# PATTERN AVOIDANCE AND RATIONAL SMOOTHNESS OF SCHUBERT VARIETIES

SARA C. BILLEY

Let $w$ be an element of the Weyl group $S_n$, and let $X_w$ be the Schubert variety associated to $w$ in the flag manifold $SL_n(\mathbb{C})/B$. Lakshmibai and Sandhya [14] showed that $X_w$ is smooth if and only if $w$ avoids the patterns 4231 and 3412. Using two tests for rational smoothness due to Carrell and Peterson [3], we show that rational smoothness of $X_w$ is characterized by pattern avoidance for types $B$, $C$ and $D$ as well. A key step in the proof of this result is a sequence of rules for factoring the Poincaré polynomials for the cohomology ring of $X_w$, generalizing the work of Gasharov [8]. Patterns can also be used to characterize actual smoothness of Schubert varieties.

## 1. Introduction

In general, let $G$ be a semisimple Lie Group, $B$ be a Borel subgroup, $w$ be an element of the associated Weyl group $W$, $X_w = \overline{BwB/B}$ be the Schubert variety indexed by $w$ in the flag manifold $G/B$. Let $\leq$ be the Bruhat order on $W$ and $p_w(t) = \sum_{v \leq w} t^{l(v)}$, where $l(v)$ is the length of $v$. Then $p_w(t^2)$ is the Poincaré polynomial of the cohomology ring of $X_w$, but we will abuse notation and refer to $p_w(t)$ as the Poincaré polynomial as well.

Roughly speaking, a Schubert variety is rationally smooth if local Poincaré duality holds. Any variety which is smooth is necessarily rationally smooth, however the reverse implication is not true in general. We take the next theorem as the formal definition of rational smoothness.

**Theorem 1.1.** [3] *For any Weyl group $W$ and any $w \in W$, a Schubert variety $X_w$ is rationally smooth if and only if either of the following hold:*

1. *The Poincaré polynomial $p_w(t)$ is symmetric.*
2. *The Bruhat graph is regular, i.e. every vertex has the same number of edges.*

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*Here the Bruhat graph of $w$ is the graph with vertices in the set $\{v \in W : v \leq w\}$ and edges between $u$ and $v$ if $u = v\sigma$ for some reflection $\sigma$ (not necessarily a simple reflection).*

The Schubert varieties $X_w$ for type $A$ are subvarieties of the flag manifold $SL_n(\mathbb{C})/B$ where $B$ are the upper triangular matrices in $SL_n(\mathbb{C})$. An excellent overview of the Schubert calculus for type $A$ appears in a recent book by Fulton [7]. The Weyl group of type A is the symmetric group $S_n$, whose elements are permutations written in *one-line* notation as $w_1 w_2 \dots w_n$. $S_n$ is generated by the simple reflections $\sigma_i$ for $1 \leq i \leq n-1$ where $w\sigma_i$ is obtained from $w$ by interchanging positions $i$ and $i+1$. For example, if $w = 612435$, then $w\sigma_1$ is 162435.

Deodhar [5] showed that rational smoothness is equivalent to smoothness in Type $A$ (extended to all simply-laced roots systems by Peterson). Lakshmibai and Sandhya [14] showed that $X_w$ is smooth if and only if $w$ avoids the patterns 4231 and 3412, *i.e.* no length 4 subsequence in $w_1 w_2 \dots w_n$ has the same relative order as 4231 or 3412. Furthermore, Gasharov [8] has shown that a Schubert variety $X_w$ of type A is (rationally) smooth if and only if $p_w(t)$ factors into polynomials of the form $1 + t + t^2 + \cdots + t^r$. We extend both of these results to type $B$ using combinatorial techniques in Theorem 1.2 below. As a consequence we obtain new proofs in the type $A$ case of both theorems.

The Schubert varieties $X_w$ for type $B$ and $C$ are subvarieties of the flag manifolds $SO_{2n+1}(\mathbb{C})/B(\mathbb{C})$ and $Sp_{2n}(\mathbb{C})/B(\mathbb{C})$ (respectively) where $B(\mathbb{C})$ is a Borel subgroup (see [10, Sect. 23]). The Weyl group of types $B$ and $C$ are the hyperoctahedral groups (or signed permutation groups) $B_n$. We write the signed permutations in one-line notation with a bar over an element with a negative sign. The group $B_n$ is generated by the simple reflections $\sigma_i$ for $1 \leq i \leq n-1$ as well as $\sigma_0$ where $w\sigma_0$ is $\overline{w_1} w_2 \dots w_n$. Note, we have chosen a different set of simple roots from that found in [9].

For any sequence $a_1, \dots, a_k$ of distinct non-zero real numbers we define $\mathrm{fl}(a_1, \dots, a_k)$ to be the element in $B_k$ with signed numbers in the same positions and same relative order of the underlying permutation.

Using this notation, we state the main theorem of this article.

**Theorem 1.2.** *If $X_w$ is a Schubert variety of type $B$ or $C$, then the following conditions are equivalent:*

1. *$X_w$ is rationally smooth.*
2. *For each subsequence $i_1 < i_2 < i_3 < i_4$, $\mathrm{fl}(w_{i_1} w_{i_2} w_{i_3} w_{i_4})$ corresponds to a rationally smooth Schubert variety.*
3. *$p_w(t)$ factors into a product of polynomials of the form $1 + t + t^2 + \cdots + t^r$.*

The proof of Theorem 1.2 follows three main steps. First in Theorem 3.3 we show that under certain conditions $p_w(t)$ factors into $(1+t+t^2+\cdots+t^r)p_v(t)$ for some $r$ and $v$ with $l(v)=l(w)-r$. Second, in Lemma 4.4, we show that at least one of these factoring conditions is always satisfied by signed permutations for which condition 2 holds, and furthermore, $v$ avoids all bad patterns as well. Therefore, if every length 4 subsequence of $w$ corresponds with a rationally smooth Schubert variety, then $p_w(t)$ factors into a product of polynomials of the form $1+t+t^2+\cdots+t^r$. Hence, $p_w(t)$ is symmetric which implies $X_w$ is rationally smooth by Theorem 1.1. Third, in Lemma 4.3 we explicitly show that if $w$ contains a bad pattern then the Bruhat graph is not regular, hence, by Carrell and Petersons' second test in Theorem 1.1, $X_w$ is not rationally smooth. The second and thirds steps are combined in Theorem 4.2.

