf{a}}{x^\alpha}$. Similarly, $\left[{\mathbf{c} \atop z^\gamma}\right]$ will denote a reduced word admissible monomial pair if $z^\gamma \in \mathcal{A}_z(\mathbf{c})$.

The notation is merely a bookkeeping device to exhibit the reduced word associated with a particular term. By our convention, when these symbols appear in a polynomial the value of $\binom{\mathbf{a}}{x^\alpha}$ is x^α. We multiply the symbols by concatenating the reduced words and multiplying the monomials. Note the use of the notation $\left[\begin{smallmatrix}\mathbf{c}\\ z^\gamma\end{smallmatrix}\right]$ implies that 0 and $\hat{1}$ do not appear in the reduced word $\mathbf{c}$, by the definition of $\mathcal{A}_z$. With this notation, (IV.24) becomes

$$\mathfrak{C}_w = \sum_{\mathbf{ab}\in R(w)} \sum_{\substack{x^\alpha\in\mathcal{A}_x(\mathbf{a})\\ z^\beta\in\mathcal{A}_z(\mathbf{b})}} 2^{i(\alpha)} \begin{pmatrix}\mathbf{a}\\ x^\alpha\end{pmatrix} \begin{bmatrix}\mathbf{b}\\ z^\beta\end{bmatrix} \tag{IV.37}$$

$$= \tilde{\sum_{\mathbf{ab}\in R(w)}} 2^{i(\alpha)} \begin{pmatrix}\mathbf{a}\\ x^\alpha\end{pmatrix} \begin{bmatrix}\mathbf{b}\\ z^\beta\end{bmatrix}. \tag{IV.38}$$

Here the symbol $\tilde{\sum}$ indicates that the sum ranges over all possible admissible monomials for each $\mathbf{a}$ and $\mathbf{b}$.

Lemma 2.6. *For any reduced word admissible monomial pair* $\left[\begin{smallmatrix}\mathbf{c}\\ z^\gamma\end{smallmatrix}\right]$, *where* $z^\gamma = z_1^{\gamma_1} z_2^{\gamma_2}\cdots$, *we have*

$$\partial_0 \begin{bmatrix}\mathbf{c}\\ z^\gamma\end{bmatrix} = \begin{cases} -\frac{1}{z_1}\left[\begin{smallmatrix}\mathbf{c}\\ z^\gamma\end{smallmatrix}\right] & \gamma_1 \text{ odd} \\ 0 & \gamma_1 \text{ even.}\end{cases} \tag{IV.39}$$

Proof. Equation (IV.39) follows immediately from the definition $\partial_0 f = (f - \sigma_0 f)/(-2z_1)$. □

Recall that a sequence $b_1 > b_2 > \cdots > b_j < \cdots < b_k$ having no peak is said to be a *vee*.

Lemma 2.7. *For all* $u \in B_\infty$,

$$\partial_0 F_u(X) = \frac{1}{z_1} \tilde{\sum_{\substack{\mathbf{ab}\in R(u)\\ k>0}}} 2^{i(\alpha)} \begin{pmatrix}\mathbf{a}\\ x^\alpha\end{pmatrix} \begin{pmatrix}\mathbf{b}\\ z_1^k\end{pmatrix}, \tag{IV.40}$$

where the notation signifies that $\mathbf{a}$ *and* $\mathbf{b}$ *range over reduced word admissible monomial pairs such that* $\mathbf{ab} \in R(u)$ *and* $\mathbf{b}$ *is a vee of length* $k > 0$.

PROOF. By Lemma 2.4,

$$\partial_0 F_u(X) = \frac{F_u(X) - F_u(z_1 + X)}{-2z_1} \tag{IV.41}$$

$$= \frac{1}{-2z_1}\Big[\sum_{\mathbf{c}\in R(u)}^{\sim} 2^{i(\gamma)}\binom{\mathbf{c}}{x^\gamma} - \sum_{\mathbf{ab}\in R(u)}^{\sim} 2^{i(\alpha)+\chi_k}\binom{\mathbf{a}}{x^\alpha}\binom{\mathbf{b}}{z_1^k}\Big], \tag{IV.42}$$

where χ_k is 1 if $k > 0$ and 0 otherwise. In the second sum $\mathbf{a}$ and $\mathbf{b}$ range over all pairs such that $\mathbf{ab} \in R(u)$ and $\mathbf{b}$ is a vee; k is the length of $\mathbf{b}$. Note that the requirement that $\mathbf{b}$ is a vee is implicit in the use of the symbol $\binom{\mathbf{b}}{z_1^k}$. The terms in the first sum are just the terms with $k = 0$ in the second sum. Hence

$$\partial_0 F_u(X) = \frac{1}{2z_1}\sum_{\substack{\mathbf{ab}\in R(u)\\ k>0}}^{\sim} 2^{i(\alpha)+1}\binom{\mathbf{a}}{x^\alpha}\binom{\mathbf{b}}{z_1^k}, \tag{IV.43}$$

which is the same as (IV.40). □

LEMMA 2.8. *Let* $\tilde{\mathfrak{C}}_w$ *denote the polynomial defined by* (IV.23) *and* (IV.24)*. For all* $w \in B_\infty$*,*

$$\partial_0 \tilde{\mathfrak{C}}_w = \frac{1}{z_1}\sum_{\substack{\mathbf{abc}\in R(w)\\ b_k=0}}^{\sim} 2^{i(\alpha)}\binom{\mathbf{a}}{x^\alpha}\begin{bmatrix} b_1 b_2 \dots b_{k-1} \\ z_1^{k-1}\end{bmatrix}\binom{b_k}{z_1}\begin{bmatrix}\mathbf{c}\\ z_2^{\gamma_2} z_3^{\gamma_3}\dots\end{bmatrix}, \tag{IV.44}$$

where the notation implies $b_1 > b_2 > \dots > b_k = 0$ *and* $\mathbf{b} = b_1 b_2 \dots b_k$ *has length* $k > 0$*.*

PROOF. From the definition, we have

$$\partial_0 \tilde{\mathfrak{C}}_w = \sum_{\substack{uv=w\\ l(u)+l(v)=l(w)\\ v\in S_\infty}} \sum_{\mathbf{c}\in R(v)}^{\sim} \partial_0\Big(F_u(X)\begin{bmatrix}\mathbf{c}\\ z^\gamma\end{bmatrix}\Big). \tag{IV.45}$$

By Lemma 2.2, we can expand (IV.45) as the sum of two polynomials. The first term of (IV.27) yields

(IV.46)
$$\sum_{\substack{uv=w\\ l(u)+l(v)=l(w)\\ v\in S_\infty}} \tilde{\sum_{\mathbf{c}\in R(v)}} (\partial_0 F_u(X))(\sigma_0 \begin{bmatrix} \mathbf{c} \\ z^\gamma \end{bmatrix})$$

(IV.47)
$$= \frac{1}{z_1} \tilde{\sum_{\substack{\mathbf{abc}\in R(w)\\ k>0}}} (-1)^{\gamma_1} 2^{i(\alpha)} \begin{pmatrix} \mathbf{a} \\ x^\alpha \end{pmatrix} \begin{pmatrix} b_1 b_2 \dots b_k \\ z_1^k \end{pmatrix} \begin{bmatrix} \mathbf{c} \\ z^\gamma \end{bmatrix}$$

(IV.48)
$$= \frac{1}{z_1} \tilde{\sum_{\substack{\mathbf{abc}'\in R(w)\\ k>0}}} (-1)^{\gamma_1} 2^{i(\alpha)} \begin{pmatrix} \mathbf{a} \\ x^\alpha \end{pmatrix} \begin{pmatrix} b_1 b_2 \dots b_k \\ z_1^k \end{pmatrix} \begin{bmatrix} b_{k+1} \dots b_m \\ z_1^{\gamma_1} \end{bmatrix} \begin{bmatrix} \mathbf{c}' \\ z_2^{\gamma_2} z_3^{\gamma_3} \dots \end{bmatrix}.$$

The second term of (IV.27) yields

(IV.49)
$$\sum_{\substack{uv=w\\ l(u)+l(v)=l(w)\\ v\in S_\infty}} \tilde{\sum_{\mathbf{c}\in R(v)}} F_u(X)(\partial_0 \begin{bmatrix} \mathbf{c} \\ z^\gamma \end{bmatrix}) = \frac{1}{z_1} \tilde{\sum_{\substack{\mathbf{ac}\in R(w)\\ \gamma_1 \text{ odd}}}} -2^{i(\alpha)} \begin{pmatrix} \mathbf{a} \\ x^\alpha \end{pmatrix} \begin{bmatrix} \mathbf{c} \\ z^\gamma \end{bmatrix}$$

