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MAXIMAL SINGULAR LOCI OF SCHUBERT VARIETIES IN $SL(n)/B$

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ABSTRACT. Schubert varieties in the flag manifold $SL(n)/B$ play a key role in our understanding of projective varieties. One important problem is to determine the locus of singular points in a variety. In 1990, Lakshmibai and Sandhya showed that the Schubert variety X_w is nonsingular if and only if w avoids the patterns 4231 and 3412. They also gave a conjectural description of the singular locus of X_w. In 1999, Gasharov proved one direction of their conjecture. In this paper we give an explicit combinatorial description of the irreducible components of the singular locus of the Schubert variety X_w for any element $w \in \mathfrak{S}_n$. In doing so, we prove both directions of the Lakshmibai-Sandhya conjecture. These irreducible components are indexed by permutations which differ from w by a cycle depending naturally on a 4231 or 3412 pattern in w. Our description of the irreducible components is computationally more efficient ($O(n^6)$) than the previously best known algorithms, which were all exponential in time. Furthermore, we give simple formulas for calculating the Kazhdan-Lusztig polynomials at the maximum singular points.

1. INTRODUCTION

Schubert varieties play an essential role in the study of the homogeneous spaces G/B for any semisimple group G and Borel subgroup B; every closed subvariety in G/B can be written as the union of Schubert varieties, the classes of Schubert varieties form a basis for the cohomology ring of G/B and the Schubert varieties correspond to the lower order ideals of a partial order associated to G/B. Specifically, this Bruhat order is an order on the T-fixed points in G/B where T is the maximal torus in B. The T-fixed points, e_w, correspond bijectively with elements in the Weyl group $W = N(T)/T$ of G and T. A tremendous amount of information about a Schubert variety can be obtained by examining the corresponding Weyl group element. Our main theorem gives a simple and efficient method for giving the irreducible components of the singular locus of a Schubert variety.[1]

In the late 1950's, Chevalley [13] showed that all Schubert varieties in G/B are nonsingular in codimension one. Since that time, many beautiful results on determining singular points of Schubert varieties have surfaced (see [3]). By definition, the Schubert variety X_w is the closure of the B-orbit of e_w. Therefore any point

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[1]While in the process of preparing this submission, the authors learned that Cortez [14], Kassel-Lascoux-Reutenauer [19] and Manivel [27] have each recently independently proved a theorem equivalent to Theorem 1.

$p \in X_w$ is singular if and only if all points in the orbit Bp are singular. Since the singular locus of a variety is closed, the singular locus of X_w is a union of Schubert varieties indexed by the maximal elements $v < w$ such that e_v is singular in X_w.

Let $\mathrm{maxsing}(X_w)$ denote the maximal set of Weyl group elements corresponding to singular points in X_w in Bruhat order, i.e. X_v is an irreducible component of the singular locus of X_w if and only if $v \in \mathrm{maxsing}(X_w)$. The goal of this paper is to give an explicit algorithm for finding $\mathrm{maxsing}(X_w)$ in the case where G is $SL_n(\mathbb{C})$, B is the set of invertible upper triangular matrices, T is the set of invertible diagonal matrices, and W is the symmetric group $\mathfrak{S}_n$. The algorithm we present is very efficient, $O(n^6)$, and removes the need to search through all nonsingular T-fixed points (as is the case with previously known techniques).

In type A (i.e., $G = SL(n)$), smoothness is equivalent to rational smoothness ([15], see also [12] in the case of ADE) so the maximal singular locus of X_w also determines the maximal permutations $x \leq w$ for which the corresponding Kazhdan-Lusztig polynomial is different from 1. We use the explicit form of $\mathrm{maxsing}(X_w)$ to compute all Kazhdan-Lusztig polynomials at maximal singular points (MSP's); they are either $1 + q + \cdots + q^k$ or $1 + q^k$ depending on whether the corresponding bad pattern is 4231 or 3412 (respectively). These formulas have also been found by Manivel [26] and Cortez [14].

2. Main results

In 1990, Lakshmibai and Sandhya [21] showed that the Schubert variety $X_w \subset SL(n)/B$ is smooth at every point if and only if the permutation matrix for w does not contain any 4×4 submatrix equal to 3412 or 4231. We use these two permutation patterns to produce the maximal permutations below w which correspond to points in the singular locus. This verifies the conjecture stated in [21] on the singular locus of X_w. (Gasharov, using a map similar to the one we introduce in Section 6, shows in [17] that the points constructed in [21] are singular. His result proves one direction of this conjecture.) In fact, our proof starts from an arbitrary maximal singular T-fixed point e_x in X_w and shows that w must contain a 4231 or 3412 pattern and x must contain a 2143 or 1324 pattern (respectively).

The main theorem below shows that elements of $\mathrm{maxsing}(X_w)$ are obtained by acting on w by certain cycles. These cycles are best absorbed graphically in terms of the permutation matrices $\mathrm{mat}(x)$ and $\mathrm{mat}(w)$. Examples are shown in Figure 1.

Theorem 1. *X_x is an irreducible component of the singular locus of X_w if and only if*

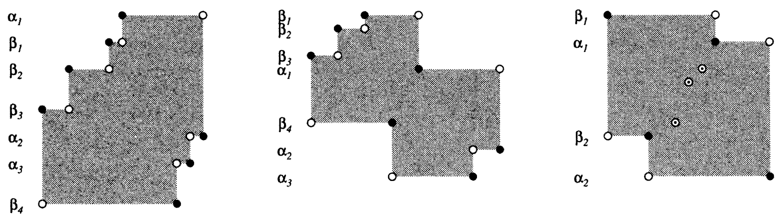
(1) $x = w \circ (\alpha_1, \ldots, \alpha_m, \beta_k, \ldots, \beta_1)$ *for disjoint sequences*

$$1 \leq \alpha_1 < \cdots < \alpha_m \leq n, \text{ with } w(\alpha_1) > \cdots > w(\alpha_m), \text{ and}$$
$$1 \leq \beta_1 < \cdots < \beta_k \leq n, \text{ with } w(\beta_1) > \cdots > w(\beta_k).$$

(2) *The permutation matrices of x and w differ in one of the three ways shown in Figure 1.*

(3) *The interiors of the shaded regions in Figure 1 do not contain any other 1's in the permutation matrix of w. In the third case, the 1's contained in the shaded region must form a decreasing sequence.*

After introducing basic notation in Section 3, we then introduce in Section 4 the pictorial characterization of the Bruhat order we rely on. In Sections 5 and



(1) 4231 Case *(2) 3412 Case* *(3) 45312 Case*

FIGURE 1. Possible differences between permutation matrices for $x \in \text{maxsing}(X_w)$ as described in Theorem 1. $\circ$'s denote 1's in $\text{mat}(w)$; $\bullet$'s denote 1's in $\text{mat}(x)$.

6, we discuss the Lakshmibai-Seshadri basis for the tangent space of a Schubert variety indexed by transpositions and the set $\mathcal{R}(x,w) = \{t : x < xt \leq w\}$. We also define a set of maps that allows us to relate $\mathcal{R}(x,w)$ and $\mathcal{R}(y,w)$ when x and y differ by a transposition. These maps will then allow us to investigate not only whether a point e_x is singular, but whether it is *maximally* singular. To describe those permutations $x \in \text{maxsing}(X_w)$, we show that related permutations $\widetilde{x}$ must, among other qualities, avoid the patterns 231, 312 and 1234. We complete the description of $\text{maxsing}(X_w)$ in Sections 8 and 9.

The remaining sections contain applications arising from our description of $\text{maxsing}(X_w)$. In Section 10, we prove the conjecture of Lakshmibai and Sandhya on the composition of $\text{maxsing}(X_w)$. Using the tools we have developed, in Section 11 we calculate the values of the Kazhdan-Lusztig polynomials at maximal singular points. In Section 12, we give some example calculations pertaining to the composition of $\text{maxsing}(X_w)$. Finally, in Section 13, we state a simple method for determining the number of elements in $\text{maxsing}(X_w)$ in terms of pattern avoidance and containment.

3. PRELIMINARIES

We begin by introducing our basic notation and terminology. Let $\mathfrak{S}_n$ denote the symmetric group on n letters. We will view elements of $\mathfrak{S}_n$ as permutations on $[1,\ldots,n]$. Let $w(i)$ be the image of i under the permutation w. We have a one-line notation for a permutation w given by writing the image of $[1,\ldots,n]$ under the action of w: $[w(1), w(2), \ldots, w(n)]$. We will also often utilize the permutation matrix for w (denoted $\text{mat}(w)$).

Let s_i denote the adjacent transposition interchanging i and $i+1$. Then $\mathcal{S} = \{s_i\}_{i\in[1,\ldots,n-1]}$ is the standard generating set for $\mathfrak{S}_n$ with relations $s_i^2 = 1$, $s_i s_j = s_j s_i$ for $|i-j| > 1$, and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $0 < i < n$. Let $\mathcal{T}$ denote the set of all *transpositions* in $\mathfrak{S}_n$. The elements of $\mathcal{T}$ are all the conjugates of elements in $\mathcal{S}$:

$$\mathcal{T} = \{t_{j,k} = s_j s_{j+1} \cdots s_{k-2} s_{k-1} s_{k-2} \cdots s_{j+1} s_j : 1 \leq j < k \leq n\}. \tag{3.1}$$

If we wish to refer to a transposition t that affects positions a and b, but the relative values of a and b are unknown, we will write $t_{\{a,b\}}$.

The *length* $l(w)$ of an element $w \in \mathfrak{S}_n$ is the minimum r for which we have an expression $w = s_{i_1} \cdots s_{i_r}$. A *reduced expression* $w = s_{i_1} \cdots s_{i_r}$ is an expression for which $l(w) = r$. It is a standard fact that

$$l(w) = \#\{1 \leq i < j \leq n : w(i) > w(j)\}. \tag{3.2}$$

Definition 2. Let $x, w \in \mathfrak{S}_n$, $p, q \in \mathbb{Z}$. Define the *rank function* for w by $r_w(p,q) = \#\{i \leq p : w(i) \geq q\}$. Also, the *difference function* for the pair x, w is defined by $d_{x,w}(p,q) = r_w(p,q) - r_x(p,q)$.

In $SL(n)/B$, Schubert varieties can be defined in terms of the rank function. To do this, fix a basis $\{u_1, u_2, \ldots, u_n\}$ of $\mathbb{C}^n$. This fixes a base flag $(F_1 \subset F_2 \subset \cdots \subset F_n)$ where F_i is the span of $\{u_n, \ldots, u_{n-i+1}\}$. We identify $SL(n)/B$ with the set of all complete flags of vector spaces $(V_1 \subset V_2 \subset \cdots \subset V_n \cong \mathbb{C}^n)$ with $\dim V_i = i$. Then

$$X_w = \{(V_1 \subset V_2 \subset \cdots \subset V_n) : \dim(V_p \cap F_{n-q+1}) \leq r_w(p,q)\}. \tag{3.3}$$

(This definition is equivalent to that given in [16].) Note that the flag

$$e_w = (\langle u_{w(n)} \rangle \subset \langle u_{w(n)}, u_{w(n-1)} \rangle \subset \cdots \subset \langle u_{w(n)}, \ldots, u_{w(1)} \rangle) \tag{3.4}$$

is an element of X_w. Furthermore, e_w is fixed by the left T action and X_w can be viewed as the closure of the orbit Be_w. Therefore, $X_v \subseteq X_w$ if and only if $e_v \in X_w$. This defines a partial order, called the Bruhat (or Bruhat-Chevalley) order, on $\mathfrak{S}_n$ by

$$v \leq w \Leftrightarrow X_v \subseteq X_w. \tag{3.5}$$

The Bruhat order has a number of characterizations (see, e.g., [18]). One of the most common definitions is as the transitive closure of the relations $vt < v$ for $t \in \mathcal{T}$ if $l(vt) < l(v)$. However, we prefer to work with a more graphical characterization which follows directly from the definition of the rank and difference functions above. The corresponding "Bruhat pictures" that we associate to each pair $x \leq w$ will be discussed in the next section. These pictures will rely on the two conclusions below.

Lemma 3. *We have $x \leq w$ if and only if $d_{x,w}$ is everywhere non-negative.*

Corollary 4. *If $x \leq y \leq w$, then $d_{x,w} - d_{y,w}$ is everywhere non-negative.*

The following fact about the Bruhat order will in useful throughout the text. An analogous left-handed version exists.