From the proof of Lemma 4.4 we can in fact make the following stronger claim which will be proven in Section 4. Let $[\hat{0},w]$ be the interval in the Bruhat order of all elements weakly below $w$.

**Corollary 1.3.** *For any $w\in S_n$ or $B_n$ such that $X_w$ is rationally smooth, the interval $[\hat{0},w]$ has a symmetric chain decomposition, i.e. contains a subposet using all vertices which is a product of chains.*

A ranked poset with maximum rank $m$ is *rank symmetric* if the number of elements of rank $i$ equals the number of elements of rank $m-i$, it is *rank unimodal* if the number of elements on each rank forms a unimodal sequence, and it is $k$-Sperner if the largest subset containing no $(k+1)$-element chain has cardinality equal to the sum of the $k$ middle ranks. In [18], Stanley has shown that $[\hat{0},w]$ has the Peck property, namely it is rank symmetric, rank unimodal and $k$-Sperner for all $k$. Using Corollary 1.3, we can extend this statement to any Schubert variety which is rationally smooth following the work of Canfield [6].

**Corollary 1.4.** *Let $w\in S_n$ or $B_n$. $X_w$ is rationally smooth if and only if the poset $[\hat{0},w]$ has the Peck property.*

From the factorization of $p_w(t)$ we have the following formula for counting the number of elements below $w$.

**Corollary 1.5.** *Let $w\in S_n$ or $B_n$ such that $X_w$ is rationally smooth, let $[\mathbf{m}]_\mathbf{t}$ denote the polynomial $(1+t+\ldots+t^m)$, and assume $p_w(t)=[\mu_\mathbf{1}]_\mathbf{t}[\mu_\mathbf{2}]_\mathbf{t}\cdots[\mu_\mathbf{k}]_\mathbf{t}$ for some $\mu_1,\ldots,\mu_k$ using Theorem 1.2. Then the number of elements weakly below $w$ in Bruhat order is $\mu_1\mu_2\cdots\mu_k$.*

Historically, there are several other tests for smoothness in various cases. Kumar [12] has given a very general test for determining both smoothness and rational smoothness of Schubert varieties in $G/B$ at any fixed point under the maximal torus action. This test works for any Kac-Moody group $G$ and depends on computing equivariant multiplicities in the nil-Hecke ring. In addition to Lakshmibai and Sandhyas' work, Lakshmibai and Song have given a characterization of smoothness in type $C$ which also depends on the 1-line notation for $w \in B_n$ as an element embedded in $S_{2n}$. This appears in Song's Ph.D thesis [17], see also [15]. Recently, Carrell has given an algorithm for determining the singular locus in a Schubert variety in $G/B$ with $G$ semisimple using the *augmented* Bruhat graph. Each of these tests is computationally more difficult than the complexity of the test we propose in the paper for rational smoothness. The complexity of determining if a signed permutation in $B_n$ avoids a finite list of patterns of length 4 is $\mathbf{O}(n^4)$.

It was known that the Poincaré polynomial of a smooth (not just rationally smooth) Schubert variety factors according to the heights of certain roots [3, Sect.5]. In Section 5, we compare our factored formula for $p_w(t)$ with Carrell's formula.

In Section 2, we define the Bruhat order and give an equivalent characterization in terms of one-line notation. We give an explicit algorithm for the factorization of $p_w(t)$ for type $A$ and $B$ in Section 3. In Section 4, we use the factoring theorems to prove that rational smoothness is equivalent to pattern avoidance in type $B$ as well. The minimal set of 26 patterns is given. We compare our factored Poincaré polynomial formula with Carrell's formula in Section 5. Finally, in Section 6 we complete the characterization of smoothness and rational smoothness for Schubert varieties of the classical groups.

## 2. Characterizing Bruhat order

The Bruhat or Bruhat-Chevalley order on a Weyl group is a highly important combinatorial tool for studying Schubert varieties [4]. There are several ways to characterize this partial order. We state two methods, the first one holds for all Weyl groups, the second one is specific to the classical groups.

Let $\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_p}$ be an expression for $w$ in terms of the Weyl group generators of minimal length. Then $v \le w$ in *Bruhat order* if and only if there exists a subsequence $i_i < i_2 < \cdots < i_q$ such that $\sigma_{a_{i_1}}\cdots\sigma_{a_{i_q}}$ equals $v$ [11, 5.10].

We say that a set $\{a_1, a_2, \ldots a_k\}$ is less than a set $\{b_1, b_2 \ldots, b_k\}$ if when the elements in the two sets are written in increasing order we have $a_i \le b_i$ for each $1 \le i \le k$. For example, $\{-2, 5, 6\} < \{1, 6, 7\}$.

The following equivalent criterion for Bruhat order was proved by Deodhar in the type $A$ case (see [11] for attribution) and extended to type $B$ (and $D$)

by Proctor [16, Thm. 5BC]: Let $v, w \in B_n$, then $v \leq w$ if and only if for each $1 \leq i \leq n$ we have $\{v_i, \ldots, v_n\} > \{w_i, \ldots, w_n\}$.

For example, take $w = \bar{6}\bar{5}4\bar{1}2\bar{3}$ and $v = \bar{4}63\bar{2}51$ (in one-line notation). Then we compute the following table of sorted lists

$$\bar{3} < 1 \tag{1}$$
$$\bar{3}2 < 15 \tag{2}$$
$$\bar{3}\bar{1}2 < \bar{2}15 \tag{3}$$
$$\bar{3}\bar{1}24 < \bar{2}135 \tag{4}$$
$$\bar{5}\bar{3}\bar{1}24 < \bar{2}1356 \tag{5}$$
$$\bar{6}\bar{5}\bar{3}\bar{1}24 < \bar{4}\bar{2}1356 \tag{6}$$

Hence, $v \leq w$ as elements of $B_6$.

We will need the following lemma in Section 4.

**Lemma 2.1.** *Say two signed permutations $v$ and $w$ agree everywhere except in positions $i_1 < \cdots < i_k$. Then $v \leq w$ if and only if* $\mathrm{fl}(v_{i_1} \ldots v_{i_k}) \leq \mathrm{fl}(w_{i_1} \ldots w_{i_k})$.