(IV.50)
$$= \frac{1}{z_1} \tilde{\sum_{\substack{\mathbf{abc}'\in R(w)\\ m \text{ odd}}}} -2^{i(\alpha)} \begin{pmatrix} \mathbf{a} \\ x^\alpha \end{pmatrix} \begin{bmatrix} \mathbf{b} \\ z_1^m \end{bmatrix} \begin{bmatrix} \mathbf{c}' \\ z_2^{\gamma_2} z_3^{\gamma_3} \dots \end{bmatrix}.$$

Next we examine the coefficient C_A of the general term

(IV.51)
$$A = \frac{1}{z_1} 2^{i(\alpha)} \begin{pmatrix} \mathbf{a} \\ x^\alpha \end{pmatrix} \left\langle \begin{matrix} b_1 \dots b_m \\ z_1^m \end{matrix} \right\rangle \begin{bmatrix} \mathbf{c} \\ z_2^{\gamma_2} z_3^{\gamma_3} \dots \end{bmatrix}$$

in the sum of (IV.48) and (IV.50). Here the bracket $\langle\rangle$ denotes the entire factor involving z_1, for which $\mathbf{b}$ is, in general, a vee followed by a decreasing sequence. From (IV.48) there is a contribution of $(-1)^{\gamma_1} = (-1)^{m-k}$ to C_A for every k such that

(IV.52)
$$\left\langle \begin{matrix} b_1 \dots b_m \\ z_1^m \end{matrix} \right\rangle = \begin{pmatrix} b_1 \dots b_k \\ z_1^k \end{pmatrix} \begin{bmatrix} b_{k+1} \dots b_m \\ z_1^{m-k} \end{bmatrix},$$

i.e., such that $b_1 \dots b_k$ is a vee, $b_{k+1} > \cdots > b_m$ and $b_m \neq 0$ unless $k = m$. From (IV.50) there is a contribution of -1 provided $b_1 > \cdots > b_m$, m is odd, and $b_m \neq 0$.

We need to verify that $C_A = 0$ unless $b_1 > \cdots > b_m = 0$, and then $C_A = 1$.

Case 1: $b_m \neq 0$

First assume there is an index i such that $b_i < b_{i+1}$, and chose i to be as large as possible. Then (IV.48) contributes two terms, for $k = i$ and $k = i+1$, which cancel, while (IV.50) contributes nothing.

Otherwise, assume $b_1 > b_2 > \cdots > b_m > 0$. For each $1 \leq k \leq m$, there is a contribution of $(-1)^{m-k}$ from (IV.48). If m is even, then $C_A = \sum_{k=1}^{m}(-1)^{m-k} = 0$. If m is odd there is also a contribution from (IV.50) so $C_A = -1+\sum_{k=1}^{m}(-1)^{m-k} = 0$. Therefore, every term A with $b_m \neq 0$ has $C_A = 0$.

Case 2: $b_m = 0$

In this case, the only contribution is from (IV.52) with $k = m$, *i.e.* $\left\langle {b_1 \ldots b_m \atop z_1^m} \right\rangle = \left({b_1 \ldots b_m \atop z_1^m} \right)$. Hence $C_A = 1$. Furthermore, $\mathbf{b}$ must be a vee so we have $b_1 > b_2 > \cdots > b_m = 0$. □

Proof of Theorem 3A. Formulae (IV.23) and (IV.24) are equivalent by (III.21) and (IV.22). To prove they give the Schubert polynomials, we take them for the moment as the definition of $\mathfrak{C}_w$ and show that $\mathfrak{C}_w$ satisfies the recurrence

$$\partial_i \mathfrak{C}_w = \begin{cases} \mathfrak{C}_{w\sigma_i} & l(w\sigma_i) < l(w) \\ 0 & l(w\sigma_i) > l(w) \end{cases} \tag{IV.53}$$

for all $i \geq 0$. For $i > 0$ we already have the recurrence by Lemma 2.3. Clearly, the constant term of $\mathfrak{C}_w$ is 0 if $w \neq 1$ and $\mathfrak{C}_1 = 1$.

It remains to prove (IV.53) for $i = 0$. By Lemma 2.8,

$$\partial_0 \mathfrak{C}_w = \frac{1}{z_1} \tilde{\sum_{\substack{\mathbf{abc} \in R(w) \\ b_k = 0}}} 2^{i(\alpha)} \begin{pmatrix} \mathbf{a} \\ x^\alpha \end{pmatrix} \begin{bmatrix} b_1 b_2 \ldots b_{k-1} \\ z_1^{k-1} \end{bmatrix} \begin{pmatrix} b_k \\ z_1 \end{pmatrix} \begin{bmatrix} \mathbf{c} \\ z_2^{\gamma_2} z_3^{\gamma_3} \ldots \end{bmatrix}. \tag{IV.54}$$

The admissibility of the monomial $z_2^{\gamma_2} z_3^{\gamma_3} \cdots$ implies each letter $c_i > 1$, hence $\sigma_0 \sigma_{c_1} \cdots \sigma_{c_m} = \sigma_{c_1} \cdots \sigma_{c_m} \sigma_0$. Hence, (IV.54) is equal to

$$\frac{1}{z_1} \tilde{\sum_{\mathbf{abc}0 \in R(w)}} 2^{i(\alpha)} \begin{pmatrix} \mathbf{a} \\ x^\alpha \end{pmatrix} \begin{bmatrix} b_1 b_2 \ldots b_{k-1} \\ z_1^{k-1} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ z_2^{\gamma_2} z_3^{\gamma_3} \ldots \end{bmatrix} \begin{pmatrix} 0 \\ z_1 \end{pmatrix}. \tag{IV.55}$$

If $l(w\sigma_0) > l(w)$ the summation is empty, while if $l(w\sigma_0) < l(w)$ it becomes

$$\text{(IV.56)} \qquad \tilde{\sum_{\mathbf{abc}\in R(w\sigma_0)}} 2^{i(\alpha)} \binom{\mathbf{a}}{x^\alpha} \begin{bmatrix} b_1 b_2 \ldots b_{k-1} \\ z_1^{k-1} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ z_2^{\gamma_2} z_3^{\gamma_3} \ldots \end{bmatrix},$$

which is $\mathfrak{C}_{w\sigma_0}$. $\square$

The Schubert polynomials of type B are defined by $\mathfrak{B}_w = 2^{-s(w)}\mathfrak{C}_w$. Since every reduced word for $w \in B_\infty$ contains $s(w)$ 0's, it is easy to see that the polynomials $\mathfrak{B}_w$ satisfy (IV.53) with ∂_0^B in place of ∂_0.

3. Proof of the formulas for type D_n

We turn now to type D Schubert polynomials. Reduced words for elements of D_∞ use the alphabet $\{\hat{1}, 1, 2, \ldots\}$. Our notation $\left[{\mathbf{c} \atop z^\gamma}\right]$ does not allow $c_i = \hat{1}$. Let us introduce a second notation $\left[{\mathbf{c} \atop z^\gamma}\right]^\wedge$ which allows $c_i = \hat{1}$ but not $c_i = 1$, and requires $z^\gamma \in \mathcal{A}_z(\hat{\mathbf{c}})$ where $\hat{\mathbf{c}}$ is the word $\mathbf{c}$ with 1's and $\hat{1}$'s interchanged. Note that $\left[{\mathbf{c} \atop z^\gamma}\right]^\wedge$ is a reduced word admissible monomial pair if and only if $\left[{\hat{\mathbf{c}} \atop z^\gamma}\right]$ is.

Lemma 3.1. *Let $\tilde{\mathfrak{D}}_w$ denote the polynomial defined by* (IV.25) *and* (IV.26). *For all $w \in D_\infty$,*

$$\text{(IV.57)} \qquad \sigma_0 \tilde{\mathfrak{D}}_w = \tilde{\sum_{\mathbf{abc}\in R(w)}} 2^{i(\alpha)-o(\mathbf{a})} \binom{\mathbf{a}}{x^\alpha} \begin{bmatrix} b_1 \ldots b_k \\ z_1^k \end{bmatrix}^\wedge \begin{bmatrix} \mathbf{c} \\ z_2^{\gamma_2} z_3^{\gamma_3} \ldots \end{bmatrix}.$$

Proof. We have

$$\text{(IV.58)} \qquad \sigma_0 \tilde{\mathfrak{D}}_w = \sum_{\substack{uv=w \\ l(u)+l(v)=l(w) \\ v\in S_\infty}} \tilde{\sum_{\mathbf{c}\in R(v)}} \sigma_0\left(E_u(X) \begin{bmatrix} \mathbf{c} \\ z^\gamma \end{bmatrix}\right)$$

$$\text{(IV.59)} \qquad = \sum_{\substack{uv=w \\ l(u)+l(v)=l(w) \\ v\in S_\infty}} \tilde{\sum_{\mathbf{c}\in R(v)}} (-1)^{\gamma_1} E_u(z_1 + X) \begin{bmatrix} \mathbf{c} \\ z^\gamma \end{bmatrix}$$

by Lemma 2.4. Expanding $E_u(z_1+X)$ in monomials by Proposition 2.10 we get

(IV.60)

$$\sigma_0\tilde{\mathfrak{D}}_w = \tilde{\sum_{\mathbf{abc}\in R(w)}} (-1)^{\gamma_1} 2^{i(\alpha)-o(\mathbf{a})+\chi_k-o(\mathbf{b})} \binom{\mathbf{a}}{x^\alpha}\binom{\mathbf{b}}{z_1^k}\begin{bmatrix}\mathbf{c}\\ z^\gamma\end{bmatrix}$$

$$= \tilde{\sum_{\mathbf{abc'}\in R(w)}} (-1)^{\gamma_1} 2^{i(\alpha)-o(\mathbf{a})+\chi_k-o(b_1\dots b_k)} \binom{\mathbf{a}}{x^\alpha}\binom{b_1\dots b_k}{z_1^k}\begin{bmatrix}b_{k+1}\dots b_m\\ z_1^{\gamma_1}\end{bmatrix}\begin{bmatrix}\mathbf{c}'\\ z_2^{\gamma_2}z_3^{\gamma_3}\dots\end{bmatrix},$$

where $m=\gamma_1+k$ and $\chi_k=1$ if $k>0$ and 0 otherwise.

