# FACTORING PEAK POLYNOMIALS

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ABSTRACT. Let $\mathfrak{S}_n$ be the symmetric group of permutations $\pi = \pi_1\pi_2\cdots\pi_n$ of $\{1, 2, \ldots, n\}$. An index $i$ of $\pi$ is a peak if $\pi_{i-1} < \pi_i > \pi_{i+1}$, and we let $P(\pi)$ denote the set of peaks of $\pi$. Given any set $S$ of positive integers, we define $\mathcal{P}_S(n) = \{\pi \in \mathfrak{S}_n : P(\pi) = S\}$. Burdzy, Sagan, and the first author showed that for all fixed subsets of positive integers $S$ and sufficiently large $n$ we have $|\mathcal{P}_S(n)| = p_S(n)2^{n-|S|-1}$ for some polynomial $p_S(x)$ depending on $S$. It is conjectured that the coefficients of $p_S(x)$ expanded in a binomial coefficient basis centered at $\max(S)$ are all positive, and we show that this is a consequence of a stronger conjecture that bounds the modulus of the zeros of $p_S(x)$. Our main results give an explicit formula for peak polynomials in the binomial basis centered at 0, show that all peaks are zeros of $p_S(x)$, and that $0, 1, 2, \ldots, i_r$ are zeros of $p_S(x)$ for any $i_r \in S$ if $i_{r+1} - i_r$ is odd. Additionally, we enumerate $|\mathcal{P}_S(n)|$ using alternating permutations for all peak sets $S$.

## 1. INTRODUCTION

Let $\mathfrak{S}_n$ be the symmetric group of all permutations $\pi = \pi_1\pi_2\ldots\pi_n$ of $[n] := \{1, 2, \ldots, n\}$. An index $i$ of $\pi$ is a *peak* if $\pi_{i-1} < \pi_i > \pi_{i+1}$, and the *peak set* of $\pi$ is defined as $P(\pi) = \{i : i \text{ is a peak of } \pi\}$. We are interested in counting the permutations of $\mathfrak{S}_n$ with a given peak set, so let us define $\mathcal{P}_S(n) = \{\pi \in \mathfrak{S}_n : P(\pi) = S\}$. We say that a set $S = \{i_1 < i_2 < \cdots < i_s\}$ is $n$-admissible if $|\mathcal{P}_S(n)| \neq 0$. Note that we insist the elements of $S$ be listed in increasing order and that $S$ is $n$-admissible if and only if $1 < i_1$, no two $i_r$ are consecutive integers, and $i_s < n$. If we make a statement about an *admissible* set $S$, we mean that $S$ is $n$-admissible for some $n$, and the statement holds for every $n$ such that $S$ is $n$-admissible. Burdzy, Sagan, and the first author recently proved the following result in [3].

**Theorem 1.1** ([3, Theorem 3]). *If $S$ is a nonempty admissible set and $m = \max(S)$, then*

$$|\mathcal{P}_S(n)| = p_S(n)2^{n-|S|-1}$$

*for $n \geq m$, where $p_S(x)$ is a polynomial of degree $m-1$ depending on $S$ such that $p_S(n)$ is an integer for all integral inputs $n$. If $S = \emptyset$, then $|\mathcal{P}_S(n)| = 2^{n-1}$, so we can set $p_\emptyset(n) = 1$.*

If $S$ is not admissible, then $|\mathcal{P}_S(n)| = 0$ for all positive integers $n$, and one defines the corresponding polynomial to be $p_S(x) = 0$. Thus, for all finite sets $S$ of positive integers, $p_S(x)$ is a well-defined polynomial, which is called the *peak polynomial* for $S$.

In this paper we study properties of peak polynomials such as their expansions into binomial bases, zeros, and relative values values at nonnegative integers. We also enumerate permutations with a given peak set using alternating permutations and connect our results to other recent work about the peak statistic [3, 5, 8, 10]. Our primary motivation comes from combinatorics, information theory, and probability theory. Peaks sets have been studied for

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decades going back to [11] and used more recently in a probabilistic project concerned with mass redistribution [2]. Below are the primary results of this paper.

**Theorem 1.2.** *Let $S = \{i_1 < i_2 < \cdots < i_s = m\}$ be admissible and nonempty. For $0 \le j \le m-1$, define the coefficients*

$$d_j^S = (-1)^{m-j-1}(-2)^{|S\cap(j,\infty)|-1} p_{S\cap[j]}(j).$$

*If there exists an index $1 \le r \le s-1$ such that $i_{r+1} - i_r$ is odd, let $b = i_r$ for the largest such $r$. Then the peak polynomial $p_S(x)$ expands in the binomial basis centered at $0$ as*

$$p_S(x) = \sum_{j=b}^{m-1} d_j^S \binom{x}{j}.$$

*Otherwise, if there are no odd gaps, then*

$$p_S(x) = \left(d_0^S - (-2)^{|S|-1}\right) + \sum_{j=1}^{m-1} d_j^S \binom{x}{j}.$$

Observe that by Theorem 1.1, $p_S(m) = 0$ using the fact that $\mathcal{P}_S(m)$ is empty, but we may have $p_S(\ell) \neq 0$ for $\ell < m$ even though $|\mathcal{P}_S(\ell)| = 0$. The next two theorems describe additional zeros of $p_S(x)$.

**Corollary 1.3.** *If $S = \{i_1 < i_2 < \cdots < i_s\}$ and $i_{r+1} - i_r$ is odd for some $1 \le r \le s-1$, then $0, 1, 2, \ldots, i_r$ are zeros of $p_S(x)$.*

**Theorem 1.4.** *We have $p_S(i) = 0$ for all $i \in S$.*

In [3] they conjecture that the coefficients of any peak polynomial are nonnegative integers in the shifted binomial basis $\binom{x-m}{j}$, where $m$ is the maximum value in the corresponding peak set. We refer to this as the "*positivity conjecture*", and we show in this paper that it is a consequence of the following conjecture. These two conjectures motivated our research, because they suggest that we look at the zeros of peak polynomials.

**Conjecture 1.5.** The complex zeros of $p_S(z)$ lie in $\{z \in \mathbb{C} : |z| \le m \text{ and } \mathrm{Re}(z) \ge -3\}$ if $S$ is admissible.

The paper is organized as follows. Section 2 covers the background material on peak polynomials and the calculus of finite differences. We formally recall the positivity conjecture from [3]. In Section 3 we prove that Conjecture 1.5 implies the positivity conjecture. Section 4 proves Theorems 1.2, 1.3, 1.4, and identifies some special peak polynomials. Section 5 demonstrates some behaviors of peak polynomials evaluated at nonnegative integers and patterns in the table of forward differences of $p_S(x)$. Section 6 develops a new method for counting the number of permutations with a given peak set using alternating permutations and the inclusion-exclusion principle. In Section 7 we relate our work to other recent results about permutations with a given peak set. We conclude with several conjectures suggested by this investigation.

## 2. Background

In this section we state results from [3] that are used throughout this paper. Additionally, we discuss the calculus of finite differences, specifically forward differences, and the positivity conjecture from [3]. Let $S$ be a nonempty admissible set of constants and $m = \max(S)$ throughout the section.