*Proof.* Using the Deodhar-Proctor criteria for Bruhat order, we know that $v \leq w$ if and only if the set $\{v_j, \ldots, v_n\}$ is greater than $\{w_j, \ldots, w_n\}$ for each $j$ under the partial order on sets. Since $v, w$ agree everywhere except positions $i_1, i_2, ...i_n$, we only need to show that given any two sets, we have $\{x, a_1, a_2, \ldots, a_k\} < \{x, b_1, \ldots, b_k\}$ if and only if $\{a_1, a_2, \ldots, a_k\} < \{b_1, \ldots, b_k\}$. Then the lemma follows from the fact that the flattening function maintains the relative order of its arguments.

Without loss of generality we can assume $a_1 \leq \cdots \leq a_k$ and $b_1 \leq \cdots \leq b_k$. Say $a_i < x \leq a_{i+1}$ and $b_j < x \leq b_{j+1}$, then the following are equivalent:

1. $\{a_1, a_2, \ldots, a_k\} < \{b_1, \ldots, b_k\}$.
2. $i \geq j$, $a_m < b_m$ for all $1 \leq m \leq j$ and all $i + 1 \leq m \leq k$, and $a_{j+1} \leq \cdots \leq a_i < x \leq b_{j+1} \leq \cdots \leq b_i$.
3. $\{x, a_1, a_2, \ldots, a_k\} < \{x, b_1, \ldots, b_k\}$.

□

## 3. Factoring Poincaré polynomials

In this section we give rules for factoring the Poincaré polynomials $p_w(t)$ as defined in the introduction. These rules are stated in Theorems 3.2 and 3.3, *i.e.* type $A$ and $B$ are stated separately. It is clear from our choice of simple reflections (see Section 1) that $S_n$ and $B_{n-1}$ are naturally embedded in $B_n$ in such a way that respects the Bruhat order. Therefore, the proof of the factoring rules in the type $A$ case is included in the type $B$ case. We give the proof of Theorem 3.3 after several lemmas; some new and some from the literature. We conclude this section with corollaries on a subposet of an interval in the Bruhat order and a recursive formula for counting the number of elements in such an interval.

**Definition 3.1.** *Let $w \in B_n$ and say $w_d = \pm n$. If $w_d = +n$ then we say $w$ contains a consecutive sequence if there exists a subsequence $d < i_1 < \cdots < i_k < n$ such that $w_{i_1} = n-1$, $w_{i_2} = n-2, \ldots, w_{i_k} = n-k$ and $w_n = n-k-1$. If $w_d = \bar{n}$ then a consecutive sequence consists of two subsequences $d > j_1 > \cdots > j_l \geq 1$ (possibly empty) and $d < i_1 < \cdots < i_k < n$ such that $w_{j_a} = \overline{n-a}$ for $1 \leq a \leq l$, $w_{i_b} = n-b-l$ for $1 \leq b \leq k$, and $w_n = n-k-l-1$.*

For example, take $w = 3\bar{2}\bar{6}\bar{7}154 \in B_7$. Then $w_4 = \bar{7}$, $d = 4$, and $w$ has a consecutive sequence in positions $3, 4, 6$, and $7$. Note, $3\bar{2}\bar{6}\bar{7}1\bar{5}4$ does not have a consecutive sequence since $\bar{5}$ is to the right of $\bar{7}$.

The following theorem first appears in the work of Gasharov [8].

**Theorem 3.2.** *Let $w \in S_n$, and assume $w_d = n$ and $w_n = e$. The Poincaré polynomial of $w$ factors in the form*

$$p_w(t) = (1 + t^1 + \cdots + t^{\mu})p_{w'}. \tag{7}$$

*under the following circumstances:*

1. *If $n = w_d > w_{d+1} > \cdots > w_n$, then $p_w$ factors with $w' = w\sigma_d \cdots \sigma_{n-1}$ and $\mu = n - d$.*
2. *If $w$ contains a consecutive sequence ending in $w_n = e$, then $p_w$ factors with $w' = \sigma_{n-1} \cdots \sigma_{e+1}\sigma_e w$ and $\mu = n - e$.*

For example, apply Theorem 3.2 to the element $w_0 = n \ldots 21$ (in one-line notation) to get the well-known formula for the Poincaré polynomial of $S_n$ [11, 3.7], namely

$$p_{w_0}(t) = \prod_{k=1}^{n-1}(1 + t + \ldots + t^k). \tag{8}$$

**Theorem 3.3.** *Let $w \in B_n$, and assume $w_d = \pm n$ and $w_n = \pm e$. The Poincaré polynomial of $w$ factors in the form*

$$p_w(t) = (1 + t^1 + \cdots + t^{\mu})p_{w'}. \tag{9}$$

*under the following circumstances:*

1. *If $w_d = +n$ and $w_d > w_{d+1} > \cdots > w_n$, then $p_w$ factors with $w' = w\sigma_d \cdots \sigma_{n-1}$ and $\mu = n - d$.*

2. *If $w$ contains a consecutive sequence ending in $w_n = e$ for $e > 0$, then $p_w$ factors with $w' = \sigma_{n-1} \cdots \sigma_{e+1}\sigma_e w$ and $\mu = n - e$.*
3. *If each $w_i$ is negative and $w_1 > w_2 > \ldots \widehat{w_d} \cdots > w_n$ (decreasing after removing $w_d$), then $p_w$ factors with $w' = w\sigma_{d-1} \cdots \sigma_1\sigma_0\sigma_1 \cdots \sigma_{n-1}$ and $\mu = d + n - 1$.*
4. *If each $w_i$ is negative and $w_1 > w_2 > \cdots > w_{n-1}$, then $p_w$ factors with $w' = \sigma_{n-1} \cdots \sigma_1\sigma_0\sigma_1 \cdots \sigma_{e-1} w$ and $\mu = e + n - 1$.*
5. *If each $w_i$ positive except for $w_d = \bar{n}$ and $w_1 > w_2 > \cdots > w_d$, then $p_w$ factors with $w' = w\sigma_{d-1} \cdots \sigma_1\sigma_0$ and $\mu = d$.*

