

EMBEDDED FACTOR PATTERNS FOR DEODHAR ELEMENTS IN KAZHDAN–LUSZTIG THEORY

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ABSTRACT. The Kazhdan–Lusztig polynomials for finite Weyl groups arise in the geometry of Schubert varieties and representation theory. It was proved very soon after their introduction that they have nonnegative integer coefficients, but no simple all positive interpretation for them is known in general. Deodhar [Deo90] has given a framework for computing the Kazhdan–Lusztig polynomials which generally involves recursion. We define embedded factor pattern avoidance for general Coxeter groups and use it to characterize when Deodhar’s algorithm yields a simple combinatorial formula for the Kazhdan–Lusztig polynomials of finite Weyl groups. Equivalently, if (W, S) is a Coxeter system for a finite Weyl group, we classify the elements $w \in W$ for which the Kazhdan–Lusztig basis element C'_w can be written as a monomial of C'_s where $s \in S$. This work generalizes results of Billey–Warrington [BW01] that identified the Deodhar elements in type A as 321-hexagon-avoiding permutations, and Fan–Green [FG97] that identified the fully-tight Coxeter groups.

1. INTRODUCTION

The Kazhdan–Lusztig polynomials for finite Weyl groups [KL79] arise as Poincaré polynomials for intersection cohomology of Schubert varieties [KL80] and as a q -analogue of the multiplicities for Verma modules [BB81, BK81]. They are defined to be the coefficients in the transition matrix for expanding the Kazhdan–Lusztig basis elements in the Hecke algebra associated to the Weyl group into the standard basis. Several algorithms exist, formulas for special cases, and interesting properties are known for these polynomials; see for example [Hum90, Deo94, MW03, Pol99, LS81, Bre04, BB05]. In particular, these polynomials have nonnegative integer coefficients but no simple all positive formula for the coefficients is known in general for all Coxeter groups.

Deodhar [Deo90] proposes a combinatorial framework for determining the Kazhdan–Lusztig polynomials of an arbitrary Coxeter group. The algorithm he describes is shown to work for all Coxeter groups where the Kazhdan–Lusztig polynomials are known to have nonnegative integer coefficients which includes Weyl groups and the Coxeter groups associated to crystallographic Kac–Moody groups. Under certain conditions, Deodhar’s algorithm for determining the Kazhdan–Lusztig polynomials turns out to be a beautiful combinatorial formula. These conditions are also equivalent to the Kazhdan–Lusztig basis element C'_w being equal to a product of C'_s ’s indexed by generators of the Coxeter group. We say that w is *Deodhar* when it satisfies these conditions. In 1999, Billey and Warrington [BW01] gave an efficient characterization of the Deodhar elements in the symmetric group as *321-hexagon avoiding* permutations. Their results extend to finite *linear* Weyl groups, types A, B, F, G . Our goal is to give a similar characterization for all finite Weyl groups.

In this paper we give two characterizations of the Deodhar elements for all finite Weyl groups. One characterization is given in terms of 1-line pattern avoidance in analogy with the type A result. This

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characterization gives a polynomial time algorithm to test for the Deodhar status of an element, but involves a long list of patterns. The second characterization is in terms of a new type of pattern called an *embedded factor*. These patterns are defined in terms of reduced expressions, and generalize containment in the 2-sided weak Bruhat order. Theorem 5.12 states that the Deodhar elements of Weyl groups can be characterized by avoiding embedded factors from the following list, as well as an additional 1-line pattern for type D .

Type	Coxeter Graph	Embedded Factor Patterns
$I_2(m)$, $m \geq 3$	$\bullet_1 \overset{m}{\text{---}} \bullet_2$	$s_1 s_2 s_1, s_2 s_1 s_2$ (short braids)
A_7	$\bullet_1 \text{---} \bullet_2 \text{---} \bullet_3 \text{---} \bullet_4 \text{---} \bullet_5 \text{---} \bullet_6 \text{---} \bullet_7$	$s_5 s_6 s_7 s_3 s_4 s_5 s_6 s_2 s_3 s_4 s_5 s_1 s_2 s_3$ (HEX)
B_7/C_7	$\bullet_0 \overset{4}{\text{---}} \bullet_1 \text{---} \bullet_2 \text{---} \bullet_3 \text{---} \bullet_4 \text{---} \bullet_5 \text{---} \bullet_6$	$s_4 s_5 s_6 s_2 s_3 s_4 s_5 s_1 s_2 s_3 s_4 s_0 s_1 s_2$ ($BHEX$)
D_6	$\bullet_{\bar{1}}$ \diagdown $\bullet_1 \text{---} \bullet_2 \text{---} \bullet_3 \text{---} \bullet_4 \text{---} \bullet_5$	$s_3 s_4 s_5 s_1 s_2 s_3 s_4 s_{\bar{1}} s_2 s_3 s_1$ (HEX_4)
D_7	$\bullet_{\bar{1}}$ \diagdown $\bullet_1 \text{---} \bullet_2 \text{---} \bullet_3 \text{---} \bullet_4 \text{---} \bullet_5 \text{---} \bullet_6$	$s_3 s_4 s_5 s_6 s_2 s_3 s_4 s_5 s_{\bar{1}} s_2 s_3 s_4 s_1 s_2 s_3$ (HEX_2) $s_4 s_5 s_6 s_2 s_3 s_4 s_5 s_1 s_2 s_3 s_4 s_{\bar{1}} s_2 s_1$ (HEX_{3a}) $s_1 s_4 s_5 s_6 s_2 s_3 s_4 s_5 s_{\bar{1}} s_2 s_3 s_4 s_1 s_2$ (HEX_{3b})
D_8	$\bullet_{\bar{1}}$ \diagdown $\bullet_1 \text{---} \bullet_2 \text{---} \bullet_3 \text{---} \bullet_4 \text{---} \bullet_5 \text{---} \bullet_6 \text{---} \bullet_7$	$s_4 s_5 s_6 s_7 s_3 s_4 s_5 s_6 s_2 s_3 s_4 s_5 s_{\bar{1}} s_1 s_2 s_3 s_4$ (diamond, to be avoided as a 1-line pattern)
E_6	\bullet_5 $ $ $\bullet_0 \text{---} \bullet_1 \text{---} \bullet_2 \text{---} \bullet_3 \text{---} \bullet_4$	$s_0 s_1 s_2 s_5 s_3 s_4 s_2 s_3 s_1 s_2 s_5 s_0 s_1$ $s_5 s_1 s_2 s_3 s_0 s_1 s_2 s_5 s_4 s_3 s_2 s_1 s_0$ $s_1 s_2 s_5 s_3 s_4 s_2 s_3 s_1 s_2 s_5 s_0 s_1 s_2$ $s_2 s_5 s_1 s_2 s_3 s_0 s_1 s_2 s_5 s_4 s_3 s_2 s_1$
E_7	\bullet_5 $ $ $\bullet_0 \text{---} \bullet_1 \text{---} \bullet_2 \text{---} \bullet_3 \text{---} \bullet_4 \text{---} \bullet_5$	$s_0 s_1 s_2 s_3 s_4 s_6 s_5 s_2 s_3 s_4 s_1 s_2 s_3 s_0 s_1$ $s_3 s_4 s_6 s_1 s_2 s_3 s_0 s_1 s_2 s_5 s_4 s_3 s_2 s_1 s_0$ $s_1 s_2 s_3 s_4 s_6 s_5 s_2 s_3 s_4 s_1 s_2 s_3 s_0 s_1 s_2$ $s_2 s_3 s_4 s_6 s_1 s_2 s_3 s_0 s_1 s_2 s_5 s_4 s_3 s_2 s_1$ $s_5 s_2 s_3 s_4 s_6 s_1 s_2 s_5 s_3 s_4 s_2 s_3 s_0 s_1 s_2 s_5$

FIGURE 1. Minimal non-Deodhar patterns

The embedded factor patterns take into account different ways of embedding one Weyl group into another as a parabolic subgroup. For example, the Weyl group of type E_8 has parabolic subgroups of types A_2, A_7, D_6, D_7, E_6 , and E_7 from this list. Therefore, a Deodhar element in the Weyl group of type E_8 cannot have any embedded factors in the form of a short braid, hexagon, HEX_i , or any of the E_6 or E_7 patterns.

In type D_n , the Deodhar elements must also avoid the ‘‘diamond’’ pattern $[\bar{1}678\bar{5}234]$ as a 1-line pattern. This single 1-line pattern encapsulates an infinite antichain of type D embedded factor patterns further discussed in Example 5.11.

We also provide a finite test in Theorem 11.5 to determine when it is possible to translate between classical 1-line pattern avoidance and the embedded factor pattern avoidance of Definition 5.5. This result generalizes a fact which is implicit in [BW01] that avoiding the hexagon embedded factor pattern can be characterized by avoiding 4 classical permutation patterns when we restrict to the fully commutative elements. Theorem 11.5 also justifies using the methods in this paper to study certain classical pattern classes, and has been extended for type A in [DJ07] to obtain new enumerative results.

In Section 2, we recall the basic definitions of Kazhdan–Lusztig polynomials and basis elements. We define the Deodhar elements and recall the theorem that inspired this name. In Section 3, we recall the heap of a reduced expression for Weyl groups of types A, B, D , and in Section 4 we review the definition of classical pattern avoidance. The heaps will be the main tool for proving the characterization in Theorem 5.12 presented in Section 5. In Section 6 we reduce the proof of the main theorem to short braid avoiding elements. In Sections 7 and 8, we define the convex elements and give a complete classification of convex Deodhar elements. Then in Section 9, we characterize the non-convex Deodhar elements in type D which completes the proof for type D . In Section 10, we complete the classification of Deodhar elements for the remaining finite Weyl groups. Section 11 gives a pattern comparison result, which shows that the Deodhar property for type D can be characterized by avoiding finitely many 1-line patterns. Finally, we close with some open problems and enumerative data in Section 12.

2. BACKGROUND AND NOTATION

In this section we will set up our notation and review some of the motivation for our main theorems. For a reader unfamiliar with Coxeter groups, we recommend either the classic text by Humphreys [Hum90] or the recent text by Björner and Brenti [BB05] for a more combinatorial treatment.

Let W be a Coxeter group with generating set S and relations of the form $(s_i s_j)^{m(i,j)} = 1$. The Coxeter graph for W is the graph on the generating set S with edges connecting s_i and s_j labeled $m(i, j)$ for all pairs i, j with $m(i, j) > 2$. For example, the table in Figure 1 shows the Coxeter graphs for the finite Weyl groups that contain minimal non-Deodhar patterns. Note that if $m(i, j) = 3$ it is customary to leave the corresponding edge unlabeled.

An *expression* is any product of generators from S . The *length* $l(w)$ of an element $w \in W$ is the minimum length of any expression for the element w . Such a minimum length expression is called *reduced*. Each element $w \in W$ can have several different reduced expressions that represent it. Given $w \in W$, we represent a reduced expressions for w in sans serif font, say $w = w_1 w_2 \cdots w_p$ where each $w_i \in S$.

It is a theorem of Tits [Tit69] that every reduced expression for an element w of a Coxeter group can be obtained from any other by applying a sequence of braid moves of the form

$$\underbrace{s_i s_j s_i s_j \cdots}_{m(i,j)} \mapsto \underbrace{s_j s_i s_j s_i \cdots}_{m(i,j)}$$

where s_i and s_j are generators in S that appear in the reduced expression for w , and each factor in the move has $m(i, j)$ letters. Let the *support* of an element $w \in W$, denoted $\text{supp}(w)$, be the set of all generators appearing in any reduced expression for w , which is well-defined by Tits' theorem. We say that the element w is *connected* if $\text{supp}(w)$ is connected in the Coxeter graph of W .

We define an equivalence relation on the set of reduced expressions for a fixed Coxeter element where two reduced expressions are in the same *commutativity class* if one can be obtained from the other by *commuting moves* of the form $s_i s_j \mapsto s_j s_i$, where $m(i, j) = 2$. In particular, if every reduced expression for w can be obtained from any other by commuting moves then we say w is *fully commutative*. By Tits' theorem, an element w is fully commutative if and only if no reduced expression for w contains a consecutive subexpression of the form $s_i s_j s_i s_j \cdots$ of length $m(i, j) \geq 3$.

We call any expression of the form $s_i s_j s_i$ for $m(i, j) \geq 3$ a *short braid*. This name reflects the fact that we are not considering any longer braid, even if $m(i, j) > 3$. We caution the reader that some authors have used the term short braid to refer to a commuting move between two entries s_i and s_j where $m(i, j) = 2$. The *short braid avoiding* elements of a Coxeter group are those with no reduced expression containing a factor $s_i s_j s_i$ where s_i and s_j are any pair of generators that do not commute (i.e. $m(i, j) \neq 2$). Hence, a short braid avoiding element is also fully commutative.

Given a Coxeter group W , we can form the *Hecke algebra* \mathcal{H} over $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ with basis $\{T_w : w \in W\}$ and relations:

$$(2.1) \quad T_s T_w = T_{sw} \text{ for } l(sw) > l(w)$$

$$(2.2) \quad (T_s)^2 = (q - 1)T_s + qT_1$$

where T_1 corresponds to the identity element. In particular, this implies that

$$T_w = T_{w_1} T_{w_2} \cdots T_{w_p}$$

whenever $w_1 w_2 \cdots w_p$ is a reduced expression for w .

Kazhdan and Lusztig [KL79] described another basis for \mathcal{H} that is invariant under the Hecke algebra involution mapping

$$q \mapsto q^{-1}$$

$$T_s \mapsto (T_s)^{-1}.$$

This basis, denoted $\{C'_w : w \in W\}$, has important applications in representation theory and algebraic geometry [KL79, KL80]. The Kazhdan–Lusztig polynomials $P_{x,w}(q)$ arise as the “change of basis” matrix between these two bases of \mathcal{H} :

$$C'_w = q^{-\frac{1}{2}l(w)} \sum_{x \in W} P_{x,w}(q) T_x.$$

The C'_w are defined uniquely to be the Hecke algebra elements that are invariant under the involution and have expansion coefficients as above where $P_{x,w}$ is a polynomial in q with

$$(2.3) \quad \text{degree } P_{x,w}(q) \leq \frac{l(w) - l(x) - 1}{2}$$

and $P_{w,w}(q) = 1$. We use the notation C'_w to be consistent with the literature because there is already a related basis denoted C_w .

For $w \in W$ and $s \in S$ with $l(sw) > l(w)$, the Kazhdan–Lusztig basis elements multiply according to the rule

$$C'_s C'_w = C'_{sw} + \sum_{sz < z < w} \mu(z, w) C'_z$$

where $\mu(z, w)$ is the coefficient of $q^{\frac{1}{2}(l(w) - l(z) - 1)}$ (the term of highest possible degree) in the Kazhdan–Lusztig polynomial $P_{z,w}(q)$. This is appreciably more complicated than the corresponding multiplication formula (2.1) in the $\{T_w\}$ basis.

Deodhar [Deo90] studied the case when C'_w can be written simply as a product of C'_{s_i} 's. In this case, he also gives nice combinatorial formulas for all the polynomials $P_{x,w}(q)$. We will describe Deodhar's defect statistic and his theorem in terms of masks on reduced expressions.

Fix a reduced expression $w = w_1 \cdots w_k$. Define a *mask* σ associated to the reduced expression w to be any binary word $\sigma_1 \cdots \sigma_k$ of length $k = l(w)$. Every mask corresponds to a subexpression of w defined by $w^\sigma = w_1^{\sigma_1} \cdots w_k^{\sigma_k}$ where

$$w_j^{\sigma_j} = \begin{cases} w_j & \text{if } \sigma_j = 1 \\ \text{id} & \text{if } \sigma_j = 0. \end{cases}$$

Each w^σ is a product of generators in a subsequence of $w_1 \cdots w_k$ so it determines an element of W that is less than w in *Bruhat order*. For $1 \leq j \leq k$, we also consider initial sequences of masks denoted $\sigma[j] = \sigma_1 \cdots \sigma_j$, and the corresponding initial subexpressions $w^{\sigma[j]} = w_1^{\sigma_1} \cdots w_j^{\sigma_j}$. For example, we have $w^{\sigma[k]} = w^\sigma$. A mask σ is *proper* if it has at least one zero.

We say that a position j (for $2 \leq j \leq k$) of w is a *defect* with respect to the mask σ if

$$(2.4) \quad l(w^{\sigma[j-1]}w_j) < l(w^{\sigma[j-1]}).$$

Note that a defect occurs in position j if $w^{\sigma[j-1]}$ satisfies the length condition above; the value of σ_j is irrelevant when determining if j is a defect. Let $d_w(\sigma)$ denote the number of defects of w for a mask σ . We will use the notation $d(\sigma) = d_w(\sigma)$ when the reduced word w is fixed.