**Corollary 2.1** ([3, Corollary 4])**.** *We have*

$$p_S(x) = p_{S_1}(m-1)\binom{x}{m-1} - 2p_{S_1}(x) - p_{S_2}(x),$$

*where* $S_1 = S \setminus \{m\}$ *and* $S_2 = S_1 \cup \{m-1\}$.

**Theorem 2.2** ([3, Theorem 6])**.** *If* $S = \{m\}$, *then*

$$p_S(x) = \binom{x-1}{m-1} - 1.$$

In the calculus of finite differences we define the *forward difference* operator $\Delta$ to be $(\Delta f)(x) = f(x+1) - f(x)$. Higher order differences are given by $(\Delta^n f)(x) = (\Delta^{n-1} f)(x+1) - (\Delta^{n-1} f)(x)$. We use the definition of the Newton interpolating polynomial to expand $p_S(x)$ in the binomial basis centered at $k$ as

$$p_S(x) = \sum_{j=0}^{m} (\Delta^j p_S)(k)\binom{x-k}{j}.$$

Notice its similarity to Taylor's theorem. Below is an example of the forward differences of $p_{\{2,6,10\}}(x)$. The $k$-th column in the table is the basis vector for the expansion of $p_{\{2,6,10\}}(x)$ in the binomial basis centered at $k$. We consider these expansions centered at both 0 and $m$ in this paper.

| $j, k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | -8 | -4 | 0 | 2 | 4 | 6 | 0 | -18 | -72 | -196 | 0 |
| 1 | 4 | 4 | 2 | 2 | 2 | -6 | -18 | -54 | -124 | 196 | 3094 |
| 2 | 0 | -2 | 0 | 0 | -8 | -12 | -36 | -70 | 320 | 2898 | 12376 |
| 3 | -2 | 2 | 0 | -8 | -4 | -24 | -34 | 390 | 2578 | 9478 | 26564 |
| 4 | 4 | -2 | -8 | 4 | -20 | -10 | 424 | 2188 | 6900 | 17086 | 36376 |
| 5 | -6 | -6 | 12 | -24 | 10 | 434 | 1764 | 4712 | 10186 | 19290 | 33324 |
| 6 | 0 | 18 | -36 | 34 | 424 | 1330 | 2948 | 5474 | 9104 | 14034 | 20460 |
| 7 | 18 | -54 | 70 | 390 | 906 | 1618 | 2526 | 3630 | 4930 | 6426 | 8118 |
| 8 | -72 | 124 | 320 | 516 | 712 | 908 | 1104 | 1300 | 1496 | 1692 | 1888 |
| 9 | 196 | 196 | 196 | 196 | 196 | 196 | 196 | 196 | 196 | 196 | 196 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

TABLE 1. Forward differences of $p_{\{2,6,10\}}(x)$

We know from Theorem 1.1 that $(\Delta^0 p_S)(m) = 0$, $(\Delta^{m-1} p_S)(k)$ is a positive integer, and $(\Delta^j p_S)(k) = 0$ for all $k \in \mathbb{Z}$ and $j \geq m$. Burdzy, Sagan, and the first author proposed the following positivity conjecture in [3].

**Conjecture 2.3** ([3, Conjecture 14])**.** Each coefficient $(\Delta^j p_S)(m)$ is a positive integer for $1 \leq j \leq m-1$ and all admissible sets $S$.

It follows from Stanley's text [13, Corollary 1.9.3] that $p_S(n)$ is an integer for all integral $n$ if and only if the coefficients in the expansion of $p_S(n)$ in a binomial basis are integral, so we only need to prove that $(\Delta^j p_S)(m)$ is positive for $1 \leq j \leq m-1$.

## 3. An approach to the positivity conjecture

The following lemmas form a chain of arguments that proves that the positivity conjecture is a consequence of Conjecture 1.5. We write $p(x)$ or $p(z)$ when we are discussing properties of all polynomials, and we use $p_S(x)$ when we are discussing peak polynomials in particular.

**Lemma 3.1.** *If $p(z)$ does not have a complex zero with real part greater than $m$, then $p'(z), p''(z), \ldots, p^{(m-1)}(z)$ do not have a complex zero with real part greater $m$, and thus, no real zero greater than $m$.*

*Proof.* We use the Gauss–Lucas theorem, which states that if $p(z)$ is a (nonconstant) polynomial with complex coefficients, then all the zeros of $p'(z)$ belong to the convex hull of the set of zeros of $p(z)$. By assumption all of the zeros of $p(z)$ lie in the half-plane $\{z \in \mathbb{C} : \mathrm{Re}(z) \leq m\}$, so then by the Gauss–Lucas theorem, all of the zeros of $p'(z)$ also lie in this half-plane. Repeating this argument, we see that $p'(z), p''(z), \ldots, p^{(m-1)}(z)$ do not have a complex zero with real part greater than $m$ and thus no real zero greater than $m$. □

**Lemma 3.2.** *If $S$ is admissible and none of $p_S(x), p_S'(x), p_S''(x), \ldots, p_S^{(m-1)}(x)$ have a real zero greater than $m$, then $p_S(x), p_S'(x), \ldots, p_S^{(m-1)}(x)$ are all positive for $x > m$.*

*Proof.* Since $S$ is admissible, $p_S(m+1)$ is a positive integer. If $p_S(x)$ is nonpositive for some $x_0 > m$, then $p_S(x)$ has a zero greater than $m$ by the intermediate value theorem, which contradicts the assumption. Therefore $p_S(x)$ is positive for $x > m$, so its leading coefficient is positive. It follows that the leading coefficients of $p_S'(x), p_S''(x), \ldots, p_S^{(m-1)}(x)$ are also positive, so all of the derivatives of $p_S(x)$ are eventually positive. Again by the intermediate value theorem, the derivatives $p_S'(x), p_S''(x), \ldots, p_S^{(m-1)}(x)$ are all positive for $x > m$. □

**Lemma 3.3.** *If $p(x)$ is a polynomial of degree $m-1$ and $p'(x), p''(x), \ldots, p^{(m-1)}(x)$ are positive for $x > m$, then all of the forward differences $(\Delta p)(m), (\Delta^2 p)(m), \ldots, (\Delta^{m-1} p)(m)$ are positive.*

*Proof.* Proposition 17 of [9] states that if $f(x)$ is $n$ times differentiable on $[m, m+n]$, then there exists $\xi \in (m, m+n)$ such that $(\Delta^n f)(x) = f^{(n)}(\xi)$. Polynomials are infinitely differentiable, so there exists $\xi \in (m, m+n)$ such that $(\Delta^n p)(m) = p^{(n)}(\xi)$. By assumption, $p'(x), p''(x), \ldots, p^{(m-1)}(x)$ are positive for $x > m$, so $p'(\xi), p''(\xi), \ldots, p^{(m-1)}(\xi)$ are positive for all $\xi > m$. Therefore, $(\Delta p)(m), (\Delta^2 p)(m), \ldots, (\Delta^{m-1} p)(m)$ are positive. □

**Theorem 3.4.** *If $S$ is admissible and $p_S(n)$ has no zero whose real part is greater than $m$, then each coefficient $(\Delta^j p_S)(m)$ is positive for $1 \leq j \leq m-1$.*

*Proof.* The proof is a consequence of Lemma 3.1, Lemma 3.2, and Lemma 3.3. □

It is clear that Conjecture 1.5 satisfies the hypothesis of Theorem 3.4, so we can prove Conjecture 2.3 if we can appropriately bound the zeros of $p_S(x)$. It is worth noting that we have checked the zeros of the peak polynomials for all admissible sets $S$ with $\max(S) \leq 15$ in [7], and they agree with Conjecture 1.5.