For example, if $w$ is $1\bar{5}\bar{6}\bar{2}43$ then $p_w$ factors under Rule 2 with $w' = 1\bar{4}\bar{5}\bar{2}36$ and $\mu = 3$. Also, after repeatedly applying either Rule 3 or 4 of Theorem 3.3 to the element $w_0^B = \bar{1}\bar{2} \cdots \bar{n}$ we get the well-known formula for the Poincaré polynomial of $B_n$ [11, 3.7], namely

$$p_{w_0^B}(t) = \prod_{k=0}^{n-1}(1 + t + \ldots + t^{2k+1}). \tag{10}$$

**Lemma 3.4.** *For any $w \in W$, the interval given by $\{v \leq w\}$ in Bruhat order is isomorphic to the interval $\{u \leq w^{-1}\}$ as posets.*

*Proof.* Recall, $v \leq w$ if and only if any reduced expression $\sigma_{a_1}\sigma_{a_2} \cdots \sigma_{a_p}$ for $w$ contains a subexpression $\sigma_{a_{i_1}} \cdots \sigma_{a_{i_q}}$ which equals $v$. The lemma follows directly from the fact that reversing product of generators gives an expression for the inverse element. □

Let $P$ be any parabolic subgroup of $B_n$ (*i.e.* $P$ is generated by a subset $I$ of the simple reflections in $B_n$.) It is well known that [11, Prop. 1.10(c)] for each $w \in B_n$ there exists unique elements $u \in P$ and $v \in \{u \in B_n : l(\sigma u) > l(u)$ for all $\sigma \in I\}$ such that $w = uv$ and $l(w) = l(u) + l(v)$. Furthermore, $v$ is the unique element of minimal length in the coset $Pw$ of $B_n$ mod $P$ where we quotient $B_n$ by $P$ on the left.

**Lemma 3.5.** *For any Weyl group $W$, any parabolic subgroup $P$, and any $w \in W$, let $w = uv$ be the unique decomposition of $w$ such that $u \in P$ and $v$ is a minimal coset representative for $Pw$. If $c \in W$ is also a minimal length element in the coset $Pc$ then $c \leq w$ if and only if $c \leq v$.*

*Proof.* This follows directly from [16, Lem.3.2]. □

**Lemma 3.6.** *All minimal coset representatives for $B_n(\mathrm{mod}\ B_{n-1})$ below $c = \sigma_{n-1}\sigma_{n-2} \cdots \sigma_d$ $(d \geq 1)$ in Bruhat order are of the form $\sigma_{n-1}\sigma_{n-2} \cdots \sigma_k$ for $d \leq k \leq n - 1$. Similarly, all minimal coset representatives for $B_n(\mathrm{mod}\ S_n)$ below $c = \sigma_0\sigma_1 \cdots \sigma_d$ in Bruhat order are of the form $\sigma_0\sigma_1 \cdots \sigma_k$ for $0 \leq k \leq d$.*

*Proof.* This follows from the fact that the generators $\sigma_i$ and $\sigma_j$ commute for all $1 \leq i, j \leq n-1$ provided $|i-j| > 1$. □

**Proposition 3.7.** [1] *Let $w \in W$ and let $P$ be any parabolic subgroup (i.e. $P$ is generated by a subset of the simple reflections in $W$.) There exists a unique maximal element $v \leq w$ such that $v \in P$.*

**Lemma 3.8.** *Given any $u, w \in W$ such that $u \leq w$ and given any simple reflection $\sigma$ such that $w\sigma \leq w$ then $u\sigma \leq w$. Similarly, if $\sigma w \leq w$ then $\sigma u \leq w$.*

*Proof.* This follows directly from [11, Lemma 7.4]. □

*Proof of Theorem 3.3.* We prove Rules 1, 3, and 5 of the theorem. Rules 2 and 4 are just the inverse statements of Rules 1 and 3 respectively so they follow by Lemma 3.4. Let $B_w$ be the set of all $v \leq w$ graded by the length of $v$, denoted $l(v)$, *i.e.* the minimal number of generators needed to express $v$. In each Rule below we will show that $B_w$ is isomorphic to $B_{w'} \times \{0, \ldots, \mu\}$ as graded sets where the rank of $(v', i)$ is $l(v') + i$. Therefore, $p_w(t) = p_{w'}(t)(1 + t + \cdots + t^{\mu})$.

Rule 1 : *Assume $w_d = n$ and $w_d > w_{d+1} > \cdots > w_n$.*

Let $c = \sigma_{n-1} \cdots \sigma_d$ and let $w' = wc^{-1}$ , then $w'_n = n$ so $w' \in B_{n-1}$, *i.e.* the parabolic subgroup generated by $\sigma_0, \ldots, \sigma_{n-2}$. Note, $w'$ is a maximal element in the Bruhat order which is in $B_{n-1}$ and below $w$. Hence, $w'$ is the unique such maximal element by Proposition 3.7.

Given $v \leq w$ write $v = v^1 v^2$ where $v^1$ is in $B_{n-1}$ and $v^2$ is the minimal coset representative for v in $B_n (\mathrm{mod}\ B_{n-1})$ (moding out on the left). By Proposition 3.7, $v^1 \leq w'$. By Lemma 3.5, we have $v^2 \leq c$, hence $v^2 = \sigma_{n-1} \cdots \sigma_{n-k}$ for some $0 \leq k \leq n-d$ by Lemma 3.6. Therefore, we can define an injective map $\varphi : B_w \to B_{w'} \times \{0, \ldots, n-d\}$ given by mapping $v$ to the pair $(v^1, l(v^2))$. Note, $l(v) = l(v^1) + l(v^2)$ so this map preserves rank.

Let $u \in B_{n-1}$ be any element such that $u \leq w'$ and $k \in \{0, \ldots, n-d\}$, then $u \leq w$ and so is $u\sigma_{n-1} \ldots \sigma_{n-k}$ by Lemma 3.8 and the assumption $w_d > w_{d+1} > \cdots > w_n$. Since $u\sigma_{n-1} \cdots \sigma_{n-k}$ is uniquely represented as a product of an element in $B_{n-1}$ times a minimal coset representative, we have $\varphi(u\sigma_{n-1} \cdots \sigma_{n-k}) = (u, k)$. Hence, $\varphi$ is surjective.