We are now in a position to define Deodhar's condition.

Definition 2.1. Let $w \in W$ be a Coxeter element with reduced expression w , and let σ be a proper mask for w . We say that a position j is a *zero-defect* if $\sigma_j = 0$ and j is also a defect in w . We say that position j in w is a *plain-zero* if $\sigma_j = 0$ and j is not a defect in w . Then, the mask σ is *Deodhar* if

$$(2.5) \quad \# \text{ of zero-defects of } \sigma < \# \text{ of plain-zeros of } \sigma.$$

Moreover, a reduced expression w is *Deodhar* if every proper mask σ on w is Deodhar.

Example 2.2. Assume $s_2s_1s_3s_2$ is a reduced expression in a Coxeter group. The word/mask pair

$$w = \begin{bmatrix} s_2 & s_1 & s_3 & s_2 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

has a zero-defect in position 4 and plain-zeros in positions 2 and 3. One can verify that (2.5) holds for all proper masks on w , so w is Deodhar.

For certain Coxeter groups, Deodhar has shown that when a reduced expression $w \in W$ satisfies this condition, the Kazhdan–Lusztig polynomials $P_{x,w}$ can be obtained as the generating function that counts masks on w with respect to the defect statistic. Equivalently, C'_w can be written as a product of C'_{s_i} 's. He also shows that the notion of being Deodhar is well-defined on Coxeter group elements and is independent of the choice of reduced word used to verify the condition. Note that Deodhar actually used a slightly different condition which is equivalent to the one given here in (2.5) [BW01, Lemma 2]. The original condition is

$$d(\sigma) \leq \frac{1}{2}(l(w) - l(w^\sigma) - 1)$$

which comes directly from the maximum degree bound of the Kazhdan–Lusztig polynomial $P_{w^\sigma, w}(q)$ in (2.3).

Theorem 2.3. [Deo90] *Let W be any Coxeter group where the Kazhdan–Lusztig polynomials are known to have nonnegative coefficients, and let $w = w_1 \cdots w_k$ be a reduced expression for some $w \in W$. Then the following are equivalent:*

- (1) *The expression w is Deodhar.*
- (2) *The element w is Deodhar.*
- (3) *The Kazhdan–Lusztig basis element C'_w is given by*

$$C'_w = q^{-\frac{1}{2}l(w)} \sum q^{d(\sigma)} T_{w^\sigma}$$

where the sum is over all masks σ on w .

- (4) *For all $x \in W$, the Kazhdan–Lusztig polynomial $P_{x,w}$ is given by*

$$P_{x,w}(q) = \sum q^{d(\sigma)}$$

where the sum is over all masks σ on w such that $w^\sigma = x$.

- (5) The Kazhdan–Lusztig basis element C'_w satisfies $C'_w = C'_{w_1} \cdots C'_{w_k}$.
- (6) The Bott–Samelson resolution of the corresponding Schubert variety X_w is small.
- (7) The Poincaré polynomial for the full intersection cohomology group of the Schubert variety X_w is

$$\sum_i \dim(\mathcal{IH}^{2i}(X_w))q^i = (1+q)^{l(w)}.$$

Remark 2.4. The equivalence of (1) through (6) are implicit in Deodhar [Deo90]. The equivalence of (4) and (7) is proved explicitly by Billey and Warrington [BW01]. Lusztig [Lus93] and Fan and Green [FG97] have studied those elements for which (5) holds. These elements are called “tight” in the terminology of those papers.

The main goal of this paper is to give an efficient way to identify Deodhar elements. In the case when W is the symmetric group, Billey and Warrington [BW01] gave a concise description of the Deodhar elements as those that are 321-hexagon avoiding, where the term “hexagon” comes from the notion of a heap on a permutation. We will describe the heap construction and classical pattern avoidance in the next two sections and return to the study of Deodhar elements in Section 5.

3. HEAPS AND STRING DIAGRAMS

Each reduced expression can be associated with a partial order called the heap that we define below. This partial order allows us to visualize a reduced expression as a set of lattice points while maintaining the pertinent information about the relations among the generators. Cartier and Foata [CF69] were among the first to study heaps of dimers, and these were generalized to other settings by Viennot [Vie89]. More recently, Stembridge has studied enumerative aspects of heaps [Ste96, Ste98] in the context of fully commutative elements. We will use the heaps to visualize the reduced expressions that appear in the table on Page 2 and prove our characterization in type D .

Definition 3.1. Suppose $w = w_1 \cdots w_k$ is a fixed reduced expression, and define a partial ordering on the indices $\{1, \dots, k\}$ by the transitive closure of the relation $i < j$ if $i < j$ and $m(w_i, w_j) \neq 2$. In particular, $i < j$ if $i < j$ and $w_i = w_j$. This partial order is called the *heap* of w . We label the element i of the poset by the corresponding generator w_i .

Remark 3.2. Observe that heaps are well defined up to commutativity class, so if u and v are two reduced expressions for w in the same commutativity class then the labeled heaps of u and v are equal. In particular, if w is fully commutative then there is a unique labeled heap poset for the element w , regardless of which reduced expression is used to generate it.

Let G denote the Coxeter graph for W . We can embed the heap poset as a set of lattice points in $G \times \mathbb{N}$. To do this, begin by reading the reduced expression w from left to right, and drop a point in the column representing each generator w_i . We can envision each point as being “fat” and under the influence of “gravity,” in the sense that the point must fall to the lowest possible position in the column over the generator corresponding to w_i in the Coxeter graph without passing any previously placed points in adjacent columns. Here, we say two columns are adjacent when they correspond to adjacent vertices in the Coxeter graph. Since generators that are adjacent in the Coxeter graph do not commute, we must place the point representing w_i at a level that is *above* the level of any other adjacent points that have already been placed. Because generators that are not adjacent in the Coxeter graph do commute, points that lie in non-adjacent columns can slide past each other or land at the same level.

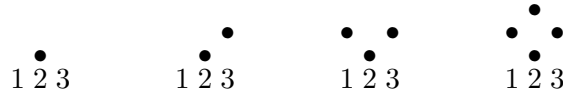
Definition 3.3. Let w be a reduced expression for a Coxeter element. We let $Heap(w)$ denote the lattice representation of the heap poset in $G \times \mathbb{N}$ constructed as described in the preceding paragraph. (We will amend this definition in Example 3.7 to account for the fork in the Coxeter graph of type D .)

We will give several examples of heaps in different Coxeter groups in the next three examples and also introduce some useful terminology for permutations and signed permutations.

Example 3.4. The Coxeter graph of type A_{n-1} is the following:



The corresponding Coxeter group is the symmetric group S_n . We may refer to elements in the symmetric group by the *1-line notation* $w = [w_1w_2w_3 \cdots w_n]$ where w is the bijection mapping i to w_i written in italic font. The generators s_1, s_2, \dots, s_{n-1} are the adjacent transpositions where s_i interchanges i and $i + 1$. For example, when multiplying a permutation on the right by s_2 , we interchange the entries in positions 2 and 3 of the 1-line notation, so for $w = [3412] \in S_4$ we have $ws_2 = [3142]$. Dually, when multiplying on the left by s_2 , we interchange the digits 2 and 3 in the 1-line notation for the element, so $s_2w = [2413]$. One reduced expression for w is $s_2s_3s_1s_2$. We build up the heap one generator at a time for the reduced expression $w = s_2s_3s_1s_2$ in type A_3 as shown below.



We can view the points in the lattice as the vertices in the Hasse diagram for the heap poset where the edges are implied by the Coxeter graph. Note that the other reduced expression $s_2s_1s_3s_2$ for $[3412]$ corresponds to a different linear extension of the heap above.

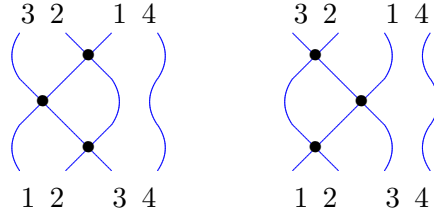
Remark 3.5. In the lattice representation of a heap poset, all of the entries of the reduced expression that correspond to the same generator lie in a column over the given generator in the Coxeter graph. Each entry will have a certain *level* in the heap, but the poset is not ranked. In the example below, the reduced expression $w = s_1s_4s_2s_3$ has a heap where the rank of the s_3 entry is not well defined and the level of the s_4 entry is an artifact of the way we imposed “gravity” in the construction.



In Section 7, we will further refine the lattice representation of the heap by coalescing the connected components so entries that are connected in $G \times \mathbb{N}$ satisfy the covering relation in the heap poset.

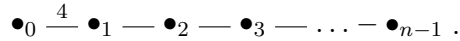
In type A , the heap construction can be combined with another combinatorial model for permutations in which the entries from the 1-line notation are represented by strings. The points where two strings cross can be viewed as adjacent transpositions of the 1-line notation. Hence, we can overlay strings on top of a heap diagram to recover the 1-line notation for the element, by drawing the strings from bottom to top so that they *cross* at each entry in the heap where they meet and *bounce* at each lattice point not in the heap. Conversely, each permutation string diagram corresponds with a heap by taking all of the points where the strings cross as the “fat” points of the heap and letting them “fall” according to the relations given by the Coxeter graph.

For example, we can overlay strings on the two heaps of $[3214]$. Note that the labels in the picture below refer to the strings, not the generators.



The string diagram helps us to visualize the relationship between the 1-line notation for a permutation and the corresponding heap. Fixing a reduced expression w for a permutation, there exists a string diagram that is obtained from $Heap(w)$ by adding strings that cross at each lattice point of $Heap(w)$. Observe that we are able to read off the 1-line notation for an element by labeling the strings $1, \dots, n$ along the bottom and then reading the corresponding labels from the top. We can also obtain reduced expressions from any string diagram by reading the string crossings as generators in any order that is consistent with the implied heap poset structure.

Example 3.6. The Coxeter graph of type B_n is of the form



From this graph, we see that the symmetric group S_n is a parabolic subgroup of this Coxeter group. That is, S_n is generated by a subset of the generators of B_n . Because of this, the elements of this group have a standard 1-line notation in which a subset of the entries are *barred*. We often think of the barred entries as negative numbers, and this group is referred to as the group of *signed permutations* or the *hyperoctahedral group*. The action of the generators on the 1-line notation is the same for $\{s_1, s_2, \dots, s_{n-1}\}$ as in type A in which ws_i interchanges the entries in positions i and $i + 1$ in the 1-line notation for w . The s_0 generator acts on the right of w by changing the sign of the first entry in the 1-line notation for w . For example, $w = [\bar{4}2\bar{3}1]$ is an element of B_4 and

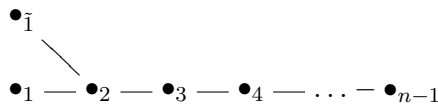
$$ws_0 = [42\bar{3}1]$$

$$ws_1 = [2\bar{4}\bar{3}1].$$

Note that because the edge in the B_n Coxeter graph connecting s_0 and s_1 is labeled 4, we have that $s_0s_1s_0s_1$ and $s_1s_0s_1s_0$ are reduced expressions for the same element denoted $[\bar{1}\bar{2}34 \dots n]$ in 1-line notation.

The heap for a type B reduced expression will look like the heap of a type A expression because its Coxeter graph is a path. As in type A , we can adorn these heap constructions with strings that represent the digits of the 1-line notation for the element. If we label the strings at the bottom of the diagram with the numbers from $1, \dots, n$ then the s_0 generator has the effect of bouncing the string back in the direction from which it came, while changing the sign of the label for the string. All other generators cross the strings as in type A .

Example 3.7. The Coxeter graph for type D_n is shown below.



The elements of type D can be viewed as the subgroup of B_n consisting of signed permutations with an even number of barred entries. The action of the generators on the 1-line notation is the same for $\{s_1, s_2, \dots\}$ as in type A in which ws_i interchanges the entries in positions i and $i + 1$ in the 1-line notation for w . The $s_{\bar{1}}$ generator acts on the right of w by marking the first two entries in the 1-line

notation for w with bars and interchanging them. For example, $v = [4\bar{2}\bar{3}\bar{1}]$ and $w = [\bar{4}\bar{2}\bar{3}\bar{1}]$ are elements of D_4 and

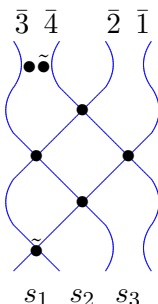
$$\begin{aligned} vs_{\bar{1}} &= [\bar{2}\bar{4}\bar{3}\bar{1}] \\ ws_{\bar{1}} &= [\bar{2}\bar{4}\bar{3}\bar{1}]. \end{aligned}$$

Although the Coxeter graph for type D has a fork, we will draw the heap of type D elements in a linearized way by allowing entries in the first column to consist of either generator or both:

$$\begin{aligned} s_1 &= \bullet \\ s_{\bar{1}} &= \tilde{\bullet} \\ s_1 s_{\bar{1}} &= \bullet \tilde{\bullet}. \end{aligned}$$

We denote this linearized lattice point representation by $\text{Heap}(w)$. Hence, if $w \in D_n$ then $\text{Heap}(w)$ is a subset of $[n-1] \times \mathbb{N}$ rather than $G \times \mathbb{N}$, where $[n-1] = \{1, 2, \dots, n-1\}$.

As in type A , we can adorn $\text{Heap}(w)$ with strings that represent the digits of the 1-line notation for the element. If we label the strings at the bottom of the diagram with the numbers from $1, \dots, n$, then the $s_{\bar{1}}$ generator crosses the strings that intersect it and changes the sign on the labels for both strings. All other generators simply cross the strings as in type A . For example, the heap of the reduced expression $s_{\bar{1}}s_2s_3s_1s_2s_{\bar{1}}s_1 = [\bar{3}\bar{4}\bar{2}\bar{1}]$ is given below.



4. CLASSICAL PATTERN AVOIDANCE

The 1-line notations for types A, B, D carry a notion of pattern containment that generalizes the following classical definition.

Definition 4.1. Let $w = [w_1 \dots w_n]$ be a permutation in S_n written in 1-line notation as described in Example 3.4. Let $p = [p_1 \dots p_k]$ be another permutation in S_k for $k \leq n$. Then we say w *contains the permutation pattern* p if there exists a subsequence $1 \leq i_1 < i_2 < \dots < i_k \leq n$ such that

$$w_{i_a} < w_{i_b} \iff p_a < p_b$$

for all $1 \leq a < b \leq k$. If w does not contain p then we say that w *avoids the permutation pattern* p .

In other words, w contains p if there exist k rows and columns in the permutation matrix for w whose common entries are the permutation matrix for p . For example, $w = [\underline{5}3\underline{2}4\underline{1}]$ contains the pattern $p = [321]$ in several ways including the underlined subsequence, and w avoids $q = [1234]$.

One of the earliest uses of permutation patterns occurred in computer science [Tar72]. A good introduction to enumerative methods in pattern avoidance can be found in [Bón04]. Several interesting properties of Schubert varieties, Kazhdan–Lusztig polynomials and Bruhat order can be characterized by pattern avoidance [LS90, BP05, BB03, Gas98, WY06b, WY06a, Ten06a, BMB06].

A property of permutations can be characterized by pattern avoidance if there is no w having the property and p not having the property such that w contains p as a pattern. Equivalently, the subset of

permutations that do not have the property must be an upper order ideal in the poset on $S_\infty = \bigcup_{n>0} S_n$ ordered by pattern containment. The property is then characterized by avoiding the minimal elements of this upper order ideal.

For example, the property of a permutation being short braid avoiding in type A is characterized by avoiding the pattern $[321]$, first noted by [BJS93]. Hence, the fully commutative elements in S_n are enumerated by the Catalan numbers [SS85]. Also, checking a permutation of length n for a subsequence of length 3 can be done in $O(n^3)$ time while looking at all reduced words for a typical permutation takes an exponential amount of time in n . A pattern avoidance characterization is most useful when the set of minimal patterns is finite, but this need not be the case [SB00].

When $w = [w_1 \dots w_n]$ is a signed permutation from type B or type D , then we say that w contains the 1-line pattern $p = [p_1 \dots p_k]$ if the underlying permutation for w , obtained by ignoring the bars in the 1-line notation, contains the underlying permutation for p , and assuming the pattern instance occurs in positions $i_1 < i_2 < \dots < i_k$, we further require that w_{i_j} is barred only if p_j is barred. For example, the type D element $w = [\bar{5}32\bar{4}1]$ contains the pattern $p = [3\bar{2}1]$ as the underlined subsequence $[\bar{5}32\bar{4}1]$, but *not* the underlined subsequence $[\bar{5}32\bar{4}1]$ because the pattern of bars does not match.

