Vol. 96, No. 1 DUKE MATHEMATICAL JOURNAL © 1999

KOSTANT POLYNOMIALS AND THE COHOMOLOGY RING FOR G/B

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The Schubert calculus for G/B can be completely determined by a certain matrix related to the Kostant polynomials introduced in [1, Sect. 5]. The polynomials are defined by vanishing properties on the orbit of a regular point under the action of the Weyl group. For each element w in the Weyl group, the polynomials also have nonzero values on the orbit points corresponding to elements that are larger than w in the Bruhat order. Our main theorem is an explicit formula for these values. The matrix of orbit values can be used to determine the cup product for the cohomology ring for G/B, using only linear algebra or as described in [14].

1. Introduction. Let G be a semisimple Lie group, H be a Cartan subgroup, W be its corresponding Weyl group with generators $\sigma_1, \sigma_2, \dots, \sigma_n$, and B be a Borel subgroup. Let $\mathbb{C}[\mathbf{h}^*]$ be the algebra of polynomial functions on the Cartan subalgebra $\mathbf{h}$ over $\mathbb{C}$. Fix a regular element $\mathbf{O} \in \mathbf{h}$ such that $\alpha_i(\mathbf{O})$ is a positive integer for all simple roots α_i. Any Weyl group element v acts on the right on $\mathbf{O}$ by the action on the Cartan subalgebra. We define the following interpolating polynomials by their values on the orbit of $\mathbf{O}$.

Definition 1. A *Kostant polynomial* K_w is any element of $\mathbb{C}[\mathbf{h}^*]$ of degree $l(w)$ (nonhomogeneous) such that

$$K_w(\mathbf{O}v) = \begin{cases} 1, & v = w, \\ 0, & l(v) \leqslant l(w) \text{ and } v \neq w. \end{cases} \tag{1.1}$$

These polynomials were defined originally by Kostant and appear in [1, Thm. 5.9] for the finite case. They were later generalized by Kostant and Kumar in [14], there denoted $\xi_{w^{-1}}$. Kostant showed that K_w is unique modulo the ideal of all elements of $\mathbb{C}[\mathbf{h}^*]$ that vanish on the orbit of $\mathbf{O}$ under the Weyl group action. Furthermore, he showed that the highest homogeneous component of a Kostant polynomial represents a Schubert class in the cohomology ring of G/B. Indeed, Carrell has shown there is a direct connection between the ring of polynomials defined on the orbit $\mathbf{O}W$ and the cohomology ring of G/B. Namely, $H^*(G/B)$ is isomorphic to the graded ring canonically associated to the polynomial ring of

Received 18 March 1997. Revision received 13 August 1997.
1991 *Mathematics Subject Classification.* 05E15, 22E60, 14M15.
Author's work supported by a National Science Foundation Postdoctoral Fellowship.

the variety given by the set of points $\mathbf{O}W$ (see [4]). This isomorphism has also been given by Kostant and Kumar [14, Thm. 2.12] via the nil-Hecke ring.

The object of study for this paper is not precisely the Kostant polynomials themselves but instead the values of the Kostant polynomials on the points in the orbit of $\mathbf{O}$ under the Weyl group action. From the definition of K_w, we know $K_w(\mathbf{O}v)$ is zero if $l(v) \leqslant l(w)$ and $v \neq w$. However, the orbit values $K_w(\mathbf{O}v)$, if $l(v) > l(w)$, are not specified (though they are completely determined). Our main result, stated in Theorem 3, is an explicit formula for computing these orbit values, namely,

$$K_w(\mathbf{O}v) = \frac{1}{\pi_w} \sum_{b_{i_1} b_{i_2} \dots b_{i_k} \in R(w)} \prod_{j=1}^{k} \sigma_{b_1} \sigma_{b_2} \cdots \sigma_{b_{i_j - 1}} \alpha_{b_{i_j}}(\mathbf{O}), \tag{1.2}$$

where $\mathbf{b} = b_1 b_2 \cdots b_p$ is any fixed reduced word for v, $R(w)$ is the set of all reduced words for w, and the sum is over all sequences $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant p$ such that $b_{i_1} b_{i_2} \dots b_{i_k} \in R(w)$. The scalar factor π_w^{-1} appears in order to normalize $K_w(\mathbf{O}w) = 1$; see Section 4 for the definition. This formula is independent of the chosen reduced word for v and exhibits the strong connection between the Kostant polynomials and the Bruhat order.

In the case where G is SL_n, the Kostant polynomials are the double Schubert polynomials (multiplied by a scalar) introduced by Lascoux and Schützenberger in [17]; see also [20]. We give the precise connection in Remark 1 of Section 8 for all of the classical groups. Note that our main theorem in this case gives formulas for evaluating double Schubert polynomials at orbit points. These values were originally studied by Lascoux and Schützenberger in [18] and [19]. Recently, Lascoux, Leclerc, and Thibon [15] have explained the interesting connection between the Yang-Baxter equations and the specializations of double Schubert polynomials, using techniques similar to those used in Section 3.

We begin with a review of a few results from the renowned paper [1] by Bernstein, Gelfand, and Gelfand. In Section 3, we introduce the nil-Coxeter algebra in order to prove that the orbit value formula is independent of the choice of reduced word. The explicit formula for the orbit values is stated as a theorem and is proved in Section 4. The orbit value formula is generalized in Theorem 4 to give explicit formulas for the ξ^v-functions given by Kostant and Kumar [14]. In Section 6 we explicitly show how the matrix of orbit values is related to the cup product in the cohomology ring for the flag manifold G/B, following Kostant's original notion, which was extended in [14]. This beautifully demonstrates the relationship between the structure constants for Schubert cycles in the cohomology ring of G/B and the D matrix (see Proposition 5.5 for the general definition). An example is given in Section 7, which may be helpful to the reader.

While we prove Theorem 4 only for the finite case (i.e., for G semisimple), the formula for the orbit values is equally valid for all Kac-Moody Lie algebras. In fact, Kumar has extended Theorem 4 to the case where W is the Weyl group of an arbitrary Kac-Moody Lie algebra. We include his proof in the appendix.

The main results of this paper were announced in [2].

2. Divided difference equations. The divided difference equations, defined by Bernstein, Gelfand, and Gelfand, are used to recursively compute the Schubert classes starting from the unique Schubert class of dimension zero and working up to higher dimensions. In this section we show that these operators also act nicely on the Kostant polynomials. The divided difference equations for Kostant polynomials also lead to a recursive method for computing the matrix of orbit values for these polynomials. Much of the theory of Schubert classes and divided difference operators was independently conceived by Demazure [5] around the same time. We cite just one source for simplicity.

Given a semisimple Lie group and a Cartan subgroup there is a root system Δ contained in some ambient vector space V with a positive definite symmetric bilinear form (α, β). (See [11] for details on Lie groups and root systems.) Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be a choice of simple roots in the root system. Let Δ_+ (respectively, Δ_-) be the positive roots (respectively, negative roots) with respect to this choice of simple roots. Let R be the ring of polynomials in the simple roots with rational coefficients, that is, $\mathbb{Q}[\alpha_1, \alpha_2, \ldots, \alpha_n]$. R can be realized as a subring of the ring of all polynomial functions on the Cartan subalgebra $\mathbb{C}[\mathbf{h}^*]$.

For each simple root α_i, there is a corresponding simple reflection σ_i over the hyperplane perpendicular to α_i. This action is given explicitly by $\sigma_i(v) = v - \langle v, \alpha_i \rangle \alpha_i$, where

$$\langle \alpha, \beta \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)}. \tag{2.1}$$

The Weyl group W corresponding to the root system Δ is generated by the simple reflections. The Weyl group of a semisimple Lie group is always finite, and there exists a unique element w_0 in W with longest length.