We need to determine the coefficient C_A of the general term of (IV.60)

$$A = 2^{i(\alpha)-o(\mathbf{a})}\binom{\mathbf{a}}{x^\alpha}\left\langle\begin{matrix}b_1\dots b_m\\ z_1^m\end{matrix}\right\rangle\begin{bmatrix}c\\ z_2^{\gamma_2}z_3^{\gamma_3}\dots\end{bmatrix}. \tag{IV.61}$$

There is a contribution of $2^{\chi_k-o(b_1\dots b_k)}(-1)^{m-k}$ to C_A for each k such that

$$\left\langle\begin{matrix}b_1\dots b_m\\ z_1^m\end{matrix}\right\rangle = \binom{b_1\dots b_k}{z_1^k}\begin{bmatrix}b_{k+1}\dots b_m\\ z_1^{k+1}\end{bmatrix}, \tag{IV.62}$$

i.e., such that $b_1\dots b_k$ is a vee, $b_{k+1}>\dots>b_m$, and $b_m\neq\hat{1}$ unless $k=m$.

Case 1: $b_m\neq 1$ *or* $\hat{1}$

First, assume there exists an index i such that $b_i<b_{i+1}$ and choose i to be as large as possible. Then there are two possibilities for k in (IV.62), namely $k=i$ and $k=i+1$. Therefore, $C_A = 2^{1-o(b_1\dots b_i)}\big[(-1)^{m-i}+(-1)^{m-i-1}\big]=0$.

Otherwise, $b_1>b_2>\dots>b_m>1$. For each $0\le k\le m$, there is a contribution to C_A; $k=0$ contributes $(-1)^m$ and each $0<k\le m$ contributes $2(-1)^{m-k}$. Thus, $C_A=(-1)^m+\sum_{k=1}^m 2(-1)^{m-k}=1$.

Case 2: $b_{m-1}b_m=\hat{1}1$ *or* $1\hat{1}$

These terms come in pairs since σ_1 and $\sigma_{\hat{1}}$ commute. For $b_{m-1}b_m=\hat{1}1$, there are two possibilities, $k=m-1$ and $k=m$, giving $C_A=-\frac{1}{2}$. For $b_{m-1}b_m=1\hat{1}$ we must have $k=m$, giving $C_A=\frac{1}{2}$. Both terms have the same underlying monomial so their net contribution to $\sigma_0\tilde{\mathfrak{D}}_w$ is zero.

Case 3: $b_m=1$ *and* $b_{m-1}\neq\hat{1}$

If $b_1\dots b_{m-1}$ has an ascent, say $b_i<b_{i+1}$, then $C_A=0$ as in Case 1. Otherwise, if $b_1>b_2>\dots>b_{m-1}>1$ then for $k=0$, $k=m$, and $0<k<m$ there are

contributions of $(-1)^m, 1$ and $2(-1)^{m-k}$ respectively. Hence, $C_A = (-1)^m + 1 + \sum_{k=1}^{m-1} 2(-1)^{m-k} = 0$.

Case 4: $b_m = \hat{1}$ *and* $b_{m-1} \neq 1$

For this case, we must have $k = m$. We must also have $b_1 > b_2 > \cdots > b_{m-1}$ since $\mathbf{b} = b_1 \ldots b_m$ must be a vee and $b_m = \hat{1}$ is its least element. Therefore $o(\mathbf{b}) = 1$ and $C_A = 1$.

Summarizing, there is a coefficient $C_A = 1$ for each A with $b_1 > \cdots > b_m$ and $b_m \neq 1$, and there is a net contribution of zero from all other terms. The terms with $C_A = 1$ are precisely those of the form

$$A = 2^{i(\alpha)-o(\mathbf{a})} \begin{pmatrix} \mathbf{a} \\ x^\alpha \end{pmatrix} \begin{bmatrix} b_1 \ldots b_m \\ z_1^m \end{bmatrix}^\wedge \begin{bmatrix} c \\ z_2^{\gamma_2} z_3^{\gamma_3} \ldots \end{bmatrix}, \tag{IV.63}$$

proving (IV.57). □

Corollary 3.2. *For all* $w \in D_\infty$,

$$\sigma_0 \tilde{\mathfrak{D}}_w = \tilde{\mathfrak{D}}_{\widehat{w}}, \tag{IV.64}$$

where $\widehat{w}$ *is the image of* w *under the involution of* D_∞ *given by interchanging* σ_1 *and* $\sigma_{\hat{1}}$.

Proof. By Lemma 3.1,

$$\sigma_0 \tilde{\mathfrak{D}}_w = \sum_{\mathbf{abc} \in R(w)}^{\sim} 2^{i(\alpha)-o(\mathbf{a})} \begin{pmatrix} \mathbf{a} \\ x^\alpha \end{pmatrix} \begin{bmatrix} \mathbf{b} \\ z_1^k \end{bmatrix}^\wedge \begin{bmatrix} \mathbf{c} \\ z_2^{\gamma_2} z_{\gamma_3}^3 \ldots \end{bmatrix} \tag{IV.65}$$

$$= \sum_{\widehat{\mathbf{abc}} \in R(\widehat{w})}^{\sim} 2^{i(\alpha)-o(\widehat{\mathbf{a}})} \begin{pmatrix} \widehat{\mathbf{a}} \\ x^\alpha \end{pmatrix} \begin{bmatrix} \widehat{\mathbf{b}} \\ z_1^k \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{c}} \\ z_2^{\gamma_2} z_{\gamma_3}^3 \ldots \end{bmatrix} \tag{IV.66}$$

$$= \tilde{\mathfrak{D}}_{\widehat{w}}. \quad \square \tag{IV.67}$$

Proof of Theorem 4A. Formulas (IV.25) and (IV.26) are equivalent by Proposition 2.10 and Proposition 2.1. To prove they give Schubert polynomials of type D, we take them for the moment as the definition of $\mathfrak{D}_w$ and show that $\mathfrak{D}_w$

then satisfies the recurrence

$$\partial_i \mathfrak{D}_w = \begin{cases} \mathfrak{D}_{w\sigma_i} & l(w\sigma_i) < l(w) \\ 0 & l(w\sigma_i) > l(w) \end{cases} \tag{IV.68}$$

for all $i \in \{\hat{1}, 1, 2, \ldots\}$. For $i \neq \hat{1}$ we already have the recurrence by Lemma 2.3. The constant term of $\mathfrak{D}_w$ is 0 if $w \neq 1$ and $\mathfrak{D}_1 = 1$.