Some early applications of pattern avoidance in types B and D occurred in [Bi98, BL98]. Type B enumerative results have been obtained by [Bec97, Sim00, MW04], and there are also some extensions to colored permutations [Man03, Man02]. Classical pattern avoidance extends to all Coxeter groups using the notion of root subsystems described in [BP05, BB03].

5. DEODHAR ELEMENTS OF COXETER GROUPS

Let W be an arbitrary Coxeter group with $w, y \in W$. We say that w contains y as a factor if there exist elements a and b in W such that $w = ayb$ and $l(w) = l(a) + l(y) + l(b)$. Equivalently, w contains y as a factor if some reduced expression for w contains some reduced expression for y as a consecutive subword, i.e. $w = w_1 w_2 \dots w_p$ and $y = w_i w_{i+1} \dots w_j$ for some $1 \leq i \leq j \leq p$. The induced partial order on Coxeter group elements is known as the *two-sided weak Bruhat order* [BB05]. We will use the following lemma to show that the Deodhar elements form a lower order ideal in the two-sided weak Bruhat order.

Lemma 5.1. *Let W be a Coxeter group. If $w \in W$ is Deodhar then w^{-1} is Deodhar.*

Proof. Let $w = w_1 \dots w_p$ be a reduced expression for w , and consider the two products

$$(5.1) \quad C'_{w_1} C'_{w_2} \cdots C'_{w_p} = q^{-\frac{p}{2}} \sum R_u(q) T_u$$

$$(5.2) \quad C'_{w_p} \cdots C'_{w_2} C'_{w_1} = q^{-\frac{p}{2}} \sum S_u(q) T_u$$

It follows directly from the symmetry in the multiplication rule (2.1) that

$$R_u(q) = S_{u^{-1}}(q).$$

By Deodhar's Theorem 2.3, w is Deodhar if and only if

$$(5.3) \quad C'_w = C'_{w_1} C'_{w_2} \cdots C'_{w_p}$$

$$(5.4) \quad \implies R_u(q) = P_{u,w}(q) \quad \text{for all } u$$

$$(5.5) \quad \implies S_{u^{-1}}(q) = P_{u,w}(q) \quad \text{for all } u$$

$$(5.6) \quad \implies \deg(S_{u^{-1}}(q)) \leq \frac{l(w^{-1}) - l(u^{-1}) - 1}{2} \text{ for all } u \text{ and } S_{w^{-1}}(q) = 1$$

$$(5.7) \quad \implies C'_{w_p} \cdots C'_{w_2} C'_{w_1} = C'_{w^{-1}}$$

by definition of $C'_{w^{-1}}$ since the product of C'_{w_i} 's is invariant under the involution. Therefore, w^{-1} is Deodhar. \square

The following proposition can easily be derived from Proposition 2.1.4 of [FG97]. We include an independent proof for completeness.

Proposition 5.2. *Let $w, y \in W$ be Coxeter group elements. If y is not Deodhar and w contains y as a factor, then w is not Deodhar either.*

Proof. Suppose y is not Deodhar and $y = y_1 \cdots y_p$ is a reduced expression. Then by (2.5) there exists a proper mask σ for $y_1 \cdots y_p$ with

$$\# \text{ of zero-defects of } \sigma \geq \# \text{ of plain-zeros of } \sigma.$$

Consider multiplying y on the right by a generator $s \in S$ such that $l(ys) > l(y)$. We can extend the mask σ by placing a 1 in the last position to obtain a mask for $y_1 \cdots y_p s$. Since the inequality above remains unchanged for the new mask, the element ys is not Deodhar.

Next, consider multiplying y on the left by a generator $s \in S$ such that $l(sy) > l(y)$ or equivalently $l(y^{-1}s) > l(y^{-1})$. By Lemma 5.1, y is not Deodhar implies y^{-1} is not Deodhar. So by the argument above, $y^{-1}s$ is not Deodhar and so neither is sy .

The theorem follows by induction on $l(w) - l(y)$. \square

We immediately obtain a combinatorial proof of a result from [FG97] which shows that the Deodhar elements are all short braid avoiding.

Corollary 5.3. *Let $w \in W$ be a Coxeter group element. If w contains a short braid then w is not Deodhar.*

Proof. Say s_i, s_j are noncommuting generators. The reduced expression/mask pair

$$\begin{bmatrix} s_i & s_j & s_i \\ 1 & 0 & 0 \end{bmatrix}$$

has a zero-defect in the last position hence the number of zero-defects equals the number of plain zeros. Consequently, $s_i s_j s_i$ is not Deodhar. Therefore, any element that contains a short braid as a factor in a reduced expression is not Deodhar. \square

Proposition 5.2 also shows that the non-Deodhar elements form an upper order ideal in the two-sided weak Bruhat order. In order to obtain an efficient generating set for this ideal, we consider a refinement of factor containment.

Definition 5.4. Let W, W' be Coxeter groups with associated Coxeter graphs G, G' respectively. Then, a *Coxeter embedding of G'* is an injective map of the generators $f : G' \rightarrow G$ that restricts to a labeled graph isomorphism onto its image.

A Coxeter embedding induces an injection of W' into W , and we will abuse notation and call this map $f : W' \rightarrow W$ a Coxeter embedding also. When phrased algebraically, a Coxeter embedding $f : W' \rightarrow W$ is an injective map of generators for the Coxeter group such that $m(s_i, s_j) = m(f(s_i), f(s_j))$ for all $s_i, s_j \in W'$. Since f is a map of generators, we can extend it to a map of Coxeter group elements by treating it as a word homomorphism on any reduced expression in W' .

Definition 5.5. Suppose W is a Coxeter group, and $w, y \in W$. Let W_y be the parabolic subgroup whose generators are determined by the support of y . If there exists a Coxeter embedding $f : W_y \rightarrow W$ such that w contains $f(y)$ as a factor, then we say that w contains y as an *embedded factor*.

This definition yields a stronger reformulation of Proposition 5.2, which enables us to characterize Deodhar elements with a shorter list of non-Deodhar patterns.

Corollary 5.6. *Let y be a Coxeter element that is not Deodhar. If w is a Coxeter element that contains y as an embedded factor, then w is not Deodhar either.*

Proof. If σ is a mask for a reduced expression y then it follows from Definition 5.4 that σ is also a mask for $f(y)$. This mask has defects in exactly the same positions as it does when it is applied to y . Hence, y is Deodhar if and only if $f(y)$ is Deodhar, for any Coxeter embedding $f : W_y \rightarrow W$. \square

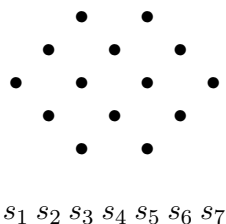
Example 5.7. In the Coxeter group of type B_4 , $[2\bar{4}51\bar{3}] = s_1 s_2 s_3 s_1 s_0 s_1 s_2 s_1 s_0 s_4 s_3 s_4 s_1$ contains the factor $s_4 s_3 s_4$ in the parabolic subgroup generated by $\{s_3, s_4\}$. This subgroup is isomorphic to S_3 and $s_4 s_3 s_4$ maps to $s_2 s_1 s_2 = [321] \in S_3$, so $[2\bar{4}51\bar{3}]$ contains $[321]$ as an embedded factor. Consequently, it is not Deodhar.

Example 5.8. In type A , the Coxeter embeddings of connected subgraphs are simply shifts of the generators along the linear Coxeter graph or reversed shifts. In particular, if the generators of S_k are labeled s_1, s_2, \dots, s_{k-1} in its Coxeter graph, then the image of the generators under a Coxeter embedding $f : S_k \rightarrow S_n$ are either of the form $s_{1+j}, s_{2+j}, \dots, s_{k-1+j}$ or $s_{k-1+j}, \dots, s_{2+j} s_{1+j}$ for some $0 \leq j \leq n - k$.

The *hexagon avoiding* elements of a Coxeter group are the ones that avoid the element

$$u = s_3 s_2 s_1 s_5 s_4 s_3 s_2 s_6 s_5 s_4 s_3 s_7 s_6 s_5 = [46718235]$$

of A_7 as an embedded factor. The name arises from the shape formed by the heap of u :



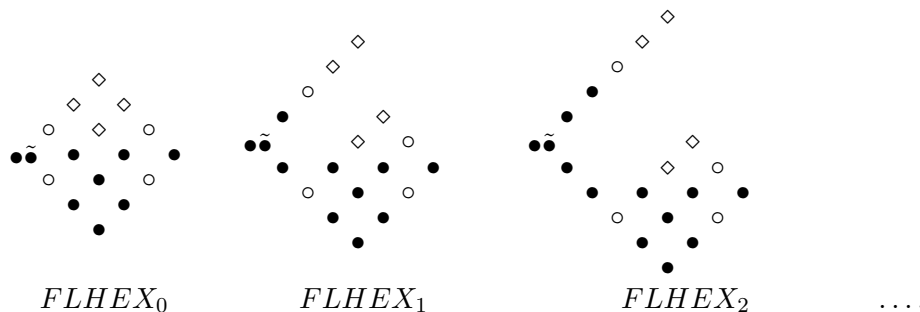
Billey and Warrington have given a complete characterization of the Deodhar elements in *linear Weyl groups* where the Coxeter graph consists of a single path. We can now state their theorem precisely.

Theorem 5.9. [BW01] *In types A_n, B_n , an element w is Deodhar if and only if w avoids short braids and hexagons as embedded factors.*

Our main theorem below generalizes this theorem. It is a concise characterization of the Deodhar condition for the other finite Weyl groups.

Remark 5.10. The techniques used in [BW01] do not easily extend to the remaining finite Weyl groups of types D and E . Among the connected short-braid avoiding heaps of type A , the notions of coalesced heap containment and embedded factor containment are essentially the same up to the orientation of the Coxeter graph. This is not the case in type D . In addition, the non-Deodhar elements of type D can have heaps containing “alcoves” or “holes” so that the heap lattice points do not form a laterally convex set in the sense of [BW01].

Example 5.11. In type D , there is an infinite antichain of non-Deodhar elements (i.e. no pair of elements from the family contain each other as embedded factors) whose heaps can contain “alcoves:”



We have drawn the heaps of these elements with decorations that indicate a particular mask, as described in Section 6. The masks shown demonstrate that these elements are not Deodhar.

This example shows that the set of type D Coxeter elements partially ordered by embedded factor containment is not well quasi-ordered. Also, there is a simple example showing that the permutations partially ordered by embedded factor containment is not well quasi-ordered. See [SB00] for an analogous example in the classical permutation pattern poset.

We can still obtain a finite characterization for the Deodhar condition in type D since all of the $FLHEX_k$ elements contain $FLHEX_0 = [\bar{1}678\bar{5}234]$ as a classical 1-line pattern, in the manner described in Section 4.

Theorem 5.12. *Let $w \in W$ be an element of a finite Weyl group. Then, the following are equivalent:*

- (1) *The element w is Deodhar.*
- (2) *The element w^{-1} is Deodhar.*
- (3) *The element w avoids the short list of embedded factor patterns given in Figure 1, as well as the $FLHEX_0$ 1-line pattern of type D .*

Proof. The equivalence of (1) and (2) follow from Lemma 5.1. The equivalence of (1) and (3) follows from Theorem 5.9 for types A and B , Theorem 9.2 below for type D , and Theorem 10.1 for the finite exceptional groups. This accounts for all irreducible finite Weyl groups. \square

The proof will occupy Sections 6 through 10. For the finite exceptional groups, a brute-force search implemented on a computer suffices. Our work is simplified by the fact that we only need to check the short braid avoiding elements of these groups. For the infinite type D family, we need to show that our list of minimal non-Deodhar elements is complete. This is shown in Theorems 8.1 and 9.1. The proof of Theorem 8.1 involves a map from the heap of a type D reduced expression to a type A heap where the Deodhar condition was already known by Theorem 5.9. We complete the classification by checking that this map preserves the Deodhar property.

6. SHORT-BRAID AVOIDING HEAPS IN TYPE D

This section develops the heap technology necessary to carry out the classification of minimal non-Deodhar elements in type D under embedded factor containment. For our work in this section, it suffices by Corollary 5.3 to consider only short braid avoiding elements. Short braid avoiding elements are fully commutative so they have a unique heap poset. We draw the heaps of type D elements in a linearized way as described in Example 3.7, with entries corresponding to both s_1 and $s_{\bar{1}}$ generators in the first column, and denote this lattice point representation of the heap by $Heap(w)$. We will consider masks to be assignments of 0's and 1's to the entries in the heap instead of 0,1-sequences associated to a particular

reduced expression. Of course, a reduced word/mask pair can be read off from the heap by reading the entries in order of any linear extension of the heap.

We decorate $Heap(w)$ according to mask-value using the following table:

Decoration	Mask-value
◇	zero-defect entry
○	plain-zero entry (not a defect)
●	mask-value 1 entry
☆	entry of the heap with unknown mask-value
●	lattice point not necessarily in the heap
*	lattice point that is definitely not in the heap
★	lattice point that is highlighted for emphasis

In the decorated heaps, we don't distinguish one-defects from plain-ones since they don't contribute to the Deodhar bound (2.5).

We can adorn our decorated heaps with strings representing the digits of the standard 1-line notation for type D elements as described in Section 3. Given a decorated heap, the strings will cross only at entries corresponding with mask-value 1. At mask-value 0 entries the strings “bounce” as if the entry were not in the heap. If the decorated heap corresponds to a reduced expression w and mask σ , then the resulting labels on the strings at the top of the heap will be the 1-line notation for w^σ . Recall that a string passing through a ● entry corresponds to a $s_{\bar{1}}$ generator so it changes sign (which is not shown explicitly in our pictures).

We can use the strings to obtain a useful test for the defect status of a particular entry in a decorated heap. Note that at every entry in the heap, two strings approach it from either side. We will call these the *left string* and the *right string* for that entry.

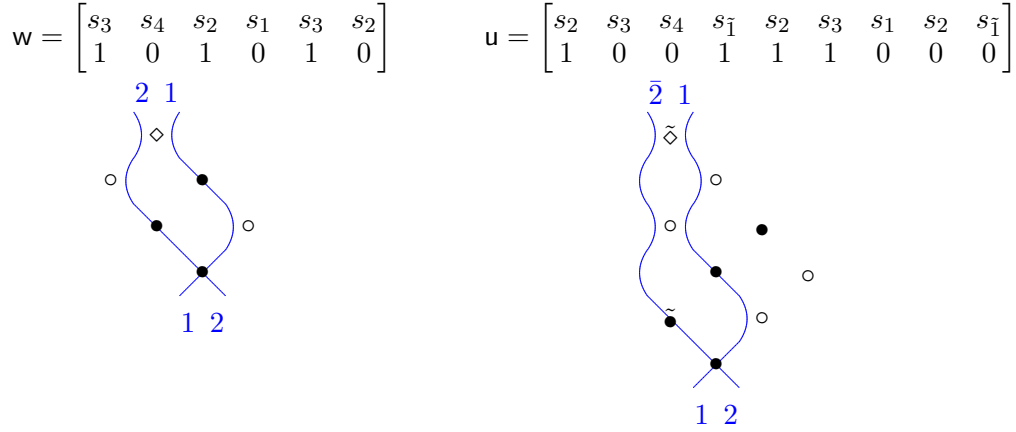
Lemma 6.1. *Consider an entry p in a heap. Draw the left and right strings emanating downward from p and label the string that ends up leftmost on the bottom by 1, and the string that ends up rightmost on the bottom by 2. The entry p is a defect if and only if the strings are labeled at the top according to the following table of patterns.*

<i>If p corresponds to the generator ...</i>	<i>... then p is a defect if and only if the strings are top-labeled</i>
$s_{\bar{1}}$	$1\bar{2}, \bar{1}\bar{2}, \bar{2}1$ or $\bar{2}\bar{1}$
<i>any other generator</i>	$1\bar{2}, \bar{1}\bar{2}, 21$ or $2\bar{1}$

Note that when p is not $s_{\bar{1}}$, this test is just a signed version of the usual type A inversion.

Proof. This follows because the length formula in type D for an even signed permutation given in one line notation $w = [w_1 w_2 \dots w_n]$ is $l(w) = \#\{i < j : w_i > w_j\} + \#\{i < j : \bar{w}_i > w_j\}$, viewing the barred entries as negative. \square

Example 6.2. The following decorated heaps with strings demonstrate the defect status of the top entry in the masked expressions below.

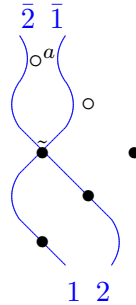


Here is an observation that is used extensively in the classification.