For each $1 \leqslant i \leqslant n$, the *divided difference operator* $\partial_i : R \to R$ acts on $f \in R$ by[1]

$$\partial_i f = \frac{f - \sigma_i f}{-\alpha_i}. \tag{2.2}$$

Let I be the ideal generated by the Weyl group invariants of positive degree. Every polynomial in I is killed by any divided difference operator. Hence, each ∂_i acts on the quotient R/I as well. If $\alpha : H^*(G/B, \mathbb{Q}) \to R/I$ is the Borel iso-

[1] This notation differs from the notation in [1] by a sign. The result is that we have interchanged the positive and the negative roots from those used in [1].

morphism and $p : H_*(G/B, \mathbb{Q}) \to H^*(G/B, \mathbb{Q})$ is the Poincaré duality, then define the *Schubert class* of w, $\mathfrak{S}_w$, in R/I to be the image of the Schubert cycle in $H_*(G/B, \mathbb{Q})$ corresponding to ww_0 under the composition of p and α.

PROPOSITION 2.1 [1, Thms. 3.14 and 3.15]. *The Schubert classes $\mathfrak{S}_w \in R/I$ have the properties*

$$\partial_i \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{w\sigma_i}, & l(w) > l(w\sigma_i), \\ 0, & l(w) < l(w\sigma_i). \end{cases} \tag{2.3}$$

Furthermore, the Schubert class $\mathfrak{S}_{w_0}$ is given by

$$\mathfrak{S}_{w_0} = \frac{(-1)^{|W|}}{|W|} \prod_{\gamma \in \Delta_+} \gamma (\operatorname{mod} I). \tag{2.4}$$

The Schubert classes and the highest homogeneous component of the Kostant polynomials are the same modulo I up to a constant. Below we give the correct statement in our notation. An example of computing this coefficient is given in Section 7.

PROPOSITION 2.2 [1]. *Let K_w^0 be the form of highest degree in K_w. Then the image of K_w^0 in R/I is equal to*

$$\left(\prod_{\gamma \in \Delta_+ \cap w\Delta_-} \gamma(\mathbf{O})^{-1} \right) \mathfrak{S}_w. \tag{2.5}$$

The next theorem shows that the divided difference operators satisfy a modified recursive formula. This theorem was also stated by Kostant and Kumar in [14] in a different form. We include the proof because it demonstrates the techniques we use to compute the Kostant polynomials.

THEOREM 1. *For $w \in W$, the divided difference operator ∂_i acts on K_w as follows:*

$$\partial_i K_w = \begin{cases} \dfrac{K_{w\sigma_i}}{\alpha_i(\mathbf{O}w\sigma_i)}, & l(w) > l(w\sigma_i), \\ 0, & l(w) < l(w\sigma_i). \end{cases} \tag{2.6}$$

Proof. The Kostant polynomials are uniquely defined by their orbit values. Therefore, we evaluate $\partial_i K_w$ at different points in the orbit. For any $v, w \in W$ and any polynomial $P \in R$, one can check that the left action of w on P corresponds with a right action on the orbit point $\mathbf{O}v$, that is,

$$P(\mathbf{O}vw) = [wP](\mathbf{O}v) = [vwP](\mathbf{O}). \tag{2.7}$$

Applying the divided difference operator, we have

$$\partial_i K_w(\mathbf{O}v) = \frac{K_w - \sigma_i K_w}{-\alpha_i}(\mathbf{O}v) \tag{2.8}$$

$$= \frac{K_w(\mathbf{O}v) - K_w(\mathbf{O}v\sigma_i)}{-\alpha_i(\mathbf{O}v)}. \tag{2.9}$$

If $l(v) < l(w) - 1$, then both terms in the numerator of (2.8) are zero by the vanishing property (1.1), so $\partial_i K_w(\mathbf{O}v) = 0$. Furthermore, if $l(v) = l(w) - 1$ and $w \neq v\sigma_i$, then $\partial_i K_w(\mathbf{O}v) = 0$. Finally, if $w = v\sigma_i$ and $l(v) = l(w) - 1$, then by (1.1) and (2.7) we have

$$\partial_i K_w(\mathbf{O}v) = \frac{-K_w(\mathbf{O}v\sigma_i)}{-\alpha_i(\mathbf{O}v)} = \frac{1}{\alpha_i(\mathbf{O}v)}. \tag{2.10}$$

Therefore, $\partial_i K_w$ has degree equal to $l(w) - 1$ and evaluates to zero on all orbit elements corresponding to v such that $l(v) < l(w)$ and $v \neq w\sigma_i$. Hence, $\partial_i K_w$ is equal to $K_{w\sigma_i}/(\alpha_i(\mathbf{O}w\sigma_i))$.

COROLLARY 2.3. *The orbit values $K_w(\mathbf{O}v)$ can be computed recursively from the top down. Namely, $K_{w_0}(\mathbf{O}v)$ is 1 if $v = w_0$ and is 0 otherwise. If $w \neq w_0$, there exists an i such that $l(w) < l(w\sigma_i)$; let u equal $w\sigma_i$. Then for any $v \in W$,*

$$K_w(\mathbf{O}v) = \alpha_i(\mathbf{O}w)\partial_i K_u(\mathbf{O}v) = \frac{K_u(\mathbf{O}v) - K_u(\mathbf{O}v\sigma_i)}{-\alpha_i(\mathbf{O}v)}\alpha_i(\mathbf{O}w). \tag{2.11}$$

Corollary 2.3 gives an algorithm to compute the values $K_w(\mathbf{O}v)$. We use this corollary in Section 4 to prove the formula for orbit values.

3. The nil-Coxeter algebra. In this section, we allow W to be the Weyl group for an arbitrary Kac-Moody Lie algebra. Let $\mathscr{A} = \mathscr{A}_W$ be the nil-Coxeter algebra for W over the field K. In other words, if W is generated by $\sigma_1, \sigma_2, \ldots, \sigma_n$ with relations given by $(\sigma_i\sigma_j)^{m_{ij}} = 1$, then $\mathscr{A}$ is generated as an algebra over $R = \mathbb{Q}[\alpha_1, \ldots, \alpha_n]$ by $u_1, u_2, \ldots, u_n$ with the relations

$$\underbrace{u_i u_j u_i u_j \ldots}_{m_{ij}\text{ factors}} = \underbrace{u_j u_i u_j u_i \ldots}_{m_{ij}\text{ factors}} \quad \text{for } i \neq j, \tag{3.1}$$

$$u_i^2 = 0. \tag{3.2}$$

As a vector space over R, a basis for $\mathscr{A}$ is given by $\{u_w : w \in W\}$, where u_w represents the equivalent products $u_{a_1}u_{a_2}\cdots u_{a_p}$ for any $a_1a_2\cdots a_p \in R(w)$. The Weyl group acts on $\mathscr{A}$ by acting on the elements in R, and the generators u_i are fixed by all elements in the Weyl group.