Lemma 5 ([18, 7.4]). *If $s \in \mathcal{S}$ and $ws < w$, then $xs \leq w \Leftrightarrow x \leq w$.*

The Bruhat graph of w is the graph with vertices labeled by $\{v \leq w\}$ and v_1 is joined to v_2 by a directed edge if $v_1 = v_2 t$ for some $t \in \mathcal{T}$ and $v_1 < v_2$ in Bruhat order. This graph plays a central role in the study of Schubert varieties. For example, Lakshmibai and Seshadri have shown that in $SL(n)/B$, the tangent space to X_w at e_x has a basis indexed by $\{t \in \mathcal{T} : xt \leq w\}$, i.e. the edges of the Bruhat graph adjacent to x. This fact forms the main criterion we will use in Section 5 for smoothness at a point. In fact, since $xt < x$ implies $xt < w$ we will just need to consider the edges "going up" from x in the Bruhat graph of w. This set will be denoted by

$$\mathcal{R}(x,w) := \{t \in \mathcal{T} : x < xt \leq w\}.$$

Over the last few years, it has become apparent that properties of the Bruhat order can often be efficiently characterized by "pattern avoidance" [2, 4, 5, 25, 30]. We say that $w = [w(1), \ldots, w(n)]$ *avoids the pattern* $v = [v(1), \ldots, v(k)]$ for $k \leq$

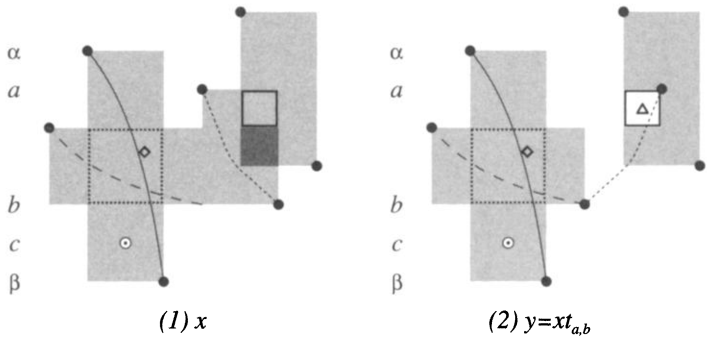


FIGURE 2. We see (among other facts) that $d_{x,w}(\diamond) \geq 1$, $d_{y,w}(\triangle) = d_{x,w}(\triangle) - 1 \geq 0$, $\mathrm{pt}_x(c) = \mathrm{pt}_w(c)$ and $t_{\alpha,\beta} \in \mathcal{R}(x, w)$.

n if we cannot find $1 \leq i_1 < \cdots < i_k \leq n$ with $w(i_1), \ldots, w(i_k)$ in the same relative order as $v(1), \ldots, v(k)$ — i.e., no submatrix of $\mathrm{mat}(w)$ on rows $i_1, \ldots, i_k$ and columns $w(i_1), \ldots, w(i_k)$ is the permutation matrix of v. Our characterization of the maximal singular locus is yet another example of the utility of this notion.

More generally, we can define pattern avoidance or containment in terms of the following flattening function. For any set $Z = \{z_1 < z_2 < \cdots < z_k\} \subseteq [1, \ldots, n]$, and $x \in \mathfrak{S}_n$, define $\mathrm{fl}_Z(x)$ to be the "flattened" permutation on $[1, \ldots, k]$ whose elements are in the same relative order as $[x(z_1), \ldots, x(z_k)]$. When the set Z is clear from context, we will abbreviate $\mathrm{fl}_Z(x)$ by $\overline{x}$. We will also write $\mathrm{fl}(i, j, \ldots, k)$ for the flattened permutation on the sequence $i, j, \ldots, k$ and write $x^{\hat{i}}$ for $\mathrm{fl}_{[1,\ldots,n]\setminus\{i\}}(x)$.

It will also be useful to have notation for an "unflattening" operator. Given a permutation $x \in \mathfrak{S}_n$, a set $Z \subseteq \{1, \ldots, n\}$ of cardinality k, and a permutation $u \in S_k$, we can define a new permutation $\mathrm{unfl}^x_Z(u) \in \mathfrak{S}_n$ by requiring that

(1) $\mathrm{fl}_Z(\mathrm{unfl}^x_Z(u)) = u$, and
(2) $x(a) = (\mathrm{unfl}^x_Z(u))(a)$ if $a \in \{1, \ldots, n\} \setminus Z$.

When x and Z are clear from context, we abbreviate $\mathrm{unfl}^x_Z(u)$ by $\widehat{u}$.

Example 6. For $x = [5, 2, 4, 1, 6, 3]$ and $Z = \{3, 5, 6\}$, we have $\mathrm{fl}_Z(x) = [2, 3, 1]$ and $(\mathrm{unfl}^x_Z)([3, 1, 2]) = [5, 2, 6, 1, 3, 4]$. Note that $x = \widehat{\mathrm{fl}_Z(x)}$.

4. BRUHAT PICTURES

The function $d_{x,w}$ affords us a graphical view of the Bruhat order. Most importantly, it lets us see the set $\mathcal{R}(x, w)$. We will now introduce the graphical notation utilized in the remainder of the paper that allows us to do this. A diagram displaying the notation we are about to describe is offered in Figure 2.

First, we plot, as black disks, all or some of the positions containing 1's in the permutation matrix $\mathrm{mat}(x)$ of x. We will sometimes overlay $\mathrm{mat}(x)$ and $\mathrm{mat}(w)$. In these cases, 1's in $\mathrm{mat}(w)$ will be marked by open circles. Points that are simultaneously in both diagrams will consist of a black disk and a larger concentric circle. Let $[a, b] \times [c, d]$ denote the set of all points $(p, q) \in \mathbb{R}^2$ such that $a \leq p \leq b$ and $c \leq q \leq d$. The following notation will be handy.

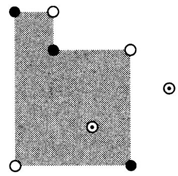


FIGURE 3. In the above configuration, $\mathcal{R}(x,w) = \{t_{1,2}, t_{2,4}, t_{2,5}, t_{4,5}\}$.

Definition 7. For $t_{p,q} \in \mathcal{R}(x,w)$, set

$$\mathcal{A}_{p,q} := \mathcal{A}_{p,q}(x) = [p+1, q-1] \times [x(p)+1, x(q)-1], \tag{4.1}$$

$$\overline{\mathcal{A}_{p,q}} := \overline{\mathcal{A}_{p,q}(x)} = [p,q] \times [x(p), x(q)], \tag{4.2}$$

$$\mathrm{pt}_x(c) := (c, x(c)) \text{ for } c \in [1, \ldots, n]. \tag{4.3}$$

Along with the points of $\mathrm{mat}(x)$, we will often shade parts of our diagram in order to specify that $d_{x,w}$ satisfies a particular inequality on a given region. Light shading on a region signifies that $d_{x,w} \geq 1$ on that region. Dark shading signifies $d_{x,w} \geq 2$. No shading places no restrictions on the values $d_{x,w}$. A region with a black border is one where $d_{x,w}$ achieves the minimum possible value allowed by the shading on that region. Dotted borders are used to demarcate regions we wish to discuss in the text. A solid or dotted curve connecting two points in $d_{x,w}$ will denote an element of $\mathcal{R}(x,w)$. A dotted curve will be used to designate t when we are particularly interested in $y = xt$. A dashed curve will be used when we wish to mark a reflection $t' \in \mathcal{R}(y,w)$. Of course, if $tt' \neq t't$, and our picture is of $d_{x,w}$, then only one of the endpoints of our dashed curve will correspond to a point of $d_{x,w}$.

As mentioned above, the great utility of these diagrams arises from being able to visualize $\mathcal{R}(x,w)$ along with the information on the Bruhat order. To see how we do this, suppose we have some reflection $t_{a,b} \in \mathcal{R}(x,w)$ (which implies $x < xt_{a,b} \leq w$). Now compare the shading (with respect to w) in $\mathrm{mat}(x)$ and $\mathrm{mat}(xt_{a,b})$. We see (as in Figure 2), that in the region $\mathcal{A}_{a,b}(x)$, $d_{xt_{a,b},w} = d_{x,w} - 1$. Hence, by Lemma 3, we can state the following:

Fact 8. Let $t_{a,b} \in \mathcal{T}$ with $x < xt_{a,b}$. The transposition $t_{a,b}$ is in $\mathcal{R}(x,w)$ if and only if it corresponds to a region on which $d_{x,w}|_{\mathcal{A}_{a,b}} \geq 1$. An example is given in Figure 3.

Note that the values of $d_{x,w}$ on the region $\overline{\mathcal{A}_{a,b}} \setminus \mathcal{A}_{a,b}$ are not considered in determining the membership of $t_{a,b}$ in $\mathcal{R}(x,w)$.

The following lemmas will be used several times in future sections. The first one allows us to infer the presence of points in $\mathrm{mat}(x)$ in a region based on a particular common pattern of shading.

Lemma 9. *Let $x < w$ and suppose $p, p', q, q' \in \mathbb{Z}$ such that*

(1) $p < p'$, $q < q'$,
(2) $d_{x,w}(p,q') = 0$,
(3) $d_{x,w}(p,q) = \alpha$, $d_{x,w}(p',q') = \beta$, $d_{x,w}(p',q) = \gamma$.

Then there exist at least $\alpha + \beta - \gamma$ values m such that $\mathrm{pt}_x(m) \in [p+1, p'-1] \times [q+1, q'-1]$ with $x(m) \neq w(m)$.

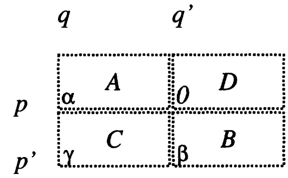


FIGURE 4. We have indicated certain values of $d_{x,w}$ at the lower left corner of each region.

Proof. Define regions A,B,C and D as in Figure 4. For $R \subset [1,n] \times [1,n]$, define

$$\Theta_{x,w}(R) = \#\{(p,q) \in R : q = w(p)\} - \#\{(p,q) \in R : q = x(p)\}. \tag{4.4}$$

Then, $d_{x,w}(p',q) = \Theta_{x,w}(A) + \Theta_{x,w}(B) + \Theta_{x,w}(C) + \Theta_{x,w}(D)$. So plugging in the specified values we have $\Theta_{x,w}(C) = -(\alpha + \beta - \gamma)$ and there are exactly $\alpha + \beta - \gamma$ more 1's of $\mathrm{mat}(x)$ than 1's of $\mathrm{mat}(w)$ in region C. $\square$

Lemma 10. *Let $u, w \in \mathfrak{S}_n$ and suppose $1 \leq i < j < k \leq n$ such that $\mathrm{fl}_{ijk}(u) = 123$. If both*

$$w \geq x = u \circ (k,j,i) \ \ (i.e.,\ \overline{x} = 312) \text{ and} \tag{4.5}$$

$$w \geq y = u \circ (i,j,k) \ \ (i.e.,\ \overline{y} = 231) \tag{4.6}$$

then $w \geq z = u \circ (i\ k)$ (i.e., $\overline{z} = 321$).

Proof. Notice that $d_{z,w}(p,\cdot) = d_{x,w}(p,\cdot)$ for $p < j$ and $d_{z,w}(p,\cdot) = d_{y,w}(p,\cdot)$ for $p \geq j$. By Lemma 3, $v \leq w$ if and only if $d_{v,w} \geq 0$. Since $x, y \leq w$, both $d_{x,w} \geq 0$ and $d_{y,w} \geq 0$. Combining this with our first observation implies that $z \leq w$. $\square$

5. A criterion for maximal smoothness

To prove Theorem 1, we start from the fact that (by definition) X_w is smooth at e_x if and only if the dimension of the Zariski tangent space at that point is equal to $l(w) = \dim(X_w)$. Lakshmibai and Seshadri, [22], describe the dimension of this tangent space in terms of the root system. Using the fact that $\#\{t \in \mathcal{T} : xt < x\} = l(x)$, we can paraphrase their result as:

Theorem 11 ([22]). *The Schubert variety $X_w \in SL(n)/B$ is smooth at e_x if and only if $\#\mathcal{R}(x,w) := \#\{t \in \mathcal{T} : x < xt \leq w\}$ equals $l(w) - l(x)$.*

This yields the following characterization of $\mathrm{maxsing}(X_w)$:

Fact 12. We have $x \in \mathrm{maxsing}(X_w)$ if and only if

(1) $\#\mathcal{R}(x,w) > l(w) - l(x)$ and
(2) for all $t \in \mathcal{R}(x,w)$, $\#\mathcal{R}(xt,w) = l(w) - l(xt)$.