Corollary 6.3. *The strings for a defect in any type D heap must cross at least once strictly below the defect.*

Proof. If the strings for an entry never cross then the right string of the entry will be labeled 2, which does not match any of the defect labelings in Lemma 6.1. □

Example 6.4. In type A, an entry is a defect if and only if its strings cross an odd number of times below. By contrast, in type D it is possible for the strings of an entry to cross below, yet not form a defect. For example,



In this case, the strings are interchanged but both negatively signed, so the values are in increasing order. Hence, a is not a defect by Lemma 6.1.

Let $w \in D_n$ be a reduced expression for a connected short-braid avoiding element. Suppose x and y are a pair of entries in $Heap(w)$ that correspond to the same generator s_i , so they lie in the same column i of the heap (setting $i = 1$ in case the generator is $s_{\bar{1}}$). Assume that x and y are a *minimal pair* in the sense that there is no other entry between them in column i . Then, for w to be reduced there must exist at least one non-commuting generator between x and y , and for w to be short braid avoiding there must actually be two non-commuting generators that lie strictly between x and y in $Heap(w)$. We call these two non-commuting generators a *resolution* of the pair x, y .

Definition 6.5. If both of the generators in a resolution lie in column $i - 1$ ($i + 1$, respectively), we call the resolution a *left (right, respectively) resolution*. If the generators lie in distinct columns, we call the resolution a *distinct resolution*.

In type D , every resolution of a minimal pair must be one of these types. Note that the s_1 and $s_{\bar{1}}$ generators lie in the same column of the type D heap so although $s_2 s_1 s_{\bar{1}} s_2$ is a fully commutative element in D_3 , the pair of s_2 entries do not have a distinct resolution. On the other hand, the pair of s_2 entries in $s_1 s_2 s_{\bar{1}} s_3 s_2 s_1$ has a distinct resolution, while the pair of s_1 generators has only a right resolution.

Recall that by Tits' theorem a permutation is fully commutative if and only if every minimal pair has a distinct resolution. Short braid avoiding is equivalent to fully commutative in type D . We establish some structural lemmas about the resolutions in $\text{Heap}(w)$ when w is short braid avoiding.

Lemma 6.6. *Let $w \in D_n$ be a short braid avoiding element. Then, no minimal pair of generators in column $i \geq 2$ in $\text{Heap}(w)$ can have a right resolution.*

Proof. Since w is short braid avoiding, if x and y are resolved by a pair of generators z_1, z_2 that lie to the right of column i then z_1 and z_2 necessarily correspond to the same generator since the Coxeter graph of type D is a path beyond column 1. Choose z_1, z_2 to be a minimal pair in column $i + 1$. Since x and y form a minimal pair, we cannot backtrack when resolving z_1 and z_2 , so z_1 and z_2 must be separated by another minimal pair of generators to the right that are non-commuting with z_1, z_2 :

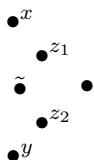


Since D_n is finitely generated, eventually there exists a minimal pair of entries in the rightmost column of $\text{Heap}(w)$ that cannot be resolved. Hence, every minimal pair of generators in column $i \geq 2$ is resolved by two generators from distinct columns or by a pair of generators to the left. \square

Lemma 6.7. *Let $w \in D_n$ be a short braid avoiding element.*

- (1) *There exists an $s_{\bar{1}}$ between every minimal pair of s_1 generators, and an s_1 between every minimal pair of $s_{\bar{1}}$ generators in $\text{Heap}(w)$.*
- (2) *If there is an entry in which both s_1 and $s_{\bar{1}}$ lie at the same level of $\text{Heap}(w)$, then column 1 contains no other entries.*
- (3) *If there exists a pair of entries in column 1 corresponding to the same generator then all of the entries in the first column must be on distinct levels of the heap, and they must alternate between the generators s_1 and $s_{\bar{1}}$.*

Proof. Part (1): Suppose that x and y are a minimal pair in column 1 corresponding to s_1 . Then they must be resolved by a pair z_1, z_2 in column 2 since s_1 commutes with every other generator. We can choose z_1, z_2 to be a minimal pair. Since z_1, z_2 cannot have a right resolution by Lemma 6.6, there exists an entry between z_1 and z_2 in column 1. By the minimality of x, y , this entry must correspond to $s_{\bar{1}}$. Moreover, z_1, z_2 cannot use a left resolution without contradicting the minimality of the pair x and y . Hence, z_1, z_2 have a distinct resolution with a heap fragment of the form:

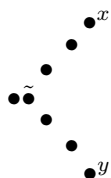


The same argument with the roles of s_1 and $s_{\bar{1}}$ reversed shows that there is an s_1 between every minimal pair of $s_{\bar{1}}$ generators.

Part (2): Suppose s_1 and $s_{\bar{1}}$ lie at the same level k of the heap. If column 1 contains another entry p , suppose without loss of generality that p forms a minimal pair with the s_1 entry from level k of the heap, and there are no other entries in column 1 between p and the s_1 entry from level k . Then, in order for the heap to correspond with a reduced expression, this pair must have a right resolution using a minimal pair of s_2 's. But resolution of the minimal pair of s_2 generators includes an entry from column 1 by Lemma 6.6, contradicting the minimality of our choice of p from column 1. Thus, there cannot be other generators in the first column.

Part (3) follows directly from Parts (1) and (2). \square

Lemma 6.8. *Let $w \in D_n$ be a short braid avoiding element. If there exists a minimal pair of entries x, y in column $i \geq 2$ with only a left resolution, then the part of the heap to the left of column i has a particular form shown below with exactly two entries in each of columns $1, \dots, i$:*



Proof. Suppose x and y are separated by a minimal pair of generators z_1, z_2 both lying in column $i - 1$. Applying reasoning similar to that in Lemma 6.6, z_1, z_2 must have a left resolution as well by minimality. Continuing on, we eventually obtain a minimal pair in column 2. In contrast to the case of a right resolution, we can resolve a minimal pair in column 2 to the left with $s_1 s_{\bar{1}}$, and by minimality we must do so. Since these generators commute, they lie at the same level in the heap.

Observe that by minimality, there can be no entries between x and y , nor between any of the other minimal pairs in the left resolution. Note also that there cannot be any other entries in column 1, by Lemma 6.7. Therefore, if there existed any other entries in columns $2, \dots, i$ above or below the minimal pairs, then they would create new minimal pairs with the existing entries and would require a left resolution or a distinct resolution by Lemma 6.6. But resolving these implied minimal pairs eventually requires additional entries in column 1, which contradicts Lemma 6.7(2). Thus, there can be no other elements in columns $2, \dots, i$. \square

Definition 6.9. Observe that if position k is a defect of $w = w_1 \dots w_p$ with respect to the mask σ , then $w^{\sigma[k-1]} w_k$ is not reduced. Hence by Tits' theorem, there must be some entry w_j with mask-value 1 that lies to the left of w_k in the reduced expression and corresponds to the same generator as w_k . We call the rightmost such position the *critical generator* for the defect k .

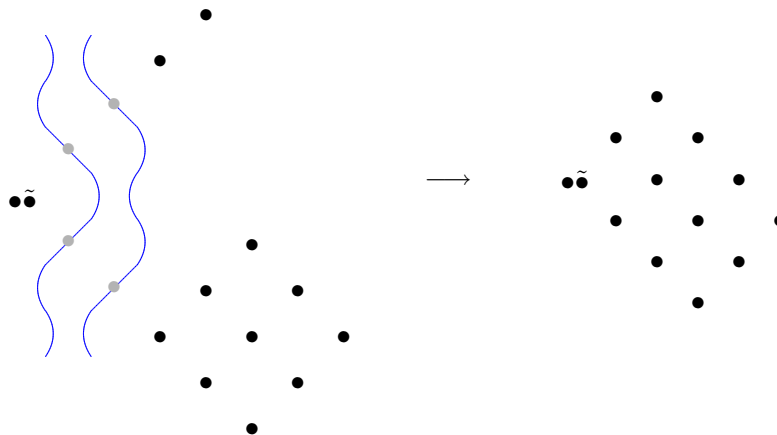
The critical generator can be viewed as an element of the heap as well. The critical generator is always the first entry in the heap below k and in the same column as k .

Lemma 6.10. *Suppose $w \in D_n$ is a connected, short braid avoiding, non-Deodhar element and $\text{Heap}(w)$ contains a minimal pair of entries with only a left resolution. Then, we can construct an element $\tilde{w} \in D_k$ with $k < n$ such that:*

- (1) w contains \tilde{w} as a 1-line pattern,
- (2) every minimal pair of entries in $\text{Heap}(\tilde{w})$ has a distinct resolution, and
- (3) \tilde{w} is a connected short-braid avoiding non-Deodhar element.

Proof. Suppose that the rightmost pair of entries that require a left resolution lie in column i . By Lemma 6.8, we have a specific form for columns $1, \dots, i$ in $\text{Heap}(w)$. To construct $\text{Heap}(\tilde{w})$ from $\text{Heap}(w)$, we remove columns 2 through i , and shift columns $i + 1, i + 2, \dots$ to the left. Hence, column 1 of $\text{Heap}(\tilde{w})$ contains an $s_1 s_{\bar{1}}$ entry leftover from the left resolution, and column $j \geq 2$ of $\text{Heap}(\tilde{w})$

contains the entries of column $j + i - 1$ of $\text{Heap}(w)$. In the example below, $i = 3$ and $\text{Heap}(\tilde{w})$ is obtained by removing the 4 gray entries.



By Lemma 6.8, the strings labeled $2, 3, \dots, i$ on the bottom of the heap end up in the same positions at the top of the heap. Therefore, the strings of $\text{Heap}(\tilde{w})$ can be canonically identified with strings $1, i + 1, \dots, n$, of $\text{Heap}(w)$, so the resulting string diagram for \tilde{w} is well-defined with w containing \tilde{w} as a 1-line pattern. Furthermore, since w is connected, so is \tilde{w} .

By the minimality assumption and Lemma 6.6, every minimal pair of entries in $\text{Heap}(w)$ to the right of column i has a distinct resolution. Therefore, by construction every pair in $\text{Heap}(\tilde{w})$ has a distinct resolution. In particular, \tilde{w} is short braid avoiding.

Finally, if w is not Deodhar then we can find a proper non-Deodhar mask σ on $\text{Heap}(w)$. We show this implies there exists a non-Deodhar mask on $\text{Heap}(\tilde{w})$. Recall from (2.5) that a mask is non-Deodhar whenever

$$\# \text{ zero-defects} \geq \# \text{ plain-zeros}$$

which we refer to as the *non-Deodhar bound* throughout the proof.

Consider the effect on the non-Deodhar bound of modifying every mask value in σ to be 1 in columns $2, \dots, i$. If we set a plain-zero that is not involved with any defect to have mask-value 1 then the mask remains non-Deodhar.

Say there exists a zero-defect at the top of column $h \leq i$. Then the strings for the defect must cross by Corollary 6.3. By Lemma 6.8, the form of the heap in columns $1, \dots, i$ is determined. Hence, the left string of the defect must travel southwest from the defect until it hits a zero in column $1 \leq g < h$, drop straight down until it hits the next entry in the heap which must also be a zero, and then continue southeast until it crosses the right string of the defect at the bottom entry in column h . Both of the entries in column g must have mask-value 0 to facilitate the string crossing for the defect. Neither entry in column g can be a zero-defect, because every defect must have a critical generator below it in the same column. As we already assumed that the mask values of both entries in column g were 0, the lower entry is not a critical generator. Thus, we find that setting the entries in columns g and h to have mask-value 1 removes a zero-defect and two plain-zeros, which preserves the non-Deodhar bound for the mask.

If there is no zero-defect in any column $j > i$ that has a string passing through column i , then the mask σ' obtained from σ by setting the mask-values of all entries in columns $1, \dots, i$ to 1 will remain non-Deodhar, since the defect status depends only on the string dynamics for the left and right strings of an entry by Lemma 6.1.

On the other hand, if there exists a zero-defect in column $j > i$ whose left string encounters column i , then in order for the strings of the defect to cross, the path of the left string must follow a similar course as described above. That is, the left string travels southwest to a plain-zero, say in column $f \leq i$,

drops straight down to another plain-zero in the same column, and continues southeast through column i again and beyond. In this case, the mask σ' is obtained from σ by setting the entries in column 1 to have mask-value 0, and setting the mask-values of entries in columns $2, \dots, i$ to 1. This will maintain the string dynamics for the left string of the defect in column j , and effectively moves the plain zeros from column f to column 1. Hence, the non-Deodhar bound is preserved.

In either of the definitions for σ' above, the strings $2, \dots, i$ cross once to the right then once to the left so remain in their original position. In particular, all the plain-zeros and zero-defects in σ' applied to w remain if we remove the strings $2, \dots, i$. Therefore, σ' restricted to columns $1, i+1, \dots, n$ determines a non-Deodhar mask for \check{w} . \square

7. CONVEX ELEMENTS IN TYPE D

In this section, we restrict our attention to a subset of the short braid avoiding elements that have the lateral convexity property introduced for type A in [BW01].

Definition 7.1. If $w \in D_n$ is short braid avoiding and every minimal pair of entries in $\text{Heap}(w)$ has a distinct resolution then we say w is *convex*.

Remark 7.2. It follows immediately from the definition that in a convex type D heap, there can only be a single generator in the rightmost column.

For example, a permutation is short braid avoiding if and only if it is convex. The element $s_2s_1s_1s_2 \in D_4$ is not convex since it does not have a distinct resolution of the s_2 's, while $s_2s_1s_3s_1s_2 \in D_4$ is convex.

It follows from [SS85] that the number of short braid avoiding elements in type A_n is the Catalan number $c_n = \frac{1}{n+1} \binom{2n}{n}$. In types B_n and C_n , the short braid avoiding (equivalently, convex) elements are also counted by Catalan numbers. In type D , Fan and Stembridge have given an explicit formula for the number of short braid avoiding elements [Ste98]. Furthermore, Stembridge characterized the short braid avoiding elements of type D in terms of 1-line patterns.

Theorem 7.3. [Ste97] *An element w in type D is short braid avoiding if and only if w avoids all 1-line patterns $[abc]$ where $|a| > b > c$ or $\bar{b} > |a| > c$.*

In type D , there is a new sequence corresponding to the number of convex elements in D_n for $n \geq 1$. The first 10 terms are

$$1, 4, 13, 44, 154, 552, 2013, 7436, 27742, 104312$$

Remark 7.4. This sequence is also characterized by 1-line patterns. The patterns are simply the patterns from Theorem 7.3 and the single additional pattern $[\bar{1}2\bar{3}] \in D_3$. It would be interesting to know more about this sequence. The notion of a convex element can be extended to other Coxeter groups where the Coxeter graph can be linearized in a meaningful way.

Definition 7.5. Given a convex element $w \in D_n$, we define an operation called *coalescing*, which connects the lattice point components of the heap as in [BW01]. Starting in column 1, if there exists an entry in column 1 below an entry in column 2 with empty lattice points between them, then allow the entry in column 1 to rise up until it is blocked by the entry in column 2. Then, allow all the entries below the column 1 entry to rise up as well, until blocked by a non-commuting generator. Work to the right, continuing to apply the same elevations until the heap is pushed together as much as possible. The resulting collection of lattice points is called the *coalesced heap* of w .

Throughout the rest of the paper, whenever we refer to a *position*, we will mean the coordinates (a, b) in the lattice \mathbb{Z}^2 containing a coalesced heap for a convex element. Here a is the column number and b is

the height of its row. Each position can be empty, contain a single generator of the heap, or contain the two generators $s_1 s_{\bar{1}}$ in the case when the entry lies in the first column of a type D heap.

Define two subsets of the heap as follows:

The *lower cone* of (a, b) : $\text{Cone}_{\wedge}(a, b) = \{(a, b) + i(-1, -1) + j(2, 0) \in \mathbb{Z}^2 : i \in [0, \infty), j \in [0, i]\}$.

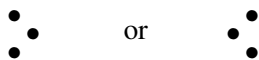
The *upper cone* of (a, b) : $\text{Cone}_{\vee}(a, b) = \{(a, b) + i(-1, 1) + j(2, 0) \in \mathbb{Z}^2 : i \in [0, \infty), j \in [0, i]\}$.

After coalescing the heap, the points in a heap that lie in $\text{Cone}_{\wedge}(a, b)$ are precisely the elements of the heap poset that are smaller than (a, b) . Similarly, the points in a heap lying in $\text{Cone}_{\vee}(a, b)$ are the elements in the heap poset that are greater than (a, b) .

Definition 7.6. A collection of lattice points L is *laterally convex* if $(a, b), (c, d) \in L$ implies $\text{Cone}_{\wedge}(a, b) \cap \text{Cone}_{\vee}(c, d) \subset L$.