Following the notation of Fomin and Kirillov [8, Sect. 1], we define the Yang-Baxter operators $h_i : R \to \mathscr{A}$ by

$$h_i(x) = e^{xu_i} = 1 + xu_i. \tag{3.3}$$

The relations among the Weyl group generators impose relations on the $h_i(x)$'s as well. It is well known that a minimal set of relations among the generators of a Weyl group are of the form $(\sigma_i\sigma_j)^{m_{ij}} = 1$. If W is the Weyl group of a semi-simple Lie algebra, then the only possibilities for m_{ij} are $2, 3, 4$, or 6. Note, if $m_{ij} = 2, 3, 4, 6$, then $\langle\alpha_i, \alpha_j\rangle\langle\alpha_j, \alpha_i\rangle$ is $0, 1, 2, 3$, respectively. If the Lie algebra is any Kac-Moody Lie algebra, then ∞ is also a possibility for m_{ij}. If $m_{ij} = \infty$, then $\langle\alpha_i, \alpha_j\rangle\langle\alpha_j, \alpha_i\rangle \geqslant 4$. See [13, Prop. 3.13] for details.

PROPOSITION 3.1 [8]. *The Yang-Baxter operators satisfy the following Yang-Baxter equations:*

$$h_i(x)h_j(y) = h_j(y)h_i(x), \qquad \textit{if } (\sigma_i\sigma_j)^2 = 1, \tag{3.4}$$

$$h_i(x)h_j(x+y)h_i(y) = h_j(y)h_i(x+y)h_j(x), \qquad \textit{if } (\sigma_i\sigma_j)^3 = 1, \tag{3.5}$$

$$\begin{aligned} &h_i(x)h_j(x+y)h_i(x+2y)h_j(y) \\ &\quad = h_j(y)h_i(x+2y)h_j(x+y)h_i(x), \qquad \textit{if } (\sigma_i\sigma_j)^4 = 1, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &h_i(x)h_j(3x+y)h_i(2x+y)h_j(3x+2y)h_i(x+y)h_j(y) \\ &\quad = h_j(y)h_i(x+y)h_j(3x+2y)h_i(2x+y)h_j(3x+y)h_i(x), \quad \textit{if } (\sigma_i\sigma_j)^6 = 1. \end{aligned} \tag{3.7}$$

It is well known (see [12, p. 14]) that the set of roots $\{\sigma_{b_1}\sigma_{b_2}\ldots\sigma_{b_{k-1}}\alpha_{b_k} : 1 \leqslant k \leqslant p\}$ is equal to $\Delta_+ \cap v\Delta_-$ and hence is independent of the choice of reduced word. We define a family of polynomials that are closely related to this set.

Definition 2. For any $v \in W$ and any reduced word $\mathbf{a} = a_1a_2\cdots a_p$ for v, define a *root polynomial* for $\mathbf{a}$ in the nil-Coxeter algebra $\mathscr{A}$ by

$$\mathfrak{R}_{\mathbf{a}} = \prod_{i=1}^{p} h_{a_i}(\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_{i-1}}\alpha_{a_i}). \tag{3.8}$$

For example, if the root system is of type A_2, the Weyl group is the symmetric group S_3. Let α_1 and α_2 be the simple roots. For $i = 1, 2$, $\sigma_i\alpha_i = -\alpha_i$ and $\sigma_i\alpha_j = \alpha_1 + \alpha_2$ for i different from j. The word 121 is a reduced word of the permutation

[3, 2, 1] (written in one-line notation). Then R_{121} is given by

$$\mathfrak{R}_{121} = (1+\alpha_1 u_1)(1+\sigma_1\alpha_2 u_2)(1+\sigma_1\sigma_2\alpha_1 u_1) \tag{3.9}$$

$$= (1+\alpha_1 u_1)(1+(\alpha_1+\alpha_2)u_2)(1+\alpha_2 u_1) \tag{3.10}$$

$$= 1+(\alpha_1+\alpha_2)(u_{[2,1,3]}+u_{[1,3,2]})+(\alpha_1^2+\alpha_1\alpha_2)u_{[2,3,1]} \tag{3.11}$$

$$+(\alpha_1\alpha_2+\alpha_2^2)u_{[3,1,2]}+(\alpha_1^2\alpha_2+\alpha_1\alpha_2^2)u_{[3,2,1]}. \tag{3.12}$$

In fact, we show in the next theorem that $\mathfrak{R}_{\mathbf{a}}$ for $\mathbf{a} \in R(v)$ depends only on the Weyl group element v and not on the choice of reduced word. Therefore, we can define the *root polynomial* for v, $\mathfrak{R}_v$ to be $\mathfrak{R}_{\mathbf{a}}$ for any $\mathbf{a} \in R(v)$.

THEOREM 2. *For any $v \in W$, choose any reduced word $\mathbf{a} = a_1a_2\cdots a_p \in R(v)$. Then*

$$\mathfrak{R}_v = \prod_{i=1}^{p} h_{a_i}(\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_{i-1}}\alpha_{a_i}) \tag{3.13}$$

is well defined.

Independently, Stembridge has shown that this theorem holds for all Coxeter groups [23]. His proof does not depend on case-by-case computations.

Proof. It is well known that the following graph is connected for any $v \in W$: *Vertices* $= R(v)$ and *Edges* $= \{(\mathbf{a},\mathbf{b}) \mid \mathbf{a},\mathbf{b}\}$ differ by a simple relation of the form $(ij)^m = (ji)^m$. Therefore, we only need to show that (3.13) is the same for two reduced words $\mathbf{a}$ and $\mathbf{b}$, which differ by a simple relation. Say $\mathbf{a} = \mathbf{r}\cdot\mathbf{s}\cdot\mathbf{t}$ and $\mathbf{b} = \mathbf{r}\cdot\mathbf{s}'\cdot\mathbf{t}$, where $\mathbf{r}$ and $\mathbf{t}$ are the initial and final subsequences that $\mathbf{a}$ and $\mathbf{b}$ have in common, $\mathbf{s} = iji\ldots$, $\mathbf{s}' = jij\ldots$, and $\mathbf{s}$, $\mathbf{s}'$ are reduced words for the same element in W.

Let $\sigma(a_1\cdots a_p)$ denote the element of W obtained from the product $\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_p}$. One can easily verify that for any position i

$$\mathfrak{R}_{a_1a_2\cdots a_p} = \mathfrak{R}_{a_1\cdots a_i}\cdot\sigma(a_1\cdots a_i)\mathfrak{R}_{a_{i+1}\cdots a_p}, \tag{3.14}$$

since $(1+\sigma_i\alpha u) = \sigma_i(1+\alpha u)$. Hence,

$$\mathfrak{R}_{\mathbf{a}} = \mathfrak{R}_{\mathbf{r}}\cdot\sigma(\mathbf{r})\mathfrak{R}_{\mathbf{s}}\cdot\sigma(\mathbf{r})\sigma(\mathbf{s})\mathfrak{R}_{\mathbf{t}}, \tag{3.15}$$

$$\mathfrak{R}_{\mathbf{b}} = \mathfrak{R}_{\mathbf{r}}\cdot\sigma(\mathbf{r})\mathfrak{R}_{\mathbf{s}'}\cdot\sigma(\mathbf{r})\sigma(\mathbf{s}')\mathfrak{R}_{\mathbf{t}}. \tag{3.16}$$

Since $\sigma(\mathbf{s}) = \sigma(\mathbf{s}')$, in order to show $\mathfrak{R}_{\mathbf{a}} = \mathfrak{R}_{\mathbf{b}}$ we only need to show $\mathfrak{R}_{\mathbf{s}} = \mathfrak{R}_{\mathbf{s}'}$.