As may be ascertained from Theorem 1, the criteria for x to be an element of $\mathrm{maxsing}(X_w)$ are local in nature. This implies that we may concentrate on only certain indices in our permutation w in order to determine $\mathrm{maxsing}(X_w)$. We now describe these indices explicitly.

Definition 13. Let

$$\Delta(x,w) = \{i, 1 \leq i \leq n : \exists\, j, 1 \leq j \leq n, \text{ with } t_{\{i,j\}} \in \mathcal{R}(x,w)\}. \tag{5.1}$$

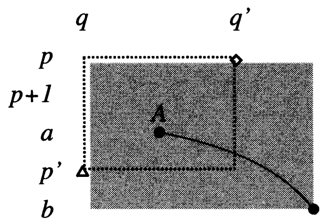


FIGURE 5. $\diamond = (p, q')$ and $\triangle = (p', q)$.

For $\Delta(x, w) = \{d_1 < d_2 < \cdots < d_k\}$, set

$$\widetilde{x} = \mathrm{fl}([x(d_1), x(d_2), \ldots, x(d_k)]) \text{ and} \tag{5.2}$$

$$\widetilde{w} = \mathrm{fl}([w(d_1), w(d_2), \ldots, w(d_k)]). \tag{5.3}$$

Note that $\widetilde{x}$ and $\widetilde{w}$ are permutations in S_k.

We now give a sufficient condition for an index b to be in $\Delta(x, w)$.

Proposition 14. *Suppose $x < w$ and $x(b) \neq w(b)$ with $1 \leq b \leq n$.*

(1) *If $w(b) < x(b)$, then $\exists\ a < b$ with $t_{a,b} \in \mathcal{R}(x, w)$ and $x(a) \neq w(a)$.*
(2) *If $w(b) > x(b)$, then $\exists\ c > b$ with $t_{b,c} \in \mathcal{R}(x, w)$ and $x(c) \neq w(c)$.*

Proof. First we prove the case of $w(b) < x(b)$. Note that

$$d_{x,w}(b-1, x(b)) = 1 + d_{x,w}(b, x(b)) \geq 1 \tag{5.4}$$

since $w(b) < x(b)$. Let $p' = b - 1$. Choose q as large as possible such that $q < x(b)$ and $d_{x,w}(p', q) = 0$ (see Figure 5). Such a q must exist since $d_{x,w}(\cdot, 0) = 0$. Now choose p as small as possible such that $p < p'$ and $d_{x,w}(g, h) \geq 1$ for all g, h with $(g, h) \in [p+1, p'] \times [q+1, x(b)]$. Then there exists a q', $q < q' \leq x(b)$ such that $d_{x,w}(p, q') = 0$. By construction,

$$d_{x,w}(p, q') = 0,\ d_{x,w}(p', q) = 0,\ d_{x,w}(p, q) \geq 0 \text{ and } d_{x,w}(p', q') \geq 1.$$

By Lemma 9, there exists an a such that $\mathrm{pt}_x(a) \in [p+1, p'] \times [q, q'-1]$ and $x(a) \neq w(a)$. Then $d_{x,w}|_{\mathcal{A}_{a,b}} \geq 1$, so by Fact 8, $t_{a,b} \in \mathcal{R}(x, w)$. This proves our claim.

To prove the case of $w(b) > x(b)$, it is easiest to use dual rank and difference functions:

$$r'_w(p, q) := \#\{i \geq p : w(i) \leq q\}, \tag{5.5}$$

$$d'_{x,w} := r'_w - r'_x. \tag{5.6}$$

One can check that $x \leq w$ if and only if $d'_{x,w} \geq 0$ and then argue as above using this new rank function. Using the dual rank function is equivalent to checking $x^{-1} < w^{-1}$. $\square$

Corollary 15. *If $x \leq w$ and $d_{x,w}(\mathrm{pt}_x(b)) > 0$, then there exists $b' < b$ with $t_{b',b} \in \mathcal{R}(x, w)$.*

Corollary 16. *Let $x \leq w$. Then $p \notin \Delta(x, w)$ if and only if both*

$$d_{x,w}(\mathrm{pt}_x(p) + (1,0)) = 0 \quad \textit{and} \quad d_{x,w}(\mathrm{pt}_x(p) + (0,1)) = 0.$$

Proposition 14 tells us that if $x(i) \neq w(i)$, then $i \in \Delta(x,w)$. It turns out that the question of whether or not $x \in \text{maxsing}(X_w)$ depends only on the pair $\widetilde{x}$, $\widetilde{w}$. This is borne out by the following simple facts. They will be used without comment in the remainder of the paper.

Lemma 17. *We have the following:*

(1) *If $x(i) = w(i)$, then $x^{\hat{i}} \leq w^{\hat{i}} \iff x \leq w$.*
(2) *$\widetilde{x} \leq \widetilde{w} \iff x \leq w$.*

Proof. The first equivalence follows from Lemma 3 by comparing $d_{x,w}$ and $d_{x^{\hat{i}},w^{\hat{i}}}$. The second follows from the first by noting that $i \in \Delta(x,w)$ whenever $x(i) \neq w(i)$. □

Proposition 18. *We have the following:*

(1) *$l(w) - l(x) = l(\widetilde{w}) - l(\widetilde{x})$.*
(2) *There exists a bijection $\mathcal{R}(\widetilde{x},\widetilde{w}) \xrightarrow{\sim} \mathcal{R}(x,w)$.*
(3) *$x \in \text{maxsing}(X_w)$ if and only if $\widetilde{x} \in \text{maxsing}(X_{\widetilde{w}})$.*

Proof. For any $i \notin \Delta(x,w)$,

$$l(w) - l(x) - (l(w^{\hat{i}}) - l(x^{\hat{i}})) = d_{x,w}(\text{pt}_x(i)) + d'_{x,w}(\text{pt}_x(i)). \tag{5.7}$$

By Corollary 15, $d_{x,w}(\text{pt}_x(i)) = 0$ and by using the dual rank function we see that $d'_{x,w}(\text{pt}_x(i))$ is also 0. This proves (1). Part (2) follows immediately from Fact 8 and the definition of $\Delta(x,w)$ by comparing $d_{x,w}$ and $d_{\widetilde{x},\widetilde{w}}$. Part (3) follows from the first two parts along with Corollary 21 (stated below). □

6. The map ϕ_t

In Fact 12 we claimed that maxsing(X_w) can be identified in terms of $\mathcal{R}(x,w)$ for $x \leq w$. To carry this out in practice, we will need to relate $\mathcal{R}(x,w)$ to $\mathcal{R}(y,w)$ when x, y differ by an element of $\mathcal{T}$. So, for every triple $yt < y \leq w$ with $t \in \mathcal{T}$, we will define a map $\phi_t^{y,w} : \mathcal{R}(y,w) \longrightarrow \mathcal{T}$. In Theorem 20 we will show that the image is actually contained in $\mathcal{R}(yt,w)$. The values of y, w are usually clear from context and we will often abbreviate $\phi_t^{y,w}$ as ϕ_t.

A similar map has been defined by Gasharov [17] for the purpose of showing that certain elements constructed by Lakshmibai and Sandhya in [21] are, in fact, singular points. (See Section 10 for details.) Theorem 20 is slightly stronger, however, than the corresponding result in [17]. We omit the proof as it follows from Lemma 10 and Table 1 by inspection.

Definition 19. Fix $yt < y \leq w$. Given some $t' \in \mathcal{R}(y,w)$, if t and t' commute, we define $\phi_t(t') = t'$. Otherwise, we can find $a < b < c$ such that $d \notin \{a,b,c\}$ implies $y(d) = yt(d) = yt'(d)$. Then we define $\phi_t^{y,w}(t')$ according to Table 1. Figure 6 shows the Bruhat pictures corresponding to Case A.

Theorem 20. *Fix $yt < y \leq w$. The map $\phi_t(\mathcal{R}(y,w)) \hookrightarrow \mathcal{R}(yt,w) \setminus \{t\}$ is injective.*

Recall from Definition 13 that

$$\Delta(x,w) = \{i, 1 \leq i \leq n : \exists\, j, 1 \leq j \leq n, \text{ with } t_{\{i,j\}} \in \mathcal{R}(x,w)\}. \tag{6.1}$$

One can check by inspecting Table 1 that we obtain the following:

Corollary 21. *For $t \in \mathcal{T}$, $yt < y \leq w$ implies that $\Delta(y,w) \subseteq \Delta(yt,w)$.*

TABLE 1. Definition of map $\phi_t^{y,w}$. We have split into cases indexed by $\mathrm{fl}_{abc}(y)$ and whether $t' = t_{a,b}$, $t' = t_{a,c}$ or $t' = t_{b,c}$. Note that the matter of inclusion of $t_{a,c}$ in $\mathcal{R}(y,w)$ is determined by the first three columns in Cases A.i, B.i, C and D. The final two columns are used in proving that ϕ_t maps $\mathcal{R}(y,w)$ into $\mathcal{R}(yt,w)$.

Case	$\overline{y}$	t	t'	$t_{a,c} \in \mathcal{R}(y,w)$	$\phi_t(t')$	$\overline{yt'}$	$\overline{yt\phi_t^{y,w}(t')}$
$A.i)$	213	$t_{a,b}$	$t_{a,c}$	✓	$t_{b,c}$	312	132
$ii)$	213	$t_{a,b}$	$t_{b,c}$	×	$t_{b,c}$	231	132
$iii)$	213	$t_{a,b}$	$t_{b,c}$	✓	$t_{a,c}$	231	321
$B.i)$	132	$t_{b,c}$	$t_{a,c}$	✓	$t_{a,b}$	231	213
$ii)$	132	$t_{b,c}$	$t_{a,b}$	×	$t_{a,b}$	312	213
$iii)$	132	$t_{b,c}$	$t_{a,b}$	✓	$t_{a,c}$	312	321
$C.i)$	312	$t_{a,b}$	$t_{b,c}$	×	$t_{a,c}$	321	231
$ii)$	312	$t_{a,c}$	$t_{b,c}$	×	$t_{b,c}$	321	231
$D.i)$	231	$t_{b,c}$	$t_{a,b}$	×	$t_{a,c}$	321	312
$ii)$	231	$t_{a,c}$	$t_{a,b}$	×	$t_{a,b}$	321	312

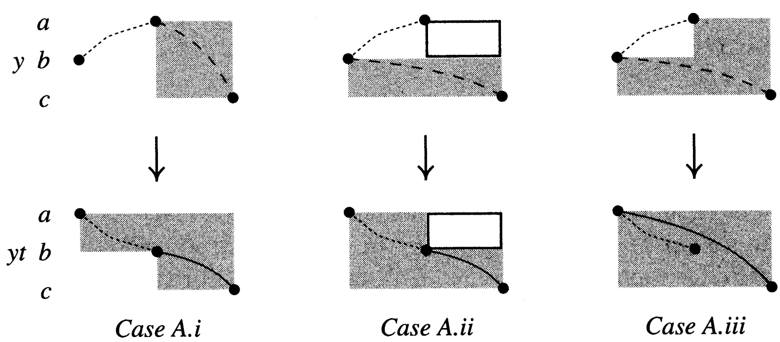


FIGURE 6. Depiction of Case A from Definition 19. The dashed (resp. dotted, solid) arcs represent t' (resp. t, $\phi_t(t')$).

For a pair of reflections t, t' where $t \in \operatorname{Im} \phi_{t'}$, it will be useful to know what we can say about the membership of t' in $\operatorname{Im} \phi_t$.

Proposition 22 (Reciprocity). *If $t, t' \in \mathcal{R}(x,w)$, $t \neq t'$, with $l(xt) = l(xt') = l(x)+1$, then $t' \in \operatorname{Im} \phi_t^{xt,w} \Leftrightarrow t \in \operatorname{Im} \phi_{t'}^{xt',w}$.*

Proof. Suppose $t \in \operatorname{Im} \phi_{t'}$. We will show that $t' \in \operatorname{Im} \phi_t$.