It remains to prove (IV.68) for $i = \hat{1}$. We shall take advantage of the symmetry between the generators σ_1 and $\sigma_{\hat{1}}$. A simple computation shows

$$\sigma_0 \partial_1 \sigma_0 f = \partial_{\hat{1}} f = \frac{f - \sigma_{\hat{1}} f}{-z_1 - z_2}. \tag{IV.69}$$

Therefore, by repeated use of Corollary 3.2 and Lemma 2.3,

$$\begin{aligned} \partial_{\hat{1}} \mathfrak{D}_w &= \sigma_0 \partial_1 \sigma_0 \mathfrak{D}_w \\ &= \sigma_0 \partial_1 \mathfrak{D}_{\widehat{w}} \\ &= \sigma_0 \begin{cases} \mathfrak{D}_{\widehat{w}\sigma_1} & l(\widehat{w}\sigma_1) < l(\widehat{w}) \\ 0 & l(\widehat{w}\sigma_1) > l(\widehat{w}) \end{cases} \\ &= \begin{cases} \mathfrak{D}_{w\sigma_{\hat{1}}} & l(w\sigma_{\hat{1}}) < l(w) \\ 0 & l(w\sigma_{\hat{1}}) > l(w). \quad \square \end{cases} \end{aligned} \tag{IV.70}$$

THEOREM 3B. *Given a partition μ with distinct parts, let $w = \overline{\mu_1}\,\overline{\mu_2} \ldots \overline{\mu_l} 12 \ldots$. Then we have*

$$\mathfrak{C}_w = Q_\mu(X), \quad \mathfrak{B}_w = P_\mu(X). \tag{IV.71}$$

PROOF. Given $w = \overline{\mu_1}\,\overline{\mu_2} \ldots \overline{\mu_l} 12 \ldots$, the only element $v \in S_\infty$ such that $uv = w$ and $l(u) + l(v) = l(w)$ is $v = 1$. Therefore $\mathfrak{C}_w = F_w$, and by Proposition 2.14 $F_w = Q_\mu$. By definition $\mathfrak{B}_w = 2^{-s(w)} \mathfrak{C}_w$ where $s(w)$ is the number of signs changed by w. Hence, $\mathfrak{B}_w = 2^{-l} Q_\mu = P_\mu$. $\square$

THEOREM 4B. *Given a partition μ with distinct parts, let $\nu_i = 1 + \mu_i$, taking $\mu_l = 0$ if necessary to make the number of parts even. Then for $w = \overline{\nu_1}\,\overline{\nu_2} \ldots \overline{\nu_l} 12 \ldots$, we have*

$$\mathfrak{D}_w = P_\mu(X). \tag{IV.72}$$

PROOF. As in the previous theorem, $\mathfrak{D}_w = E_w$, and $E_w = P_\mu$ by Proposition 2.13. □

Next we show that the polynomials $\mathfrak{B}_w$, $\mathfrak{C}_w$ and $\mathfrak{D}_w$ are integral bases of the rings in which they lie. We do this by identifying their leading terms with respect to an appropriate ordering.

DEFINITION. Given two shifted shapes λ and μ and two compositions $\alpha = (\alpha_1, \alpha_2, \dots)$ and $\beta = (\beta_1, \beta_2, \dots)$ with $m = \sum \alpha_i$ and $n = \sum \beta_j$, we say $z^\alpha Q_\lambda < z^\beta Q_\mu$ if any of the following hold:

(1) $m < n$.
(2) $m = n$ and $\alpha <_r \beta$ in reverse lexicographic order.
(3) $\alpha = \beta$ and $\lambda < \mu$ in an arbitrarily chosen total ordering.

Reverse lexicographic order means $\alpha <_r \beta$ if $\alpha_i < \beta_i$ for some i and $\alpha_l = \beta_l$ for all $l > i$.

DEFINITION. [**13**] Given $w \in S_n$, for each $i \geq 1$ let $c_i(w) = |\{j \mid j > i \text{ and } w(j) < w(i)\}|$. The composition

$$c(w) = (c_1(w), c_2(w), \dots, c_n(w)) \tag{IV.73}$$

is the *code* of w.

LEMMA 3.3. *Under the ordering $<$, the leading term of $\mathfrak{S}_w$ is distinct for each $w \in S_\infty$ and is given by $z^{c(w)}$.*

PROOF. The lemma follows by induction from the transition equation for Schubert polynomials of type A, formula (4.16) of [**13**]. □

LEMMA 3.4. *For every monomial $z^\alpha Q_\mu$ there is a unique $w \in B_\infty$ such that $z^\alpha Q_\mu$ is the leading term of $\mathfrak{C}_w$ under the ordering $<$ defined above. For this same w, $z^\alpha P_\mu$ is the leading term of $\mathfrak{B}_w$.*

PROOF. Let $w = w(1)w(2)\dots w(n)$ in one line notation. Let u_w be the increasing arrangement of the numerals $w(1), w(2), \dots w(n)$, and let $v_w = u_w^{-1} w$. Then $l(u_w) + l(v_w) = l(w)$ and $l(v_w) > l(v)$ for any other $v \in S_\infty$ such that $uv = w$ and $l(u) +$

$l(v) = l(w)$. Therefore, the leading term of $\mathfrak{C}_w$ comes from the expansion of $F_{u_w}\mathfrak{S}_{v_w}$, by Theorem 3A.

Let μ_w be the shape such that $\mathfrak{C}_{u_w} = F_{u_w} = Q_{\mu_w}$. This shape μ_w exists by Theorem 3B. By Lemma 3.3, $z^{c(v_w)}$ is the leading term of $\mathfrak{S}_{v_w}$. Therefore, $z^{c(v_w)}Q_{\mu_w}$ is the leading term of $\mathfrak{C}_w$.

Given any $z^\alpha Q_\mu$, let $v \in S_\infty$ be the unique permutation such that $c(v) = \alpha$. Define $u \in B_\infty$ by $u = \overline{\mu_1}\ \overline{\mu_2}\dots\overline{\mu_l}12\dots$. Then for $w = uv \in B_n$, $u_w = u$ and $v_w = v$, so $\mathfrak{C}_w$ has $z^\alpha Q_\mu$ as its leading term. This w is unique since μ determines u_w and α determines v_w.

From the description of u we see that $s(w) = l(\mu)$, so the leading term of $\mathfrak{B}_w = 2^{-s(w)}\mathfrak{C}_{\mathfrak{w}}$ is $z^\alpha P_\mu$. □

Lemma 3.5. *For every monomial $z^\alpha Q_\mu$ there is a unique $w \in D_\infty$ such that $z^\alpha Q_\mu$ is the leading term of $\mathfrak{D}_w$ under the ordering $>$ defined above.*

Proof. The only difference between this proof and the previous one is the computation of the leading term. Given $w = w(1)\dots w(n) \in D_n$, again let u_w be the increasing rearrangement of the $w(i)$, so $u_w = \overline{\nu_1}\ \overline{\nu_2}\dots\overline{\nu_l}12\dots$ for some partition ν. Let $v_w = u_w^{-1}w$. The leading term of $\mathfrak{D}_w$ is $z^{c(v_w)}P_\mu$ where $\mu = (\nu_1 - 1, \nu_2 - 1, \dots, \nu_l - 1)$. □

Lemma 3.6. *The Schubert polynomials $\mathfrak{B}_w$ lie in the ring $\mathbb{Z}[z_1, z_2, \dots; P_\mu]$.*

Proof. Consider a general term $\binom{\mathbf{a}}{x^\alpha}\left[{\mathbf{b} \atop z^\gamma}\right]$ occurring in $\mathfrak{C}_w$, where $\mathbf{ab} \in R(w)$. There are $s(w)$ 0's in $\mathbf{a}$, with at least one peak between each consecutive pair of them. This forces x^α to contain at least $s(w)$ distinct variables with non-zero exponent.

Every term $z^\alpha Q_\mu$ occurring in $\mathfrak{C}_w$ has positive coefficient, so no monomials cancel among terms. In particular, $Q_\mu(X)$ cannot contain any monomial involving fewer than $s(w)$ distinct variables. This forces $l(\mu) \geq s(w)$, and hence the corresponding term in $\mathfrak{B}_w = 2^{-s(w)}\mathfrak{C}_w$ is an integral multiple of $z^\alpha P_\mu$. □

Theorem 3C. *The Schubert polynomials $\mathfrak{C}_w$ of type C are a $\mathbb{Z}$-basis for the ring $\mathbb{Z}[z_1, z_2, \dots; Q_\mu]$. The polynomials $\mathfrak{B}_w$ are a $\mathbb{Z}$-basis for the ring $\mathbb{Z}[z_1, z_2, \dots; P_\mu]$.*

PROOF. By Proposition 2.1, the sets $\{z^\alpha Q_\mu\}$ and $\{z^\alpha P_\mu\}$ are $\mathbb{Z}$-bases for the rings $\mathbb{Z}[z_1, z_2, \ldots ; Q_\mu]$ and $\mathbb{Z}[z_1, z_2, \ldots ; P_\mu]$, respectively.

Since the $\mathfrak{C}_w$ have distinct leading terms, they are linearly independent. They span the ring $\mathbb{Z}[z_1, z_2, \ldots ; Q_\mu]$ since every monomial $z^\alpha Q_\mu$ occurs as the leading term of some $\mathfrak{C}_w$. Analogous remarks apply to the $\mathfrak{B}_w$. □

THEOREM 4C. *The Schubert polynomials of type D are a $\mathbb{Z}$-basis for the ring* $\mathbb{Z}[z_1, z_2, \ldots ; P_\mu]$.

PROOF. Same as the preceding proof. □

The formulas we have given for Schubert polynomials of types B, C, and D, though fully explicit, are ill-suited to practical computation because of the difficulty of using the Edelman–Greene correspondences to evaluate $F_u(X)$ and $E_u(X)$. An alternative method is to compute Schubert polynomials by applying iterated divided difference operators to the 'top' polynomials $\mathfrak{C}_{w_0^B}$ and $\mathfrak{D}_{w_0^D}$. This method is facilitated by the use of Corollary 2.5, together with convenient expressions for $\mathfrak{C}_{w_0^B}$ and $\mathfrak{D}_{w_0^D}$ which we now derive.