Lemma 7.7 (Convexity Lemma). *The coalesced heap of a connected convex element in type A, B or D is laterally convex.*

Proof. Every minimal pair in a convex element is resolved by two distinct elements x, y . If these elements appear on different levels in the coalesced heap then there must be a chain of elements preventing the lower element, say x , from rising without raising y . But, this would imply there is an alcove of the form



consisting of a minimal pair without a distinct resolution, contradicting that the heap is convex. Therefore, locally every minimal pair is only two rows apart $(a, b), (a, b + 2)$, and its distinct resolution is on adjacent points $(a - 1, b + 1), (a + 1, b + 1)$, given pictorially as

$$(7.1) \quad \begin{array}{c} \bullet \\ \bullet \quad \bullet \\ \bullet \end{array}$$

Suppose the positions $(a, b), (c, d)$ contain entries of the coalesced heap, and $(a, b) \in \text{Cone}_{\vee}(c, d)$ and $(c, d) \in \text{Cone}_{\wedge}(a, b)$. Since the heap is connected, there must exist a chain of adjacent non-commuting generators connecting the two points in the coalesced heap. Every minimal pair along the path must have a distinct resolution which by the argument above looks locally like (7.1). Filling out the diamond for each minimal pair recursively implies that every point in $\text{Cone}_{\wedge}(a, b) \cap \text{Cone}_{\vee}(c, d)$ is contained in the heap. \square

Remark 7.8. In the classification of Section 8, we will consider only connected convex elements w , and we will assume that they have been embedded in the lattice so that $\text{Heap}(w)$ is coalesced. Then, these lattice points satisfy lateral convexity by Lemma 7.7. Hence, there is a *minimal resolution* of any type D heap fragment that arises in this fashion, as a subset of lattice points of some $\text{Heap}(w)$. In particular, we can add lattice points to the heap fragment to resolve minimal pairs of entries over columns $2, \dots, n$ distinctly as in the classical lateral convexity of [BW01], and we may resolve minimal pairs of entries in the first column as prescribed by Lemma 6.7. We frequently invoke the Convexity Lemma to speak of *the resolution by convexity* of some collection of entries inside a connected convex non-Deodhar type D heap, and deduce the existence of heap entries that were not explicitly given.

Lemma 7.9. *Let w be a connected convex element of type A, B , or D with coalesced $\text{Heap}(w)$. Then, the point y covers x in the heap poset for w if and only if x has heap coordinates (i, j) and y has heap coordinates $(i \pm 1, j + 1)$.*

Proof. Suppose x has heap coordinates (i, j) and y has heap coordinates $(i \pm 1, j + 1)$. Then, x and y do not commute, and since the level difference is 1, we have that y covers x in the heap poset.

On the other hand, if x has heap coordinates (i, j) but y has heap coordinates $(i \pm \delta_1, j + \delta_2)$ for $\delta_2 \geq \delta_1 \geq 1$ with $\delta_1 + \delta_2 > 2$, then all of the points in $C = \text{Cone}_\wedge(i \pm \delta_1, j + \delta_2) \cap \text{Cone}_\vee(i, j)$ are in $\text{Heap}(w)$ by the lateral convexity proved in Lemma 7.7. In particular, $z = (i \pm 1, j + 1)$ is in $C \subset \text{Heap}(w)$, so we have $x < z < y$ in the heap poset, hence y does not cover x in the heap poset.

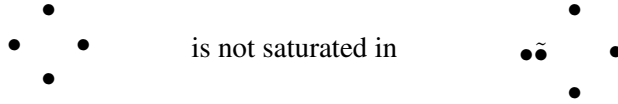
If y has heap coordinates $(i \pm \delta_1, j + \delta_2)$ for $\delta_1 > \delta_2$, then x and y are unrelated in the heap poset. \square

Let $M_{(x,y)}^w$ be the number of entries located at position (x, y) in the coalesced heap for w . These occupation numbers can take the values 1 or 2 in type D , but the value 2 can only appear in column 1 and must obey the rules in Lemma 6.7.

Definition 7.10. Let w and p be connected convex elements of type A , B or D with coalesced heaps $\text{Heap}(w)$ and $\text{Heap}(p)$, respectively. We say that $\text{Heap}(w)$ contains $\text{Heap}(p)$ as a *saturated subset* of lattice points, if there exist offsets $i, j \in \mathbb{Z}$ and a Coxeter embedding f such that for all occupied positions (x, y) in $\text{Heap}(f(p))$, we have

$$M_{(x,y)+i(2,0)+j(1,1)}^w = M_{(x,y)}^p$$

Example 7.11. Let $p = s_2 s_1 s_3 s_2$ and $w = s_2 s_1 s_{\bar{1}} s_3 s_2$. Note that



because $M_{(1,1)}^p = 1$ while $M_{(1,1)}^w = 2$.

On the other hand, $\text{Heap}(w)$ is saturated in the heap of any w' containing $s_2 s_1 s_{\bar{1}} s_3 s_2$ as a factor.

Lemma 7.12. Let w and p be connected convex elements of type A , B , or D with coalesced heaps $\text{Heap}(w)$ and $\text{Heap}(p)$, respectively. Then, the following are equivalent:

- (1) w contains p as an embedded factor.
- (2) $\text{Heap}(p)$ is contained in $\text{Heap}(w)$ as a saturated subset of lattice points.

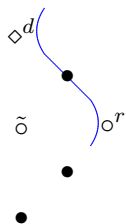
Proof. (1) \implies (2). Recall from [Sta97] that a subposet Q of some poset R is called *convex* if $b \in Q$ whenever $a < b < c$ in R and $a, c \in Q$. Take a reduced expression for w of the form $w = uf(p)v$ where f is a Coxeter embedding for p . By Definition 3.1, the labeled heap poset of $w = uf(p)v$ contains the labeled heap poset of $f(p)$ as a convex subposet. Lemma 7.9 shows that the Hasse diagram for the heap poset of w can be embedded into $[n] \times \mathbb{N}$ where $(i, j) < (i \pm 1, j + 1)$ is the corresponding cover relation, and this embedding is $\text{Heap}(w)$. Hence, the convex subposet corresponding to the labeled heap poset of $f(p)$ appears in $\text{Heap}(w)$ as a saturated subset of lattice points with shape $\text{Heap}(f(p))$.

(2) \implies (1). If $\text{Heap}(f(p))$ is contained as a saturated subset of the lattice points in $\text{Heap}(w)$ for some Coxeter embedding f , then we can build $\text{Heap}(w)$ from $\text{Heap}(f(p))$ by sequentially adding lattice points. Moreover we can only add points that are maximal or minimal entries of the intermediate heap since $\text{Heap}(f(p))$ is saturated in $\text{Heap}(w)$. This operation is equivalent to multiplying $f(p)$ on the left or right by the respective generators. Hence, w contains $f(p)$ as a factor. \square

We now generalize the notion from [BW01] of a *right critical zero* associated to a defect for convex heaps in type D .

Definition 7.13. The first zero encountered by the right string of a defect along the southeast diagonal containing the defect is the *right critical zero*.

For example, the defect d has right critical zero r :



Lemma 7.14 (Right Critical Zero Lemma). *Suppose $w \in D_n$ is convex, and σ is a mask on w . Then every defect in $\text{Heap}(w)$ has a right critical zero.*

Proof. Suppose the right string for a defect never encounters an entry with mask-value 0 as it travels southeast. It eventually turns southwest when it hits the first entry (a, b) not in the heap along that diagonal. As the string travels down, it cannot encounter another heap element in column $a - 1$ before encountering one in column a by convexity. If an element is encountered in position $(a, b - i)$ as the string travels down, then by lateral convexity (a, b) must exist in the heap contradicting our hypothesis. If no element is encountered in position $(a, b - i)$ as the string travels down, then it cannot cross the left string below, contradicting the fact that the original position was a defect by Corollary 6.3. \square

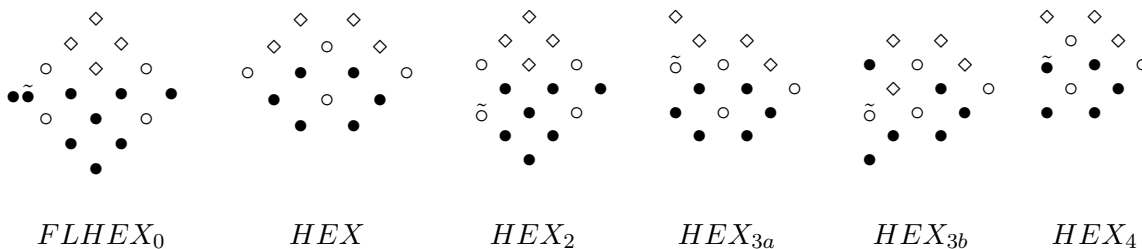
Note that defects in type D need not have a corresponding left critical zero, since a string leaving the heap on the left can re-enter the heap without a mask-value 0 entry. See Example 6.2.

8. CLASSIFICATION OF CONVEX TYPE D PATTERNS

In this section, we show that among the set of convex elements of type D , there are precisely six that are minimally non-Deodhar under the partial order of embedded factor containment. Recall that the convex elements are short braid avoiding, hence they have unique heaps. These minimal heaps have the property that if we remove either an entry that is minimal or maximal with respect to the heap poset structure, then the resulting heap corresponds to a Deodhar element. Furthermore, we will always consider the heaps to be coalesced as described in Definition 7.5.

Theorem 8.1. *Below is the complete list of the heaps of convex minimal non-Deodhar embedded factor patterns in type D up to the Coxeter graph isomorphism that interchanges s_1 and s_1^{-1} :*

(8.1)



Each heap above is decorated with a mask that demonstrates the non-Deodhar condition using the notation described in Section 6. It is straightforward to verify by computer that these heaps are minimal among the convex non-Deodhar elements in type D using embedded factor containment. Our goal is to show that this list is complete. Note that we could reduce the list further by requiring both w and w^{-1} to

avoid the pattern HEX_{3a} since HEX_{3b} is its inverse and by Lemma 5.1, w is non-Deodhar if and only if w^{-1} is non-Deodhar. All the other heaps in (8.1) have $w = w^{-1}$.

Note that the pattern HEX can embed on any 7 connected columns corresponding to a path of length 7 in the Coxeter graph, including $s_{\tilde{1}}, s_2, \dots, s_7$. The other patterns are tied to the initial segment of the type D Coxeter graph, since they include both of the generators $s_1, s_{\tilde{1}}$ from column 1. These patterns are the heaps for the reduced words that appear in the table on Page 2 in the type D case.

Remark 8.2. Recall from Lemma 7.12 that factor containment is expressed in the heap setting by deleting a sequence of entries that are minimal or maximal entries of the intermediate heap at each step. In particular, HEX is not a saturated subset of $FLHEX_0$ by Definition 7.10.

Let D_n^C and A_∞^C denote the convex elements in type D_n and type A on any finite number of generators, respectively. We will prove Theorem 8.1 by constructing a pair of projection maps

$$\begin{aligned} \pi_{NE} : D_n^C &\rightarrow A_\infty^C \\ \pi_{SE} : D_n^C &\rightarrow A_\infty^C \end{aligned}$$

that we can use to compare minimal patterns in type D with the hexagon in type A . We will show that these maps have the following properties:

- a. If $w \in D_n^C$ is non-Deodhar in type D , then at least one of the two projections sends w to a non-Deodhar element in type A .
- b. If $\pi_{NE}(w)$ or $\pi_{SE}(w)$ is non-Deodhar in type A , then w contains one of the patterns listed above as an embedded factor.

Therefore, if w is any minimal non-Deodhar element in type D , then it appears on the list. Hence, the list of convex minimal non-Deodhar elements is complete. In particular, all convex minimal non-Deodhar patterns occur in D_8 .

We will define the projections $\pi_{NE}(w), \pi_{SE}(w)$ in terms of three other maps. First, define the map

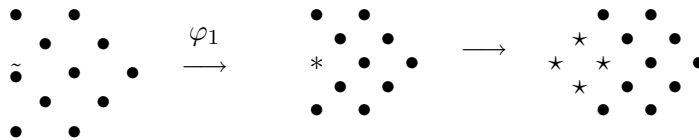
$$\varphi_i : D_n^C \rightarrow A_{n-1} \quad \text{for } i \in \{1, \tilde{1}\}$$

as the projection of w onto a type A heap obtained by removing all the entries that correspond to the \tilde{i} generator, where \tilde{i} is the generator $\{1, \tilde{1}\} \setminus \{i\}$. That is, φ_i assigns the s_i generator from column 1 of the type D heap to s_1 in column 1 of the type A heap, and removes all the entries from column 1 that correspond to the other generator. This generally leaves a heap fragment in type A that is *not* convex.

How might we construct an associated convex element in type A ? We consider two constructions. We could take an *additive approach*, denoted

$$\mathbf{add}_i : D_n^C \rightarrow A_\infty^C,$$

where we first project the element into type A by the map φ_i , then extend the Coxeter graph linearly to the left beyond the first column and add entries as necessary in order to resolve all minimal pairs of entries that do not have a distinct resolution. By Remark 7.8 restricted to type A , this operation is unique. For example, the composition shown below is \mathbf{add}_1 .



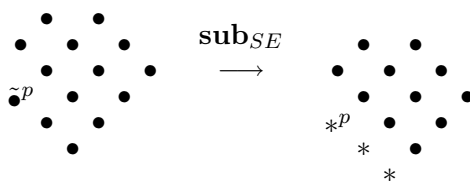
Here, $*$ denotes a point not in the heap, and \star denotes a highlighted point in the heap.

The second construction is a *subtractive approach* that applies when there are at least two entries in the first column of the type D heap. Define

$$\mathbf{sub}_{SE} : D_n^C \rightarrow D_n^C$$

$$\mathbf{sub}_{NE} : D_n^C \rightarrow D_n^C$$

where \mathbf{sub}_{NE} removes entries in the northeast (NE) diagonal up to the first lattice point not in the heap. Similarly, \mathbf{sub}_{SE} removes the southeast (SE) diagonal. This process is referred to as “shaving off a diagonal” in the proofs that follow. For example,



Note that the result of applying \mathbf{sub}_{NE} or \mathbf{sub}_{SE} is always again convex because we are removing entries that are maximal or minimal with respect to the heap poset, so the entries cannot be used in a distinct resolution.

We now define $\pi_{SE}(w)$ and $\pi_{NE}(w)$ in cases depending on how many levels the coalesced $Heap(w)$ contains in the first column. The rough idea that motivates the following technical definition is to use the minimal patterns listed in (8.1) as a guide, taking a subtractive approach when we have a heap fragment that is Deodhar and taking an additive approach when we have a heap fragment that is non-Deodhar.

Definition 8.3. Let $w \in D_n^C$. Suppose that the first column of $Heap(w)$ has entries in $k(w)$ distinct levels. Let $i \in \{1, \tilde{1}\}$ be the generator corresponding to the top entry in column 1. Let dir denote the direction SE or NE. Then, we define $\pi_{dir} : D_n^C \rightarrow A_\infty^C$ as follows:

$$\pi_{dir}(w) = \begin{cases} w & \text{if } k(w) = 0. \\ \mathbf{add}_i(w) & \text{if } k(w) = 1 \text{ and there is a single entry in column 1.} \\ \mathbf{add}_1(w) & \text{if } k(w) = 1 \text{ and there are two entries in column 1,} \\ & \text{with at least 4 entries in column 4.} \\ \mathbf{sub}_{dir}(w) & \text{if } k(w) = 1 \text{ and there are two entries in column 1,} \\ & \text{with at most 3 entries in column 4.} \\ \mathbf{add}_1(w) & \text{if } k(w) = 2 \text{ and there are at least 4 entries in column 3.} \\ \mathbf{sub}_{dir}(w) & \text{if } k(w) = 2 \text{ and there are at most 3 entries in column 3.} \\ \mathbf{add}_i(w) & \text{if } k(w) = 3. \\ \pi_{dir} \circ \mathbf{sub}_{dir}(w) & \text{if } k(w) = 4. \\ \mathbf{add}_1(w) & \text{if } k(w) \geq 5. \end{cases}$$

This definition relies on the following observations.

- (1) When there are entries in more than one level of column 1, the entries of the first column must alternate by Lemma 6.7.
- (2) In the case $k(w) = 1$, if there two entries in column 1 with less than 4 entries in column 4, then column 1 contains both s_1 and $s_{\tilde{1}}$ on the same level in column 1. We define the map π_{dir} to remove both of these generators in column 1, and shave any remaining entries from the chosen diagonal.

- (3) In the case $k(w) = 4$, observe that π_{dir} is defined by shaving the top or bottom entry of column 1, and then reducing to the case when $k(w) = 3$.

Note that the map π_{dir} always preserves the occupied lattice points that are present in the type D heap, with the possible exception that they may remove the northeast diagonal of maximal entries or southeast diagonal of minimal entries, starting from the first column and working to the right.