First, consider the case where $tt' = t't$. From the definition of ϕ, we see that $\phi_{t'}^{-1}(t) = t$. So $w \geq xt't = xtt'$. This implies that $t' \in \mathcal{R}(xt,w)$ and therefore $\phi_t(t') = t'$.

Now we suppose $tt' \neq t't$. So $a < b < c$ are determined such that $d \notin \{a,b,c\}$ implies $x(d) = xt(d) = xt'(d)$. Let $\overline{x} = \mathrm{fl}_{abc}(x)$. Note that:

(1) By hypothesis, $l(xt) = l(xt') = l(x)+1$.

(2) If $\overline{x} \in \{231, 312, 321\}$, then at most one of $t_{a,b}, t_{a,c}, t_{b,c} \in \mathcal{R}(x, w)$, not two.

We will prove the case $\overline{x} = 132$ and leave the cases $\overline{x} = 213$ and 123 to the reader. Let $\overline{x} = 132$. Then $\{t, t'\} = \{t_{a,b}, t_{a,c}\}$ and $w \geq \widehat{312}, \widehat{231}$. By Lemma 10, $w \geq \widehat{321}$. So $t_{a,b} \in \mathcal{R}(xt_{a,c}, w)$ and, as we are in Case D.ii of Definition 19, $t_{a,b} = \phi_{t_{a,c}}(t_{a,b})$. Similarly, $t_{b,c} \in \mathcal{R}(xt_{a,b}, w)$ and, as we are in Case C.i of Definition 19, $t_{a,c} = \phi_{t_{a,b}}(t_{b,c})$. □

7. Incompatible edges and restrictions on $\mathcal{R}(x, w)$

Let $x < xt \leq w$. We make the following observation from Theorem 11: If $\#\phi_t(\mathcal{R}(xt, w)) < \#\mathcal{R}(x, w) - l(x) + l(xt)$, then e_x is a singular point of X_w. The above fact is most conveniently expressed in terms of the following notation:

Definition 23. For $x < w$ and $t \in \mathcal{R}(x, w)$, let

$$\mathcal{E}_t(x, w) = \mathcal{R}(x, w) \setminus (\{t\} \cup \phi_t(\mathcal{R}(xt, w))) \tag{7.1}$$

denote the set of "extra" reflections corresponding to x and t. We often write $\mathcal{E}_{a,b}(x, w)$ for $\mathcal{E}_{t_{a,b}}(x, w)$. If $t' \in \mathcal{E}_t(x, w)$, then we say that t and t' are *incompatible* edges (in the Bruhat graph).

The elements of $\mathcal{E}_t(x, w)$ are "extra" edges in the sense that they correspond to an increase in the dimension of the Zariski tangent space.

Fact 24. If $t, t' \in \mathcal{R}(x, w)$ with $t' \in \mathcal{E}_t(x, w)$ and $l(xt) = l(x) + 1$, then $x < w$ is singular.

Fact 25. $x \leq w$ is an MSP (maximal singular point) implies that for every $t \in \mathcal{R}(x, w)$ with $l(xt) = l(x) + 1$, $\mathcal{E}_t(x, w) \neq \emptyset$.

Note, however, that if x is a singular point, but not an MSP, then it is possible that $\mathcal{E}_t(x, w) = \emptyset$. An example is afforded by $w = [4, 2, 3, 1]$, $x = [1, 2, 3, 4]$ and $t = t_{1,2}$. Conversely, if $l(xt) > l(x) + 1$, then we may have $\mathcal{E}_t(x, w) \neq \emptyset$ even if X_w is entirely smooth. Take, for example, $w = [3, 2, 1]$, $x = [1, 2, 3]$, and $t = t_{1,3}$.

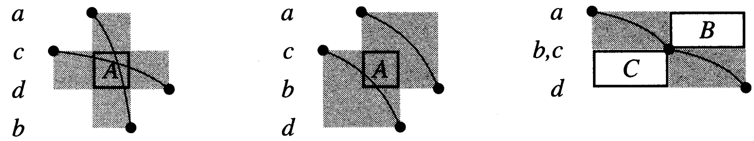
Proposition 26. *All pairs of incompatible edges $t_{a,b}, t_{c,d} \in \mathcal{R}(x, w)$ can be classified into the following two types:*

1. *Patch Incompatibility (e.g., Figure 7(1), (2)):*
   If $t_{a,b}, t_{c,d} \in \mathcal{R}(x, w)$ with $\{a, b\} \cap \{c, d\} = \emptyset$, then $t_{a,b} \in \mathcal{E}_{c,d}(x, w)$ if and only if $\min(d_{x,w}|_A) = 1$ where $A = \mathcal{A}_{a,b} \cap \mathcal{A}_{c,d}$.
2. *Link Incompatibility (e.g., Figure 7(3)):*
   If $t_{a,b}, t_{b,c} \in \mathcal{R}(x, w)$, then

$$\begin{aligned} t_{a,b} \in \mathcal{E}_{b,c}(x, w) &\iff t_{b,c} \in \mathcal{E}_{a,b}(x, w) \\ &\iff \min(d_{x,w}|_B) = \min(d_{x,w}|_C) = 0. \end{aligned} \tag{7.2}$$

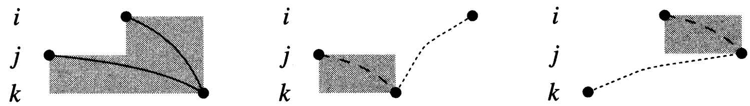
If $x < w$ is an MSP, *then all possible arrangements of $\overline{\mathcal{A}_{a,b}}$ and $\overline{\mathcal{A}_{c,d}}$ are drawn in Figure 7.*

Proof. The proof of Patch Incompatibility is clear. To prove Link Incompatibility, it suffices to consider Cases A and B of Definition 19. The proof that these are all possible incompatible pairs follows from Lemmas 27, 28, and 29 below. □



(1) Patch (2) Patch (3) Link

FIGURE 7



(1) x (2) $xt_{i,k}$ (3) $xt_{j,k}$

FIGURE 8

Lemma 27 (Ell Lemma). *Let $1 \leq i < j < k \leq n$.*

(1) *If $\mathrm{fl}_{ijk}(x) = 213$ and $t_{i,k}, t_{j,k} \in \mathcal{R}(x, w)$, then $t_{i,k} \in \operatorname{Im} \phi_{t_{j,k}}^{xt_{j,k},w}$ and $t_{j,k} \in \operatorname{Im} \phi_{t_{i,k}}^{xt_{i,k},w}$ so $t_{i,k}, t_{j,k}$ are not incompatible.*
(2) *If $\mathrm{fl}_{ijk}(x) = 132$ and $t_{i,j}, t_{i,k} \in \mathcal{R}(x, w)$, then $t_{i,j} \in \operatorname{Im} \phi_{t_{i,k}}^{xt_{i,k},w}$ and $t_{i,k} \in \operatorname{Im} \phi_{t_{i,j}}^{xt_{i,j},w}$ so $t_{i,j}, t_{i,k}$ are not incompatible.*

Proof. We only prove (1) as the proof for (2) is entirely analogous.
Diagrams for x, $xt_{i,k}$ and $xt_{j,k}$ are given in Figure 8.

We see that $t_{i,k} \in \mathcal{R}(x, w)$ implies $w \geq \widehat{312}$ and $t_{j,k} \in \mathcal{R}(x, w)$ implies $w \geq \widehat{231}$. So, by Lemma 10, $xt_{i,k}t_{j,k} = xt_{j,k}t_{i,j} = \widehat{321} \leq w$. Equivalently, $t_{j,k} \in \mathcal{R}(xt_{i,k}, w)$ and $t_{i,j} \in \mathcal{R}(xt_{j,k}, w)$. So, (Case C.ii of Definition 19) $\phi_{t_{i,k}}^{xt_{i,k},w}(t_{j,k}) = t_{j,k}$ and (Case D.i of Definition 19) $\phi_{t_{j,k}}^{xt_{j,k},w}(t_{i,j}) = t_{i,k}$. $\square$

Lemma 28. *Let $t_{a,b}, t_{c,d} \in \mathcal{R}(x, w)$. If $\overline{\mathcal{A}_{a,b}} \cap \overline{\mathcal{A}_{c,d}} = \emptyset$, then $t_{a,b} \notin \mathcal{E}_{c,d}(x, w)$ (i.e., $t_{a,b} \in \operatorname{Im} \phi_{t_{c,d}}$).*

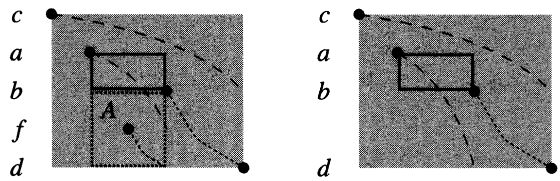
Proof. Since $\overline{\mathcal{A}_{a,b}} \cap \overline{\mathcal{A}_{c,d}} = \emptyset$, we have $t_{a,b} \in \mathcal{R}(xt_{c,d}, w)$ and $t_{a,b}t_{c,d} = t_{c,d}t_{a,b}$ so $t_{a,b} \in \operatorname{Im} \phi_{t_{c,d}}$. $\square$

There will be numerous instances in the remainder of the paper where we do the following:

(1) Assume we have an MSP x for X_w.
(2) Construct some $y = xt'' > x$.
(3) Conclude that $y < w$ from the fact that $\mathcal{A}_{t''}(x)$ is shaded.
(4) Find incompatible edges t, t' as in Fact 24 to conclude that y is also a singular point of X_w.
(5) Obtain a contradiction with our first assumption.

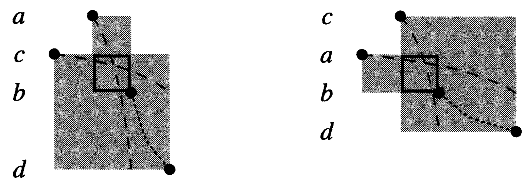
The previous technique will allow us to significantly pare down the possibilities for what $\widetilde{x}$ looks like for x an MSP. The following lemma is the first example of this strategy.

Lemma 29. *Let $x < w$ be an MSP, and assume $t_{a,b} \in \mathcal{R}(x, w)$, $l(xt_{a,b}) = l(x) + 1$, and $t_{c,d} \in \mathcal{E}_{a,b}(x, w)$. Then $\mathrm{pt}_x(a), \mathrm{pt}_x(b) \notin \mathcal{A}_{c,d}(x)$.*



(1) (2)

FIGURE 9



(1) $c > a$, $x(c) < x(a)$ (2) $c < a$, $x(c) > x(a)$

FIGURE 10

Proof. First consider the case where both $\mathrm{pt}_x(a), \mathrm{pt}_x(b) \in \mathcal{A}_{c,d}(x)$.

Suppose there is a point $\mathrm{pt}_x(f)$ in region A of Figure 9(1). Choose such an f as small as possible. Then we see that $t_{c,b}$ and $t_{a,f}$ are patch incompatible reflections for $x' = xt_{b,d}t_{f,d} \leq w$ and $l(x't_{a,f}) = l(x') + 1$. By Fact 24, x' is then singular. This contradicts the fact that x is an MSP for w.

Now suppose region A of Figure 9(1) is empty — this is shown in Figure 9(2). Then $t_{c,b}$ and $t_{a,d}$ are incompatible reflections for $xt_{b,d} \leq w$ and $l(xt_{b,d}t_{a,d}) = l(xt_{b,d}) + 1$. Since $x < xt_{b,d}$, this contradicts the fact that x is an MSP for w.

We now argue the case of $\mathrm{pt}_x(b) \in \mathcal{A}_{c,d}(x) \not\ni \mathrm{pt}_x(a)$. (The arguments for $\mathrm{pt}_x(b) \notin \mathcal{A}_{c,d}(x) \ni \mathrm{pt}_x(a)$ are parallel.)

Clearly $d > b$ and $x(d) > x(b)$. There are four possibilities with regard to the position of $\mathrm{pt}_x(c)$.

(1) $c = a$.
We are in Case A.iii of the definition of ϕ. Hence, $t_{c,d} \in \mathrm{Im}\,\phi_{t_{a,b}}$, which is a contradiction. So this case cannot occur.

(2) $c > a$, $x(c) > x(a)$.
This case cannot occur as it violates $l(xt_{a,b}) = l(x) + 1$.