Assuming W is the Weyl group for a Kac-Moody Lie algebra, we know that $\mathbf{s}$ and $\mathbf{s}'$ must be of one of the following forms:

$$\begin{aligned} &\mathbf{s} = ij, &&\mathbf{s}' = ji,\\ &\mathbf{s} = iji, &&\mathbf{s}' = jij,\\ &\mathbf{s} = ijij, &&\mathbf{s}' = jiji,\\ &\mathbf{s} = ijijij, &&\mathbf{s}' = jijiji. \end{aligned} \tag{3.17}$$

For each case we compute

$$\mathfrak{R}_{\mathbf{s}} = h_i(\alpha_i)h_j(\sigma_i\alpha_j)h_i(\sigma_i\sigma_j\alpha_i)\cdots, \tag{3.18}$$

$$\mathfrak{R}_{\mathbf{s}'} = h_j(\alpha_j)h_i(\sigma_j\alpha_i)h_j(\sigma_j\sigma_i\alpha_j)\cdots \tag{3.19}$$

and then use Proposition 3.1 to show equality. One could alternatively expand the equations below and compare terms. For these computations, we rely heavily on the table in [11, Ch. 3, p. 45], on the formula for a reflection $\sigma_i v = v - \langle v, \alpha_i\rangle\alpha_i$ for any vector v in the span of the simple roots, and on the fact that $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_{p-1}}\alpha_{i_p} = \alpha_{i_1}$ if $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_p} = \sigma_{i_2}\cdots\sigma_{i_{p-1}}$ (see [12, Thm. 1.7]).

Case 1: ($\mathbf{s} = ij$, $\mathbf{s}' = ji$). If $(\sigma_i\sigma_j)^2 = 1$, then $\langle\alpha_i, \alpha_j\rangle = \langle\alpha_j, \alpha_i\rangle = 0$. Hence, $\sigma_i\alpha_j = \alpha_j$ and $\sigma_j\alpha_i = \alpha_i$. Therefore,

$$\mathfrak{R}_{\mathbf{s}} = h_i(\alpha_i)h_j(\alpha_j), \tag{3.20}$$

$$\mathfrak{R}_{\mathbf{s}'} = h_j(\alpha_j)h_i(\alpha_i), \tag{3.21}$$

so by (3.4), we have $\mathfrak{R}_{\mathbf{s}} = \mathfrak{R}_{\mathbf{s}'}$.

Case 2: ($\mathbf{s} = iji$, $\mathbf{s}' = jij$). If $(\sigma_i\sigma_j)^3 = 1$, then $\langle\alpha_i, \alpha_j\rangle = \langle\alpha_j, \alpha_i\rangle = -1$. Hence, $\sigma_i\alpha_j = \alpha_i + \alpha_j$ and $\sigma_j\alpha_i = \alpha_i + \alpha_j$. Therefore,

$$\mathfrak{R}_{\mathbf{s}} = h_i(\alpha_i)h_j(\alpha_i + \alpha_j)h_i(\alpha_j), \tag{3.22}$$

$$\mathfrak{R}_{\mathbf{s}'} = h_j(\alpha_j)h_i(\alpha_i + \alpha_j)h_j(\alpha_i), \tag{3.23}$$

so by (3.5), we have $\mathfrak{R}_{\mathbf{s}} = \mathfrak{R}_{\mathbf{s}'}$.

Case 3: ($\mathbf{s} = ijij$, $\mathbf{s}' = jiji$). Assume without loss of generality that α_i is the short root, then $\langle\alpha_i, \alpha_j\rangle = -1$ and $\langle\alpha_j, \alpha_i\rangle = -2$. Using the fact that $\sigma_i\sigma_j\sigma_i\alpha_j = \alpha_j$, we compute the following table:

$$\begin{aligned} &\sigma_i\sigma_j\sigma_i\alpha_j = \alpha_j, &&\sigma_j\sigma_i\sigma_j\alpha_i = \alpha_i,\\ &\sigma_j\sigma_i\alpha_j = \sigma_i\alpha_j = 2\alpha_i + \alpha_j, &&\sigma_i\sigma_j\alpha_i = \sigma_j\alpha_i = \alpha_i + \alpha_j. \end{aligned} \tag{3.24}$$

Therefore,

$$\mathfrak{R}_{\mathbf{s}} = h_i(\alpha_i)h_j(2\alpha_i + \alpha_j)h_i(\alpha_i + \alpha_j)h_j(\alpha_j), \tag{3.25}$$

$$\mathfrak{R}_{\mathbf{s}'} = h_j(\alpha_j)h_i(\alpha_i + \alpha_j)h_j(2\alpha_i + \alpha_j)h_i(\alpha_i), \tag{3.26}$$

so by (3.6), we have $\mathfrak{R}_{\mathbf{s}} = \mathfrak{R}_{\mathbf{s}'}$.

Case 4: ($\mathbf{s} = ijijij$, $\mathbf{s}' = jijiji$). Assume without loss of generality that α_i is the short root, then $\langle \alpha_i, \alpha_j \rangle = -1$ and $\langle \alpha_j, \alpha_i \rangle = -3$. Again we compute the expansion of certain positive roots into the sum of α_i and α_j:

$$\begin{array}{ll} \sigma_i\sigma_j\sigma_i\sigma_j\sigma_i\alpha_j = \alpha_j, & \sigma_j\sigma_i\sigma_j\sigma_i\sigma_j\alpha_i = \alpha_i, \\ \sigma_j\sigma_i\sigma_j\sigma_i\alpha_j = \sigma_i\alpha_j = 3\alpha_i + \alpha_j, & \sigma_i\sigma_j\sigma_i\sigma_j\alpha_i = \sigma_j\alpha_i = \alpha_i + \alpha_j, \\ \sigma_i\sigma_j\sigma_i\alpha_j = \sigma_j\sigma_i\alpha_j = 3\alpha_i + 2\alpha_j, & \sigma_j\sigma_i\sigma_j\alpha_i = \sigma_i\sigma_j\alpha_i = 2\alpha_i + \alpha_j. \end{array} \tag{3.27}$$

Therefore,

$$\mathfrak{R}_{\mathbf{s}} = h_i(\alpha_i)h_j(3\alpha_i + \alpha_j)h_i(2\alpha_i + \alpha_j)h_j(3\alpha_i + 2\alpha_j)h_i(\alpha_i + \alpha_j)h_j(\alpha_j), \tag{3.28}$$

$$\mathfrak{R}_{\mathbf{s}'} = h_j(\alpha_j)h_i(\alpha_i + \alpha_j)h_j(3\alpha_i + 2\alpha_j)h_i(2\alpha_i + \alpha_j)h_j(3\alpha_i + \alpha_j)h_i(\alpha_i), \tag{3.29}$$

so by (3.7), we have $\mathfrak{R}_{\mathbf{s}} = \mathfrak{R}_{\mathbf{s}'}$. □

4. Orbit value formula. In this section, we prove the orbit value formula (1.2) in a slightly altered form. Instead of using the Kostant polynomials as originally defined, we prefer to work with a modified version $\tilde{K}_w$. We begin with some preliminary notation, state the relationship between $\tilde{K}_w$ and K_w, and prove the orbit value theorem for $\tilde{K}_w$.

Let $v \in W$, and fix a reduced word $b_1b_2 \dots b_p \in R(v)$. Recall that the roots in the set $\Delta_+ \cap v\Delta_-$ are given by $\{\sigma_{b_1}\sigma_{b_2}\cdots\sigma_{b_{i-1}}\alpha_{b_i} : 1 \leqslant i \leqslant p\}$. In other words, for each initial sequence of the chosen reduced word, $\sigma_{b_1}\sigma_{b_2}\dots\sigma_{b_{j-1}}\alpha_{b_j}$ is a positive root in the set $\Delta_+ \cap v\Delta_-$. Let

$$r_{\mathbf{b}}(j) = r_{b_1b_2\dots b_p}(j) = \sigma_{b_1}\sigma_{b_2}\dots\sigma_{b_{j-1}}(\alpha_{b_j}) \tag{4.1}$$

denote the jth positive root. Let π_v be the polynomial in $\mathbb{Q}[\mathbf{h}^*]$ obtained as

$$\pi_v = \prod_{\gamma \in \Delta_+ \cap v\Delta_-} \gamma. \tag{4.2}$$

Note that π_v is equal to the product $r_{\mathbf{b}}(1)r_{\mathbf{b}}(2)\cdots r_{\mathbf{b}}(p)$ if $\mathbf{b}$ has length p.