## 4. Zeros of peak polynomials

Our main theorems from the introduction are proved here in Subsection 4.1. In particular, we give an explicit formula for $p_S(x)$ in the binomial basis centered at 0. In Subsection 4.2 we look at peak polynomials with only integral zeros, and the results in Subsection 4.3 show that if $S$ has a gap of 3, then $p_S(x)$ is independent of the peaks to the left of this gap up to a constant. All of the results in this section assume that $S$ is admissible, though not explicitly stated in the hypothesis.

4.1. **Main results.** The following recurrence relation is very efficient for computation and is the foundation of every result in this section.

**Lemma 4.1.** *If $S = \{i_1 < i_2 < \cdots < i_s = m < m + k\}$ and $k \geq 2$, then*

$$p_S(x) = -2p_{S_1}(x)\chi(k \text{ } even) + \sum_{j=1}^{k-1}(-1)^{k-1-j}p_{S_1}(m+j)\binom{x}{m+j}.$$

*Proof.* We induct on $k$ and use Corollary 2.1. In the base case $k = 2$, and

$$p_S(x) = -2p_{S_1}(x) + p_{S_1}(m+1)\binom{x}{m+1}.$$

By induction,

$$\begin{aligned}
p_S(x) &= p_{S_1}(m+k-1)\binom{x}{m+k-1} - 2p_{S_1}(x) - p_{S_2}(x) \\
&= p_{S_1}(m+k-1)\binom{x}{m+k-1} - 2p_{S_1}(x) \\
&\quad - \left[-2p_{S_1}(x)\chi(k-1 \text{ even}) + \sum_{j=1}^{k-2}(-1)^{k-2-j}p_{S_1}(m+j)\binom{x}{m+j}\right] \\
&= -2p_{S_1}(x)\chi(k \text{ even}) + \sum_{j=1}^{k-1}(-1)^{k-1-j}p_{S_1}(m+j)\binom{x}{m+j}.
\end{aligned}$$

□

**Corollary 4.2.** *If $S = \{i_1 < i_2 < \cdots < i_s = m < m + k\}$ and $k \geq 2$, then*

$$|\mathcal{P}_S(n)| = -\chi(k \text{ } even)|\mathcal{P}_{S_1}(n)| + \sum_{j=1}^{k-1}(-1)^{k-1-j}\binom{n}{m+j}|\mathcal{P}_{S_1}(m+j)| \cdot |\mathcal{P}_\emptyset(n-(m+j))|.$$

*Proof.* Apply Theorem 1.1 to Lemma 4.1. □

We can interpret Corollary 4.2 combinatorially. Choose $m + k - 1$ of the $n$ elements and arrange them such that their peak set is $S_1$. Arrange the remaining $n-(m+k-1)$ elements so that there are no peaks, and append this sequence to the previous one. In the combined sequence there is either a peak at $m + k$,$m + k - 1$, or no peak after $m$. Since $m + k \in S$,

$$|\mathcal{P}_S(n)| = \binom{n}{m+k-1}|\mathcal{P}_{S_1}(m+k-1)| \cdot |\mathcal{P}_\emptyset(n-(m+k-1))| - |\mathcal{P}_{S_2}(n)| - |\mathcal{P}_{S_1}(n)|.$$

We repeat this procedure for $|\mathcal{P}_{S_2}(n)|$ to count all the permutations whose peak set is $S_1 \cup \{m + k - 1\}$, but this also counts permutations whose peak set is $S_1 \cup \{m + k - 2\}$ and $S_1$. We repeat this process until we count permutations whose peak set is $S_1 \cup \{m+1\}$, but this peak set is inadmissible and terminates the procedure. Notice that $|\mathcal{P}_{S_1}(n)|$ telescopes because it is included in each iteration with an alternating sign.

We now present the proof of an explicit formula for peak polynomials with nonempty peak sets in the binomial basis centered at 0. The results about zeros due to odd gaps and peaks follow.

*Proof of Theorem 1.2.* The proof follows by iterating Lemma 4.1. In the case that there no odd gaps, we have

$$p_S(x) = (-2)^{|S|-1}\left[\binom{x-1}{i_1-1} - 1\right] + \sum_{j=i_1}^{m-1} d_j^S \binom{x}{j},$$

and then use Vandermonde's identity to shift the $p_{\{i_1\}}(x)$ term to the binomial basis centered at 0. $\square$

**Corollary 4.3.** *If $S = \{i_1 < i_2 < \cdots < i_s\}$ and $i_{r+1} - i_r$ is odd for some $1 \le r \le s-1$, then $0, 1, \ldots, i_r$ are zeros of $p_S(x)$.*

*Proof.* The proof follows from Theorem 1.2. $\square$

**Corollary 4.4.** *If $S$ contains an odd peak, then $p_S(0) = 0$. Otherwise, $p_S(0) = (-2)^{|S|}$.*

*Proof.* The proof follows from Theorem 1.2. $\square$

**Theorem 4.5.** *We have $p_S(i) = 0$ for $i \in S$.*

*Proof.* We induct on $|S|$ for all nonempty admissible sets $S$. In the base case $|S| = 1$, and $p_{\{m\}}(m) = 0$ by Theorem 2.2. In the inductive step, let $m = \max(S)$. If $i \in S_1$, then $p_{S_1}(i) = 0$ by the induction hypothesis, so $p_S(i) = 0$ by Lemma 4.1. We also know that $p_S(m) = 0$ by Theorem 1.1, so $p_S(i) = 0$ for all $i \in S$. $\square$

4.2. **Peak polynomials with only integral zeros.** All of the peak polynomials in this subsection are completely factored and have all nonnegative integral zeros. As a result, they satisfy Conjecture 2.3 by Theorem 3.4, because we have bounded the real part of their zeros by $\max(S)$. In the next two lemmas, the leading coefficient is all that is recursively defined, and it depends solely on the structure of $\{i_1 < i_2 < \cdots < i_s\}$. In Conjecture 7.5 we classify all the peak polynomials with only integral zeros.

**Lemma 4.6.** *If $S = \{i_1 < i_2 < \cdots < i_s = m < m+3\}$, then*

$$p_S(x) = \frac{p_{S_1}(m+1)}{2(m+1)!}(x - (m+3))\prod_{j=0}^{m}(x-j).$$

*Proof.* Using Lemma 4.1, we see that

$$\begin{aligned}
p_S(x) &= \sum_{j=1}^{2}(-1)^{2-j} p_{S_1}(m+j)\binom{x}{m+j} \\
&= \frac{\prod_{j=0}^{m}(x-j)}{(m+1)!}\left[\frac{p_{S_1}(m+2)}{m+2}\left(x - \left(m+1+\frac{p_{S_1}(m+1)(m+2)}{p_{S_1}(m+2)}\right)\right)\right],
\end{aligned}$$

but $m+3$ is also a zero of $p_S(x)$ by Theorem 4.5. Equating the two roots, we have

$$p_{S_1}(m+2) = \frac{(m+2)p_{S_1}(m+1)}{2},$$

so then

$$p_S(x) = \frac{p_{S_1}(m+1)}{2(m+1)!}(x - (m+3))\prod_{j=0}^{m}(x-j). \qquad \square$$

**Lemma 4.7.** *If $S = \{i_1 < i_2 < \cdots < i_s = m < m+3 < m+5\}$, then*
$$p_S(x) = \frac{p_{S\setminus\{m+3,m+5\}}(m+1)}{12(m+1)!}(x-(m+5))(x-(m+3))(x-(m-2))\prod_{j=0}^{m}(x-j).$$

*Proof.* The proof follows from Corollary 2.1 and Lemma 4.6. □

The next two corollaries show how $p_S(x)$ grows from $x_0$ to $x_0+1$ for any $x_0 \in \mathbb{R}$, and they demonstrate how the zeros shift when translating $p_S(x)$ to $p_S(x+1)$.