DEFINITION. Let $\lambda \subseteq \mu$ be partitions of length at most k. The corresponding *skew multi-Schur function* is defined by

(IV.74)
$$S_{\mu/\lambda}(z_1, z_1 + z_2, \ldots, z_1 + z_2 + \cdots + z_k) = \det\left[h_{\mu_i - \lambda_j + j - i}(z_1, z_2 \ldots, z_i)\right]_{i,j=1}^{k}$$
(IV.75)
$$= \sum_T z^T,$$

where h_m denotes the complete homogeneous symmetric function of degree m, and T ranges over column-strict tableaux of shape μ/λ in which entries in row i do not exceed i.

The equivalence of formulas (IV.74) and (IV.75) is due to Gessel [**8**]—see also [**19**], since [**8**] is unpublished.

Proposition 3.7. *Let* v_0, w_0^B, w_0^D *denote the longest element in* S_n, B_n*, and* D_n *respectively. Let* $\delta_k = (k, k-1, \ldots, 1)$*. Then*

(IV.76)
$$\mathfrak{C}_{w_0^B} = \sum_\lambda Q_{\delta_n+\lambda}(X) S_{\delta_{n-1}/\lambda'}(z_1, z_1+z_2, \ldots, z_1+z_2+\cdots+z_{n-1}),$$

(IV.77)
$$\mathfrak{D}_{w_0^D} = \sum_\lambda P_{\delta_{n-1}+\lambda}(X) S_{\delta_{n-1}/\lambda'}(z_1, z_1+z_2, \ldots, z_1+z_2+\cdots+z_{n-1}).$$

Here λ' *denotes the conjugate partition to* λ*.*

Proof. For every $v \in S_n$, we have $l(w_0^B v^{-1}) + l(v) = l(w_0^B)$ and $l(w_0^D v^{-1}) + l(v) = l(w_0^D)$. Hence by Proposition 2.16,

(IV.78)
$$\mathfrak{C}_{w_0^B} = \sum_{v \in S_n} F_{w_0^B v^{-1}}(X) \mathfrak{S}_v(Z) = \sum_{v \in S_n} \sum_\lambda g^\lambda_{v_0 v^{-1}} Q_{\delta_n+\lambda}(X) \mathfrak{S}_v(Z),$$

(IV.79)
$$\mathfrak{D}_{w_0^D} = \sum_{v \in S_n} E_{w_0^D v^{-1}}(X) \mathfrak{S}_v(Z) = \sum_{v \in S_n} \sum_\lambda g^\lambda_{v_0 v^{-1}} P_{\delta_{n-1}+\lambda}(X) \mathfrak{S}_v(Z).$$

It remains to prove, for each λ,

(IV.80)
$$\sum_{v \in S_n} g^\lambda_{v_0 v^{-1}} \mathfrak{S}_v(Z) = S_{\delta_{n-1}/\lambda'}(z_1, z_1+z_2, \ldots, z_1+z_2+\cdots+z_{n-1}).$$

Equations (4.9) and (7.14) of [**13**] show that

(IV.81)
$$S_{\delta_{n-1}}(Y+z_1, Y+z_1+z_2, \ldots, Y+z_1+z_2+\cdots+z_{n-1})$$

(IV.82)
$$= \mathfrak{S}_{1^m \times v_0}(y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_{n-1})$$

(IV.83)
$$= \sum_{v \in S_n} \sum_\lambda g^\lambda_{v v_0} s_\lambda(Y) \mathfrak{S}_v(Z),$$

where $Y = y_1 + y_2 + \cdots + y_m$. Using the identity $g^\lambda_w = g^{\lambda'}_{w^{-1}}$ of [**5**] and replacing λ by its conjugate in the summation, the last expression becomes

(IV.84)
$$\sum_{v \in S_n} \sum_\lambda g^\lambda_{v_0 v^{-1}} s_{\lambda'}(Y) \mathfrak{S}_v(Z).$$

We also have by a general identity for skew multi-Schur functions

$$\begin{aligned} &S_{\delta_{n-1}}(Y+z_1, Y+z_1+z_2, \ldots, Y+z_1+z_2+\cdots+z_{n-1}) \\ &\qquad = \sum_{\lambda} s_\lambda(Y) S_{\delta_{n-1}/\lambda}(z_1, z_1+z_2, \ldots, z_1+z_2+\cdots z_{n-1}). \end{aligned} \tag{IV.85}$$

Equating coefficients of $s_{\lambda'}(Y)$ in (IV.84) and (IV.85) gives (IV.80). □

The text of Section 2 of this chapter is a reprint of material as it appears in *Schubert polynomials for the classical groups* to appear in the Journal of the AMS, co-authored with Mark Haiman. I was primary author of Chapter 4, Section 2 and each author contributed to the research.

CHAPTER V

Open Problems

As we know the Schubert polynomials form an integral basis for $\mathbb{Z}[x_1, x_2, \ldots]$. One of the long standing open problems in the theory of Schubert polynomials is to find a combinatorial proof of the following theorem.

THEOREM 7. *In the product expansions*

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_w c_{uv}^w \mathfrak{S}_w \tag{V.1}$$

and like expansions for types B, C, and D, the coefficients c_{uv}^w are non-negative.

These coefficients are the analogs of the Littlewood-Richardson coefficients in the theory of Schur functions. Monk's rule is a special case of this problem.

Our investigations led to 2 conjectures for multiplying Schubert polynomials in special cases. Let $r[b,d] = [1, 2, \ldots, b-1, b+d, b, b+1, \ldots]$. Note that $D_{bot}(r[b,d])$ is a single row. The permutation $r[b,d]$ is a special type called Grassmannian, and the Schubert polynomial $\mathfrak{S}_{r[b,d]} = h_d(x_1, x_2, \ldots, x_b)$, the homogeneous symmetric function of degree d.

CONJECTURE 1. *Given any $w \in S_\infty$ and any $r[b,d]$*

$$\mathfrak{S}_w \mathfrak{S}_{r[b,d]} = \sum \mathfrak{S}_{w'}, \tag{V.2}$$

where the sum runs over all $w' = w t_{k_1 l_1} t_{k_2 l_2} \cdots t_{k_d l_d}$ such that $k_i \leq b < l_i$ for $1 \leq i \leq d$, and if we let $w^{(i)} = w^{(i-1)} t_{k_i l_i}$ with $w^{(0)} = w$, then $\ell(w^{(i)}) = \ell(w^{(i-1)}) + 1$ and $w_{k_1}^{(1)} < w_{k_2}^{(2)} < \ldots < w_{k_d}^{(d)}$.

REMARK 0.8. *It is remarkable that this multiplication is multiplicity free!*

Let $c[b,d] = [1,2,\ldots,b-d,b-d+2,\ldots,b+1,b-d+1,b+2,b+3,\ldots]$. Note that the diagram of the permutation $D(c[b,d])$ is a single column. The permutation $c[b,d]$ is a also Grassmannian, and the Schubert polynomial $\mathfrak{S}_{r[b,d]} = e_d(x_1,x_2,\ldots,x_b)$, the elementary symmetric function.

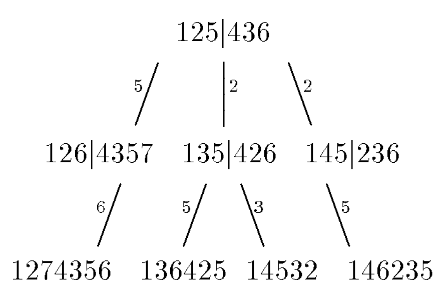
CONJECTURE 2. *Given any $w \in S_\infty$ and any $c[b,d]$*

$$\mathfrak{S}_w \mathfrak{S}_{c[b,d]} = \sum \mathfrak{S}_{w'}, \tag{V.3}$$

where the sum runs over all $w' = wt_{k_1l_1}t_{k_2l_2}\cdots t_{k_dl_d}$ such that $k_i \le b < l_i$ for $1 \le i \le d$, and if we let $w^{(i)} = w^{(i-1)}t_{k_il_i}$ with $w^{(0)} = w$, then $\ell(w^{(i)}) = \ell(w^{(i-1)}) + 1$ and $w^{(1)}_{k_1} > w^{(2)}_{k_2} > \ldots > w^{(d)}_{k_d} > 0$.

The conjectures have both been computer verified for all permutations $w, r[b,d]$, and $c[b,d]$ in S_7. We have found computations in S_8 to be beyond the capacity of our current technology, Sparc 10. These conjectures would greatly speed up any algorithm for expanding products (V.2) and (V.3).