Next, we introduce a variation on the Kostant polynomial $\tilde{K}_v$, which equals $\pi_v(\mathbf{O})K_v$. This allows us to get rid of the denominator in (1.2). In other words, we define $\tilde{K}_v$ in analogy with the Kostant polynomials by

$$\tilde{K}_v(\mathbf{O}w) = \begin{cases} \pi_v(\mathbf{O}), & w = v, \\ 0, & l(w) \leqslant l(v) \text{ and } w \neq v. \end{cases} \tag{4.3}$$

THEOREM 3. *Let $v, w \in W$, and fix a reduced word $\mathbf{b} = b_1 b_2 \ldots b_p$ for w. The orbit values of $\tilde{K}_v$ are given by*

$$\tilde{K}_v(\mathbf{O}w) = \sum_{b_{i_1} b_{i_2} \ldots b_{i_k} \in R(v)} r_{\mathbf{b}}(i_1) r_{\mathbf{b}}(i_2) \cdots r_{\mathbf{b}}(i_k)|_{\mathbf{O}}, \tag{4.4}$$

where $r_{\mathbf{b}}(j)$ is defined by (4.1), the sum is over all sequences $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant p$ such that $b_{i_1} b_{i_2} \ldots b_{i_k} \in R(v)$, and we evaluate the right-hand side at $\mathbf{O}$. Furthermore, the sum in (4.4) is independent of the choice of $\mathbf{b} \in R(w)$.

The proof follows two key lemmas.

LEMMA 4.1. *For $v, w \in W$, choose any reduced word $\mathbf{b} = b_1 b_2 \cdots b_p \in R(w)$, and define*

$$\zeta(v, w) = \sum_{b_{i_1} b_{i_2} \ldots b_{i_k} \in R(v)} r_{\mathbf{b}}(i_1) r_{\mathbf{b}}(i_2) \cdots r_{\mathbf{b}}(i_k). \tag{4.5}$$

$\zeta(v, w)$ is independent of the choice of $\mathbf{b} \in R(w)$.

Proof. The sum in (4.5) is the coefficient of u_v in the root polynomial $\mathfrak{R}_w$ from Section 3. It was shown in Theorem 2 that $\mathfrak{R}_w$ is well defined for any choice of reduced word. Therefore, the coefficient of u_v in $\mathfrak{R}_w$ is also independent of our choice of reduced word for w. □

LEMMA 4.2. *Let $\zeta(v, w)$ be the polynomial defined in (4.5), and let $\mathscr{A}_i$ act on $\zeta(v, w)$ by*

$$\mathscr{A}_i \zeta(v, w) = \frac{\zeta(v, w) - \zeta(v, w\sigma_i)}{-w\alpha_i}. \tag{4.6}$$

Then we have

$$\mathscr{A}_i \zeta(v, w) = \begin{cases} \zeta(v\sigma_i, w), & v > v\sigma_i, \\ 0, & v < v\sigma_i. \end{cases} \tag{4.7}$$

Proof. Equation (4.7) can be obtained easily from the root polynomial. From (3.14), we have

$$\mathfrak{R}_w = \mathfrak{R}_{w\sigma_i} \cdot w\sigma_i(1 + \alpha_i u_i) = \mathfrak{R}_{w\sigma_i} \cdot (1 - w\alpha_i u_i). \tag{4.8}$$

Comparing coefficients of u_v, we obtain

$$\zeta(v,w) = \begin{cases} \zeta(v, w\sigma_i) - (w\alpha_i)\zeta(v\sigma_i, w\sigma_i), & v > v\sigma_i, \\ \zeta(v, w\sigma_i), & v < v\sigma_i. \end{cases} \tag{4.9}$$

Now, (4.7) follows from (4.9) by rearranging terms and substituting $\zeta(v\sigma_i, w)$ for $\zeta(v\sigma_i, w\sigma_i)$ in the case $v > v\sigma_i$. □

Proof of Theorem 3. We show that (4.4) holds by decreasing induction on the length of v. For the longest element in W, w_0, we know $\tilde{K}_{w_0}(\mathbf{O}w_0) = \pi_{w_0}(\mathbf{O})$ and $\tilde{K}_{w_0}(\mathbf{O}w) = 0$ for all $w \in W$ such that $l(w) < l(w_0)$. This agrees with (4.4) since for any $\mathbf{b} = b_1 b_2 \ldots b_p \in R(w_0)$, there is exactly one term in the sum

(4.10)
$$\sum_{b_{i_1} b_{i_2} \ldots b_{i_k} \in R(w_0)} r_{\mathbf{b}}(i_1) r_{\mathbf{b}}(i_2) \cdots r_{\mathbf{b}}(i_k) = \begin{cases} \pi_{w_0}, & b_1 b_2 \ldots b_p \in R(w_0), \\ 0, & b_1 b_2 \ldots b_p \in R(w) \text{ and } w \neq w_0. \end{cases}$$

Therefore, we can assume by induction that (4.4) holds for all $u \in W$ such that $l(u) > l(v)$. Let u be $v\sigma_i$ for some fixed i such that $l(u) = l(v) + 1$. Then we have the following chain of equalities:

$$\tilde{K}_v(\mathbf{O}w) = \partial_i \tilde{K}_u(\mathbf{O}w) \qquad \text{(by Corollary 2.3)} \tag{4.11}$$

$$= \frac{\tilde{K}_u(\mathbf{O}w) - \tilde{K}_u(\mathbf{O}w\sigma_i)}{-w\alpha_i(\mathbf{O})} \tag{4.12}$$

$$= \left. \frac{\zeta(u,w) - \zeta(u, w\sigma_i)}{-w\alpha_i} \right|_{\mathbf{O}} \qquad \text{(by induction)} \tag{4.13}$$

$$= \zeta(v,w)|_{\mathbf{O}} \qquad \text{(by Lemma 4.2)} \tag{4.14}$$

$$= \sum_{b_{i_1} b_{i_2} \ldots b_{i_k} \in R(v)} r_{\mathbf{b}}(i_1) r_{\mathbf{b}}(i_2) \cdots r_{\mathbf{b}}(i_k)|_{\mathbf{O}} \qquad \text{(by definition (4.5)).} \tag{4.15}$$ □

COROLLARY 4.3. *The value $\tilde{K}_v(\mathbf{O}w) = \pi_v(\mathbf{O}) K_v(\mathbf{O}w)$ is a nonnegative integer, provided $\alpha_i(\mathbf{O})$ is a positive integer for each simple root α_i.*

Proof. Each positive root γ is the sum of simple roots, so the value $\gamma(\mathbf{O})$ is also a nonnegative integer. Therefore, (4.4) evaluated at $\mathbf{O}$ is the sum of products of nonnegative integers. □

The following corollaries were also shown in [14].