**Corollary 4.8.** *If $S = \{i_1 < i_2 < \cdots < i_s = m < m+3\}$, then*
$$p_S(x+1) = \lim_{t\to x}\frac{(t+1)(t-(m+2))}{(t-m)(t-(m+3))}p_S(t).$$

*Proof.* Write $p_S(x+1)/p_S(x)$ using Lemma 4.6 and apply Theorem 4.5. □

**Corollary 4.9.** *If $S = \{i_1 < i_2 < \cdots < i_s = m < m+3 < m+5\}$, then*
$$p_S(x+1) = \lim_{t\to x}\frac{(t+1)(t-(m-3))(t-(m+2))(t-(m+4))}{(t-(m-2))(t-m)(t-(m+3))(t-(m+5))}p_S(t).$$

*Proof.* Write $p_S(x+1)/p_S(x)$ using Lemma 4.7 and apply Theorem 4.5. □

We now derive closed-form formulas for $p_S(x)$ when $S = \{m, m+3, \ldots, m+3k\}$ and $S = \{m, m+3, \ldots, m+3k, m+3k+2\}$ for $k \geq 1$. These formulas are direct consequences of Lemma 4.6 and Lemma 4.7

**Corollary 4.10.** *If $S = \{m, m+3, \ldots, m+3k\}$ for $k \geq 1$, then*
$$p_S(x) = \frac{(m-1)(x-(m+3k))}{2(m+1)!(12^{k-1})}\prod_{j=0}^{m+3(k-1)}(x-j).$$

*Proof.* We induct on $k$. In the base case, $k=1$ and $S = \{m, m+3\}$. Using Lemma 4.6 and Theorem 2.2, we have
$$\begin{aligned}p_{\{m,m+3\}}(x) &= \frac{p_{\{m\}}(m+1)}{2(m+1)!}(x-(m+3))\prod_{j=0}^{m}(x-j)\\ &= \frac{(m-1)(x-(m+3))}{2(m+1)!}\prod_{j=0}^{m}(x-j).\end{aligned}$$

In the inductive step, $S = \{m, m+3, \ldots, m+3k\}$. We use Lemma 4.6 again, because $p_{S_1}(m+3k-2)$ by the inductive hypothesis, and it follows that
$$\begin{aligned}p_S(x) &= \frac{p_{S_1}(m+3k-2)}{2(m+3k-2)!}(x-(m+3k))\prod_{j=0}^{m+3(k-1)}(x-j)\\ &= \frac{(m-1)(m+3k-2)!}{2(m+1)!(12^{k-2})3!}\left[\frac{(x-(m+3k))}{2(m+3k-2)!}\prod_{j=0}^{m+3(k-1)}(x-j)\right]\\ &= \frac{(m-1)(x-(m+3k))}{2(m+1)!(12^{k-1})}\prod_{j=0}^{m+3(k-1)}(x-j).\end{aligned}$$
□

**Corollary 4.11.** *If $S = \{m, m+3, \dots, m+3k, m+3k+2\}$ for $k \geq 1$, then*

$$p_S(x) = \frac{(m-1)(x-(m+3k+2))(x-(m+3k))(x-(m+3k-5))}{(m+1)!(12^k)} \prod_{j=0}^{m+3(k-1)} (x-j).$$

*Proof.* The proof follows from Lemma 4.7 and Theorem 4.10. □

4.3. **Gap of three independence.** The following theorem shows that if $S$ has a gap of three anywhere, then $p_S(x)$ is independent of the peaks to the left of that gap up to a constant. Furthermore, the complex zeros of $p_S(x)$ depend only on the peaks to the right of the gap of three and where this gap occurs. Corollaries of this result follow.

**Theorem 4.12.** *Let $S_L = \{i_1 < i_2 < \cdots < i_\ell = m\}$ and $S_R = \{2 < j_2 < \cdots < j_r\}$. If $S = \{i_1 < i_2 < \cdots < m < m+3 < (m+1)+j_2 < \cdots < (m+1)+j_r\}$, then*

$$p_S(x) = \frac{p_{S_L}(m+1)}{2(m+1)!} p_{S_R}(x-(m+1)) \prod_{k=0}^{m} (x-k).$$

*Proof.* We first prove the corresponding statement in terms of permutations with a given peak set. Fix a positive integer $n > (m+1)+j_r$. Choose $m+1$ of the $n$ elements in $[n]$, and arrange them so that their peak set is $S_L$. Now arrange the remaining $n-(m+1)$ elements so that their peak set is $S_R$. This construction produces all of the permutations in $\mathfrak{S}_n$ whose peak set is $S$ without repetition, because $m+1$ and $m+2$ cannot be peaks since $m$ and $m+3$ are. Thus we have

$$|\mathcal{P}_S(n)| = \binom{n}{m+1} |\mathcal{P}_{S_L}(m+1)| \cdot |\mathcal{P}_{S_R}(n-(m+1))|. \tag{1}$$

Using Theorem 1.1,

$$p_S(n) 2^{n-|S|-1} = \binom{n}{m+1} p_{S_L}(m+1) 2^{(m+1)-|S_L|-1} p_{S_R}(n-(m+1)) 2^{(n-(m+1))-|S_R|-1}.$$

and since $|S| = |S_L| + |S_R|$, we have

$$p_S(n) = \frac{p_{S_L}(m+1)}{2(m+1)!} p_{S_R}(n-(m+1)) \prod_{k=0}^{m} (n-k).$$

This proves the theorem because we have shown that the polynomial on the right and the left agree on an infinite number of values. □

From the factorization in (1), we clearly see that $0, 1, 2, \dots, m$ are zeros of $p_S(z)$, and the zeros of $p_{S_R}(z)$ are zeros of $p_S(z)$ when translated to the right by $m+1$ in the complex plane. Note that $\deg(p_S(x)) = m + j_r$ because $\max(S) = (m+1)+j_r$, but we also see this by counting the $m+1$ leftmost integer roots and then the $j_r - 1$ roots of $p_{S_R}(x)$. Theorem 4.12 also implies Lemma 4.6 when $S_R = \{2\}$ for all $S_L$ because $p_{\{2\}}(x) = x-2$. The plots and corollaries below demonstrate this independence.