(3) $c > a$, $x(c) < x(a)$.
This case is depicted in Figure 10(1). Suppose $l(xt_{b,d}t_{a,d}) = l(xt_{b,d}) + 1$. Then $t_{a,d}$ and $t_{c,b}$ are patch incompatible for $xt_{b,d} \leq w$. This contradicts the fact that x is an MSP for w. If $l(xt_{b,d}t_{a,d}) > l(xt_{b,d}) + 1$, then we can argue as in Figure 9(1) to obtain our contradiction.

(4) $c < a$, $x(c) > x(a)$.
See Figure 10(2). This is analogous to the previous case. □

Proposition 30 below gives us our first non-trivial restriction regarding the composition of $\mathcal{R}(x, w)$. This proposition will greatly reduce the amount of work we need to do later on to determine possibilities for $\widetilde{x}$.

Proposition 30. *Let $x < w$ be an* MSP. *If $t \in \mathcal{R}(x, w)$, then $l(xt) = l(x) + 1$.*

Proof. Suppose that $t \in \mathcal{R}(x,w)$ and $l(xt) > l(x)+1$. We will obtain a contradiction by following the strategy on page 3926.

Let $t = t_{a,c}$. Choose b as large as possible such that $\mathrm{pt}_x(b) \in \mathcal{A}_{a,c}(x)$. Note that $t_{a,b}, t_{b,c} \in \mathcal{R}(x,w)$ and $l(xt_{b,c}) = l(x)+1$. Since x is an MSP, we can invoke Fact 25 to find a $t_{e,f} \in \mathcal{E}_{b,c}(x,w)$.

Suppose $\overline{\mathcal{A}_{a,b}(x)} \cap \overline{\mathcal{A}_{e,f}(x)} = \emptyset$. Since $t_{a,c} \in \mathcal{R}(x,w)$, $\mathcal{A}_{a,c}$ is shaded so $t_{b,c} \in \mathcal{R}(xt_{a,b},w)$. Hence $t_{b,c}$ and $t_{e,f}$ are incompatible for $xt_{a,b} \leq w$ and $l(xt_{a,b}t_{b,c}) = l(xt_{a,b})+1$. This contradicts $x \in \mathrm{maxsing}(X_w)$.

Otherwise, $\overline{\mathcal{A}_{e,f}}$ overlaps both $\overline{\mathcal{A}_{a,b}}$ and $\overline{\mathcal{A}_{b,c}}$, so, by Lemma 29, we are in one of the following two scenarios.

(1) $e = b$.
By choice of b, $f \notin \mathcal{A}_{b,c}(x)$. (Note that $f \neq c$.) So either $f > c$, $x(f) < x(c)$ or $f < c$, $x(f) > x(c)$ (the latter case is shown in Figure 11(1)). In either case, we can apply the Ell Lemma 27 to conclude that $t_{e,f} \in \mathrm{Im}\,\phi_{t_{b,c}}$. This contradicts the choice of $t_{e,f}$.

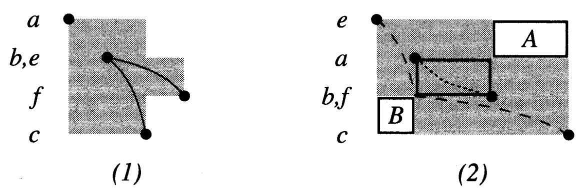


FIGURE 11

(2) $f = b$.
Since $t_{a,c} \in \mathcal{R}(x,w)$, for $t_{e,f}$ to be an element of $\mathcal{E}_{b,c}(x,w)$, we need $e < a$, $x(e) < x(a)$ and $d_{x,w} = 0$ for some point in each of regions A and B in Figure 11(2). But then $t_{e,b}$ and $t_{b,c}$ are link incompatible for $xt_{a,b} \leq w$. Furthermore, by having chosen b as large as possible, we ensure that $l(xt_{a,b}t_{b,c}) = l(xt_{a,b})+1$. This contradicts $x \in \mathrm{maxsing}(X_w)$.

□

This greatly simplifies our future investigations. We now use Proposition 26 and Lemma 9 to prove the following crucial lemmas describing the shading on $d_{x,w}$.

Lemma 31 (Cross Lemma). *Let $x < w$ be an* MSP *and suppose $1 \leq i < j < k < l \leq n$ such that $\mathrm{fl}_{ijkl}(x) = 2143$. If $t_{j,k} \in \mathcal{R}(x,w)$ and $t_{i,l} \in \mathcal{E}_{j,k}(x,w)$, then $t_{i,k}, t_{j,l} \in \mathcal{R}(x,w)$.*

Proof. We can visualize the situation as in Figure 12(1). Since $t_{i,l} \in \mathcal{E}_{j,k}(x,w)$, there is necessarily a point $\triangle$ in region A for which $d_{x,w}(\triangle) = 1$. Suppose $t_{i,k} \notin \mathcal{R}(x,w)$. Then there is a point $\square$ in region B such that $d_{x,w}(\square) = 0$. Then we can apply Lemma 9 (with $\alpha, \beta \geq 1$, $\gamma = 1$) and Proposition 30 to conclude that there is a point $\mathrm{pt}_x(p)$ of $\mathrm{mat}(x)$ in region C (see Figure 12(2)). If we choose $\square$ to be as low as possible in our diagram, then $d_{x,w}|_D \geq 1$ (see Figure 12(3)). But then $t_{i,l}$ and $t_{j,k}$ are patch incompatible for $xt_{p,k} \leq w$. This contradicts $x \in \mathrm{maxsing}(X_w)$. Therefore $t_{i,k} \in \mathcal{R}(x,w)$ and we can shade the entire region B.

To shade the lower left corner, apply this argument to x^{-1} and w^{-1}. □

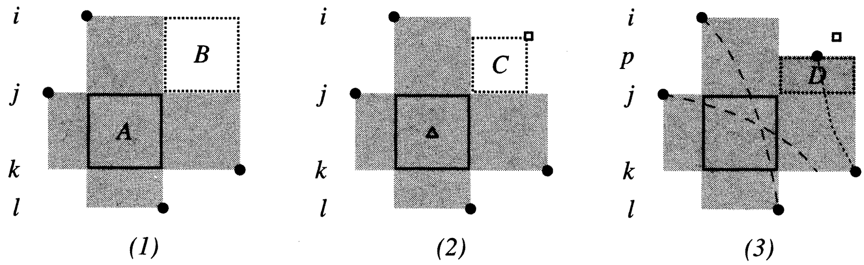


FIGURE 12

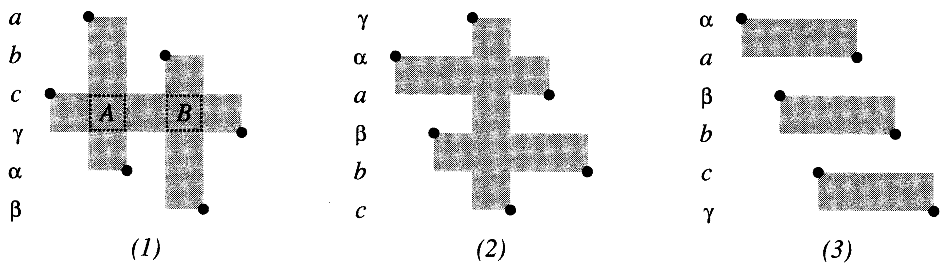


FIGURE 13

Lemma 32. *If $x < w$ is an* MSP, *then our diagram for the pair x, w does not contain any of the configurations in Figure 13 (regardless of whether or not these reflections are incompatible).*

Proof. The inadmissibility of these configurations is proved using the strategy on page 3926. We arrive at contradictions using Proposition 30 and Lemma 31. We will just prove the case depicted in Figure 13(3).

Since x is an MSP for w, there exists some $t_{d,\delta} \in \mathcal{E}_{c,\gamma}(x, w)$. Clearly, if

$$\overline{\mathcal{A}_{d,\delta}} \cap \overline{\mathcal{A}_{\alpha,a}} = \emptyset, \tag{7.3}$$

then $t_{c,\gamma}$ and $t_{d,\delta}$ are (patch or link) incompatible reflections for $xt_{\alpha,a} \leq w$. This would contradict $x \in \text{maxsing}(X_w)$.

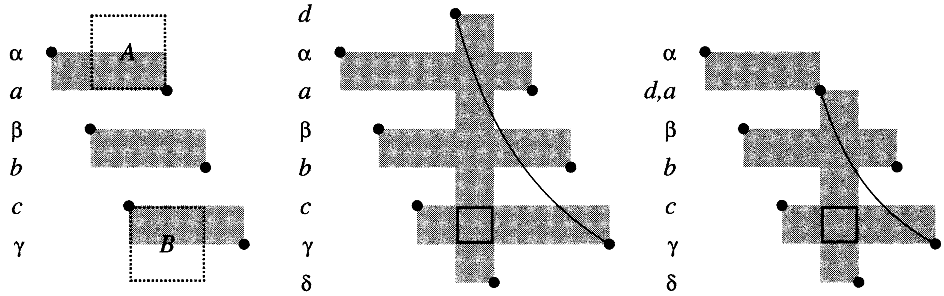
So, to ensure that (7.3) does not hold, we need $\text{pt}_x(d)$ in region A of Figure 14(1) and $\text{pt}_x(\delta)$ in region B. Here we are including the possibilities that $d = a$ or $\delta = c$. Note that (as is shown in Figure 14(1)) Proposition 30 requires that $x(\delta) < x(b)$ and $x(d) > x(\beta)$. Clearly if $d = a$, then $\delta \neq c$ and vice versa. Hence, by symmetry, we can treat only the cases where $\delta \neq c$. These two cases are illustrated in Figures 14(2) and 14(3).

For both cases, we can apply the Cross Lemma 31 to $t_{d,\delta}$ and $t_{c,\gamma}$ to conclude that $t_{d,\gamma} \in \mathcal{R}(x, w)$. Then $l(xt_{d,\delta}) > l(x)+1$, which contradicts Proposition 30. □

8. RESTRICTIONS ON $\widetilde{x}$ AND $\widetilde{w}$

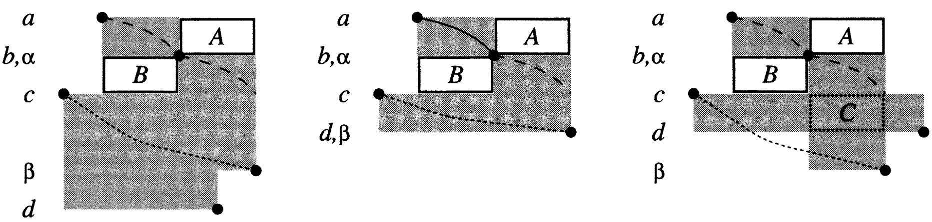
Recall that $\widetilde{x}$ and $\widetilde{w}$ are the restrictions of x and w to those positions in $\Delta(x, w)$ (see Definition 13). In order to determine the structure of $\text{maxsing}(X_w)$, we first prove the following necessary conditions on $\widetilde{x}$ for any MSP x for w.

Theorem 33. *If $x < w$ is an* MSP, *then $\widetilde{x}$ avoids the patterns* 231, 312 *and* 1234.



(1) (2) (3)

FIGURE 14



(1) $d > \beta$ (2) $d = \beta$ (3) $d < \beta$

FIGURE 15

Proof. By passing to inverses, it is enough to prove that $\widetilde{x}$ is either 231-avoiding or 312-avoiding. So choose $a, b, c \in \Delta(x, w)$ with $1 \leq a < b < c \leq n$ such that $\mathrm{fl}_{abc} = 231$.

Case 1. Assume $t_{a,b} \in \mathcal{R}(x, w)$.

By definition of $\Delta(x, w)$, there exists a $d \in \Delta(x, w)$ with $t_{c,d} \in \mathcal{R}(x, w)$. We'll assume that $c < d$ as all cases where $d < c$ are analogous to one of the cases we cover by transposing over the antidiagonal. Clearly $\overline{\mathcal{A}_{a,b}} \cap \overline{\mathcal{A}_{c,d}} = \emptyset$.

Since x is an MSP, there exists a $t_{\alpha,\beta} \in \mathcal{E}_{a,b}(x, w)$. We can assume that

$$\overline{\mathcal{A}_{\alpha,\beta}} \cap \overline{\mathcal{A}_{c,d}} \neq \emptyset \tag{8.1}$$

or else $t_{a,b}$ and $t_{\alpha,\beta}$ are (patch or link) incompatible for $xt_{c,d} \leq w$ contradicting $x \in \mathrm{maxsing}(X_w)$. There are two cases according to whether $t_{a,b}$ and $t_{\alpha,\beta}$ are link or patch incompatible. We only describe the arguments explicitly in the case of link incompatibility — the arguments are similar in the latter case. Also, we will argue only $b = \alpha$ as the case of $\beta = a$ is analogous.