Rule 3: *Assume each entry of $w$ is negative and decreasing after removing $\bar{n}$.*

Let $c = \sigma_{n-1} \cdots \sigma_1 \sigma_0 \sigma_1 \cdots \sigma_{d-1}$ and let $w' = wc^{-1}$ , then $w'_n = n$ so $w' \in B_{n-1}$. Note, $w'$ is the unique maximal element in $B_{n-1}$, namely $[\bar{1}, \bar{2}, \ldots, \overline{n-1}]$. Given $v \leq w$, write $v = v^1 v^2$ as in the previous case where $v^1 \in B_{n-1}$ and $v^2$

is the minimal coset representative for $v$, *i.e.*

$$v^2 = \begin{cases} \sigma_{n-1}\sigma_{n-2}\cdots\sigma_1\sigma_0\sigma_1\cdots\sigma_{k-1} & v_k = \bar{n} \\ \sigma_{n-1}\sigma_{n-2}\cdots\sigma_k & v_k = n. \end{cases} \tag{11}$$

Note, since $v \leq w$ we must have $l(v^2) \leq n + d - 1 = l(c)$. Define a map from $B_w$ to $B_{n-1} \times \{0, 1, \ldots, n + d - 1\}$ by mapping $v$ to $(v^1, l(v^2))$. This map is clearly bijective and rank preserving.

Rule 5: *Assume $w$ is all positive except $w_d = \bar{n}$ and $w_1 > w_2 > \cdots > w_{d-1} > w_d$.*

The proof in this case is similarly to Rule 1, so we will simply set up the notation. Let $c = \sigma_0 \cdots \sigma_{d-1}$ and let $w' = wc^{-1}$, then $w' \in S_n$, *i.e.* the parabolic subgroup generated by $\sigma_1, \ldots, \sigma_{n-1}$. Again, $w'$ is the unique maximal element in $S_n$ that lies below $w$ in Bruhat order. Hence, given $v \leq w$ let $v = v^1v^2$ be the minimal coset representation of $v$, then the map which sends $v$ to the pair $(v^1, l(v^2))$ is well defined, rank preserving and bijective from $B_w$ to $B_{w'} \times \{0, \ldots, d\}$. □

From the proof above we have as a corollary the proof for Theorem 3.2 as well by using only Rules 1 and 2.

From the proof of Theorem 3.3 we get the following corollaries.

**Corollary 3.9.** *Let $[\hat{0}, w]$ be the interval below $w$ in the Bruhat order with the induced poset structure. Assume $p_w(t)$ factors as $p_{w'}(1 + t + \cdots + t^\mu)$ using one of the rules in Theorem 3.3.*

1. *$[\hat{0}, w]$ contains $\tilde{B}_{w'} \times [\mu+1]$ as a subposet with the same number of vertices, where $[\mu + 1]$ is the chain of length $\mu + 1$.*
2. *The number of elements weakly below $w$ in Bruhat order is $p_w(1) = p_{w'}(1)(\mu + 1)$.*

## 4. Pattern avoidance and rational smoothness

In this section we prove that rational smoothness of $X_w$ can be determined by a simple test on the signed permutation $w \in B_n$. This test was motivated by the result of V. Lakshmibai and B. Sandhya in the $A_n$ case which states that $X_w$ is smooth if and only if $w$ avoids the patterns 4231 and 3412. We first define the notion of pattern avoidance in the hyperoctahedral group. Then we state the main theorem of this section, namely that rational smoothness is equivalent to avoiding certain patterns in the $B_n$ case as well. We conclude with a corollary on the existence of a subposet of the interval below $w$ in Bruhat order which factors into a product of chains and a formula for counting the number elements in an interval of the Bruhat order.

We define pattern avoidance in terms of the following function which *flattens* any subsequence into a signed permutation.

**Definition 4.1.** *For any sequence $a_1a_2\dots a_k$ of distinct non-zero real numbers, define $\mathrm{fl}(a_1a_2\dots a_k)$ to be the unique element $b = b_1b_2\dots b_k$ in $B_k$ such that*

- *For all $j$, both $a_j$ and $b_j$ have the same sign.*
- *For all $i, j$, we have $|b_i| < |b_j|$ if and only if $|a_i| < |a_j|$.*

For example, $\mathrm{fl}(\bar{6}, 3, \bar{7}, 1) = [\bar{3}, 2, \bar{4}, 1]$. Any word containing the subsequence $\bar{6}, 3, \bar{7}, 1$ does not avoid the pattern $\bar{3}2\bar{4}1$. In particular, $w = 8\bar{6}\bar{2}3\bar{7}451$ (in one-line notation) does not avoid $\bar{3}2\bar{4}1$.

Another way to describe pattern avoidance is with the signed permutation matrices. Namely, a signed permutation matrix $w$ avoids the pattern $v$ if no submatrix of $w$ is the matrix $v$.

**Theorem 4.2.** *Let $w \in B_n$, the Schubert variety $X_w$ is rationally smooth if and only if for each subsequence $i_1 < i_2 < i_3 < i_4$, $\mathrm{fl}(w_{i_1}w_{i_2}w_{i_3}w_{i_4})$ corresponds with a rationally smooth Schubert variety, i.e. $w$ avoids the following patterns:*

$$\begin{array}{ccccccc}
\bar{1}2\bar{3} & 1\bar{2}\bar{3} & 12\bar{3} & 1\bar{3}\bar{2} & \bar{2}\bar{1}\bar{3} & \bar{2}1\bar{3} & 2\bar{1}\bar{3} \\
2\bar{3}\bar{1} & \bar{3}1\bar{2} & \bar{3}\bar{2}\bar{1} & \bar{3}\bar{2}1 & \bar{3}2\bar{1} & 3\bar{2}\bar{1} & 3\bar{2}1 \\
\bar{2}\bar{4}31 & 2\bar{4}31 & \bar{3}\bar{4}\bar{1}\bar{2} & \bar{3}4\bar{1}2 & \bar{3}412 & 34\bar{1}2 & 3412 \\
4\bar{1}3\bar{2} & 413\bar{2} & \bar{4}231 & 423\bar{1} & 4231 & &
\end{array} \tag{12}$$

The proof of Theorem 4.2 follows directly from Lemmas 4.3 and 4.4 below. The proof of Lemma 4.3 depends on the Carrell-Peterson theorem that rational smoothness is equivalent to the Bruhat graph being regular and on the Deodhar-Proctor criteria for Bruhat order. The proof of Lemma 4.4 follows from the Carrell-Peterson theorem that rational smoothness is equivalent to the Poincaré polynomial being symmetric.