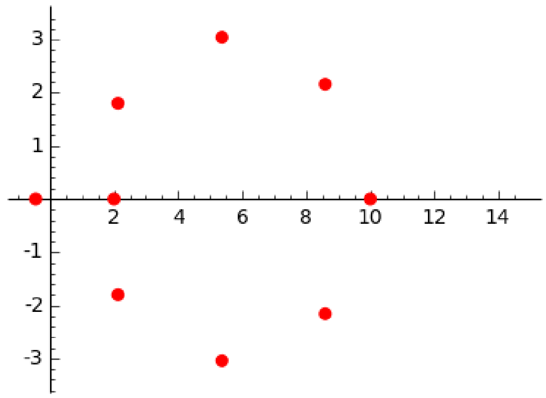


FIGURE 1. Zeros of $p_{\{2,10\}}(z)$

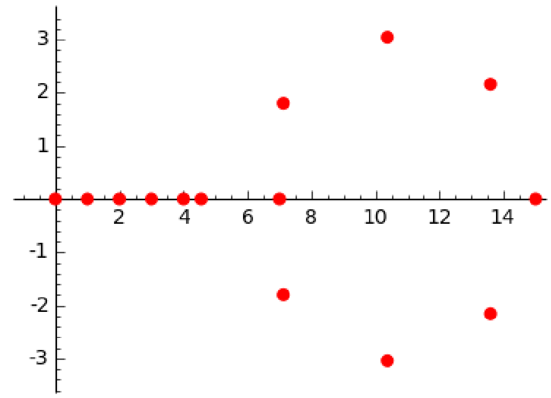


FIGURE 2. Zeros of $p_{\{4,7,15\}}(z)$

**Corollary 4.13.** *Let $S_L = \{i_1 < i_2 < \cdots < i_\ell = m\}$, $S_R = \{j_1 = 2 < j_2 < \cdots < j_r\}$, and $S = \{i_1 < i_2 < \cdots < m < m+3 < (m+1)+j_2 < \cdots < (m+1)+j_r\}$. If $S_R$ has no zero with real part greater than $j_r$, then $p_S(x)$ has no zero with real part greater than $\max(S)$.*

*Proof.* The proof follows from Theorem 4.12. $\square$

If we want to verify that Conjecture 2.3 holds for a peak set $S$ with a gap of three, then it suffices to check that it holds for $S_R$ by Corollary 4.13.

**Corollary 4.14.** *Let $S_L = \{i_1 < i_2 < \cdots < i_\ell = m\}$, $S_R = \{j_1 = 2 < j_2 < \cdots < j_r\}$, and $S = \{i_1 < i_2 < \cdots < m < m+3 < (m+1)+j_2 < \cdots < (m+1)+j_r\}$. If we define $S+1 = \{i+1 : i \in S\}$, then*

$$p_{S+1}(x) = C(S)p_S(x-1)x,$$

*where*

$$C(S) = \frac{p_{S_L+1}(m+2)}{(m+2)p_{S_L}(m+1)}$$

*is a constant depending only on $S$.*

*Proof.* Using Theorem 4.12, we see that

$$p_S(x-1) = \frac{p_{S_L}(m+1)}{2(m+1)!}p_{S_R}(x-(m+2))\prod_{k=0}^{m}(x-(k+1))$$

and

$$p_{S+1}(x) = \frac{p_{S_L+1}(m+2)}{2(m+2)!}p_{S_R}(x-(m+2))\prod_{k=0}^{m+1}(x-k).$$

Solving for $p_{S+1}(x)$, we have

$$p_{S+1}(x) = C(S)p_S(x-1)x,$$

where

$$C(S) = \frac{p_{S_L+1}(m+2)}{(m+2)p_{S_L}(m+1)}$$

depends only on $S$. $\square$

Observe that Corollary 4.14 shifts all of the zeros of $p_S(z)$ in the complex plane to the right by one and then picks up a new root at 0 since $C(S)$ is a constant. The plots below illustrate this behavior.

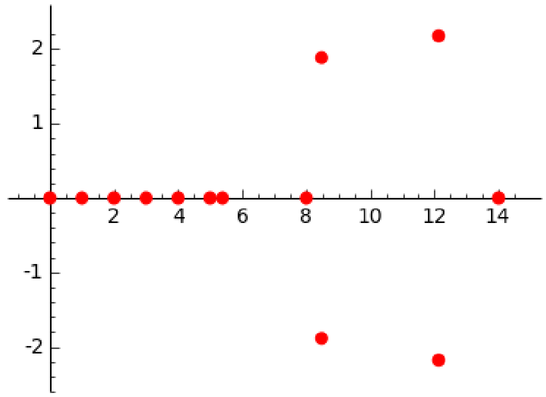


FIGURE 3. Zeros of $p_{\{3,5,8,14\}}(z)$

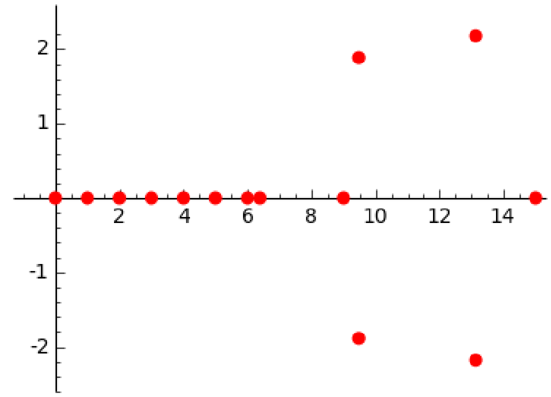


FIGURE 4. Zeros of $p_{\{4,6,9,15\}}(z)$

## 5. Evaluating $p_S(x)$ at nonnegative integers

In the previous section we identified integral zeros of $p_S(x)$, so now we will try to understand the behavior of $p_S(x)$ at nonnegative integers $j$ when $p_S(j) \neq 0$. We prove that there is a curious symmetry between column and row 0 in the table of forward differences of $p_S(x)$ (see Table 2), and that the nonzero values of $|p_S(j)|$ are weakly increasing for $j \in [\max(S) - 1]$ when $\min(S) \geq 4$. Again, assume that $S$ is a nonempty admissible set in the following hypotheses.

**Lemma 5.1.** *Let $S \neq \emptyset$ and $m = \max(S)$. For $k \geq 0$, we have*

$$\sum_{j=1}^{k-1} (-1)^{k-1-j} p_S(m+j) \binom{m+k}{m+j} = 2 p_S(m+k) \chi(k \text{ even}).$$

*Proof.* Let $T = S \cup \{m+k\}$. We know from Theorem 1.1 that $p_T(m+k) = 0$, and then apply Lemma 4.1. $\square$

**Lemma 5.2.** *For $S = \{i_1 < i_2 < \cdots < i_s = m < m+k\}$ and $\ell \in [k-1]$, we have $p_S(m+\ell) = -p_{S_1}(m+\ell)$.*

*Proof.* Using Lemma 4.1 and Lemma 5.1, observe that

$$\begin{aligned}
p_S(m+\ell) &= -2p_{S_1}(m+\ell)\chi(k \text{ even}) + \sum_{j=1}^{k-1} (-1)^{k-1-j} p_{S_1}(m+j) \binom{m+\ell}{m+j} \\
&= -2p_{S_1}(m+\ell)\chi(k \text{ even}) + (-1)^{k-\ell} \sum_{j=1}^{\ell-1} (-1)^{\ell-1-j} p_{S_1}(m+j) \binom{m+\ell}{m+j} \\
&\quad + (-1)^{k-1-\ell} p_{S_1}(m+\ell) \\
&= -2p_{S_1}(m+\ell)\chi(k \text{ even}) + (-1)^{k-\ell} 2 p_{S_1}(m+\ell)\chi(\ell \text{ even}) + (-1)^{k-1-\ell} p_{S_1}(m+\ell).
\end{aligned}$$