COROLLARY 4.4. *The orbit values $K_v(\mathbf{O}w)$ and $K_v(\mathbf{O}w\sigma_i)$ (respectively, $\zeta(v,w)$ and $\zeta(v,w\sigma_i)$) are equal if and only if $l(v) < l(v\sigma_i)$.*

Proof. This follows directly from Lemma 4.2. □

COROLLARY 4.5. *The orbit value $K_v(\mathbf{O}w)$ (respectively, $\zeta(v,w)$) is different from zero if and only if $v \leqslant w$ in the Bruhat order.*

Proof. By the definition of Bruhat order, any reduced word for w has a subsequence that is a reduced word for v if and only if $v \leqslant w$ in Bruhat order. Therefore, the corollary follows directly from Theorem 3 (respectively, the definition of $\zeta(v,w)$ given in (4.5)). □

5. Kostant and Kumar's ξ-functions. In this section we introduce the family of functions ξ^v for $v \in W$ defined by Kostant and Kumar. Using the orbit value formula in Theorem 3, we give explicit formulas for the values of ξ^v on elements in W. The ξ^v-functions are interesting because their product expansions are related to the product expansion of the $\tilde{K}_v$'s, yet they are defined independently from a choice of orbit point or any quotient ideal. We use the same notation as in [14] wherever possible. The exception to this rule is that we have interchanged the roles of w and w^{-1} in all of their formulas.

Definition 3. Let $\mathbf{Q}$ be the field of rational functions on the Cartan subalgebra $\mathbf{h}$, that is, the quotient field of $R = \mathbb{Q}[\alpha_1,\dots,\alpha_n]$. Let Ω be the $\mathbf{Q}$-module of all functions from W to $\mathbf{Q}$.

PROPOSITION 5.1 [14, Props. 4.20 and 4.24]. *There exists a family of functions $\xi^v : W \to R$ for $v \in W$ with the following properties.*

(i) *$\xi^v(w)$ equals zero unless $v \leqslant w$ and $\xi^w(w) = \prod_{\gamma\in\Delta_+\cap w\Delta_-}\gamma = \pi_w$.*

(ii) *Let $\mathscr{A}_i$ act on $\psi \in \Omega$ by*

$$(\mathscr{A}_i\psi)w = \frac{\psi(w) - \psi(w\sigma_i)}{-w\alpha_i}. \tag{5.1}$$

Then we have

$$\mathscr{A}_i\xi^v = \begin{cases} \xi^{v\sigma_i}, & v > v\sigma_i, \\ 0, & v < v\sigma_i. \end{cases} \tag{5.2}$$

In [14], the functions ξ^v were defined to be dual to their $\bar{x}_v$ elements. However, properties (i) and (ii) in Proposition 5.1 characterize the ξ^v-functions uniquely for the finite Weyl groups. We state an explicit formula for $\xi^v(w)$ and show that the formula satisfies properties (i) and (ii) as well.

THEOREM 4. *Let W be a finite Weyl group. For any $v \in W$, the function $\xi^v : W \to R$ defined in Proposition 5.1 is given explicitly by*

$$\xi^v(w) = \sum_{b_{i_1}b_{i_2}\cdots b_{i_k}\in R(v)} r_{\mathbf{b}}(i_1)r_{\mathbf{b}}(i_2)\cdots r_{\mathbf{b}}(i_k), \tag{5.3}$$

where $w \in W$, $\mathbf{b}$ is any reduced word for w, and $r_{\mathbf{b}}(j)$ is the jth positive root defined by (4.1).

Proof. By Corollary 4.5, the right-hand side of (5.3) vanishes if $v \not\leqslant w$. Also, if $v = w$, the right-hand side is π_w as defined in (4.2). Assuming W is a finite group, there exists a unique longest element w_0. Applying Proposition 5.1, we see that (5.3) holds for $v = w_0$. Now, the proof follows by decreasing induction from Lemma 4.2 and property (ii) of Proposition 5.1, as in the proof of Theorem 3. □

The following corollary originally appeared in [14] and also follows directly from Theorem 4.

COROLLARY 5.2. *For any $v, w \in W$, $\xi^v(w)$ is homogeneous of degree $l(v)$.*

In Section 6, we use the ξ-functions to arrive at a uniform approach to expanding products of Kostant functions and hence Schubert classes. We include the relevant facts about expanding products of ξ-functions here. From the definition of ξ^v in Proposition 5.1, one can see that the set $\{\xi^v : v \in W\}$ is a $\mathbf{Q}$-basis for all functions in Ω.

LEMMA 5.3. *Let*

$$\xi^u \xi^v = \sum p_{uv}^w \xi^w \tag{5.4}$$

be the expansion of the product into the basis $\{\xi^i\}$. The coefficients p_{uv}^w can be determined recursively by

$$p_{uv}^w = \frac{1}{\pi_w} \xi^u(w) \xi^v(w) - \frac{1}{\pi_w} \sum_{t<w} p_{uv}^t \xi^t(w) \tag{5.5}$$

starting with $w = u$ and going up in length.

Proof. By Proposition 5.1(i), one has $\xi^v(w) = 0$ for all $w \in W$ such that $l(w) \leqslant l(v)$ and $v \neq w$. Therefore, the expansion of any function $f : W \to R$ into the basis $\{\xi^v\}$ is given by $f = \sum_{w \in W} c_w \xi^w$, where

$$c_w = \frac{1}{\pi_w} \left(f(w) - \sum_{t<w} c_v \xi^t(w) \right). \tag{5.6}$$ □

The next proposition says that in fact the rational functions in (5.5) can be simplified to polynomials.

PROPOSITION 5.4 [14, Prop. 5.2]. *The coefficients p_{uv}^w are homogeneous polynomials in R of degree $l(u) + l(v) - l(w)$. In particular, if $l(u) + l(v) = l(w)$, then p_{uv}^w is a constant.*

Kostant and Kumar have shown that the coefficients p_{uv}^w in (5.4) can also be completely determined as coefficients in a product of matrices. Let $D = [d_{uv}]$ be the matrix with entries indexed by $u, v \in W$ and let the entry d_{uv} be defined as $\xi^u(v)$.[2] If the elements of W are ordered in a way that respects the length function, then D is upper-triangular with nonzero entries along the diagonal.

PROPOSITION 5.5 [14]. *Fix $u \in W$. Let D_u be the diagonal matrix with d_{uv} along the diagonal. Let P_u be the matrix of coefficients $[p_{uv}^w]$ from (6.1). Then*

$$P_u = D \cdot D_u \cdot D^{-1}. \tag{5.7}$$

6. Determination of the cup product in the cohomology ring of G/B. In this section we describe the main application of Theorems 3 and 4. The highest homogeneous component of a Kostant polynomial represents a Schubert class. Therefore, the highest homogeneous component of the product of Kostant polynomials represents the product of Schubert classes. We show that one can find the expansion of products of Kostant polynomials in the basis of Kostant polynomials by using the vectors of orbit values. This method of computing the cup product is much more efficient than previously known techniques that involved multiplying polynomials and possibly reducing modulo the ideal of invariants. Also, since it extends to the exceptional root systems, it is more complete than the existing theory of Schubert polynomials defined by Lascoux and Schützenberger [16]; see also [3], [7], [9], [20], [21], and many more.

We define the *orbit value function* $\xi_{\mathbf{O}}^v : W \to \mathbb{Z}$ to be the function with value $\tilde{K}_v(\mathbf{O}w) = \xi^v(w)|_{\mathbf{O}}$ on the point $w \in W$. Applying Corollary 4.3, we see that $\xi_{\mathbf{O}}^v(w)$ is always a nonnegative integer. Fix a total order on the Weyl group elements that respects the partial order determined by length. Let $\mathbb{Q}^{|W|}$ denote the set of functions from W to the rational numbers $\mathbb{Q}$. Note the set $\{\xi_{\mathbf{O}}^v : v \in W\}$ is a basis for $\mathbb{Q}^{|W|}$ since $\xi_{\mathbf{O}}^v(v) \neq 0$ and $\xi_{\mathbf{O}}^v(w) = 0$ for all $v < w$ in the chosen total order.