**Lemma 4.3.** *If $w \in B_n$ contains a pattern corresponding with a Schubert variety in $B_4$ which is not rationally smooth, then $X_w$ is not rationally smooth. Hence, $X_w$ is not smooth.*

*Proof.* Let $d_w(v)$ be the *degree* of the vertex $v$ in the Bruhat graph for $w$, *i.e.* the number of edges incident to $v$. We will show that the Bruhat graph is not regular by showing that there exists an explicit $v \leq w$ such that $d_w(v)$ is strictly greater than $d_w(w) = l(w)$.

The reflections in any Weyl group are the set of all elements of the form $u\sigma_i u^{-1}$ for any $u \in W$ and any simple reflection $\sigma_i$. In particular, the reflections in $B_n$ are transpositions $t_{ij}$ and signed transpositions $s_{ij}$: for $i < j$ and $w = w_1 \dots w_i \dots w_j \dots w_n$,

$$wt_{ij} = \dots w_j \dots w_i \dots \tag{13}$$

$$ws_{ij} = \dots \overline{w_i} \dots \overline{w_j} \dots \tag{14}$$

$$ws_{ii} = \dots \overline{w_i} \dots . \tag{15}$$

For any $u \in B_n$ such that $u \leq w$, define $E_w(u)$ to be the set of all transpositions or signed transpositions $\widetilde{t_{ij}}$ such that $u\widetilde{t_{ij}} \leq w$. $E_w(u)$ is isomorphic to the set of edges emanating from $u$ in the Bruhat graph. Furthermore, define

$$d_w^6(u) = |\{\widetilde{t_{ij}} \in E_w(u) : |\{i, j, i_1, i_2, i_3, i_4\}| = 6\}| \tag{16}$$

$$d_w^5(u) = |\{\widetilde{t_{ij}} \in E_w(u) : |\{i, j, i_1, i_2, i_3, i_4\}| = 5\}| \tag{17}$$

$$d_w^4(u) = |\{\widetilde{t_{ij}} \in E_w(u) : |\{i, j, i_1, i_2, i_3, i_4\}| = 4\}|. \tag{18}$$

Then, the degree of $u$ in the Bruhat graph breaks up into three summands as follows:

$$d_w(u) = d_w^4(u) + d_w^5(u) + d_w^6(u). \tag{19}$$

Say $w$ contains a bad pattern in positions $i_1 < i_2 < i_3 < i_4$. Let $w'$ be the signed permutation $\mathrm{fl}(w_{i_1} w_{i_2} w_{i_3} w_{i_4})$ in $B_4$. By computer verification on $B_4$, there exists an element $v' \leq w'$ such that the number of edges incident to $v'$ in the Bruhat graph is greater than the number of edges incident to $w'$. Now, define $v \in B_n$ to be the signed permutation which agrees with $w$ everywhere except in positions $i_1 i_2 i_3 i_4$ and $\mathrm{fl}(v_{i_1} v_{i_2} v_{i_3} v_{i_4})$ equals $v'$.

By Lemma 2.1 and the definition of $v$ above, one sees that in order to determine if $v\widetilde{t_{ij}} \leq w$ we only need to compare the flattened signed permutations in positions $i, j, i_1, i_2, i_3, i_4$. For each such pair of flattened elements, $v'', w'' \in B_6$, a computer verification has shown that $d_{w''}^4(v'') > d_{w''}^4(w'')$, $d_{w''}^5(v'') \geq d_{w''}^5(w'')$, and $d_{w''}^6(v'') = d_{w''}^6(w'')$. From this one can see that $d_w^4(v) > d_w^4(w)$, $d_w^5(v) \geq d_w^5(w)$, and $d_w^6(v) = d_w^6(w)$ by examining the disjoint summands: for pairs $i < j$ such that $|\{i, j, i_1, i_2, i_3, i_4\}| = 6$ and distinct $i$ such that $|\{i, j, i_1, i_2, i_3, i_4\}| = 5$. Hence, $d_w(v)$ is strictly greater than $d_w(w)$.

Therefore, by Theorem 1.1, $X_w$ is not rationally smooth if $w$ contains a bad pattern. □

**Lemma 4.4.** *If $w \in B_n$ avoids all patterns in* (12) *then $p_w(t)$ factors completely into symmetric factors of the form $(1+t+...+t^{\mu})$, hence $X_w$ is rationally smooth.*

*Proof.* First, if $w$ avoids all bad patterns then $p_w(t)$ factors according to at least one of the rules in Theorem 3.3. This follows from a careful analysis of cases:

1. If $w_d = \bar{n}$, then $w_1 > w_2 > \cdots > w_d$. Everything else is forbidden by the patterns:

$$\begin{array}{ccc} \bar{1}2\bar{3} & 1\bar{2}\bar{3} & 12\bar{3} \\ \bar{2}\bar{1}\bar{3} & \bar{2}1\bar{3} & 2\bar{1}\bar{3} \end{array}$$

2. If $w_d = \bar{n}$, either all $w_i$ are negative or all positive if $i < d$. From the above list of forbidden sequences one can see the only allowable patterns of length 3 ending in $\bar{3}$ are $21\bar{3}$ and $\bar{1}\bar{2}\bar{3}$.
3. If $w_d = \bar{n}$ and all $w_i$ are positive for $i < d$, then $w_i > 0$ for all $i > d$ also if $w$ avoids $1\bar{3}\bar{2}$ and $2\bar{3}\bar{1}$. By Rule 2 above, we also have $w_1 > w_2 > \cdots > w_d$. Hence, $p_w(t)$ factors using Rule 5.
4. If $w_d = \bar{n}$ and all $w_i$ are negative then $w_{d+1} > w_{d+2} > \cdots > w_n$ if $w$ avoids $\bar{3}\bar{2}\bar{1}$. By Rule 1, we also have $w_1 > w_2 > \cdots > w_d$, hence, $p_w(t)$ factors using Rule 3 or Rule 4.
5. If $w_d = n$ and $w_{d+1} > w_{d+2} > \cdots > w_n$ then $p_w(t)$ factors using Rule 1.
6. If $w_d = n$, $w$ is not decreasing after position $d$ and $w$ avoids all bad patterns then the patterns containing $n, n-1, \ldots, w_n$ must all be one of the following forms:

$$\begin{array}{ccc} \tilde{1}4\tilde{2}3 & \tilde{2}4\tilde{1}3 & 4\tilde{1}\tilde{2}3 \\ 4\tilde{1}32 & 42\tilde{1}3 & 43\tilde{1}2 \end{array} \tag{20}$$