Considering all possible parities of $k$ and $\ell$, we see that $p_S(m+\ell) = -p_{S_1}(m+\ell)$. $\square$

**Theorem 5.3.** *Let $S \neq \emptyset$ and $m = \max(S)$. If $j \in \{0, 1, \ldots, m\}$, then*

$$(\Delta^j p_S)(0) = (-1)^{m+j} p_S(j).$$

*Proof.* We induct on $|S|$. In the base case $|S| = 1$, and we use Lemma 2.2 and Vandermonde's identity to observe

$$p_{\{m\}}(x) = \left[\sum_{j=0}^{m-1} \binom{-1}{m-1-j}\binom{x}{j}\right] - 1.$$

It follows that,

$$(\Delta^j p_{\{m\}})(0) = \begin{cases} (-1)^{m-1} - 1 & \text{if } j = 0, \\ (-1)^{m-1-j} & \text{if } j \in [m-1], \\ 0 & \text{if } j = m. \end{cases}$$

Similarly, we use Lemma 2.2 to evaluate

$$\begin{aligned} (-1)^{m+j} p_S(j) &= (-1)^{m+j}\left[\binom{j-1}{m-1} - 1\right] \\ &= \begin{cases} (-1)^{m+1} - 1 & \text{if } j = 0, \\ (-1)^{m+j+1} & \text{if } j \in [m-1], \\ 0 & \text{if } j = m, \end{cases} \end{aligned}$$

which proves the base case.

In the inductive step $|S| \geq 2$, so let $S = \{i_1 < i_2 < \cdots < i_s = m < m+k\}$ for $k \geq 2$. Using Lemma 4.1 and expanding $p_{S_1}(x)$ in the binomial basis centered at 0,

$$\begin{aligned} p_S(x) &= -2p_{S_1}(x)\chi(k \text{ even}) + \sum_{j=m+1}^{m+k-1} (-1)^{k-1-(j-m)} p_{S_1}(j)\binom{x}{j} \\ &= -2\left[\sum_{j=0}^{m} (\Delta^j p_{S_1})(0)\binom{x}{j}\right]\chi(k \text{ even}) + \sum_{j=m+1}^{m+k-1} (-1)^{k-1-(j-m)} p_{S_1}(j)\binom{x}{j}. \end{aligned} \tag{2}$$

Assume the case that $j \in \{0, 1, \ldots, m\}$. Considering both possible parities of $k$, we use (2) and the induction hypothesis to see that

$$\begin{aligned} (\Delta^j p_S)(0) &= -2(\Delta^j p_{S_1})(0)\chi(k \text{ even}) \\ &= -2(-1)^{m+j} p_{S_1}(j)\chi(k \text{ even}) \\ &= (-1)^{(m+k)+j} p_S(j), \end{aligned}$$

because $p_S(j) = -2p_{S_1}(j)\chi(k \text{ even})$ by Lemma 4.1. Now let $j \in \{m+1, m+2, \ldots, m+k-1\}$. Using Lemma 5.2 and (2), we have

$$\begin{aligned} (\Delta^j p_S)(0) &= (-1)^{k-1-(j-m)} p_{S_1}(j) \\ &= (-1)^{(m+k)+j} p_S(j). \end{aligned}$$

Lastly, $(\Delta^m p_S)(0) = 0$ because $\deg(p_S(x)) = m - 1$, which completes the proof. $\square$

For example, if $j > 0$ is between the largest odd gap and $m$, then by this symmetry property and Theorem 1.2 one can observe that

$$p_S(j) = (-1)^{m+j}(\Delta^j p_S)(0) = -(-2)^{|S\cap(j,\infty)|-1} p_{S\cap[j]}(j).$$

If $S$ has no odd gaps, then the equation above holds for all $j \in [m]$.

**Lemma 5.4.** *If $S \neq \emptyset$ and $m = \max(S)$, then $p_S(j) < p_S(j+1)$ for $j \geq m$.*

*Proof.* We prove the result by splitting into two cases. When $|S| = 1$, we have $p_{\{m\}}(x)$, which increases on $(m-1, \infty)$ by Theorem 2.2 and proves our claim. In the second case, let $|S| \geq 2$. We want to show that $p_S(j) < p_S(j+1)$, which is equivalent to showing $2|\mathcal{P}_S(j)| < |\mathcal{P}_S(j+1)|$, so we need to construct more than twice as many permutations in $\mathfrak{S}_{j+1}$ with peak set $S$ than there are in $\mathfrak{S}_j$. Note that $p_S(m) = 0$ and $p_S(m+1) > 0$, so we need only consider $\mathfrak{S}_j$ for $j \geq m+1$. First, let $\pi \in \mathfrak{S}_j$ and append $j+1$ to $\pi$. This gives us $|\mathcal{P}_S(j)|$ permutations in $\mathfrak{S}_{j+1}$. Now construct $|\mathcal{P}_S(j)|$ different permutations by inserting $j+1$ between positions $m-1$ and $m$, so that $j+1$ becomes the final peak. Lastly, place $j+1$ at the first peak position (reading left to right), $j$ at the next peak position, etc., and then fill the empty indices from left to right with $1, 2, \ldots, j+1-|S|$, respectively. Each of the $2|\mathcal{P}_S(n)|+1$ constructed permutations is distinct and has peak set $S$, so $p_S(j) < p_S(j+1)$. $\square$

**Theorem 5.5.** *Let $S = \{i_1 < i_2 < \cdots < i_s = m\}$. For integers $1 \leq j < k$, we have $|p_S(j)| \leq |p_S(k)|$ provided $p_S(k) \neq 0$, except for the case $\{2\} \subsetneq S$ where $p_S(1) = 2p_S(3) = -(-2)^{|S|-1}$.*

*Proof.* If $|p_S(j)| = 0$, then the claim is trivially true, so assume that $|p_S(j)| > 0$ which implies $S \cap (j, \infty)$ has no odd gaps. If $S = \emptyset$ or not admissible then the statement holds so assume $S \neq \emptyset$, admissible, and $m = \max(S)$. We first consider the cases where $j < k < m$. We use these assumptions along with Theorem 1.2 and Corollary 5.3 to observe that

$$|p_S(j)| = 2^{|S\cap(j,\infty)|-1} |p_{S\cap[j]}(j)|. \tag{3}$$

Consider the case $p_S(j+1) \neq 0$. Then $j+1 \notin S$ by Theorem 4.5, and

$$\begin{aligned} |p_S(j+1)| &= 2^{|S\cap(j+1,\infty)|-1} |p_{S\cap[j+1]}(j+1)| \\ &= 2^{|S\cap(j,\infty)|-1} |p_{S\cap[j]}(j+1)|. \end{aligned}$$

To show that $|p_S(j)| \leq |p_S(j+1)|$ it suffices to show that $|p_{S\cap[j]}(j)| \leq |p_{S\cap[j]}(j+1)|$. If $S \cap [j] = \emptyset$, then we know $p_\emptyset(x) = 1$ from Theorem 1.1. Otherwise, we may use Lemma 5.4 because $S \neq \emptyset$ and $j \geq \max(S \cap [j])$. In both cases, $|p_S(j)| \leq |p_S(j+1)|$ when $|p_S(j+1)| > 0$.