By Proposition 30, there are three possibilities for the relative positions of $\mathrm{pt}_x(d)$ and $\mathrm{pt}_x(\beta)$. They are displayed in Figure 15.

In the case of Figure 15(3), if the dotted transposition $t \notin \mathcal{R}(x, w)$, then $t_{a,b}$ and $t_{\alpha,\beta}$ are link incompatible for $xt_{c,d} \leq w$. Otherwise, in all three cases, $t \in \mathcal{R}(x, w)$ and $t_{a,b}$, $t_{b,c}$ are link incompatible for $xt_{c,\beta} \leq w$. This contradicts $x \in \mathrm{maxsing}(X_w)$.

Case 2. Assume $t_{a,b} \notin \mathcal{R}(x, w)$.

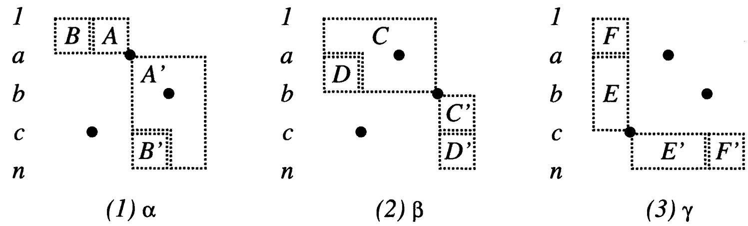


FIGURE 16

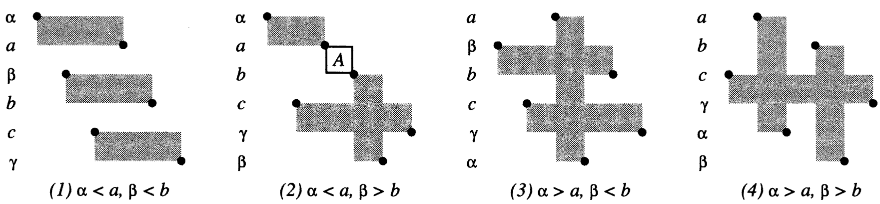


FIGURE 17

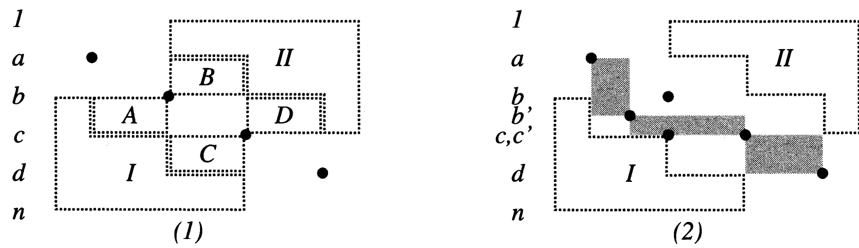


FIGURE 18

Since $a, b, c \in \Delta(x, w)$, we can find α, β, γ such that $t_{\{a,\alpha\}}, t_{\{b,\beta\}}, t_{\{c,\gamma\}} \in \mathcal{R}(x, w)$.

In light of Figure 16 and Proposition 30, we must have $\mathrm{pt}_x(\alpha) \in B \cup B'$, $\mathrm{pt}_x(\beta) \in D \cup D'$, and $\mathrm{pt}_x(\gamma) \in F \cup F'$ or else we can reduce to the previous case (of $t_{a,b} \in \mathcal{R}(x, w)$). Furthermore, by symmetry we can assume $\mathrm{pt}_x(\gamma) \in F'$. That leaves four cases depending on whether $\mathrm{pt}_x(\alpha) \in B, B'$ and whether $\mathrm{pt}_x(\beta) \in D, D'$ as pictured in Figure 17.

Configurations (1), (3), and (4) contradict Lemma 32. So consider the case of Figure 17(2). Since x is an MSP, there exists some $t_{d,\delta} \in \mathcal{E}_{b,\beta}(x, w)$. Since $\min(d_{x,w}|_A) = 0$, one can see that $\overline{\mathcal{A}_{d,\delta}} \cap \overline{\mathcal{A}_{\alpha,a}} = \emptyset$. Hence, $t_{d,\delta}$ and $t_{b,\beta}$ are (patch or link) incompatible for $xt_{\alpha,a} \leq w$.

This completes the proof that $\widetilde{x}$ is 231- and 312-avoiding.

Next we will show $\widetilde{x}$ is 1234-avoiding. Suppose we have $a < b < c < d$ with $a, b, c, d \in \Delta(x, w)$ and $\mathrm{fl}_{abcd}(x) = 1234$. We will obtain a contradiction.

By Theorem 33, no points of $\mathrm{mat}(\widetilde{x})$ may occur in regions I or II of Figure 18(1).

Since $a, d \in \Delta(x, w)$, there exist b', c' such that $t_{\{a,b'\}}, t_{\{c',d\}} \in \mathcal{R}(x, w)$. As x is an MSP, there also exists some $t_{e,f} \in \mathcal{E}_{a,b'}(x, w)$. Using Proposition 30, it is easy to check that if

$$b' \notin \overline{A \cup B} \text{ or } c' \notin \overline{C \cup D} \text{ or } e \neq b' \text{ or } f \neq c', \tag{8.2}$$

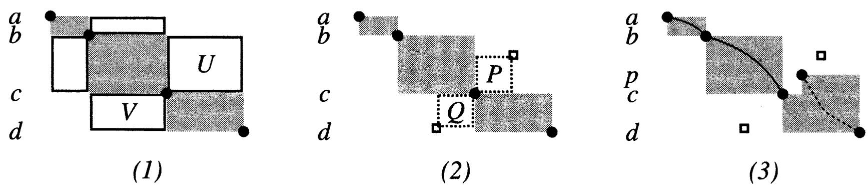


FIGURE 19

then $t_{e,f}$ and $t_{a,b'}$ are incompatible for $xt_{c',d} \leq w$. This contradicts $x \in \text{maxsing}(X_w)$. So, in particular, $t_{e,f} = t_{b',c'} \in \mathcal{R}(x,w)$. A typical (allowable) pair of positions for b' and c' is shown in Figure 18(2).

We will assume that b and c were chosen initially such that $t_{a,b}$, $t_{b,c}$, $t_{c,d} \in \mathcal{R}(x,w)$ and $t_{a,b} \in \mathcal{E}_{b,c}(x,w)$. Suppose $t_{b,c} \notin \mathcal{E}_{c,d}(x,w)$. Then $t_{a,b}$ and ($t_{b,c}$ or $t_{b,d}$) are link incompatible for $xt_{c,d} \leq w$. This contradicts $x \in \text{maxsing}(X_w)$.

So our diagram looks like that pictured in Figure 19(1) and we have $t_{a,b} \in \mathcal{E}_{b,c}(x,w), t_{b,c} \in \mathcal{E}_{c,d}(x,w)$. Therefore, we can find a point in each of the regions U and V such that $d_{x,w} = 0$. Choose the point in region U to be as low as possible. Choose the point in region V to be as far right as possible. Such points are shown in Figure 19(2). Apply Lemma 9 to the rectangle determined by these two points with $\alpha, \beta \geq 1$ and $\gamma = 0$. This, along with Proposition 30, implies that there is another point $\text{pt}_x(p)$ in either region P or Q. Without loss of generality, assume it is in region P. By having chosen the point in region U as low as possible, we find that $t_{p,d} \in \mathcal{R}(x,w)$ (see Figure 19(3)). Hence, $t_{a,b}$ and $t_{b,c}$ are link incompatible for $xt_{p,d} \leq w$. This contradicts $x \in \text{maxsing}(X_w)$. $\square$

By Theorem 33, we see that if $x \in \text{maxsing}(X_w)$, then

$$\widetilde{x} = [k, \ldots, 1, k+l, \ldots, k+1, k+l+m, \ldots, k+l+1] \tag{8.3}$$

for some $k, l, m \geq 0$. If two out of three of k, l, m are 0, then $\widetilde{x}$ is strictly decreasing, so $x \leq w$ implies that $x = w$. But then x cannot possibly be an MSP. So now we determine the possible values of k, m in Proposition 34 when $l = 0$ and the possible values of k, l, m in Proposition 35 when $k, l, m > 0$. In each proposition, we also determine what $\widetilde{w}$ must be to allow $\widetilde{x}$ to be singular.

We know from Proposition 18 that $x \in \text{maxsing}(X_w)$ iff $\widetilde{x} \in \text{maxsing}(X_{\widetilde{w}})$. Hence, for the remainder of this section, we will only consider the case where $\widetilde{x} = x$ and $\widetilde{w} = w$.

8.1. Two decreasing sequences in $\widetilde{x}$.

Proposition 34. *Let $x \in \text{maxsing}(X_w)$ with $\widetilde{x} = x$ and $\widetilde{w} = w$. Suppose that x consists of exactly two decreasing sequences:*

$$x = [k, \ldots, 1, k+m, \ldots, k+1], \tag{8.4}$$

for some $k, m \geq 1$. Then

(1) $k, m \geq 2$ *and*
(2) $w = [k+m, k, \ldots, 2, k+m-1, \ldots, k+1, 1]$.

Proof. For brevity in the following, we use the convention that $\alpha, a, a' \in [1, \ldots, k]$ and $\beta, b, b' \in [k+1, \ldots, k+m]$.

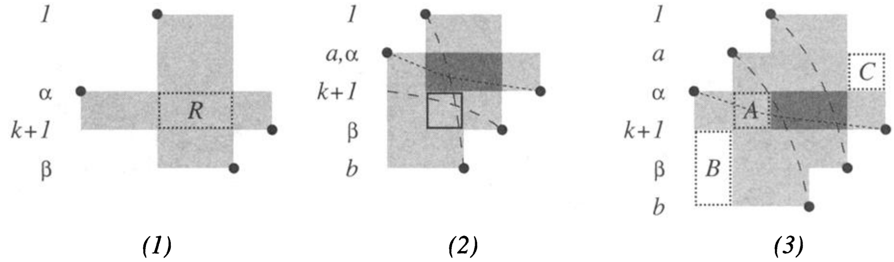


FIGURE 20

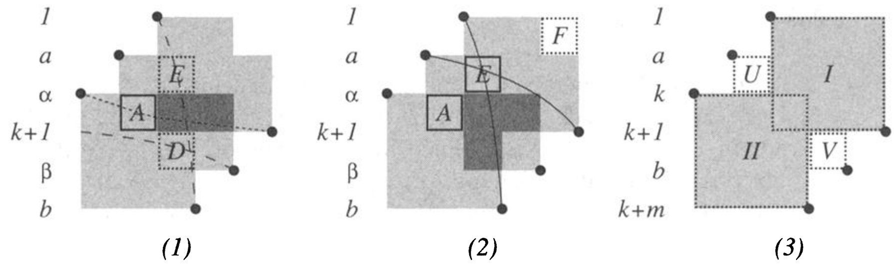


FIGURE 21

If $k = 1$ or $m = 1$, then by Lemma 27, $\mathcal{E}_{1,k+1}(x,w) = \emptyset$. This contradicts Fact 25. Hence, we obtain $k, m \geq 2$.

We proceed in steps to prove the particular form for w given above.

Step 1. $t_{1,k+1}, t_{k,k+m} \in \mathcal{R}(x,w)$.

Assume $t_{1,k+1} \notin \mathcal{R}(x,w)$. We will obtain a contradiction.

By Proposition 14, we can find α, β such that $t_{1,\beta}, t_{\alpha,k+1} \in \mathcal{R}(x,w)$ (see Figure 20(1)). Choose α as large as possible and β as small as possible subject to this restriction.

If $t_{1,\beta} \in \mathcal{E}_{\alpha,k+1}(x,w)$, then an application of the Cross Lemma 31 would offer the desired contradiction. So assume that this is not the case (i.e., assume $d_{x,w} \geq 2$ on region R of Figure 20(1)).