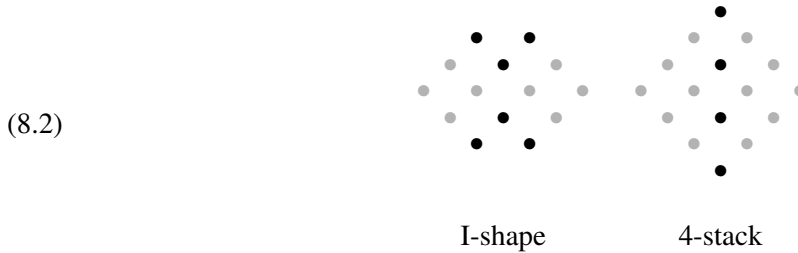
The π_{dir} projections satisfy a useful property with respect to taking inverses that we will exploit in the classification.

Lemma 8.4. *For all $w \in D_n^C$, we have*

$$\pi_{SE}(w^{-1}) = \pi_{NE}(w)^{-1}$$

Proof. Observe that the heap of w^{-1} is obtained from the heap of w by flipping the diagram upside down. In particular, the number of entries in each column of w and w^{-1} is the same. Therefore, w^{-1} falls into the same case as w when computing π_{dir} . Flipping the heap has the effect of switching all the SE and NE diagonals, while the additive resolutions remain the same. \square

Consider the following partial heaps drawn with black dots called the *I-shape* and the *4-stack*:



Some of the shaded dots will also appear in any convex heap containing either the *I-shape* or the *4-stack*. However, if the shape appears in the first 4 columns then some of the shaded dots on the left hand side of the picture will get truncated in the heap. Note that the 4-stack consists of 4 adjacent copies of the same generator if it appears in column 1. Therefore, a “4-stack” in column 1 of a convex heap necessarily contains entries from 7 distinct levels. The following result is used frequently in the classification.

Lemma 8.5. (Shape Lemma) *Suppose $w \in D_n^C$, $Heap(w)$ contains an *I-shape* or a *4-stack*, and the first column of $Heap(w)$ has entries from at least two distinct levels. Then, w is non-Deodhar and there is a choice of projection $\pi_{NE}(w)$ or $\pi_{SE}(w)$ that contains a hexagon, and so preserves the non-Deodhar condition.*

Proof. Consider the convex resolution of either the *I-shape* or the *4-stack*. By Remark 7.8, the points in the convex resolution must appear in $Heap(w)$. This implies that all of the gray dots shown in (8.2) to the right of the shape and a subset of the gray dots to the left of the shape must be in the heap, depending on how far the shape lies from the first column. By considering the columns where the shape lies, one can verify that no matter where the *I-shape* or *4-stack* appear, the convex resolution must contain one of the patterns in (8.1) as a set of points in the lattice. Since $Heap(w)$ contains entries from at least two distinct levels, we have that $Heap(w)$ contains one of the minimal heaps in (8.1) as a saturated set of lattice points by Lemma 6.7, hence w also contains it as an embedded factor by Lemma 7.12. Therefore, w is non-Deodhar.

Furthermore, it can be verified that Definition 8.3 has been chosen precisely so that if $Heap(w)$ contains one of the heaps in $\{HEX, HEX_2, HEX_{3a}, HEX_{3b}, HEX_4\}$ as an embedded factor, then there is always a choice of direction NE or SE so that the projection π_{dir} contains a hexagon in type A .

To carry out the verification, suppose that $Heap(w)$ contains HEX using column 1. Then, if there are two entries in column 1, we either project additively (if there is also a 4-stack in column 3), or we

choose to shave the entry of column 1 that is not being used in the hexagon, and its diagonal. If there are 3 entries in column 1, then we always project additively, so the hexagon is preserved. If there are 4 entries in column 1, then we shave one of the extremal entries that is not used in the hexagon, and project additively. If there are more than 4 entries in column 1, then we project additively so the hexagon is preserved. If the first column of the HEX factor occurs in column 2 of $Heap(w)$, or further to the right, and we are taking a subtractive resolution in $\pi_{dir}(w)$, then there is always a diagonal to shave that does not intersect the hexagon. Otherwise, we take an additive resolution, so the hexagon is preserved.

The shapes $\{HEX_2, HEX_{3a}, HEX_{3b}, HEX_4\}$ must all occur in column 1, so it suffices to check that there is a choice of projection that preserves the shape, regardless of how many entries column 1 contains. Note that column 1 must contain at least two entries by hypothesis, and if it contains more than 4 entries, then we resolve additively, so the shapes are preserved. The cases left to verify are then that there exists a non-Deodhar projection $\pi_{NE}(w)$ or $\pi_{SE}(w)$ if $Heap(w)$ contains:

- HEX_2 with 2, 3 or 4 entries in column 1,
- HEX_{3a} with 3 or 4 entries in column 1,
- HEX_{3b} with 3 or 4 entries in column 1, or
- HEX_4 with 3 or 4 entries in column 1.

This verification is finite and straightforward, so we omit it. □

We now prove Property (b) for π_{dir} .

Proposition 8.6. *Let $w \in D_n^C$. If $\pi_{dir}(w) \in A_\infty^C$ is non-Deodhar, then w contains a pattern from (8.1) as an embedded factor.*

Proof. Suppose $\pi_{dir}(w)$ is non-Deodhar. Then, by Theorem 5.9 the heap of $\pi_{dir}(w)$ contains a hexagon. We begin by considering the various ways in which the first column of the type D heap can be shifted under the π_{dir} map. Let $\sigma(w)$ be the number of the column that contains the image of column 1 under $\pi_{dir}(w)$. This can be computed by determining how many columns must be added to the left when applying the **add** map or subtracted if the entire first column is removed.

Case for $\pi_{dir}(w)$	Shift $\sigma(w)$
$k(w) = 0$	1
$k(w) = 1$ and there is a single entry in column 1	1
$k(w) = 1$ and there are two entries in column 1, with at least 4 entries in column 4	1
$k(w) = 1$ and there are two entries in column 1, with less than 4 entries in column 4	0
$k(w) = 2$ and there are at least 4 entries in column 3	2
$k(w) = 2$ and there are less than 4 entries in column 3	1
$k(w) = 3$	3
$k(w) = 4$	3
$k(w) \geq 5$	$k(w)$

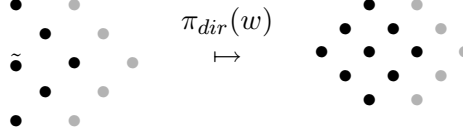
If the hexagon appears weakly to the right of column $\sigma(w)$ then $Heap(w)$ contains the shape HEX in (8.1), since π_{dir} only adds points to the left of column $\sigma(w)$. In particular, if $\sigma(w) = 0$, then from the table above we see $k(w) = 1$ and column 1 contains $\{s_1, s_{\bar{1}}\}$ on the same level, with less than 4 entries in column 4. In this case, $\pi_{dir}(w)$ removes all of the entries in the first column, and shaves a diagonal. If there exists a hexagon in $\pi_{dir}(w)$, we have that there is a hexagon in $Heap(w)$ that lies strictly to the right of column 1 in $Heap(w)$, so w contains HEX as an embedded factor.

If $\sigma(w) = 1$ then from the table above we see that $k(w) = 1$ or $k(w) = 2$ and there are less than 4 entries in column 3. In each of these cases the π_{dir} projection doesn't add any points to the heap.

Therefore, if $\pi_{dir}(w)$ contains a hexagon that uses column 1, then $Heap(w)$ contains either HEX or $FLHEX_0$ as an embedded factor.

If $\sigma(w) = 2$ then $k(w) = 2$ and there is a 4 stack in column 3, so by Lemma 6.7 and Remark 7.8, $Heap(w)$ contains HEX_2 as an embedded factor.

If $\sigma(w) = 3$, then $k(w) = 3$ or $k(w) = 4$. In the case $k(w) = 3$, $\pi_{dir}(w)$ adds three new points to the left of $Heap(w)$. If a hexagon in $\pi_{dir}(w)$ uses column 1, then $Heap(w)$ must contain HEX_4 .



If a hexagon in $\pi_{dir}(w)$ uses column 2 (but not column 1) then $Heap(w)$ contains HEX_{3a} or HEX_{3b} as an embedded factor. If a hexagon appears in column 3 or further to the right, then $Heap(w)$ contains HEX as an embedded factor as well.

In the case $\sigma(w) = 3$ and $k(w) = 4$ then $\pi_{dir}(w)$ shaves a maximal or minimal diagonal and adds back three points as above. The same analysis of the placement of a hexagon in $\pi_{dir}(w)$ implies the existence of a HEX_{3a} , HEX_{3b} or a HEX_4 as embedded factors.

If $\sigma(w) = s \geq 5$, we again must have at least 5 entries in column 1 of w , so by convexity w contains HEX_4 . Note that $\sigma(w) = 4$ never occurs. \square

We now turn to the proof of Property (a) for π_{dir} . In preparation for the proof, recall from Corollary 6.3 that the strings for a defect must cross below the defect. The first time the two strings meet again will be called the *initial string crossing* for the defect or just the *string crossing*, for short.

Proposition 8.7. *Let $w \in D_n^C$. If w is non-Deodhar, then there is some choice of projection $\pi_{NE}(w)$ or $\pi_{SE}(w)$ that is non-Deodhar.*

Proof. To prove this, we will consider the various cases given in Definition 8.3 for π_{dir} , breaking on the number of levels in the first column of w denoted $k(w)$. Recall from (2.5) that a proper mask is non-Deodhar whenever

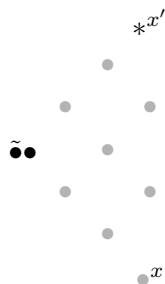
$$\# \text{ zero-defects} \geq \# \text{ plain-zeros}$$

which we refer to as the *non-Deodhar bound* throughout the proof. The general strategy is to either show that w contains a 4-stack or an I-shape from Lemma 8.5 when $k(w) \geq 2$, or to assume a proper non-Deodhar mask on w , and then show how to adjust it to obtain a mask on $\pi_{dir}(w)$ that retains the non-Deodhar bound by deleting at least as many plain-zeros as zero-defects.

Case $k(w) = 0$ or 1. First, suppose that w has at most a single entry in the first column so π_{dir} is just the embedding map into type A . Then we can interpret w as an element of type A by projecting the unique generator in the first column to the generator s_1 in type A . Hence, w is non-Deodhar if and only if it contains a hexagon if and only if $\pi_{dir}(w)$ contains a hexagon, so π_{dir} preserves the non-Deodhar condition in this case.

Second, suppose that w has two entries lying at the same level in the first column and there are 4 entries in column 4. Then $\pi_{NE} = \pi_{SE} = \varphi_1$ just removes the single $s_{\bar{1}}$ entry, and by lateral convexity $\pi_{dir}(w)$ contains a hexagon.

Third, if there exist two entries in column 1 and there are fewer than 4 entries in column 4, then at least one of the points labeled x or x' is not in the heap:



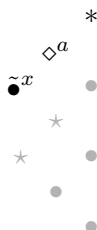
By Lemma 5.1, we have that w is non-Deodhar if and only if w^{-1} is non-Deodhar. Hence, by Lemma 8.4 we may assume that the entry x' is not in the heap, by taking a non-Deodhar mask on w^{-1} if necessary, whose heap is obtained from $Heap(w)$ by flipping the diagram upside down. In this case, we choose the projection $\pi_{NE}(w)$ so $Heap(\pi_{NE}(w))$ is obtained from $Heap(w)$ by shaving off the maximal NE diagonal up to x' .

We need to verify that the non-Deodhar bound holds on the mask restricted to $\pi_{NE}(w)$. By convexity, there are at most three occupied positions in the NE diagonal as in the figure above since x' is not in the heap. There cannot be a defect for w in column 1 since the defect would have no critical generator so there can be at most two defects along the maximal NE diagonal. If there exists a defect whose left string touches either entry in the first column, then both entries in the first column must have mask value 0, for otherwise the left string of the defect would exit the heap without crossing the right string, or be labeled negatively. There could be a single defect on the NE diagonal in column 3 whose left string does not touch the entries in column 1, but then it must intersect a plain zero along the NE diagonal in column 2. Therefore, shaving off the NE diagonal removes at least as many plain-zeros as zero-defects, so the Deodhar bound is preserved.

Case $k(w) = 2$. By Lemma 6.7, whenever the first column contains entries on more than one distinct level, we must have that each level contains a unique entry, and the entries alternate between the s_1 and $s_{\bar{1}}$ generators. Without loss of generality, we can assume the top entry in column 1 is $s_{\bar{1}}$.

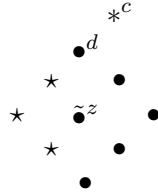
First, suppose $k(w) = 2$ and there is a 4-stack in column 3. By the Shape Lemma 8.5, there is a hexagon in $Heap(\pi_{dir}(w))$.

Second, suppose that there are at most 3 entries in column 3. Then we can further assume that the maximal NE diagonal from the top entry in column 1 contains at most two entries by taking a non-Deodhar mask on w^{-1} if necessary using Lemma 5.1 and Lemma 8.4. The projection $\pi_{NE}(w)$ shaves off the NE diagonal, so we need to show that this removes at least as many plain-zeros as zero-defects to preserve the bound. There cannot be a defect in column 1, because the defect would have no critical generator, so there can be at most one defect in the maximal NE diagonal, located in position a :



If x has mask value 1, then the left string of a is negatively signed while the right string has a positive label, in which case a is not a defect by Lemma 6.1. Hence, if a is a zero-defect, then x must be a plain-zero so removing both x and a from the heap will preserve the non-Deodhar bound.

Case $k(w) = 3$. Suppose that w has entries on 3 distinct levels in the first column and assume without loss of generality that the top entry corresponds to s_1 . The projection $\pi_{NE} = \pi_{SE}$ adds the three entries marked as \star to the left of $Heap(w)$:



If w contains a 4-stack in column 2, then $\pi_{NE}(w)$ contains a hexagon by the Shape Lemma 8.5. Hence, we can assume there are at most 3 entries in column 2. By considering w^{-1} if necessary, and applying Lemma 5.1 and Lemma 8.4, we can assume that the point marked c in column 2 is not in the heap.

Fix a non-Deodhar mask for w . Since there are exactly three alternating entries in the first column, there is at most a single defect in the first column, and it must occur in position d because any defect requires a critical generator below it. If d is not a defect then extend the mask to $\pi_{NE}(w)$ by setting the mask-values of the three new points to 1:

(8.3)

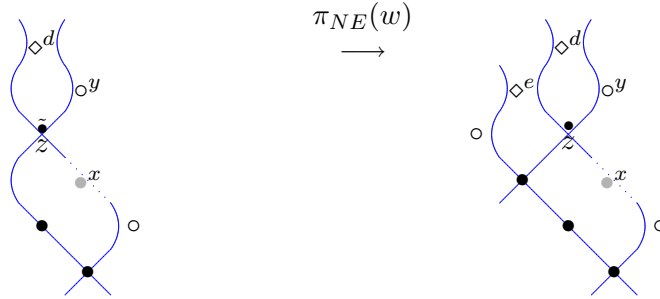
If a string passes through point z in $Heap(w)$, it changes sign and so it cannot be the left string of any defect. Therefore the mask assignment in (8.3) preserves all zero-defects and plain-zeros, hence it preserves the non-Deodhar bound.

Now, suppose that d is a defect in the first column. Then we have the following heap fragment where the critical generator below d must have mask-value 1, but the mask-values of entries x , y and z are variable:



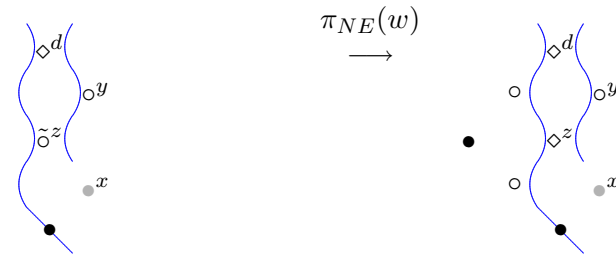
Breaking into cases on the mask-values of y and z , we demonstrate how to extend the given non-Deodhar mask of w to a non-Deodhar mask of $\pi_{NE}(w)$.