Now assume that $p_S(j+1) = 0$. Combining Theorem 1.1, Corollary 5.3, and the assumption that $|p_S(j)| > 0$, this implies $|p_{S\cap[j+1]}(j+1)| = 0$ which in turn implies $j+1 \in S$ by Lemma 5.4. Since $S$ is admissible $j+2 \notin S$ so $p_{S\cap[j+1]}(j+2) = p_{S\cap[j+2]}(j+2) > 0$. By (3) this implies $|p_S(j+2)| > 0$. To show that $|p_S(j)| \leq |p_S(j+2)|$, we will show that

$$2^{|S\cap(j,\infty)|-1} |p_{S\cap[j]}(j)| \leq 2^{|S\cap(j+2,\infty)|-1} |p_{S\cap[j+2]}(j+2)|, \tag{4}$$

assuming $j+1 \in S$. Let $R = S \cap [j+2]$, and $R_1 = R \setminus \{j+1\}$. Using Theorem 1.1, (4) is true if and only if

$$4|\mathcal{P}_{R_1}(j)| \leq |\mathcal{P}_R(j+2)|. \tag{5}$$

To prove (5), observe that one can choose any $j$ elements from $[j+1]$, arrange them to have peak set $R_1$ in $|\mathcal{P}_{R_1}(j)|$ ways, and then append $j+2$ and the remaining element to this sequence in decreasing order. The resulting permutation has peak set $R$, and doing this in all possible ways yields $(j+1)|\mathcal{P}_{R_1}(j)|$ distinct permutations in $\mathfrak{S}_{j+2}$. If $j+1 \geq 4$, then (5) holds so $|p_S(j)| \leq |p_S(k)|$ when $|p_S(j+1)| = 0$. Observe that the exact same argument proves the theorem for the case $m > 3$, $j = m-1$, and $k = m+1$.

If $j+1 \in \{2,3\}$, then by (3) we can complete the proof using the fact that $p_{\emptyset}(x) = 1$, and by computing the values of $p_{\{2\}}(n)$ and $p_{\{3\}}(n)$ for $n = 0,1,2,3,4$, we have

$$S = \{2\} \implies (-2,-1,0,1,2)$$

and

$$S = \{3\} \implies (0,-1,-1,0,2).$$

In fact, using that data and Theorem 1.2 we see $p_S(1) = -(-2)^{|S|-1}$ for all nonempty admissible sets $S$ with no odd gaps and 0 otherwise. Similarly,

$$p_S(2) = \begin{cases} 0 & \text{if } 2 \in S \text{ or } S \text{ has an odd gap,} \\ 1 & \text{if } S = \emptyset, \\ -(-2)^{|S|-1} & \text{otherwise,} \end{cases}$$

and

$$p_S(3) = \begin{cases} 0 & \text{if } 3 \in S \text{ or } S \text{ has an odd gap after } 3, \\ 1 & \text{if } S \subset [2], \\ -(-2)^{|S|-2} & \text{if } \{2\} \subsetneq S, \\ -(-2)^{|S|-1} & \text{otherwise,} \end{cases}$$

which proves the special case of the theorem where the inequality does Not hold. For completeness,

$$p_S(4) = \begin{cases} 0 & \text{if } 4 \in S \text{ or } S \text{ has an odd gap after } 4, \\ 1 & \text{if } S = \emptyset, \\ 2 & \text{if } S = \{2\} \text{ or } S = \{3\}, \\ -(-2)^{|S|-1} & \text{if } \{2,3\} \cap S = \emptyset, |S| > 1, \text{ and } S \text{ has no odd gaps,} \\ (-2)^{|S|-1} & \text{otherwise.} \end{cases}$$

For $n > 4$, the values of $|p_S(n)|$ are not typically powers of 2.

Finally, the theorem holds for all remaining cases with $m < j < k$ by Lemma 5.4 and transitivity. $\square$

The previous proof also implies the following statement.

**Corollary 5.6.** *Let $S$ be a set of positive integers and $j$ be a positive integer such that $p_S(j) \neq 0$. Let $k \geq j$ integer. If $p_S(k) = 0$ then $k \in S$.*

## 6. Connections to alternating permutations

In this section we enumerate permutations with a given peak set using alternating permutations and tangent numbers instead of the recurrence given by Lemma 4.1. Alternating permutations allow us to easily count the number of permutations whose peak set is a superset of $S$, so we combine this idea with the inclusion-exclusion principle to evaluate $|\mathcal{P}_S(n)|$.

Assume that $S$ is a nonempty admissible peak set and that $m = \max(S)$. Let $\mathcal{Q}_S(n) = \{\pi \in \mathfrak{S}_n : S \subseteq P(\pi)\}$ be the set of permutations $\pi \in \mathfrak{S}_n$ whose peak set contains $S = \{i_1 < i_2 < \cdots < i_s\}$, and let us partition $S$ into runs of alternating substrings. An *alternating* *substring* is a maximal size subset $A_r$ such that $A_r = \{i_r, i_r+2, \ldots, i_r+2(k-1)\} \subseteq S$, where $i_r - i_{r-1} \geq 3$ if $i_{r-1} \in S$, and we call $A_r$ an alternating substring because

$$\pi_{i_r-1} < \pi_{i_r} > \pi_{i_r+1} < \pi_{i_r+2} > \cdots < \pi_{i_r+2(k-1)} > \pi_{i_r+2(k-1)+1}$$

is an alternating permutation in $\mathfrak{S}_{2k+1}$ under an order-preserving map. Alternating permutations have peaks at every even index, and there are $E_{2k+1}$ of them in $\mathfrak{S}_{2k+1}$. The numbers $E_{2k+1}$ are the tangent numbers given by the generating function

$$\begin{aligned}\tan x &= \sum_{k=0}^{\infty} \frac{E_{2k+1}}{(2k+1)!} x^{2k+1} \\ &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \ldots\end{aligned}$$

André proved this result in [1] using a generating function that satisfies a differential equation. See [12] for more background on alternating permutations.

Now let $\mathcal{A}(S)$ be the partition of an admissible set $S$ into maximal alternating substrings. For example, if $S = \{2, 5, 9, 11, 19, 21, 23, 26\}$, then

$$\mathcal{A}(S) = \{A_1, A_2, A_3, A_5, A_8\} = \{\{2\}, \{5\}, \{9, 11\}, \{19, 21, 23\}, \{26\}\}.$$

The following results demonstrate how we can use $\mathcal{Q}_S(n)$ to enumerate permutations with a given peak set.