To see the efficiency of this rule, let us give an example. Suppose we want to expand $\mathfrak{S}_{r[3,2]}\mathfrak{S}_{[1,2,5,4,3]}$ in the basis of Schubert polynomials. Let $b = 3$, $d = 2$, and construct a rooted tree as follows:



The top of the tree is the initial permutation. We assume there are an infinite number of fixed points beyond what is written. We have inserted a vertical line after the position $b = 3$. To find the children of the root, we find all transpositions that switch numbers across the vertical line so that the lengths increase by exactly one. We label the edge from the root to a child by the smallest of the two numbers switched. Of course the smallest number will always come from the left. This constructs the first generation of the tree. For the next generation, repeat the process above but only allowing the transpositions for which the smallest number is bigger than the label on the edge of this node. Repeat the last step $d = 2$ times. The leaves of the tree are precisely the permutations w' which appear in the expansion in (V.2).

The text in this chapter pertaining to the conjectures appears in *RC-graphs and Schubert polynomials* to appear in Experimental Mathematics, co-authored with Nantel Bergeron.

CHAPTER VI

Tables

w	$\mathfrak{S}_w$
1 2 3 4=1	1
1 2 4 3=σ_3	$z_3 + z_2 + z_1$
1 3 2 4=σ_2	$z_2 + z_1$
1 3 4 2=$\sigma_2\sigma_3$	$z_2z_3 + z_1z_3 + z_1z_2$
1 4 2 3=$\sigma_3\sigma_2$	$z_2^2 + z_1z_2 + z_1^2$
1 4 3 2=$\sigma_3\sigma_2\sigma_3$	$z_2^2z_3 + z_1z_2z_3 + z_1^2z_3 + z_1z_2^2 + z_1^2z_2$
2 1 3 4=σ_1	z_1
2 1 4 3=$\sigma_1\sigma_3$	$z_1z_3 + z_1^2 + z_1z_2$
2 3 1 4=$\sigma_1\sigma_2$	z_1z_2
2 3 4 1=$\sigma_1\sigma_2\sigma_3$	$z_1z_2z_3$
2 4 1 3=$\sigma_1\sigma_3\sigma_2$	$z_1z_2^2 + z_1^2z_2$
2 4 3 1=$\sigma_1\sigma_3\sigma_2\sigma_3$	$z_1z_2^2z_3 + z_1^2z_2z_3$
3 1 2 4=$\sigma_2\sigma_1$	z_1^2
3 1 4 2=$\sigma_2\sigma_1\sigma_3$	$z_1^2z_3 + z_1^2z_2$
3 2 1 4=$\sigma_2\sigma_1\sigma_2$	$z_1^2z_2$
3 2 4 1=$\sigma_2\sigma_1\sigma_2\sigma_3$	$z_1^2z_2z_3$
3 4 1 2=$\sigma_2\sigma_1\sigma_3\sigma_2$	$z_1^2z_2^2$
3 4 2 1=$\sigma_2\sigma_1\sigma_3\sigma_2\sigma_3$	$z_1^2z_2^2z_3$
4 1 2 3=$\sigma_3\sigma_2\sigma_1$	z_1^3
4 1 3 2=$\sigma_3\sigma_2\sigma_1\sigma_3$	$z_1^3z_3 + z_1^3z_2$
4 2 1 3=$\sigma_3\sigma_2\sigma_1\sigma_2$	$z_1^3z_2$
4 2 3 1=$\sigma_3\sigma_2\sigma_1\sigma_2\sigma_3$	$z_1^3z_2z_3$
4 3 1 2=$\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2$	$z_1^3z_2^2$
4 3 2 1=$\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2\sigma_3$	$z_1^3z_2^2z_3$

TABLE VI.1. Type A Schubert polynomials for $w \in S_4$

w	$\mathfrak{B}_w$
$1\,2\,3=1$	1
$\overline{1}\,2\,3=\sigma_0$	P_1
$2\,1\,3=\sigma_1$	$2\,P_1 + z_1$
$\overline{2}\,1\,3=\sigma_1\sigma_0$	P_2
$2\,\overline{1}\,3=\sigma_0\sigma_1$	$P_2 + P_1\,z_1$
$\overline{2}\,\overline{1}\,3=\sigma_0\sigma_1\sigma_0$	P_{21}
$1\,\overline{2}\,3=\sigma_1\sigma_0\sigma_1$	$P_3 + P_2\,z_1$
$\overline{1}\,\overline{2}\,3=\sigma_1\sigma_0\sigma_1\sigma_0$	$P_{31} + P_{21}\,z_1$
$1\,3\,2=\sigma_2$	$2\,P_1 + z_1 + z_2$
$\overline{1}\,3\,2=\sigma_2\sigma_0$	$2\,P_2 + P_1\,z_1 + P_1\,z_2$
$3\,1\,2=\sigma_2\sigma_1$	$2\,P_2 + 2\,P_1\,z_1 + {z_1}^2$
$\overline{3}\,1\,2=\sigma_2\sigma_1\sigma_0$	P_3
$3\,\overline{1}\,2=\sigma_2\sigma_0\sigma_1$	$P_3 + 2\,P_{21} + 2\,P_2\,z_1 + P_1\,{z_1}^2$
$\overline{3}\,\overline{1}\,2=\sigma_2\sigma_0\sigma_1\sigma_0$	P_{31}
$1\,\overline{3}\,2=\sigma_2\sigma_1\sigma_0\sigma_1$	$P_4 + P_3\,z_1$
$\overline{1}\,\overline{3}\,2=\sigma_2\sigma_1\sigma_0\sigma_1\sigma_0$	$P_{41} + P_{31}\,z_1$
$2\,3\,1=\sigma_1\sigma_2$	$2\,P_2 + 2\,P_1\,z_1 + 2\,P_1\,z_2 + z_1\,z_2$
$\overline{2}\,3\,1=\sigma_1\sigma_2\sigma_0$	$P_3 + 2\,P_{21} + P_2\,z_1 + P_2\,z_2$
$3\,2\,1=\sigma_1\sigma_2\sigma_1$	$2\,P_3 + 4\,P_{21} + 4\,P_2\,z_1 + 2\,P_1\,{z_1}^2 + 2\,P_2\,z_2 + 2\,P_1\,z_1\,z_2 + {z_1}^2\,z_2$
$\overline{3}\,2\,1=\sigma_1\sigma_2\sigma_1\sigma_0$	$P_4 + 2\,P_{31} + P_3\,z_1 + P_3\,z_2$
$3\,\overline{2}\,1=\sigma_1\sigma_2\sigma_0\sigma_1$	$2\,P_{31} + P_3\,z_1 + 2\,P_{21}\,z_1 + P_2\,{z_1}^2$
$\overline{3}\,\overline{2}\,1=\sigma_1\sigma_2\sigma_0\sigma_1\sigma_0$	P_{32}
$2\,\overline{3}\,1=\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1$	$2\,P_{41} + P_4\,z_1 + 2\,P_{31}\,z_1 + P_3\,{z_1}^2$
$\overline{2}\,\overline{3}\,1=\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1\sigma_0$	$P_{42} + P_{32}\,z_1$
$2\,3\,\overline{1}=\sigma_0\sigma_1\sigma_2$	$P_3 + P_2\,z_1 + P_2\,z_2 + P_1\,z_1\,z_2$
$\overline{2}\,3\,\overline{1}=\sigma_0\sigma_1\sigma_2\sigma_0$	$P_{31} + P_{21}\,z_1 + P_{21}\,z_2$
$3\,2\,\overline{1}=\sigma_0\sigma_1\sigma_2\sigma_1$	$P_4 + 2\,P_{31} + 2\,P_3\,z_1 + 2\,P_{21}\,z_1 + P_2\,{z_1}^2 + P_3\,z_2 + 2\,P_{21}\,z_2 + 2\,P_2\,z_1\,z_2 + P_1\,{z_1}^2\,z_2$
$\overline{3}\,2\,\overline{1}=\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0$	$P_{32} + P_{41} + P_{31}\,z_1 + P_{31}\,z_2$
$3\,\overline{2}\,\overline{1}=\sigma_0\sigma_1\sigma_2\sigma_0\sigma_1$	$P_{32} + P_{31}\,z_1 + P_{21}\,{z_1}^2$
$\overline{3}\,\overline{2}\,\overline{1}=\sigma_0\sigma_1\sigma_2\sigma_0\sigma_1\sigma_0$	P_{321}
$2\,\overline{3}\,\overline{1}=\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1$	$P_{42} + P_{32}\,z_1 + P_{41}\,z_1 + P_{31}\,{z_1}^2$
$\overline{2}\,\overline{3}\,\overline{1}=\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1\sigma_0$	$P_{421} + P_{321}\,z_1$
$1\,3\,\overline{2}=\sigma_1\sigma_0\sigma_1\sigma_2$	$P_4 + P_3\,z_1 + P_3\,z_2 + P_2\,z_1\,z_2$
$\overline{1}\,3\,\overline{2}=\sigma_1\sigma_0\sigma_1\sigma_2\sigma_0$	$P_{41} + P_{31}\,z_1 + P_{31}\,z_2 + P_{21}\,z_1\,z_2$
$3\,1\,\overline{2}=\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1$	$2\,P_{41} + P_4\,z_1 + 2\,P_{31}\,z_1 + P_3\,{z_1}^2 + 2\,P_{31}\,z_2 + P_3\,z_1\,z_2 + 2\,P_{21}\,z_1\,z_2 + P_2\,{z_1}^2\,z_2$
$\overline{3}\,1\,\overline{2}=\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0$	$P_{42} + P_{32}\,z_1 + P_{32}\,z_2$
$3\,\overline{1}\,\overline{2}=\sigma_1\sigma_0\sigma_1\sigma_2\sigma_0\sigma_1$	$P_{42} + P_{32}\,z_1 + P_{41}\,z_1 + P_{31}\,{z_1}^2 + P_{32}\,z_2 + P_{31}\,z_1\,z_2 + P_{21}\,{z_1}^2\,z_2$
$\overline{3}\,\overline{1}\,\overline{2}=\sigma_1\sigma_0\sigma_1\sigma_2\sigma_0\sigma_1\sigma_0$	$P_{421} + P_{321}\,z_1 + P_{321}\,z_2$
$1\,\overline{3}\,\overline{2}=\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1$	$P_{43} + P_{42}\,z_1 + P_{32}\,{z_1}^2$
$\overline{1}\,\overline{3}\,\overline{2}=\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1\sigma_0$	$P_{431} + P_{421}\,z_1 + P_{321}\,{z_1}^2$
$1\,2\,\overline{3}=\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2$	$P_5 + P_4\,z_1 + P_4\,z_2 + P_3\,z_1\,z_2$
$\overline{1}\,2\,\overline{3}=\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_0$	$P_{51} + P_{41}\,z_1 + P_{41}\,z_2 + P_{31}\,z_1\,z_2$
$2\,1\,\overline{3}=\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1$	$2\,P_{51} + P_5\,z_1 + 2\,P_{41}\,z_1 + P_4\,{z_1}^2 + 2\,P_{41}\,z_2 + P_4\,z_1\,z_2 + 2\,P_{31}\,z_1\,z_2 + P_3\,{z_1}^2\,z_2$
$\overline{2}\,1\,\overline{3}=\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0$	$P_{52} + P_{42}\,z_1 + P_{42}\,z_2 + P_{32}\,z_1\,z_2$
$2\,\overline{1}\,\overline{3}=\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_0\sigma_1$	$P_{52} + P_{42}\,z_1 + P_{51}\,z_1 + P_{41}\,{z_1}^2 + P_{42}\,z_2 + P_{32}\,z_1\,z_2 + P_{41}\,z_1\,z_2 + P_{31}\,{z_1}^2\,z_2$
$\overline{2}\,\overline{1}\,\overline{3}=\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_0\sigma_1\sigma_0$	$P_{521} + P_{421}\,z_1 + P_{421}\,z_2 + P_{321}\,z_1\,z_2$
$1\,\overline{2}\,\overline{3}=\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1$	$P_{53} + P_{43}\,z_1 + P_{52}\,z_1 + P_{42}\,{z_1}^2 + P_{43}\,z_2 + P_{42}\,z_1\,z_2 + P_{32}\,{z_1}^2\,z_2$
$\overline{1}\,\overline{2}\,\overline{3}=\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1\sigma_0$	$P_{531} + P_{431}\,z_1 + P_{521}\,z_1 + P_{421}\,{z_1}^2 + P_{431}\,z_2 + P_{421}\,z_1\,z_2 + P_{321}\,{z_1}^2\,z_2$