Multiplication of Kostant polynomials corresponds to pointwise multiplication of orbit value functions as we see in the next statement.

LEMMA 6.1. *Say $\xi^u\xi^v = \sum p_{uv}^w \xi^w$. Then we have*

$$\xi_{\mathbf{O}}^u \cdot \xi_{\mathbf{O}}^v = \sum p_{uv}^w(\mathbf{O})\xi_{\mathbf{O}}^w, \tag{6.1}$$

and the product of Kostant polynomials $\tilde{K}_u\tilde{K}_v$ (modulo the ideal of all polynomials that vanish on the orbit of $\mathbf{O}$) expands with the same coefficients

$$\tilde{K}_u \cdot \tilde{K}_v = \sum p_{uv}^w(\mathbf{O})\tilde{K}_w, \tag{6.2}$$

where $p_{uv}^w(\mathbf{O})$ is the evaluation of p_{uv}^w on the point $\mathbf{O} \in \mathbf{h}$.

[2] Note that our d_{uv} corresponds with $d_{u^{-1}v^{-1}}$ in [14].

Proof. The expansion of $\xi^u_{\mathbf{O}} \cdot \xi^v_{\mathbf{O}}$ follows by simply evaluating $\xi^u \xi^v = \sum p^w_{uv} \xi^w$ on any point $x \in W$ and then evaluating at $\mathbf{O}$.

The expansion of any polynomial into Kostant polynomials (modulo the ideal of functions that vanish on the orbit) is completely determined by its orbit values. Since the orbit values are the same as the corresponding values of $\xi_{\mathbf{O}}$-functions, the expansions have the same coefficients. $\square$

It is easy to compute the expansion of any function in $\mathbb{Q}^{|W|}$ into the sum of the $\xi^v_{\mathbf{O}}$ because of their upper-triangular form. The expansion only involves linear algebra on vectors of nonnegative integers. Furthermore, each of the formulas in (5.5) and (5.7) can be evaluated at $\mathbf{O}$ to obtain a similar formula for $p^w_{uv}(\mathbf{O})$. The evaluation can be computed at each step in the recursive formula analogous to (5.5), therefore polynomial division never needs to be computed in the expansion of $\xi^u_{\mathbf{O}} \xi^v_{\mathbf{O}}$. Hence, we propose that one compute the coefficients c^w_{uv} in the expansion of $\mathfrak{S}_u \mathfrak{S}_v = \sum c^w_{uv} \mathfrak{S}_w$ by computing $\xi^u_{\mathbf{O}} \cdot \xi^v_{\mathbf{O}} = \sum p^w_{uv}(\mathbf{O}) \xi^w_{\mathbf{O}}$.

COROLLARY 6.2. *If we have*

$$\xi^u_{\mathbf{O}} \cdot \xi^v_{\mathbf{O}} = \sum_{w \in W} p^w_{uv}(\mathbf{O}) \xi^w_{\mathbf{O}}, \tag{6.3}$$

then the product of Schubert classes $\mathfrak{S}_u$ and $\mathfrak{S}_v$ expand as

$$\mathfrak{S}_u \cdot \mathfrak{S}_v = \sum_{l(w)=l(u)+l(v)} p^w_{uv}(\mathbf{O}) \mathfrak{S}_w. \tag{6.4}$$

Proof. First note that the coefficient $p^w_{uv}(\mathbf{O})$ is a constant if $l(w) = l(u) + l(v)$ by Proposition 5.4. Hence, the sum in (6.4) is independent of the choice of $\mathbf{O} \in \mathbf{h}$. By Proposition 2.2 we know that the highest homogeneous component of $\tilde{K}_v$ is equal to the Schubert class $\mathfrak{S}_v$ (modulo I). Therefore, $\mathfrak{S}_u \mathfrak{S}_v$ is equal to the highest homogeneous component of $\tilde{K}_u \tilde{K}_v$. The highest homogeneous component in the expansion of $\tilde{K}_u \tilde{K}_v$ is the sum of all terms in the expansion with $l(w) = l(u) + l(v)$. Therefore, the corollary follows from Lemma 6.1. $\square$

7. An example. In this section we compute the Kostant polynomial for the permutation $[3, 1, 2]$ (written in one-line notation). This establishes our conventions for the action of the Weyl group for different modules. We need to make several choices in order to begin our computations.

Let $\{\varepsilon_i : 1 \leqslant i \leqslant 3\}$ be unit coordinate vectors in the ambient vector space. For $i = 1, 2$, let α_i be the simple roots $\varepsilon_{i+1} - \varepsilon_i$. Let $\mathbf{O}$ be an indeterminate point (o_1, o_2, o_3). Let $v \in S_3$; then evaluation of a polynomial $f(x_1, x_2, x_3)$ on the point $\mathbf{O}v$ gives $f(o_{v(1)}, o_{v(2)}, o_{v(3)})$.

Next we compute $K_{[3,1,2]}$. One can easily verify that the polynomial $(x_1 - o_1)(x_1 - o_2)$ vanishes on the four permutations $[1, 2, 3]$, $[2, 1, 3]$, $[1, 3, 2]$, [2, 3, 1] and not on [3, 1, 2]. Normalizing this polynomial we have

$$K_{[3,1,2]} = \frac{(x_1 - o_1)(x_1 - o_2)}{(o_3 - o_1)(o_3 - o_2)}. \tag{7.1}$$

Note, $\Delta_+ \cap [3,1,2]\Delta_- = \{\varepsilon_3 - \varepsilon_1, \varepsilon_3 - \varepsilon_2\}$, so the denominator in (7.1) is equal to

$$\pi_{[3,1,2]} = \prod_{\gamma \in \Delta_+ \cap [3,1,2]\Delta_-} \gamma(\mathbf{O}) = (o_3 - o_1)(o_3 - o_2). \tag{7.2}$$

Lascoux and Schützenberger have shown that there exists a particularly nice set of representatives called Schubert polynomials for the Schubert classes of SL_n/B. They established this theory in [16] and then extended it, along with several others, notably Macdonald in [20]. Lascoux and Schützenberger's choice of the top Schubert polynomial in S_n is $\mathfrak{S}_{w_0} = x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$. These representatives are then stable under the inclusion of S_n into S_{n+1}.

To continue our example, we can check that $\mathfrak{S}_{[3,1,2]}$ is $\partial_2 x_1^2 x_2 = x_1^2$. Comparing this with the highest homogeneous component of $K_{[3,1,2]}$, we see

$$K^0_{[3,1,2]} = \frac{1}{\pi_{[3,1,2]}}\mathfrak{S}_{[3,1,2]}. \tag{7.3}$$

In Table 1 we give the orbit values for S_4 on the orbit point $\mathbf{O} = (1,2,3,4)$. Using the table, one can verify that

$$\xi_{\mathbf{O}}^{[3,1,2,4]}\xi_{\mathbf{O}}^{[1,3,4,2]} = \xi_{\mathbf{O}}^{[3,1,4,2]} + \xi_{\mathbf{O}}^{[3,2,4,1]} + \xi_{\mathbf{O}}^{[4,1,3,2]}. \tag{7.4}$$

(Note that the permutations on the left-hand side of (7.4) do not all have the same length.) Therefore,

$$\mathfrak{S}_{[3,1,2,4]}\mathfrak{S}_{[1,3,4,2]} = \mathfrak{S}_{[3,2,4,1]} + \mathfrak{S}_{[4,1,3,2]}. \tag{7.5}$$

8. Concluding remarks.

Remark 1. The Kostant polynomials and the double Schubert polynomials defined by Lascoux and Schützenberger [17] are closely related in the case of SL_n. Let $\mathfrak{S}_w(X,Y)$ be the double Schubert polynomial indexed by w on the two alphabets X and Y. Then $\tilde{K}_w(x_1, x_2, \ldots, x_n) = \mathfrak{S}_w(X, \mathbf{O})$. This fact can be proven in several ways: using the vanishing properties of the double Schubert polynomials as originally proven by Lascoux and Schützenberger [17], using the combinatorial interpretation for the terms in a double Schubert polynomial defined by Fomin and Kirillov [6], or by divided difference equations as shown by Shimozono [22].