**Lemma 6.1.** *For $n \geq m+1$, we have*

$$|\mathcal{Q}_S(n)| = n! \prod_{A_r \in \mathcal{A}(S)} \frac{E_{2|A_r|+1}}{(2\,|A_r|+1)!}.$$

*Proof.* The formula is easily checked in the case $S = \emptyset$, so assume $S \neq \emptyset$. Assume the theorem is true by induction for all sets $S'$ such that $|\mathcal{A}(S')| < |\mathcal{A}(S)|$. Say $A_1 = \{i_1, i_1 + 2, \ldots, i_1 + 2(k-1)\} \in \mathcal{A}(S)$. We count the number of permutations $\pi \in \mathfrak{S}_n$ such that $A_1 \subseteq P(\pi)$ by choosing $2k+1$ of the $n$ elements, arranging them such that their peak set is $A_1$ in $E_{2k+1}$ ways, then appending any permutation of the remaining $n-(2k+1)$ elements arranged to have peak set contained in $S' = S \setminus A_1$. The result now follows by induction. $\square$

**Lemma 6.2.** *For $n \geq m+1$, we have*

$$|\mathcal{P}_S(n)| = \sum_{T \supseteq S} (-1)^{|T-S|} |\mathcal{Q}_T(n)|.$$

*Proof.* The proof follows the inclusion-exclusion principle. $\square$

Call an index $i$ a *free index* of peak set $S$ if $i \in [m+2]$ and $i$ is neither a peak nor adjacent to a peak in $S$. The following theorem gives us a closed-form expression of tangent numbers for $|\mathcal{P}(m+1)|$ and $|\mathcal{P}(m+2)|$ when $S$ has no free indices. Note that if $S$ has no free indices, then it can be thought of as separate independent alternating permutations that are concatenated to each other, similar to the independence in Theorem 4.12.

**Corollary 6.3.** *If $S$ has no free indices and $k \in [2]$, then*

$$|\mathcal{P}_S(m+k)| = (m+k)! \prod_{A_r \in \mathcal{A}(S)} \frac{E_{2|A_r|+1}}{(2|A_r|+1)!}.$$

*Proof.* We observe that $S$ is the only admissible superset of $S$ and use Lemma 6.1 and Lemma 6.2. $\square$

## 7. Related work and conjectures

In this final section we relate our work to other recent results about permutations with a given peak set, and we also restate some conjectures that stemmed from our work. Kasraoui characterized in [10] which peak sets $S$ maximize $|\mathcal{P}_S(n)|$ for $n \geq 6$ and explicitly computed $|\mathcal{P}_S(n)|$ for such sets $S$. We compute the maximum $|\mathcal{P}_S(n)|$ in a different way using alternating permutations.

**Theorem 7.1** ([10, Theorem 1.1, 1.2])**.** *For $n \geq 6$, the sets $S$ that maximize $|\mathcal{P}_S(n)|$ are*

$$S = \begin{cases} \{3,6,9,\dots\} \cap [n-1] \text{ and } \{4,7,10,\dots\} \cap [n-1] & \text{if } n \equiv 0 \pmod 3, \\ \{3,6,9,\dots,3s,3s+2,3s+5,\dots\} \cap [n-1] \text{ for } 1 \leq s \leq \lfloor \frac{n}{3} \rfloor & \text{if } n \equiv 1 \pmod 3, \\ \{3,6,9,\dots\} \cap [n-1] & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

**Theorem 7.2** ([10, Theorem 1.2])**.** *Suppose $n \geq 6$ and $S$ maximizes $|\mathcal{P}_S(n)|$. Set $\ell = \lfloor \frac{n}{3} \rfloor$. Then we have*

$$|\mathcal{P}_S(n)| = \begin{cases} \frac{1}{5} 3^{2-\ell} n! & \text{if } n \equiv 0 \pmod 3, \\ \frac{2}{5} 3^{1-\ell} n! & \text{if } n \equiv 1 \pmod 3, \\ 3^{-\ell} n! & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

*Alternative proof.* We work by cases using Theorem 7.1. When $n \equiv 0 \pmod 3$, there is only one admissible superset of $S$, which we call $T$. Using Theorem 6.1 and Lemma 6.2,

$$\begin{aligned} |\mathcal{P}_S(n)| &= |\mathcal{Q}_S(n)| - |\mathcal{Q}_T(n)| \\ &= n! \left(\frac{1}{3}\right)^{\ell-1} - n! \left(\frac{1}{3}\right)^{\ell-2} \left(\frac{2}{15}\right) \\ &= \frac{1}{5} 3^{2-\ell} n!, \end{aligned}$$

as desired. We use Corollary 6.3 to prove the cases $n \equiv 1, 2 \pmod 3$, which are simpler because there are no admissible supersets of $S$. $\square$

Another new result in [5] shows that the number of permutations with the same peak set for signed permutations can be enumerated using the peak polynomial $p_S(x)$ for unsigned permutations. Again, we present an alternate proof, and it can be used to reduce many signed permutation statistic problems to unsigned permutation statistic problems. We denote the group of signed permutations as $B_n$.

**Theorem 7.3** ([5, Theorem 2.7])**.** *Let $|\mathcal{P}^*_S(n)|$ be the number of signed permutations $\pi \in B_n$ with peak set $S$. We have $|\mathcal{P}^*_S(n)| = p_S(n) 2^{2n-|S|-1}$, where $p_S(x)$ is the same peak polynomial used to count unsigned permutations $\pi \in \mathfrak{S}_n$ with peak set $S$.*

*Alternative proof.* We naturally partition $B_n$ by the signage of the permutations, which gives us $2^n$ copies of $\mathfrak{S}_n$ under an order-preserving map, and then we work in each copy of $\mathfrak{S}_n$ separately. For example, $B_3 = \{\mathfrak{S}_{+++}, \mathfrak{S}_{++-}, \mathfrak{S}_{+-+}, \mathfrak{S}_{+--}\mathfrak{S}_{-++}, \mathfrak{S}_{-+-}, \mathfrak{S}_{--+}, \mathfrak{S}_{---}\}$ and $\mathfrak{S}_{++-}$ are the permutations of $\{1, 2, -3\}$. It follows that $|\mathcal{P}^*_S(n)| = 2^n |\mathcal{P}_S(n)|$, so $|\mathcal{P}^*_S(n)| = p_S(n) 2^{2n-|S|-1}$ by Theorem 1.1. $\square$

Now we restate some conjectures. In [7] we checked Conjecture 7.4 for all admissible peak sets $S$ where $\max(S) \leq 15$, and this conjecture implies the truth of Conjecture 2.3, which we explained in Section 3. We have also shown in Subsection 4.2 that the peak sets listed in Conjecture 7.5 have only integral zeros, but we have not proven the other direction. Conjecture 7.6 is an observation that is related to Conjecture 7.4, and we have proved it for all integral $x_0$ using Lemma 5.2 and Lemma 5.4, but not all real $x_0$.

**Conjecture 7.4.** The complex zeros of $p_S(n)$ lie in $\{z \in \mathbb{C} : |z| \leq m \text{ and } \mathrm{Re}(z) \geq -3\}$ if $S$ is admissible.

**Conjecture 7.5.** If $S = \{i_1 < i_2 < \cdots < i_s\}$ is admissible and all of the roots of $p_S(n)$ are real, then all of the roots of $p_S(n)$ are integral. Furthermore, $p_S(n)$ has all real roots if and only if $S = \{2\}$, $S = \{2,4\}$, $S = \{3\}$, $S = \{3,5\}$, $S = \{i_1 < i_2 < \cdots < i_s < i_s + 3\}$, or $S = \{i_1 < i_2 < \cdots < i_s < i_s + 3 < i_s + 5\}$.

**Conjecture 7.6.** Let $S$ be admissible and $|S| \geq 2$. If $p_S(x_0) = 0$ for $x_0 \in \mathbb{R}$, then $x_0 > \max(S_1)$ if and only if $x_0 = \max(S)$.

**Question 7.7.** What does $p_S(n)$ count for $n > \max(S)$?

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