If z has mask-value 1, then y must have mask-value 0 or else the right string for d will be labeled positively, while the left string will be labeled negatively, contradicting that d is a defect. In this case, the strings for d cross at z , and both become negatively labeled, so must cross once more below z . Thus, we can set the mask-values of the additional entries to the left as shown:



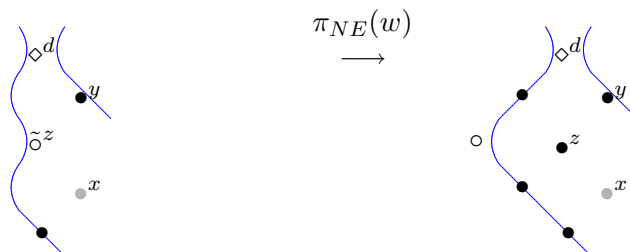
The string dynamics show that d and e are defects in $Heap(\pi_{NE}(w))$. The new mask has one additional plain-zero and zero-defect so the non-Deodhar bound is maintained.

If both y and z have mask-value 0, then the strings for d must cross below x in the heap poset. In this case, we can set the mask-values of the new entries as shown:



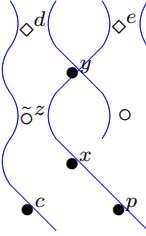
The string dynamics for d are preserved in the new mask, so d remains a defect, and z becomes a defect since it has exactly the same string dynamics as d . Note that z is not a defect for w since it corresponds to the $s_{\bar{1}}$ generator in $Heap(w)$. This mask is non-Deodhar because the original one was; we've changed a plain zero to a defect and added two plain zero entries, maintaining the non-Deodhar bound.

If z has mask-value 0 and y has mask-value 1, then either there is another defect e whose left string touches z , or there is not. If not, we can just move the zero at z to the left to preserve the string dynamics for d and extend the mask to $\pi_{NE}(w)$ as shown:



This mask is non-Deodhar assuming the original one was since we have not changed the number of zero-defects or plain-zeros.

On the other hand, if z is touched by the left string of a defect at or above e , then we claim $Heap(w)$ also contains an entry p below x in column 3. Hence, the heap contains an I-shape with corners c, d, e and p , so the Shape Lemma 8.5 implies that $\pi_{NE}(w)$ is non-Deodhar. For example,



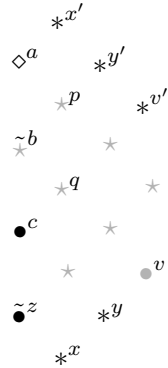
To verify the existence of $p \in Heap(w)$, observe that the highest possible crossing for the strings of d is at the point c , and consider two cases. If the strings of d do not cross at c , they must cross at a point below c and p so by convexity $p \in Heap(w)$. For the strings of d to cross at c , the point x must have mask-value 1 and the right critical zero of d is located as shown, in which case the strings for the other defect e cross at or below the point p , so again $p \in Heap(w)$.

Thus, for all possible mask-values of x, y the projection π_{NE} preserves the non-Deodhar condition, finishing the case $k(w) = 3$.

Case $k(w) = 4$. Suppose that w has entries on 4 distinct levels in the first column.

First, we reduce to the case where w contains a decorated heap fragment of the form:

(8.4)



Then we study cases corresponding to the mask values in the gray star positions.

If there is a point at x or x' , then there is a 4-stack in the second column. If there is a point at y or y' , we obtain an I-shape with the other entries that exist by convexity. In either case, the Shape Lemma 8.5 implies $\pi_{dir}(w)$ contains a hexagon. Similarly, if both v and v' exist, then we obtain an I-shape, so at least one of them is not in the heap. By considering w^{-1} if necessary, we can assume that v' is not in the heap.

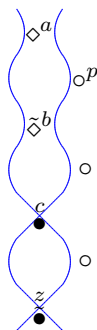
By Definition 8.3 for π_{dir} , we must first choose one of the extremal diagonals to shave, and then additively resolve the remaining heap fragment, in such a way that the non-Deodhar bound is maintained. Hence, choosing a projection π_{NE} or π_{SE} really just amounts to choosing the top or bottom entry from the first column to remove in such a way that the non-Deodhar bound is maintained. Once we choose the entry and shave it, we will have a non-Deodhar heap with only three entries in the first column, so we can appeal to the previous case $k(w) = 3$ when we apply π_{dir} the second time.

If z has mask-value 0, then it can play no critical role in any defect: it cannot be a defect itself since there is no critical generator below, and it cannot be a critical-zero for a defect since both strings meeting at z leave the heap below. Therefore, we choose to shave z in this case which maintains the non-Deodhar bound.

Hence, we can assume that z has mask-value 1 and that it is a string crossing for some zero-defect, since we can again remove it if it is not. Note that if a is not a zero-defect, we can choose to shave it without changing the Deodhar bound. Therefore, we can also assume that a is a zero-defect, which forces c to have mask-value 1 since it must be the critical generator for a . Thus, we have a heap fragment of the form in (8.4).

Suppose that b is a plain zero. Then, the strings that cross at z must both be labeled negatively everywhere above z since there is no other active $s_{\bar{1}}$ generator to change the signs of the strings. Therefore, no defect above z can be removed by removing z since the strings remain in increasing order as they pass through z and all defects in this case must correspond with one of the type A generators, s_1, s_2, \dots, s_{n-1} . Hence, we can shave z without affecting the defect status of any defect whose strings cross at z , and so we maintain the Deodhar bound in this case.

Next, suppose b is a zero-defect. Then its strings must cross at z since this is its critical generator, and there are no lower entries. Moreover, the right critical zero for a must occur at p , or the strings for a will never cross. Hence, the paths of the strings from b are completely prescribed and we have the heap fragment:



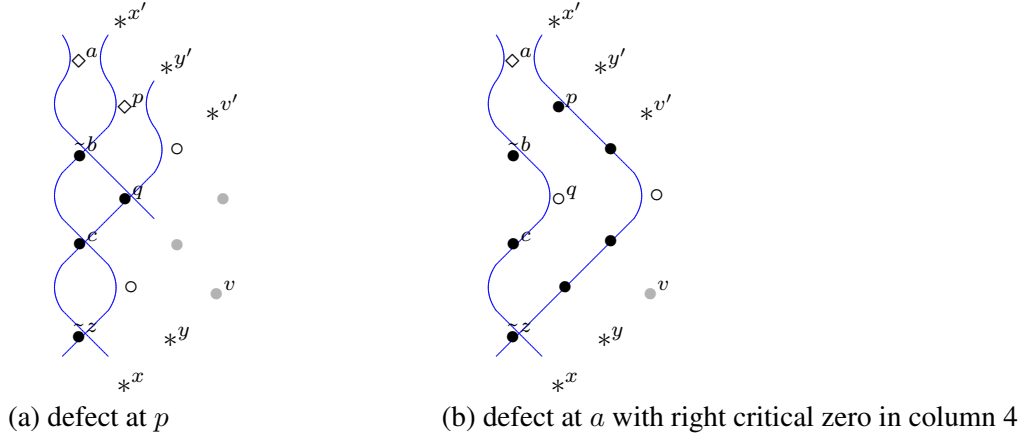
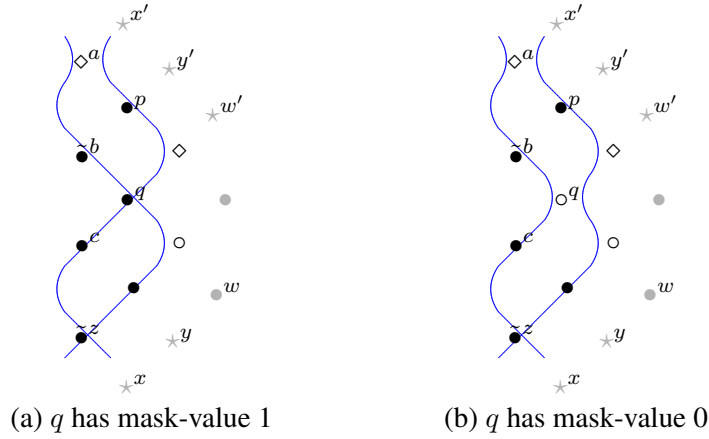
Since p cannot be a defect (because it lacks a critical generator), and it cannot enable a left string crossing for any defect above p (or we introduce an I-shape), removing a and changing the mask-value at p to 1 maintains the Deodhar bound.

Finally, suppose that b has mask-value 1, and consider the possible locations of the zero-defect whose strings cross at z ; call it d_z . If d_z is located on the NE diagonal from c , then its left string is labeled $\bar{2}$, which is a contradiction. If d_z is located at p , then the remaining mask values are forced, and in particular, q must have mask-value 1 as shown in Figure 2(a). Then, we obtain the contradiction that a cannot be a defect in this case since its left string is labeled $\bar{2}$, while its right string is labeled 1. Thus $d_z = a$ is the only viable possibility.

Suppose $d_z = a$. If the right critical zero for a occurs in column 4, then the mask-values are forced and in particular q must be a plain-zero as shown in Figure 2(b). Hence, we can shave a and change the mask-value of q to 1, since q is not itself a defect, nor can any other defect use q to effect a string crossing. This choice of projection preserves the Deodhar bound.

If the right critical zero for a is located in column 2 or 3 and it is a plain-zero, then we can shave a and change the mask-value of the right critical zero to 1, since no other defect can use the right critical zero of a to effect a string crossing. Hence, we may assume that the right critical zero of a is itself a zero-defect.

If the right critical zero for a is in column 3, then we have one of the cases shown in Figure 3. Once


 FIGURE 2. Locations of defects with strings crossing at z

 FIGURE 3. a has a right-critical zero that is a defect in column 3

we choose a mask-value for q , the rest of the mask-values are determined. In each case, the right critical zero for a cannot be a defect because its strings do not cross, which is a contradiction.

If the right critical zero for a is in column 2, then we have the cases shown in Figure 4. If q has mask-value 1 then the strings for a cannot cross at z , and if q has mask-value 0, then the right critical zero for a cannot be a defect because its strings do not cross.

Thus, in all cases there is a choice of projection π_{NE} or π_{SE} which preserves the non-Deodhar condition.

Case $k(w) = 5$. Suppose that w has entries on 5 or more distinct levels in the first column. Then, w contains the pattern HEX_4 and so the projections $\pi_{dir}(w)$ contain a hexagon.

We have shown in all cases that there is a choice of projection $\pi_{NE}(w)$ or $\pi_{SE}(w)$ that remains non-Deodhar if w is non-Deodhar, concluding the proof. \square

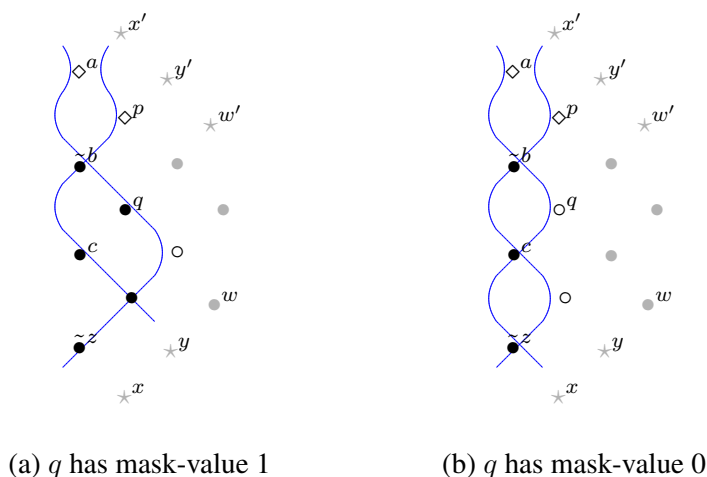


FIGURE 4. a has a right-critical zero that is a defect in column 2

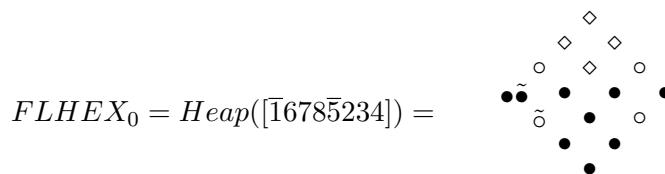
9. PROOF OF THE TYPE D CHARACTERIZATION THEOREM

In this section, we complete the classification of the minimal non-Deodhar embedded factors for type D .

Theorem 9.1. *Suppose w is a short braid avoiding, type D element that is not convex. Then, w is non-Deodhar if and only if w contains $[\bar{1}678\bar{5}234]$ as a 1-line pattern.*

Proof. Since w is not convex, we have that $Heap(w)$ contains a minimal pair of entries that require a left resolution.

Suppose w is non-Deodhar. By Lemma 6.8, $Heap(w)$ must have an $s_1s_{\bar{1}}$ entry. By Lemma 6.10, w contains a convex non-Deodhar element \tilde{w} as a 1-line pattern. Moreover, the construction of \tilde{w} preserves the $s_1s_{\bar{1}}$ entry. The only convex minimally non-Deodhar heap from Theorem 8.1 that has an entry with an $s_1s_{\bar{1}}$ entry is:



Hence, \tilde{w} must contain a $FLHEX_0$ factor. Note the relative order of the 8 strings passing through a $FLHEX_0$ gives rise to a $[\bar{1}678\bar{5}234]$ pattern in \tilde{w} . The relative order of these strings cannot change due to multiplication by additional generators above or below in a short braid avoiding heap since any additional adjacent transposition applied on the left or right of $[\bar{1}678\bar{5}234]$ would create a short braid. Therefore, both \tilde{w} and w contain $[\bar{1}678\bar{5}234]$ as a 1-line pattern.

Conversely, suppose that w is a short braid avoiding type D element that contains $[\bar{1}678\bar{5}234]$ as a 1-line pattern. We will reduce to the case where $w_1 = \bar{1}$ and the values in the pattern representing $678\bar{5}234$ occur consecutively in the 1-line notation for w . Then, we will be able to give a reduced factorization for w that contains a $FLHEX_k$ which is known to be non-Deodhar; see Example 5.11.

Consider the 1-line notation for w and highlight a pattern instance for $[\bar{1}678\bar{5}234]$ on values $a_1 < a_2 < \dots < a_8$ and in positions $p_1 < \dots < p_8$:

$$w = [\dots \bar{a}_1 \dots a_6 \dots a_7 \dots a_8 \dots \bar{a}_5 \dots a_2 \dots a_3 \dots a_4 \dots]$$

Since all 5 of the signed permutations

$$[\overline{123}], [21\overline{3}], [\overline{132}], [\overline{231}], [\overline{312}]$$

with at least one ascent and all entries negative are forbidden by Theorem 7.3, we have that all of the entries besides $\overline{a_1}, \overline{a_5}$ must be positive. In addition, since $[\overline{123}], [\overline{21\overline{3}}]$ and $[\overline{2\overline{31}}]$ are all forbidden by Theorem 7.3, we have that $a_1 = 1$. Hence, if we let $i = p_1$ then w contains $w' = ws_{i-1}s_{i-2}\cdots s_1$ as a factor, and w' also contains the 1-line pattern $[\overline{1678\overline{5}234}]$ with $w'_1 = \overline{1}$. Therefore, we only need to consider the case when $w_1 = a_1 = \overline{1}$.

Next, if there exist entries in w between $\overline{a_5}$ and a_2 , let $t = p_6 - 1$ be the position just to the left of a_2 . By Theorem 7.3, $a_8 > w_t > a_2$ would imply a forbidden $[321]$ pattern, so either

- (1) $w_t > a_8$ in which case $w' = ws_t$ also has the pattern and is a factor of w , or
- (2) $1 < w_t < a_2$ in which case we can change the highlighted pattern to choose a pattern instance where the a_2 is closer to the $\overline{a_5}$.

Therefore, we only need to consider the case when $\overline{a_5}a_2$ are consecutive. Furthermore, by the same argument in which the role of a_2 is replaced by a_3 and a_4 , respectively, we can assume $\overline{a_5}a_2a_3a_4$ are consecutive in w . Note also that all of the entries in positions $> p_8$ must have values $> a_4$, for otherwise we obtain a forbidden $[321]$ instance, with $a_8 > a_4$.

By Theorem 7.3, w avoids the patterns $[21\overline{3}], [31\overline{2}], [32\overline{1}]$ so the values in w between the $\overline{1}$ and $\overline{a_5}$ are increasing positive numbers. By changing the highlighted pattern instance if necessary, we can assume the values $a_6a_7a_8$ are the biggest three among these so $a_6a_7a_8\overline{a_5}$ are in consecutive positions. Consider the value in position $p_2 - 1$, just to the left of a_6 . If $a_5 > w_{p_2-1} > a_2$ then w contains the 1-line pattern $[\overline{2\overline{31}}]$ which is forbidden by Theorem 7.3. If $w_{p_2-1} > a_5$ then we move the a_8 out of the pattern to the right, and change the highlighted pattern so that w_{p_2-1} becomes a_6 , a_6 becomes a_7 , and a_7 becomes a_8 . Specifically, if $i = p_4$ then w contains $w' = ws_i s_{i+1} s_{i+2} s_{i+3}$ as a factor and w' also contains $[\overline{1678\overline{5}234}]$ as a 1-line pattern in the manner described. Hence, we only need to consider the case when the entries in w between $\overline{1}$ and a_6 all have values less than a_2 .