   where $\tilde{i}$ is either $i$ or $\bar{i}$. Therefore, $w$ must contain a consecutive sequence ending in $w_n$ and $w_n$ must be positive. Hence, $p_w(t)$ factors using Rule 2.
7. If $w_d = \bar{n}$, $w_1, \ldots, \widehat{w_d}, \ldots w_n$ are not all positive or all negative, and $w$ avoids all bad patterns then the patterns containing $n, n-1, \ldots w_n$ must all be one of the forms given in (20) or one of the forms below:

$$\begin{array}{cccc} \bar{1}\bar{2}\bar{4}3 & \bar{1}\bar{3}\bar{4}2 & \bar{1}\bar{4}\tilde{2}3 & \bar{1}\bar{4}32 \\ \bar{2}\bar{3}\bar{4}1 & \bar{2}\bar{4}\tilde{1}3 & \bar{3}\bar{4}\tilde{1}2 & \bar{3}\bar{4}21 \\ \bar{4}\tilde{1}23 & \bar{4}\bar{1}\bar{2}3 & \bar{4}\tilde{1}32 & \bar{4}213 \\ \bar{4}312 & \bar{4}321 & & \end{array} \tag{21}$$

   In particular, $w$ must contain a consecutive sequence ending in $w_n$ and $w_n$ must be positive. Hence, $p_w(t)$ factors using Rule 2.

Second, if $w$ avoids all bad patterns then so does $w'$ where $p_w(t) = (1 + \cdots + t_\mu)p_{w'}$ from Theorem 3.3. This fact follows from the construction of $w'$ in each case. In Rules 1, 2, 3 and 4, all flattened patterns in $w'$ appear as patterns in $w$. In Rule 5, new patterns are created in $w'$ all starting with $n$. Examining the list of patterns in (12) one sees that 4231 is the only bad pattern that could have been created in $w'$ since $w' \in S_n$. If the pattern 4231 appears in $w'$ then one of the following patterns must have been in $w$: $\bar{4}231$, $2\bar{4}31$, $23\bar{4}1$, or $231\bar{4}$. However, each of these four patterns are bad patterns themselves or they contain the bad pattern $12\bar{3}$, contradicting the assumption that $w$ avoids all bad patterns. □

**Corollary 4.5.** *For any $w \in B_n$ such that $X_w$ is rationally smooth, the poset $B_w$ contains a subposet which is a product of chains.*

*Proof.* Corollary 3.9 shows that each time $p_w(t)$ factors there is a subposet of $[\hat{0}, w]$ on all of the vertices which is of the form $\tilde{B}_{w'} \times [k]$ where $[k]$ is a chain. Lemma 4.4 shows that if $w$ avoids the patterns in (12) then $p_w$ factors as well as $p_{w'}$ etc. □

## 5. Comparing factored formulas

In this section, we compare our factorizations for $p_w(t)$ if $X_w$ is rationally smooth with Carrell's generalization of the Kostant-Macdonald formula.

**Theorem 5.1.** [3, Sect. 5, Thm. I] *Let $W$ be any Weyl group, $w \in W$ and assume $X_w$ is smooth (not just rationally smooth). Then*

$$p_w(t) = \prod_{\beta \in R_+ \sigma_\beta \leq w} \frac{1 - t^{\mathrm{ht}(\beta)+1}}{1 - t^{\mathrm{ht}(\beta)}}. \tag{22}$$

*where $R_+$ are the positive roots and $\mathrm{ht}(\beta) = \sum k_i$ if $\beta = \sum k_i \alpha_i$ in terms of a basis of simple roots $\{\alpha_i\}$.*

**Theorem 5.2.** *Fix $w \in S_n$. Let $h_i$ be the number of positive roots in the set $\{\beta \in R_+ : \mathrm{ht}(\beta) = i$ and $\sigma_\beta \leq w\}$. If $p_w(t) = (1 + t + \cdots + t^{\mu_1})(1 + t + \cdots + t^{\mu_2}) \dots (1 + t + \cdots + t^{\mu_k})$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$, then the partition $\mu = \mu_1 \dots \mu_k$ is conjugate to the partition $h_1 \dots h_k$ and $h_i = 0$ for all $i > k$.*

*Proof.* The root system of type $A_{n-1}$ is given by the vectors $\pm(e_i - e_j)$ for $1 \leq i < j \leq n$. We chose a basis so that the vectors $e_i - e_j$ for $i < j$ are in the positive span, hence these are the positive roots. The height of $e_i - e_j$ is $j - i$. The reflection corresponding with $e_i - e_j$ is the transposition $t_{ij}$.

For any $u \in S_n$ such that $p_u(t) = (1+t+\cdots+t^{\mu_1})(1+t+\cdots+t^{\mu_2}) \dots (1+t+\cdots+t^{\mu_k})$, let $\mu(u)$ be the partition obtained by arranging $\mu_1, \dots, \mu_k$ in decreasing order. Let $h(u)$ be the partition given by $h_1, h_2, \dots, h_k$ where $h_i = \#\{\beta \in R_+ : \mathrm{ht}(\beta) = i$ and $\sigma_\beta \leq w\}$.

The theorem is clearly true for $S_1$, so assume it holds for all smooth (or equivalently rationally smooth) elements in $S_{n-1}$ by induction.

Note that for any reflection $\sigma$ we have $\sigma \leq w$ if and only if $\sigma \leq w^{-1}$ since $\sigma$ equals $\sigma^{-1}$. Therefore, we can assume $p_w$ factors according to the first rule of Theorem 3.2. Say $n = w_d > w_{d+1} > \ldots > w_n$ and $p_w(t) = (1 + t + \cdots + t^{n-d})p_v(t)$, then the Ferrers shape $\mu(w)$ is obtained from the shape $\mu(v)$ by adding a row of length $n - d$.

On the other hand, using Deodhar's rule for Bruhat order and the fact that $n = w_d > w_{d+1} > \cdots > w_n$, we have $t_{in} \leq w$ if and only if $d \leq i < n$. Furthermore, $t_{ij} \leq w$ for $1 \leq i < j < n$ if and only if $t_{ij} \leq v$. Hence, if the partition $h(v)$ is conjugate to $\mu(v)$, and $h(w)$ is obtained from $h(v)$ by adding 1 to each $h_i$ for $1 \leq i \leq n - d$, then $\mu(w)$ is the conjugate of $h(w)$. $\square$

## 6. Smoothness of Schubert Varieties

The following theorems complete the characterization of smoothness and rational smoothness for Schubert varieties for the classical groups.