TABLE VI.2. Type B Schubert polynomials for $w \in B_3$

w	$\mathfrak{C}_w$
$1\,2\,3=1$	1
$\bar{1}\,2\,3=\sigma_0$	Q_1
$2\,1\,3=\sigma_1$	$Q_1 + z_1$
$\bar{2}\,1\,3=\sigma_1\sigma_0$	Q_2
$2\,\bar{1}\,3=\sigma_0\sigma_1$	$Q_2 + Q_1\,z_1$
$\bar{2}\,\bar{1}\,3=\sigma_0\sigma_1\sigma_0$	Q_{21}
$1\,\bar{2}\,3=\sigma_1\sigma_0\sigma_1$	$Q_3 + Q_2\,z_1$
$\bar{1}\,\bar{2}\,3=\sigma_1\sigma_0\sigma_1\sigma_0$	$Q_{31} + Q_{21}\,z_1$
$1\,3\,2=\sigma_2$	$Q_1 + z_1 + z_2$
$\bar{1}\,3\,2=\sigma_2\sigma_0$	$2\,Q_2 + Q_1\,z_1 + Q_1\,z_2$
$3\,1\,2=\sigma_2\sigma_1$	$Q_2 + Q_1\,z_1 + {z_1}^2$
$\bar{3}\,1\,2=\sigma_2\sigma_1\sigma_0$	Q_3
$3\,\bar{1}\,2=\sigma_2\sigma_0\sigma_1$	$Q_3 + Q_{21} + 2\,Q_2\,z_1 + Q_1\,{z_1}^2$
$\bar{3}\,\bar{1}\,2=\sigma_2\sigma_0\sigma_1\sigma_0$	Q_{31}
$1\,\bar{3}\,2=\sigma_2\sigma_1\sigma_0\sigma_1$	$Q_4 + Q_3\,z_1$
$\bar{1}\,\bar{3}\,2=\sigma_2\sigma_1\sigma_0\sigma_1\sigma_0$	$Q_{41} + Q_{31}\,z_1$
$2\,3\,1=\sigma_1\sigma_2$	$Q_2 + Q_1\,z_1 + Q_1\,z_2 + z_1\,z_2$
$\bar{2}\,3\,1=\sigma_1\sigma_2\sigma_0$	$Q_3 + Q_{21} + Q_2\,z_1 + Q_2\,z_2$
$3\,2\,1=\sigma_1\sigma_2\sigma_1$	$Q_3 + Q_{21} + 2\,Q_2\,z_1 + Q_1\,{z_1}^2 + Q_2\,z_2 + Q_1\,z_1\,z_2 + {z_1}^2\,z_2$
$\bar{3}\,2\,1=\sigma_1\sigma_2\sigma_1\sigma_0$	$Q_4 + Q_{31} + Q_3\,z_1 + Q_3\,z_2$
$3\,\bar{2}\,1=\sigma_1\sigma_2\sigma_0\sigma_1$	$Q_{31} + Q_3\,z_1 + Q_{21}\,z_1 + Q_2\,{z_1}^2$
$\bar{3}\,\bar{2}\,1=\sigma_1\sigma_2\sigma_0\sigma_1\sigma_0$	Q_{32}
$2\,\bar{3}\,1=\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1$	$Q_{41} + Q_4\,z_1 + Q_{31}\,z_1 + Q_3\,{z_1}^2$
$\bar{2}\,\bar{3}\,1=\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1\sigma_0$	$Q_{42} + Q_{32}\,z_1$
$2\,3\,\bar{1}=\sigma_0\sigma_1\sigma_2$	$Q_3 + Q_2\,z_1 + Q_2\,z_2 + Q_1\,z_1\,z_2$
$\bar{2}\,3\,\bar{1}=\sigma_0\sigma_1\sigma_2\sigma_0$	$Q_{31} + Q_{21}\,z_1 + Q_{21}\,z_2$
$3\,2\,\bar{1}=\sigma_0\sigma_1\sigma_2\sigma_1$	$Q_4 + Q_{31} + 2\,Q_3\,z_1 + Q_{21}\,z_1 + Q_2\,{z_1}^2 + Q_3\,z_2 + Q_{21}\,z_2 + 2\,Q_2\,z_1\,z_2 + Q_1\,{z_1}^2\,z_2$
$\bar{3}\,2\,\bar{1}=\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0$	$Q_{32} + Q_{41} + Q_{31}\,z_1 + Q_{31}\,z_2$
$3\,\bar{2}\,\bar{1}=\sigma_0\sigma_1\sigma_2\sigma_0\sigma_1$	$Q_{32} + Q_{31}\,z_1 + Q_{21}\,{z_1}^2$
$\bar{3}\,\bar{2}\,\bar{1}=\sigma_0\sigma_1\sigma_2\sigma_0\sigma_1\sigma_0$	Q_{321}
$2\,\bar{3}\,\bar{1}=\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1$	$Q_{42} + Q_{32}\,z_1 + Q_{41}\,z_1 + Q_{31}\,{z_1}^2$
$\bar{2}\,\bar{3}\,\bar{1}=\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1\sigma_0$	$Q_{421} + Q_{321}\,z_1$
$1\,3\,\bar{2}=\sigma_1\sigma_0\sigma_1\sigma_2$	$Q_4 + Q_3\,z_1 + Q_3\,z_2 + Q_2\,z_1\,z_2$
$\bar{1}\,3\,\bar{2}=\sigma_1\sigma_0\sigma_1\sigma_2\sigma_0$	$Q_{41} + Q_{31}\,z_1 + Q_{31}\,z_2 + Q_{21}\,z_1\,z_2$
$3\,1\,\bar{2}=\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1$	$Q_{41} + Q_4\,z_1 + Q_{31}\,z_1 + Q_3\,{z_1}^2 + Q_{31}\,z_2 + Q_3\,z_1\,z_2 + Q_{21}\,z_1\,z_2 + Q_2\,{z_1}^2\,z_2$
$\bar{3}\,1\,\bar{2}=\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0$	$Q_{42} + Q_{32}\,z_1 + Q_{32}\,z_2$
$3\,\bar{1}\,\bar{2}=\sigma_1\sigma_0\sigma_1\sigma_2\sigma_0\sigma_1$	$Q_{42} + Q_{32}\,z_1 + Q_{41}\,z_1 + Q_{31}\,{z_1}^2 + Q_{32}\,z_2 + Q_{31}\,z_1\,z_2 + Q_{21}\,{z_1}^2\,z_2$
$\bar{3}\,\bar{1}\,\bar{2}=\sigma_1\sigma_0\sigma_1\sigma_2\sigma_0\sigma_1\sigma_0$	$Q_{421} + Q_{321}\,z_1 + Q_{321}\,z_2$
$1\,\bar{3}\,\bar{2}=\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1$	$Q_{43} + Q_{42}\,z_1 + Q_{32}\,{z_1}^2$
$\bar{1}\,\bar{3}\,\bar{2}=\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1\sigma_0$	$Q_{431} + Q_{421}\,z_1 + Q_{321}\,{z_1}^2$
$1\,2\,\bar{3}=\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2$	$Q_5 + Q_4\,z_1 + Q_4\,z_2 + Q_3\,z_1\,z_2$
$\bar{1}\,2\,\bar{3}=\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_0$	$Q_{51} + Q_{41}\,z_1 + Q_{41}\,z_2 + Q_{31}\,z_1\,z_2$
$2\,1\,\bar{3}=\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1$	$Q_{51} + Q_5\,z_1 + Q_{41}\,z_1 + Q_4\,{z_1}^2 + Q_{41}\,z_2 + Q_4\,z_1\,z_2 + Q_{31}\,z_1\,z_2 + Q_3\,{z_1}^2\,z_2$
$\bar{2}\,1\,\bar{3}=\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0$	$Q_{52} + Q_{42}\,z_1 + Q_{42}\,z_2 + Q_{32}\,z_1\,z_2$
$2\,\bar{1}\,\bar{3}=\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_0\sigma_1$	$Q_{52} + Q_{42}\,z_1 + Q_{51}\,z_1 + Q_{41}\,{z_1}^2 + Q_{42}\,z_2 + Q_{32}\,z_1\,z_2 + Q_{41}\,z_1\,z_2 + Q_{31}\,{z_1}^2\,z_2$
$\bar{2}\,\bar{1}\,\bar{3}=\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_0\sigma_1\sigma_0$	$Q_{521} + Q_{421}\,z_1 + Q_{421}\,z_2 + Q_{321}\,z_1\,z_2$
$1\,\bar{2}\,\bar{3}=\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1$	$Q_{53} + Q_{43}\,z_1 + Q_{52}\,z_1 + Q_{42}\,{z_1}^2 + Q_{43}\,z_2 + Q_{42}\,z_1\,z_2 + Q_{32}\,{z_1}^2\,z_2$
$\bar{1}\,\bar{2}\,\bar{3}=\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1\sigma_0$	$Q_{531} + Q_{431}\,z_1 + Q_{521}\,z_1 + Q_{421}\,{z_1}^2 + Q_{431}\,z_2 + Q_{421}\,z_1\,z_2 + Q_{321}\,{z_1}^2\,z_2$