TABLE 1
Type A orbit value matrix S_4

1234	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1243	0	1	0	0	2	1	1	0	0	2	3	1	2	0	1	3	3	2	2	1	3	3	2	3
1324	0	0	1	0	1	2	0	2	1	2	2	3	1	2	2	3	2	4	2	3	4	3	4	4
2134	0	0	0	1	0	0	1	1	2	0	1	1	2	2	3	1	2	2	3	3	2	3	3	3
1342	0	0	0	0	2	0	0	0	0	2	6	0	2	0	0	6	6	2	2	0	6	6	2	6
1423	0	0	0	0	0	2	0	0	0	2	0	3	0	0	2	3	0	6	2	3	6	3	6	6
2143	0	0	0	0	0	0	1	0	0	0	3	1	4	0	3	3	6	4	6	3	6	9	6	9
2314	0	0	0	0	0	0	0	2	0	0	2	3	0	2	0	3	2	6	0	3	6	3	6	6
3124	0	0	0	0	0	0	0	0	2	0	0	0	2	2	6	0	2	2	6	6	2	6	6	6
1432	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	6	0	6	2	0	12	6	6	12
2341	0	0	0	0	0	0	0	0	0	0	6	0	0	0	0	6	6	0	0	0	6	6	0	6
2413	0	0	0	0	0	0	0	0	0	0	0	3	0	0	0	3	0	12	0	3	12	3	12	12
3142	0	0	0	0	0	0	0	0	0	0	0	0	4	0	0	0	6	4	6	0	6	12	6	12
3214	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	2	6	0	6	6	6	12	12
4123	0	0	0	0	0	0	0	0	0	0	0	0	0	0	6	0	0	0	6	6	0	6	6	6
2431	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	6	0	0	0	0	12	6	0	12
3241	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	6	0	0	0	6	12	0	12
3412	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	12	0	0	12	0	12	12
4132	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	6	0	0	12	6	12
4213	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	6	0	6	12	12
3421	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	12	0	0	12
4231	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	12	0	12
4312	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	12	12
4321	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	12

In [19], Lascoux and Schützenberger show that all the vanishing properties for the double Schubert polynomials follow from two facts. First, the double Schubert polynomials satisfy the divided difference equations, and second, the double Schubert polynomial indexed by the longest element of the symmetric group vanishes on all other permutations and evaluates to the product of the positive roots on itself. Theorem 1 generalizes this statement for the other semi-simple Lie groups.

Remark 2. The Kostant polynomials for the other classical groups are related to the double Schubert polynomials as defined by Fulton [9] and also those defined by Pragacz and Ratajski [21]. In both cases, Graham [10] has shown that the vanishing properties for the double Schubert polynomials, indexed by the longest element in the Weyl group, follow from their geometric construction. The vanishing properties for the other classes then follow from Theorem 1. The vanishing properties for Fulton's top double Schubert polynomial have also been shown independently by Shimozono and myself, using only theory of symmetric functions.

Fulton has shown that certain specializations of double Schubert polynomials determine the degree of a degeneracy locus. Is it possible to show that the orbit values in Theorem 3 always determine the degree of some variety? This would give geometric meaning to these nonnegative integers.

Remark 3. It is well known that the structure constants for G/B are nonnegative integers. We have observed in our calculations that in fact all of the coefficients $p_{uv}^{w}(\mathbf{O})$ are nonnegative integers. Can one give formulas analogous to (4.4) for these coefficients that would prove they are nonnegative? Is there a geometrical meaning to the coefficients that would prove nonnegativity?

Remark 4. Carrell [4] has shown that $H^*(G/B)$ is isomorphic to the graded ring canonically associated to the polynomial ring of the variety given by the set of points $\mathbf{O}W$. Furthermore, the cohomology for a Schubert variety $H^*(X_w)$ is isomorphic to the graded ring associated to the polynomial ring of the variety given by the set of points $\{\mathbf{O}u : w \leqslant w^{-1}\}$ (see [4]). Therefore, the $\xi_w^{\mathbf{O}}$-functions can be used to compute the cup product in any $H^*(X_w)$ in a similar way to that described in Section 6.

Appendix

Here we outline the proof by Kumar that extends Theorem 4 to any Kac-Moody Lie algebra. The critical difference in his proof is to start the induction from the identity element and give a recursive formula for building up the orbit values from there.

Lemma A.1. *Let W be the Weyl group for an arbitrary Kac-Moody Lie algebra. Let v and w be any two elements of W. Then we have*

$$\xi^v(w) = \begin{cases} \xi^v(w\sigma_i) - w\alpha_i\xi^{v\sigma_i}(w\sigma_i), & v > v\sigma_i, \\ \xi^v(w\sigma_i), & v < v\sigma_i. \end{cases} \tag{A.1}$$

Proof. Proposition 5.1(ii) implies $\xi^v(w) = \xi^v(w\sigma_i)$ if $v < v\sigma_i$. Using this fact, the proof follows by rearranging terms (5.2). □

THEOREM 5. *Let W be the Weyl group of an arbitrary Kac-Moody Lie algebra. Using the notation from Lemma A.1, we have*

$$\xi^v(w) = \zeta(v, w), \tag{A.2}$$

where $\zeta(v, w)$ is the polynomial defined in (4.5).

Proof. If w is the identity element in W, there is nothing to prove. Otherwise, pick a reflection σ_i such that $w > w\sigma_i$. First assume $v > v\sigma_i$, then by Lemma A.1

$$\xi^v(w) = \xi^v(w\sigma_i) - (w\alpha_i)\xi^{v\sigma_i}(w\sigma_i). \tag{A.3}$$

By (upward) induction on the length of w, we have the validity of the theorem for $\xi^v(w\sigma_i)$ and $\xi^{v\sigma_i}(w\sigma_i)$. From (A.3) and (4.9), it is easy to see that we get the validity of the theorem for $\xi^v(w)$.

So consider the case $v < v\sigma_i$ (still assuming $w\sigma_i < w$). In this case, Lemma A.1 implies

$$\xi^v(w) = \xi^v(w\sigma_i). \tag{A.4}$$

Again by induction the theorem is true for $\xi^v(w\sigma_i)$, and hence we get the validity of the theorem for $\xi^v(w)$.

Observe that we have implicitly used Lemma 4.1 in the proof above since we use the fact that one can choose a reduced word for w that ends in i. $\square$

Acknowledgments. I am very grateful to Bertram Kostant for inspiring this work. Also, I would like to thank Sergey Fomin, Adriano Garsia, Mark Haiman, Shrawan Kumar, Mark Shimozono, Richard Stanley, David Vogan, and Nolan Wallach for many enlightening suggestions.

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