**Theorem 6.1.** *Let $w$ be any element of $B_n$.*

1. *In type $B$, $X_w$ is smooth if and only if $X_w$ is rationally smooth and $w$ avoids $\bar{2}\bar{1}$.*
2. *In type $C$, $X_w$ is smooth if and only if $X_w$ is rationally smooth and $w$ avoids $1\bar{2}$.*

The proof will appear in [2] and relies on the characterization of rational smoothness given in Theorem 4.2 and the work of Lakshmibai [13] who has given a basis of the tangent space of a Schubert variety at a $T$-fixed point in terms of positive roots. Note that Theorem 6.1 does not substantially reduce the minimal number of patterns one needs to check in each case.

Dale Peterson (unpublished) has shown that rational smoothness is equivalent to smoothness of Schubert varieties in types $A$, $D$ and $E$.

**Theorem 6.2.** *Let $w \in D_n$, the Schubert variety $X_w$ is smooth (or equivalently rationally smooth) if and only if $w$ avoids the following list of patterns:*

$$
\begin{array}{ccccccc}
12\bar{3} & \bar{1}2\bar{3} & \bar{1}\bar{3}\bar{2} & 1\bar{3}\bar{2} & \bar{2}\bar{1}\bar{3} & \bar{3}\bar{2}\bar{1} & \\
\bar{1}4\bar{3}2 & \bar{2}1\bar{3}\bar{4} & 2\bar{1}\bar{3}\bar{4} & 21\bar{3}\bar{4} & \bar{2}\bar{3}1\bar{4} & 2\bar{3}1\bar{4} & \\
2\bar{4}31 & \bar{2}\bar{4}3\bar{1} & \bar{2}4\bar{3}\bar{1} & 24\bar{3}\bar{1} & 2\bar{4}3\bar{1} & \bar{2}4\bar{3}1 & \\
\bar{2}\bar{4}31 & 3\bar{1}\bar{2}\bar{4} & 31\bar{2}\bar{4} & 3\bar{2}1\bar{4} & 3\bar{2}\bar{4}1 & \bar{3}\bar{4}1\bar{2} & \\
3\bar{4}\bar{1}\bar{2} & \bar{3}412 & 34\bar{1}2 & \bar{3}4\bar{1}2 & 3412 & \bar{3}\bar{4}\bar{1}\bar{2} & \\
3\bar{4}1\bar{2} & \bar{3}\bar{4}\bar{2}1 & 34\bar{2}\bar{1} & \bar{3}4\bar{2}1 & 3\bar{4}\bar{2}1 & \bar{4}\bar{1}\bar{3}2 & \\
413\bar{2} & \bar{4}\bar{1}3\bar{2} & 4\bar{1}3\bar{2} & \bar{4}13\bar{2} & 4\bar{1}\bar{3}2 & \bar{4}1\bar{3}2 & 4\bar{2}1\bar{3} \\
4\bar{2}\bar{3}\bar{1} & \bar{4}\bar{2}\bar{3}1 & \bar{4}231 & 423\bar{1} & \bar{4}23\bar{1} & 4231 & 4\bar{2}\bar{3}1 \\
4\bar{3}\bar{1}\bar{2} & 4\bar{3}\bar{1}2 & \bar{4}\bar{3}12 & 4\bar{3}1\bar{2} & 4\bar{3}\bar{2}1 & &
\end{array}
\tag{23}
$$

This list of 55 patterns are all of lengths 3 or 4 as in the $B_n$ case. However, some of the patterns are not elements $D_4$ since they have a odd number of signs. The proof of Theorem 6.2 follows similar lines to the proof of Theorem 4.2; one can show that if $w \in D_n$ contains a bad pattern then there exists an explicit node in the Bruhat graph whose degree is too large, and conversely if w does not contain any bad patterns then its Poincaré polynomial factors into a product of symmetric factors. Below is a sufficient set of factoring rules used in the proof of Theorem 6.2.

**Proposition 6.3.** *Let $w \in D_n$, and assume $w_d = n$ and $w_n = e$. The Poincaré polynomial of $w$ factors in the form*

$$p_w(t) = (1 + t^1 + \cdots + t^\mu)p_{w'}. \tag{24}$$

*under the following circumstances:*

1. *If $w = w_0$ is the longest element in $D_n$, namely $\pm 1\bar{2}\bar{3}\dots\bar{n}$, then*

$$p_{w_0} = \prod_{k=1}^{n-1}(1 + t + \cdots + t^{k-1} + 2t^k + t^{k+1} + \cdots + t^{2k+1}) \tag{25}$$

2. *If $n = w_d > w_{d+1} > \cdots > w_n$, then $p_w$ factors with $w' = w\sigma_d \cdots \sigma_{n-1}$ and $\mu = n - d$.*
3. *If $w$ contains a consecutive sequence ending in $w_n = e$ and $e > 0$, then $p_w$ factors with $w' = \sigma_{n-1} \cdots \sigma_{e+1}\sigma_e w$ and $\mu = n - e$.*
4. *If $w$ contains exactly two negatives, say $w_1 < 0$ and $w_d < 0$ one of which is necessarily $\bar{n}$, and $w_2 > w_2 > \cdots > w_d$ then $p_w(t)$ factors with $w' = w\sigma_{d-1}\sigma_{d-2}\dots\sigma_2\sigma_{-1}$ and $\mu = d - 1$.*
5. *If $w$ contains exactly two negatives including $w_e = \bar{1}$, there exists $1 < d < e$ such that $w_d < 0$ and $w_1 > \cdots > w_d$, then $w' = w\sigma_{d-1}\sigma_{d-2}\dots\sigma_2\sigma_1$ and $\mu = d - 1$.*
6. *If $\bar{1}$ and $\bar{n}$ are the only two negatives, $\bar{n}$ appears further to the right than $\bar{1}$, and $w_1 > w_2 > \cdots > w_d = \bar{n}$, then $w' = w\sigma_{d-1}\sigma_{d-2}\dots\sigma_2\sigma_1$ and $\mu = d - 1$.*

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Author's address: Dept. of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139

*E-mail address*: billey@math.mit.edu