TABLE VI.3. Type C Schubert polynomials for $w \in B_3$

w	$\mathfrak{D}_w$
$1\,2\,3=1$	1
$2\,1\,3=\sigma_1$	$P_1 + z_1$
$\overline{2}\,\overline{1}\,3=\sigma_{\hat{1}}$	P_1
$\overline{1}\,\overline{2}\,3=\sigma_1\sigma_{\hat{1}}$	$P_2 + P_1\,z_1$
$1\,3\,2=\sigma_2$	$2\,P_1 + z_1 + z_2$
$3\,1\,2=\sigma_2\sigma_1$	$P_2 + 2\,P_1\,z_1 + {z_1}^2$
$\overline{3}\,\overline{1}\,2=\sigma_2\sigma_{\hat{1}}$	P_2
$\overline{1}\,\overline{3}\,2=\sigma_2\sigma_1\sigma_{\hat{1}}$	$P_3 + P_2\,z_1$
$2\,3\,1=\sigma_1\sigma_2$	$P_2 + P_1\,z_1 + P_1\,z_2 + z_1\,z_2$
$3\,2\,1=\sigma_1\sigma_2\sigma_1$	$P_3 + P_{21} + 2\,P_2\,z_1 + P_1\,{z_1}^2 + P_2\,z_2 + 2\,P_1\,z_1\,z_2 + {z_1}^2\,z_2$
$\overline{3}\,\overline{2}\,1=\sigma_1\sigma_2\sigma_{\hat{1}}$	P_{21}
$\overline{2}\,\overline{3}\,1=\sigma_1\sigma_2\sigma_1\sigma_{\hat{1}}$	$P_{31} + P_{21}\,z_1$
$\overline{2}\,3\,\overline{1}=\sigma_{\hat{1}}\sigma_2$	$P_2 + P_1\,z_1 + P_1\,z_2$
$3\,\overline{2}\,\overline{1}=\sigma_{\hat{1}}\sigma_2\sigma_1$	$P_{21} + P_2\,z_1 + P_1\,{z_1}^2$
$\overline{3}\,2\,\overline{1}=\sigma_{\hat{1}}\sigma_2\sigma_{\hat{1}}$	$P_3 + P_{21} + P_2\,z_1 + P_2\,z_2$
$2\,\overline{3}\,\overline{1}=\sigma_{\hat{1}}\sigma_2\sigma_1\sigma_{\hat{1}}$	$P_{31} + P_3\,z_1 + P_{21}\,z_1 + P_2\,{z_1}^2$
$\overline{1}\,3\,\overline{2}=\sigma_1\sigma_{\hat{1}}\sigma_2$	$P_3 + P_2\,z_1 + P_2\,z_2 + P_1\,z_1\,z_2$
$3\,\overline{1}\,\overline{2}=\sigma_1\sigma_{\hat{1}}\sigma_2\sigma_1$	$P_{31} + P_3\,z_1 + P_{21}\,z_1 + P_2\,{z_1}^2 + P_{21}\,z_2 + P_2\,z_1\,z_2 + P_1\,{z_1}^2\,z_2$
$\overline{3}\,1\,\overline{2}=\sigma_1\sigma_{\hat{1}}\sigma_2\sigma_{\hat{1}}$	$P_{31} + P_{21}\,z_1 + P_{21}\,z_2$
$1\,\overline{3}\,\overline{2}=\sigma_1\sigma_{\hat{1}}\sigma_2\sigma_1\sigma_{\hat{1}}$	$P_{32} + P_{31}\,z_1 + P_{21}\,{z_1}^2$
$\overline{1}\,2\,\overline{3}=\sigma_2\sigma_1\sigma_{\hat{1}}\sigma_2$	$P_4 + P_3\,z_1 + P_3\,z_2 + P_2\,z_1\,z_2$
$2\,\overline{1}\,\overline{3}=\sigma_2\sigma_1\sigma_{\hat{1}}\sigma_2\sigma_1$	$P_{41} + P_4\,z_1 + P_{31}\,z_1 + P_3\,{z_1}^2 + P_{31}\,z_2 + P_3\,z_1\,z_2 + P_{21}\,z_1\,z_2 + P_2\,{z_1}^2\,z_2$
$\overline{2}\,1\,\overline{3}=\sigma_2\sigma_1\sigma_{\hat{1}}\sigma_2\sigma_{\hat{1}}$	$P_{41} + P_{31}\,z_1 + P_{31}\,z_2 + P_{21}\,z_1\,z_2$
$1\,\overline{2}\,\overline{3}=\sigma_2\sigma_1\sigma_{\hat{1}}\sigma_2\sigma_1\sigma_{\hat{1}}$	$P_{42} + P_{32}\,z_1 + P_{41}\,z_1 + P_{31}\,{z_1}^2 + P_{32}\,z_2 + P_{31}\,z_1\,z_2 + P_{21}\,{z_1}^2\,z_2$

TABLE VI.4. Type D Schubert polynomials for $w \in D_3$

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