Since x is an MSP, by Fact 25, we can find some $t_{a,b} \in \mathcal{E}_{1,\beta}(x,w)$. Recall that we chose α as large as possible such that $t_{\alpha,k+1} \in \mathcal{R}(x,w)$. It follows then that $a \leq \alpha$. Similarly, our choice of β as small as possible such that $t_{1,\beta} \in \mathcal{R}(x,w)$, in conjunction with the Cross Lemma 31 and Ell Lemma 27, implies that $b > \beta$. Suppose $a = \alpha$. This is depicted in Figure 20(2). We see that $t_{k+1,\beta}$ and $t_{1,b}$ are patch incompatible for $xt_{\alpha,k+1} \leq w$. This contradicts $x \in \operatorname{maxsing}(X_w)$. So we may assume $a < \alpha$ as in Figure 20(3).

Suppose that $d_{x,w} \geq 2$ on region A. Then $t_{a,b}$ and $t_{1,\beta}$ are patch incompatible for $xt_{\alpha,k+1} \leq w$. This contradicts $x \in \operatorname{maxsing}(X_w)$. So there is at least one point in region A for which $d_{x,w}$ has value 1. Now we can apply the Cross Lemma 31 to the patch incompatible pair $t_{\alpha,k+1}, t_{a,b}$ to conclude that $d_{x,w} \geq 1$ on regions B and C. We display this knowledge in Figure 21(1).

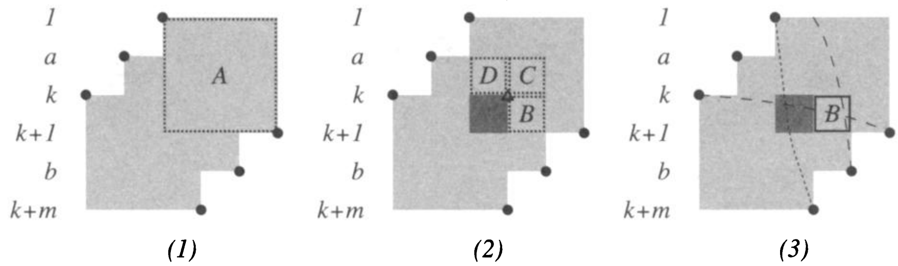


FIGURE 22. We have displayed the case of $a < k$, but the argument holds for $a = k$ too.

Now suppose that there is a point in region D for which $d_{x,w} = 1$. Then $t_{k+1,\beta}$ and $t_{1,b}$ are patch incompatible reflections for $xt_{\alpha,k+1} \leq w$. This contradicts $x \in \text{maxsing}(X_w)$. Since, by construction, $t_{a,b} \in \mathcal{E}_{1,\beta}(x,w)$, the only possibility left is that $\min(d_{x,w}|_E) = 1$ (as in Figure 21(2)). We can now apply the Cross Lemma 31 to $t_{1,b}$ and $t_{a,k+1}$ to conclude that $d_{x,w} \geq 1$ on region F. Hence $t_{1,k+1} \in \mathcal{R}(x,w)$ as claimed.

The proof that $t_{k,k+m} \in \mathcal{R}(x,w)$ is entirely analogous when one uses $d'_{x,w}$ from (5.6).

Step 2. $t_{a,b} \in \mathcal{R}(x,w)$ for all $1 \leq a \leq k$ and $k+1 \leq b \leq k+m$.
By the previous step, we know that we can shade rectangles I and II in Figure 21(3). For every a, b, by the definition of $\Delta(x,w)$, we can shade the corresponding regions U and V, respectively.

Step 3. $d_{x,w} \leq 1$ on region A in Figure 22(1).
Suppose, on the contrary, that $d_{x,w} \geq 2$ for some point in region A.

Since $d_{x,w}$ is non-decreasing as we move down or left in region A, we can assume that $d_{x,w}(\triangle) \geq 2$ for $\triangle = (k, k+1)$. But then there must be some a, b with $1 < a \leq k$ and $k+1 < b \leq k+m$ with either $a < k$ or $b < k+m$ and $t_{a,b} \in \mathcal{E}_{1,k+1}(x,w)$ (see Figure 22(2)). Note that $\min(d_{x,w}|_{B \cup C \cup D}) = 1$ by choice of $t_{a,b}$. If $\min(d_{x,w}|_B) = 1$, then $t_{k,k+1}$ and $t_{1,b}$ are patch incompatible for $xt_{1,k+m} \leq w$ (see Figure 22(3)). This contradicts $x \in \text{maxsing}(X_w)$. So we can assume $d_{x,w}|_B \geq 2$. Since $\min(d_{x,w}|_{B \cup C \cup D}) = 1$, and $d_{x,w}$ is non-decreasing in region A as we move down or left, we can now assume that $\min(d_{x,w}|_C) = 1$. But then $t_{1,b}$ and $t_{a,k}$ are patching incompatible for $xt_{k,k+1} \leq w$. This contradicts $x \in \text{maxsing}(X_w)$. So $d_{x,w}|_A \leq 1$ as claimed.

Step 4. By the previous step, there is at most one point of $\text{mat}(w)$ in region A. But as $w > x$, $\widetilde{x} = x$ and $\widetilde{w} = w$, this fixes the remaining points and we have $w = [k+m, k, \ldots, 2, k+m-1, \ldots, k+1, 1]$, as claimed.

□

8.2. **Three decreasing sequences in $\widetilde{x}$.** We repeat the task of the previous section when $\widetilde{x}$ consists of three decreasing sequences rather than two.

Proposition 35. *Let $x \in \text{maxsing}(X_w)$ with $\widetilde{x} = x$ and $\widetilde{w} = w$. Suppose that x consists of exactly three decreasing subsequences:*

$$x = [k, \ldots, 1, k+l, \ldots, k+1, k+l+m, \ldots, k+l+1], \tag{8.5}$$

for some $k, l, m \geq 1$. Then

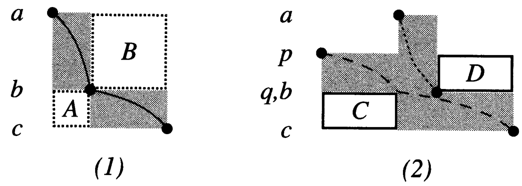


FIGURE 23

(1) $l \geq 2$,
(2) $l = 2$ *if* $k > 1$ *or* $m > 1$,
(3) *(shown in Figure 29(2))*

$$w = [k+l, k, \ldots, 2, k+l+m, k+l-1, \ldots, k+2,$$
$$1, k+l+m-1, \ldots, k+l+1, k+1].$$

Proof. Again, we'll assume throughout this proof that $a, a', \alpha \in [1, \ldots, k]$, $b, b', \beta \in [k+1, \ldots, k+l]$ and $c, \gamma \in [k+l+1, \ldots, k+l+m]$. We now prove a series of claims elucidating the structure of $\mathcal{R}(x, w)$.

Step 1. There exist a, b, c such that $t_{a,b}, t_{b,c} \in \mathcal{R}(x, w)$.
Suppose there is no such triple of indices. Then by the definition of $\Delta(x, w)$, for given a, c there exist b, b' ($b \neq b'$) such that $t_{a,b}, t_{b',c} \in \mathcal{R}(x, w)$. By the Ell Lemma 27, along with the assumptions that x is an MSP and that such triples do not exist, we can find $\alpha \neq a$ and $\beta \neq b'$ such that $t_{\alpha,\beta} \in \mathcal{E}_{a,b}(x, w)$. But then $t_{a,b}$ and $t_{\alpha,\beta}$ are patch incompatible for $xt_{b',c} \leq w$.

Step 2. If $t_{a,b}, t_{b,c} \in \mathcal{R}(x, w)$, then $t_{a,b} \in \mathcal{E}_{b,c}(x, w)$.
Suppose, on the contrary, that $t_{a,b} \in \operatorname{Im} \phi_{t_{b,c}}$. Then $d_{x,w} \geq 1$ on either all of region A or all of region B in Figure 23(1).

Assume that $d_{x,w} \geq 1$ on region A. Now, since x is an MSP, by Fact 25 there exists $t_{p,q} \in \mathcal{E}_{b,c}(x, w)$. We now consider the two possibilities for the relative positions of $t_{p,q}$ and $t_{b,c}$.

Suppose that $t_{p,q}$ and $t_{b,c}$ are link incompatible — i.e., we have $q = b$ (Figure 23(2)). For $t_{p,q}$ to be link incompatible with $t_{b,c}$, we need $p > a$ (as depicted in Figure 23(2)) since we are assuming $\min(d_{x,w}|_A) \geq 1$. Additionally, as $t_{p,q} \in \mathcal{E}_{b,c}(x, w)$, $d_{x,w}$ must have value 0 for at least one point on each of regions C and D. Thus $t_{p,b}$ and $t_{b,c}$ are link incompatible for $xt_{a,b} \leq w$.

On the other hand, $t_{p,q}$ and $t_{b,c}$ may be patch incompatible. Then there are four possibilities for the relative positions of $\mathrm{pt}_x(p)$, $\mathrm{pt}_x(b)$, $\mathrm{pt}_x(q)$ and $\mathrm{pt}_x(c)$ depending on whether $p < b$ and whether $q < c$ (see Figure 24). In each situation, $t_{p,q}$ and $t_{b,c}$ are patch incompatible for $xt_{a,b} \leq w$.

We have obtained a contradiction of $x \in \operatorname{maxsing}(X_w)$ for every scenario in which $d_{x,w} \geq 1$ on region A. Arguing similarly if $d_{x,w} \geq 1$ on region B, we conclude that $t_{a,b} \in \mathcal{E}_{b,c}(x, w)$.

Step 3. Given β, there exist α, γ such that $t_{\alpha,\beta}, t_{\beta,\gamma} \in \mathcal{R}(x, w)$.
By Step 1, there exist a, b, c such that $t_{a,b}, t_{b,c} \in \mathcal{R}(x, w)$. If $b = \beta$, then we are done — so assume not. We can at least find a q with $t_{\{\beta,q\}} \in \mathcal{R}(x, w)$. Without loss of generality, assume $q = \gamma$ for some $\gamma > \beta$. We split into cases according to whether $\gamma < c$, $\gamma = c$ or $\gamma > c$. These are depicted in Figure 25.

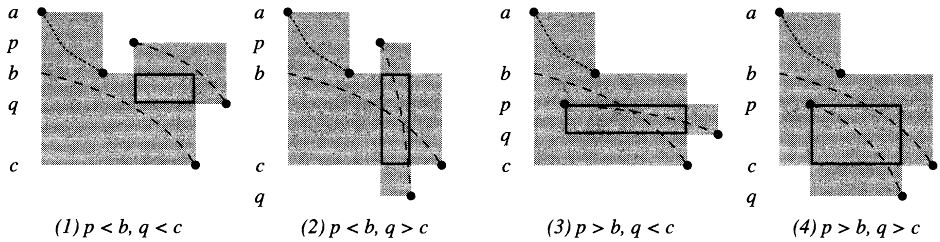


FIGURE 24

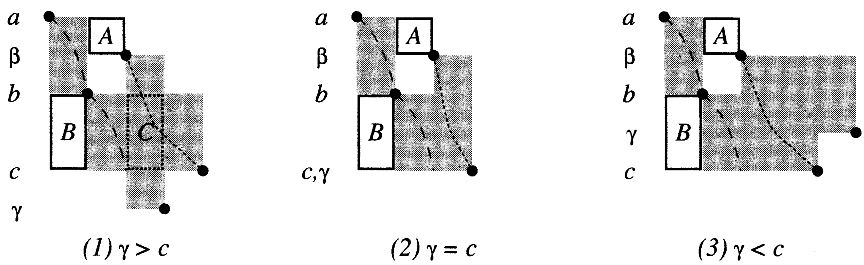


FIGURE 25. We have displayed the case of $\beta < b$, but the proof of Step 3 also holds for $\beta > b$.

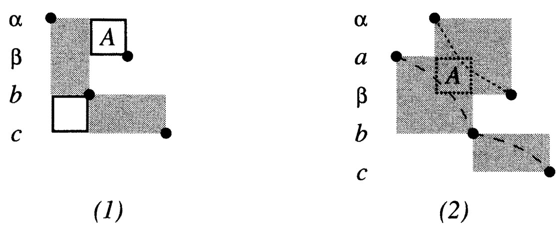


FIGURE 26

Note that by the previous step, $t_{a,b} \in \mathcal{E}_{b,c}(x, w)$, so $\min(d_{x,w}|_B) = 0$. In addition, if $\min(d_{x,w}|_A) \geq 1$, then $t_{a,\beta} \in \mathcal{R}(x, w)$ as desired. So in the following arguments (and Figure 25), we assume $\min(d_{x,w}|_A) = 0$ and derive a contradiction of $x \in \text{maxsing}(X_w)$.