Summarizing, we can assume the 1-line notation for w is of the form $[\overline{1} \dots a_6 a_7 a_8 \overline{a_5} a_2 a_3 a_4 \dots]$ with the following conditions:

- (1) The elements in the first dotted sequence are increasing, all with positive values less than a_2 .
- (2) All entries in the second dotted sequence have value greater than $a_4 > 0$.

When we draw the string diagram corresponding to these rules, we find that w contains $FLHEx_k$ as a factor.

Alternatively, we obtain a reduced factorization of w containing a non-Deodhar factor as follows. Let $1 < y_1 < \dots < y_j < a_5$ be the values that appear to the right of $\overline{a_5}$ in w . Note that a_2, a_3, a_4 are the smallest three of the y_i 's. Let u be the permutation with values a_6, a_7, a_8 moved left and consecutively stacked adjacent to a_5 and with values y_1, \dots, y_k moved right and consecutively stacked adjacent to a_5 , i.e.

$$u = [123 \cdots a_2 a_3 a_4 y_4 \cdots y_j a_5 a_6 a_7 a_8 \dots n].$$

Next, move the block $y_4 \cdots y_j$ across the block $a_5 a_6 a_7 a_8$ to get

$$u' = [123 \cdots a_2 a_3 a_4 a_5 a_6 a_7 a_8 y_4 \cdots y_j \dots n].$$

Then, we have

$$w = u' \cdot FLHEx_k \cdot v$$

for some permutation v that arranges the entries to the right of a_4 in their final order. Here $k = a_5 - 2 - j$ is the number of strings strictly between 1 and a_5 that remain between positions 1 and p_5 . The $FLHEx_k$ pattern represents the operation that starting from u' moves a_5 to position 2, shifts the block $a_2 a_3 a_4$ to the right of $a_6 a_7 a_8$, applies $s_1 s_{\overline{1}}$ to change the signs on 1 and a_5 in positions 1 and 2, and finally moves $\overline{a_5}$ back to position $k + 3$. \square

Theorem 9.2. *In type D , we have that w is Deodhar if and only if it avoids the 1-line pattern $[\bar{1}678\bar{5}234]$, and the six embedded factors*

- $s_1s_2s_1$ (short braid)
- $s_5s_6s_7s_3s_4s_5s_6s_2s_3s_4s_5s_1s_2s_3$ (HEX)
- $s_3s_4s_5s_6s_2s_3s_4s_5s_1s_2s_3s_4s_1s_2s_3$ (HEX_2)
- $s_4s_5s_6s_2s_3s_4s_5s_1s_2s_3s_4s_1s_2s_1$ (HEX_{3a})
- $s_1s_4s_5s_6s_2s_3s_4s_5s_1s_2s_3s_4s_1s_2$ (HEX_{3b})
- $s_3s_4s_5s_1s_2s_3s_4s_1s_2s_3s_1$ (HEX_4).

Proof. If w contains a short braid then w is not Deodhar by Corollary 5.3. If w is convex, then w is Deodhar if and only if it avoids the patterns $HEX, HEX_2, HEX_{3a}, HEX_{3b}, HEX_4$ by Theorem 8.1. If w is short braid avoiding and not convex, then w is Deodhar if and only if it avoids $[\bar{1}678\bar{5}234]$ by Theorem 9.1. □

10. DEODHAR ELEMENTS OF EXCEPTIONAL WEYL GROUPS

The minimal non-Deodhar patterns for the other finite Weyl groups are computed by software that implements a game of K. Eriksson described in [Eri95] and [BB05]. We give the minimal lists for each type that account for Coxeter graph isomorphisms and patterns contained in parabolic subgroups. For example, a Deodhar element of E_8 must avoid the short braid on each pair of non-commuting generators, the hexagon pattern of length 14 contained in the type A parabolic subgroup, the type D patterns, and all of the patterns in E_6 and E_7 , the former of which can be embedded in two different ways.

Theorem 10.1. *Below is the complete list of minimal non-Deodhar embedded factor patterns in the Weyl groups of type E . The only minimal non-Deodhar elements in types G_2 and F_4 are short braids.*

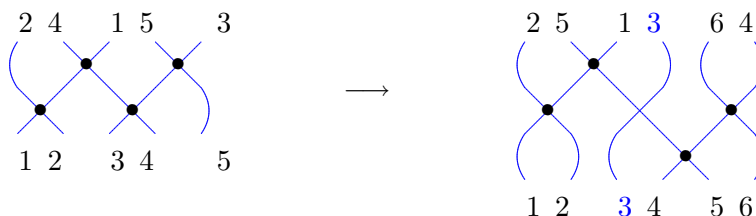
<i>Lie type</i>	<i>Coxeter graph</i>	<i>Reduced expression patterns</i>
E_6	<pre> ●5 ●0 — ●1 — ●2 — ●3 — ●4 </pre>	$s_0s_1s_2s_5s_3s_4s_2s_3s_1s_2s_5s_0s_1$ $s_5s_1s_2s_3s_0s_1s_2s_5s_4s_3s_2s_1s_0$ $s_1s_2s_5s_3s_4s_2s_3s_1s_2s_5s_0s_1s_2$ $s_2s_5s_1s_2s_3s_0s_1s_2s_5s_4s_3s_2s_1$
E_7	<pre> ●5 ●0 — ●1 — ●2 — ●3 — ●4 — ●6 </pre>	$s_0s_1s_2s_3s_4s_6s_5s_2s_3s_4s_1s_2s_3s_0s_1$ $s_3s_4s_6s_1s_2s_3s_0s_1s_2s_5s_4s_3s_2s_1s_0$ $s_1s_2s_3s_4s_6s_5s_2s_3s_4s_1s_2s_3s_0s_1s_2$ $s_2s_3s_4s_6s_1s_2s_3s_0s_1s_2s_5s_4s_3s_2s_1$ $s_5s_2s_3s_4s_6s_1s_2s_5s_3s_4s_2s_3s_0s_1s_2s_5$

Proof. These groups are finite, so this list of minimal patterns is verifiable by computer. The code used is available at <http://www.math.washington.edu/~brant/liberikson.html>. □

11. PATTERNS OF CONVEX ELEMENTS

In this section, we prove that the Deodhar property can be characterized by avoiding finitely many 1-line patterns in types A , B and D . In general, properties characterized by 1-line pattern avoidance are not equivalent to properties characterized by embedded factor avoidance, even if the embedded factors are convex, as seen in the example below. However, we describe a finite test for when we may translate between the two types of pattern avoidance on convex elements. Although it is not a direct generalization, this idea is related to a type A result of Tenner [Ten06b]. In [DJ07] the main result of this section has been extended for type A to cases where the elements may not be convex.

Example 11.1. Consider the subset of permutations $S_n(p)$ avoiding $p = s_1s_3s_2s_4 = [24153]$ as an embedded factor. Then, the element $w = s_4s_1s_3s_5s_2 = [251364]$ avoids p as an embedded factor, so $w \in S_n(p)$, yet it contains p as a 1-line pattern. The string diagrams below depict how the extra string is added to p to get w :



Let $W^C = \bigcup_{n \geq 1} W_n^C$ denote the convex elements in one of the classical families of irreducible Weyl groups, type A , B or D . Let $W^C(p)$ be the subset of the convex elements in W that is characterized by avoiding a single embedded factor pattern p .

Let $r(p)$ be the rank of the Weyl group containing p , and let $U^C(p)$ be the set of all convex elements in $W_{r(p)}$ that contain p as a factor, i.e. $U^C(p)$ are the convex elements in the upper order ideal generated by p in the two-sided weak order on $W_{r(p)}$. We will show that when p satisfies an additional hypothesis called the *ideal* condition, avoiding p as an embedded factor is equivalent to avoiding the elements of $U^C(p)$ as 1-line patterns. To carry this out, we will make frequent use of the linearized, coalesced heap and string diagrams on connected, convex elements.

We say that a Coxeter embedding on W *reverses orientation* if the labels on the corresponding linear Coxeter graph are reversed under the embedding. Otherwise, we say that it *preserves orientation*. If w contains an embedded factor p under an orientation preserving Coxeter embedding then we say that w contains p as an *orientated embedded factor*.

Proposition 11.2. *If $w \in W^C$ contains $p \in W^C$ as an oriented embedded factor, then w contains an element of $U^C(p)$ as a 1-line pattern.*

Proof. By Lemma 7.12, we have that $Heap(w)$ contains a shifted copy of $Heap(p)$ as a saturated set of lattice points. Furthermore, we can build $Heap(w)$ from the shifted copy of $Heap(p)$ by sequentially adding lattice points that are maximal or minimal with respect to the intermediate heap as follows. Suppose that the shifted copy of $Heap(p)$ occupies columns $i, i + 1, \dots, k$ in $Heap(w)$. Then, we begin with the set of strings $S = \{i, i + 1, \dots, k + 1\}$ that appear in the shifted copy of $Heap(p)$. These strings initially correspond to the 1-line pattern p , and we show by induction that S continues to encode a 1-line pattern from U_p as we add minimal or maximal lattice points to the heap.

Consider the relative order of the strings in S when we add a maximal lattice point in column j to the heap. If the new point crosses a pair of strings that are both in S then the new string configuration on S corresponds to an element in the upper order ideal $U^C(p)$. If the new point crosses a pair x, y of strings such that at most one is contained in S then the string configuration on S is unchanged. Similarly,

the relative order of the strings in S corresponds to an element in $U^C(p)$ when we add a minimal lattice point in column j .

Hence, at the end of this inductive construction $Heap(w)$ contains the 1-line pattern encoded by the strings in S , and the element corresponding to this 1-line pattern contains p as a factor. \square

Example 11.1 shows that the converse of Proposition 11.2 can fail in general. However, on the special patterns defined below a converse can be stated.

Definition 11.3. We say that $p \in W^C$ is an *ideal embedded factor pattern* if for every $q \in W_{r(p)+1}^C$ containing p as a 1-line pattern, we have that q contains p as an oriented embedded factor.

Proposition 11.4. *If $p \in W^C$ is an ideal embedded factor pattern and $w \in W^C$ contains p as a 1-line pattern, then w contains p as an oriented embedded factor.*

Proof. Consider the case that $w \in W_{r(p)+1}$. By Definition 11.3, we have that w contains p as an oriented embedded factor. By Lemma 7.12, this implies that $Heap(w)$ contains $Heap(p)$ as a saturated subset, so we can highlight an instance of $Heap(p)$ inside $Heap(w)$.

Now by induction, assume the proposition holds for all convex elements in $\cup_{k=1}^n W_k^C$ and let $w \in W_{n+1}^C$. Then if w contains p as a 1-line pattern then w contains some $p' \in W_n^C$ that also contains p as a 1-line pattern. By induction, $Heap(p')$ contains a shifted copy of $Heap(p)$ as a saturated subset of lattice points, we want to show that $Heap(w)$ must also contain a copy of $Heap(p)$.

The string diagram imposed on $Heap(w)$ can be obtained from the string diagram on $Heap(p')$ by adding one additional string. The additional string will add extra points to the heap at each crossing. This string may cut through the copy C of $Heap(p)$, but since p is ideal, the extra points added along with C must also contain a shifted copy of $Heap(p)$ as a saturated subset of lattice points by Definition 11.3. Therefore by Lemma 7.12, w contains p as an oriented embedded factor. \square

Thus combining Proposition 11.2 and Proposition 11.4, we have shown the following result.

Theorem 11.5. *Suppose P is the subset of W^C characterized by avoiding a finite combination of oriented convex embedded factors F and 1-line patterns G . If each of the elements in $F' = \bigcup_{p \in F} U^C(p)$ is an ideal pattern, then P is characterized by avoiding the permutations in $F' \cup G$ as 1-line patterns.*

Corollary 11.6. *Under the hypotheses of Theorem 11.5, there is a polynomial time algorithm available to test an element of W^C for membership in P .*

Remark 11.7. Note that the existence of a polynomial time algorithm is not evident from the embedded factor version of the characterization, because a typical element can have exponentially many reduced expressions by [Sta84].

Remark 11.8. Recall that the fully commutative elements in types A and B are automatically convex. In particular, the theorem applies to properties on [321]-avoiding permutations that are characterized by avoiding finitely many embedded factors. Similarly, any pattern class that includes the fully commutative basis elements from Theorem 7.3 together with $[\bar{1}2\bar{3}]$ is convex in type D .

Corollary 11.9. *There exist a finite number of patterns of rank less than 9 in each Weyl group family of types A, B, D that characterize the Deodhar elements. Therefore, there exists an $O(n^8)$ test for the Deodhar condition in all finite Weyl groups of rank n .*

Proof. It is straightforward to verify that the embedded factor patterns from Theorem 8.1 are ideal using a computer. The corollary then follows from Theorem 11.5. \square

We have verified by computer that the embedded factor patterns characterizing the Deodhar elements correspond to 75 type D 1-line patterns.

It is not known how to define a pattern system so that a finite characterization of the Deodhar condition in other Coxeter groups may be obtained, but Example 5.11 shows that something stronger than embedded factor containment is required in general.

Question 11.10. *Can Deodhar elements in other Coxeter groups be characterized by avoiding a finite number of root subsystem patterns?*

12. TOWARD ENUMERATING DEODHAR ELEMENTS

Stankova and West [SW04] found a homogeneous linear recurrence relation with constant coefficients that gives the number of 321-hexagon avoiding permutations.

Theorem 12.1. [SW04] *The number c_n of 321-hexagon-avoiding permutations in S_n satisfies the recurrence*

$$c_{n+1} = 6c_n - 11c_{n-1} + 9c_{n-2} - 4c_{n-3} - 4c_{n-4} + c_{n-5}$$

for all $n \geq 6$ with initial conditions $c_1 = 1$, $c_2 = 2$, $c_3 = 5$, $c_4 = 14$, $c_5 = 42$, $c_6 = 132$.

Also, Vatter [Vat05] has obtained an enumeration scheme for these elements automatically using the WilfPlus Maple package. Do elegant enumerative formulas such as this exist for counting the Deodhar elements in type D ?

Figure 5 shows the number of Deodhar elements for the finite Weyl groups as a fraction of the fully commutative elements. The latter were enumerated by [Ste98].

Type/Rank	2	3	4	5	6	7	8
$A/B/G/F$	5	14	42	132	429	1426/ 1430	4806/ 4862
D			48	167	575/ 593	1976/ 2144	6791/ 7864
E					642/ 662	2341/ 2670	8305/ 10846

FIGURE 5. Enumeration of Deodhar elements

It would be interesting if one could find an enumerative formula for the number of Deodhar elements in type D . More generally, what can be said about enumerating families avoiding a list of embedded factor patterns?

It is possible to show that in type A , the embedded factor pattern classes satisfy a Stanley–Wilf bound using a theorem of Tenner.

Theorem 12.2. [Ten06b] *If $p \in S_k$ avoids [2143] and $w \in S_n$ contains p as a 1-line permutation pattern, then w contains p as an oriented embedded factor.*

We denote the set of elements from S_n that avoid p as an embedded factor by $S_n(p)$, and the set of elements from S_n that avoid $[p]$ as a 1-line permutation pattern by $S_n[p]$.

Corollary 12.3. *For all permutations p , there exists a constant $c = c_p$ such that $|S_n(p)| \leq c^n$.*

Proof. We first show how to construct a permutation $q \in S_k$ that contains p as a factor, but avoids [2143]. Suppose $p \in S_k$ contains a [2143] instance. Begin by choosing the leftmost position i in the 1-line notation for p from the set of all positions that play the role of 4 in any [2143] instance of p . Then, we can multiply p on the right by the adjacent transposition s_{i-1} to move p_i one position to the left. Note

that $p_{i-1} < p_i$ or else we could have chosen p_{i-1} to play the role of 4 in any [2143] instance in which p_i participates, contradicting that p_i was chosen to be leftmost.

By continuing to move the entry p_i to the left in a reduced fashion, we can eventually move it past the leftmost entry that plays the role of 1 in any [2143] instance in which p_i plays the role of 4. Having removed all of the [2143] instances where the entries that play the role of 4 occur in positions $\leq i$, we choose the next leftmost position that plays the role of 4 in some [2143] instance and repeat the argument. The resulting element q contains p as a factor, and contains no [2143] instances.

Hence, if w contains q as an embedded factor, then it contains p as an embedded factor, so contrapositively $S_n(p) \subset S_n(q)$ and by Theorem 12.2 we have $S_n(q) \subset S_n[q]$. We can apply the Marcus–Tardos Theorem [MT04] to $S_n[q]$ since it is expressed as a 1-line permutation pattern class, and we obtain the upper bound. \square

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