Assume $\gamma > c$. If $\min(d_{x,w}|_C) \geq 2$, then $t_{a,b}, t_{b,c}$ are link incompatible for $xt_{\beta,\gamma} \leq w$. Otherwise, by the Cross Lemma 31, $t_{\beta,c} \in \mathcal{R}(x, w)$. Then $t_{a,b}, t_{b,c}$ are link incompatible for $xt_{\beta,c} \leq w$.

If $\gamma \leq c$, then $t_{\beta,c} \in \mathcal{R}(x, w)$ and we get a contradiction as above.

Step 4. For every α, β, γ, we have $t_{\alpha,\beta}, t_{\beta,\gamma} \in \mathcal{R}(x, w)$.

Suppose $t_{\alpha,\beta} \notin \mathcal{R}(x, w)$. By the definition of $\Delta(x, w)$ and the fact that $\widetilde{x} = x$, we know that there exists b such that $t_{\alpha,b} \in \mathcal{R}(x, w)$. Now we can apply the previous step to obtain a c such that $t_{b,c} \in \mathcal{R}(x, w)$. Note that by Step 2, $t_{\alpha,b}$ and $t_{b,c}$ are link incompatible. So our situation is as depicted as in Figure 26(1).

Using the logic of the previous step, we see that $\min(d_{x,w}|_A) = 0$ contradicts $x \in \text{maxsing}(X_w)$. Hence $t_{\alpha,\beta} \in \mathcal{R}(x, w)$ as desired.

The argument for showing $t_{\beta,\gamma} \in \mathcal{R}(x, w)$ is analogous.

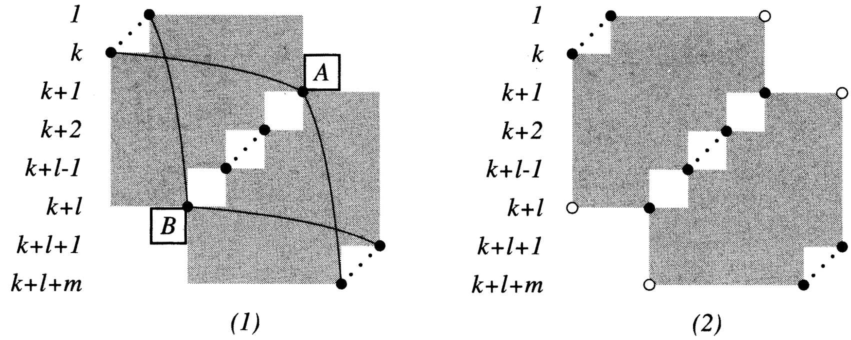


FIGURE 27

Step 5. $d_{x,w} = 1$ in the shaded region of Figure 27(1).

By Steps 2, 3 and 4, we see $d_{x,w} \geq 1$ on the shaded region of Figure 27(1).

We will show $d_{x,w} \leq 1$ also. If $a \neq \alpha$ and $b \neq \beta$, then $t_{a,b}, t_{\alpha,\beta} \in \mathcal{R}(x,w)$ implies $t_{\alpha,\beta} \in \mathcal{E}_{a,b}(x,w)$. By Step 4, $t_{b,c} \in \mathcal{E}_{a,b}(x,w)$ for some c (see Figure 26(2).

Suppose $t_{\alpha,\beta} \notin \mathcal{E}_{a,b}(x,w)$ — i.e., $d_{x,w} \geq 2$ on region A. Then $t_{a,b}$ and $t_{b,c}$ are link incompatible for $xt_{\alpha,\beta} \leq w$.

A similar argument can be used to show that if $b \neq \beta$ and $c \neq \gamma$, then $t_{b,c}, t_{\beta,\gamma} \in \mathcal{R}(x,w)$ implies $t_{\beta,\gamma} \in \mathcal{E}_{b,c}(x,w)$. The claim of $d_{x,w} \leq 1$ then follows by inspection from these two facts and the given explicit form of x.

Step 6. Condition (1) in Proposition 35 holds, namely $l \geq 2$.

By Step 2, $t_{k,k+1}$ and $t_{k+1,k+l+m}$ are link incompatible. This implies that $\min(d_{x,w}|_A) = 0$. Similarly, $\min(d_{x,w}|_B) = 0$. It then follows from our explicit description of x that the values of $w(i)$ for $i = 1, k+1$ are as shown in Figure 27(2). Arguing with $d'_{x,w}$ (see (5.6)) and region B, we see that $w(i)$ for $i = k+l, k+l+m$ is as shown in the same figure. But this means that $w^{-1}(k+l) = 1$ and $w^{-1}(k+1) = k+l+m$. This can only happen if $l > 1$.

Step 7. w is as stated in condition (3).

Step 5 tells us that we can conclude that $d_{x,w} = 1$ on all shaded areas of Figure 27. Therefore, $w(i) = k+2-i$ for $2 \leq i \leq k$. A similar argument to that in Step 5 shows that $w(i) = 2(k+l)+m-i$ for $k+l+1 \leq i < k+l+m$. So we need only investigate the values of $w(i)$ for $k+1 < i < k+l$. To do this, assume that $w(i) = x(i)$ for $k+1 < i \leq j$ for some j with $k+1 \leq j < k+l-1$. Then, as in Figure 28(1), we see that $d_{x,w} = 0$ on region B.

So, $t_{k,k+1}$ and $t_{k+1,j+2}$ are link incompatible for $xt_{j+2,k+l+m} \leq w$. This contradicts $x \in \operatorname{maxsing}(X_w)$. Hence $w(i) = x(i)$ for all i with $k+1 < i < k+l$. So

$$w = [k+l, k, \ldots, 2, k+l+m, k+l-1, \ldots, k+2, \\ 1, k+l+m-1, \ldots, k+l+1, k+1], \tag{8.6}$$

as desired.

Step 8. Condition (2) in Proposition 35 holds.

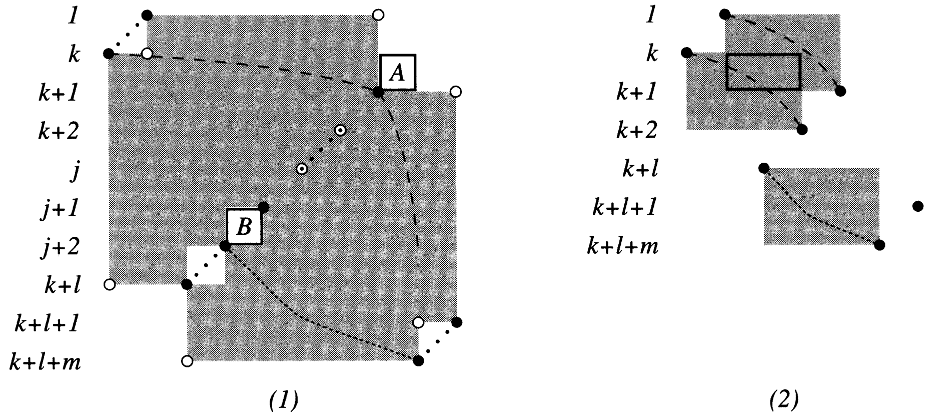


FIGURE 28

We need to show that if $l > 2$, then $k, m = 1$. So assume $k > 1$. By Steps 4 and 5, $t_{1,k+1}$ and $t_{2,k+2}$ are patch incompatible reflections for $xt_{k+l,k+l+m} \leq w$ (see Figure 28(2)). This contradicts $x \in \text{maxsing}(X_w)$. The argument showing that $m = 1$ is analogous.

This completes the proof of Proposition 35. □

9. MAXIMAL SINGULARITY OF CANDIDATES

We now finish the proof of Theorem 1 by showing that the restrictions we have discovered for $\widetilde{x}$ in Propositions 34 and 35 are sufficient to show that these points correspond to MSP's in the appropriate Schubert variety. This task consists of two steps:

(1) Show that the points x are singular points.
(2) Show that any cover of x that is still below w is a smooth point.

So that we can describe maxsing(X_w) succinctly, we introduce the following notation:

Definition 36. For $k, m \geq 2$, define

$$x_{k,m} = [k, \ldots, 1, k+m, \ldots, k+1], \tag{9.1}$$

$$w_{k,m} = [k+m, k, \ldots, 2, k+m-1, \ldots, k+1]. \tag{9.2}$$

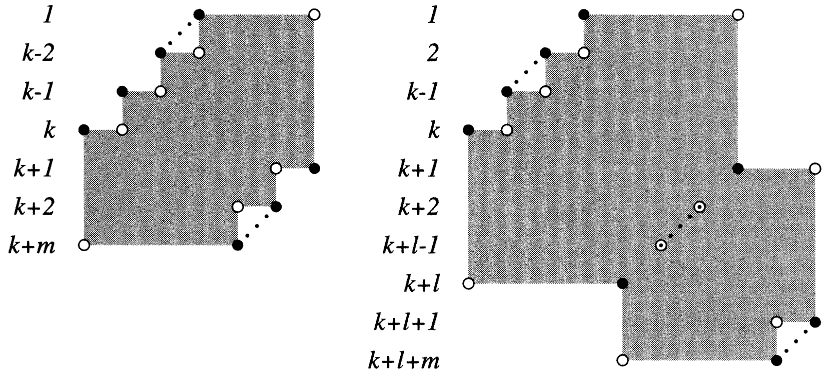
For $k, m \geq 1$ and $l \geq 2$, define

$$x_{k,l,m} = [k, \ldots, 1, k+l, \ldots, k+1, k+l+m, \ldots, k+l+1], \tag{9.3}$$

$$\begin{aligned} w_{k,l,m} = [&k+l, k, \ldots, 2, k+l+m, k+l-1, \ldots, k+2, \\ &1, k+l+m-1, \ldots, k+l+1, k+1]. \end{aligned} \tag{9.4}$$

Theorem 37 (Rephrasing Theorem 1). *x is an* MSP *of X_w if and only if*

(1) *$t \in \mathcal{R}(x, w)$ implies $l(xt) = l(x) + 1$.*
(2) (a) *For some $k, m \geq 2$, we have $\widetilde{x} = x_{k,m}$ and $\widetilde{w} = w_{k,m}$*
or
(b) *For some $k, m \geq 1$, $l = 2$ or $k = m = 1, l \geq 2$, we have $\widetilde{x} = x_{k,l,m}$ and $\widetilde{w} = w_{k,l,m}$.*



(1) (2)

FIGURE 29

Proof. Proposition 30 tells us that condition (1) is necessary. Propositions 34 and 35 tell us that conditions (2a) and (2b) are necessary. So all we need to show is sufficiency.

Let t be a reflection such that $x < y = xt \leq w$. As ϕ_t is injective, to calculate $\#\mathcal{R}(y, w)$ from $\mathcal{R}(x, w)$ we need only count how many reflections in $\mathcal{R}(x, w)$ are not in the image of ϕ_t. Note by Proposition 18 that $\#\mathcal{R}(x, w) = \#\mathcal{R}(\widetilde{x}, \widetilde{w})$ and $l(w) - l(x) = l(\widetilde{w}) - l(\widetilde{x})$.

Consider first the case shown in Figure 29(1) of two decreasing sequences for $\widetilde{x} = x_{k,m}$. Note that

$$l(w_{k,m}) = \binom{k}{2} + \binom{m}{2} + k + m - 1, \tag{9.5}$$

$$l(x_{k,m}) = \binom{k}{2} + \binom{m}{2}, \tag{9.6}$$

$$\#\mathcal{R}(x, w) = k \cdot m \text{ and} \tag{9.7}$$

$$l(w) - l(x) - \#\mathcal{R}(x, w) = k + m - km - 1. \tag{9.8}$$

Since $k, m \geq 2$, (9.8) is negative. So by Theorem 11, e_x is a singular point of X_w.

To prove that it is a maximal singular point, we consider some $t_{a,b} \in \mathcal{R}(x, w)$ and let $y = xt_{a,b}$.

Then, viewing Figure 30(1), it is easily seen that $\#\mathcal{R}(y, w) = (k-1) + (m-1) = k + m - 2$. Since $l(y) = l(x) + 1$, by Theorem 11 and (9.8), y is a smooth point of X_w. Since y was chosen as an arbitrary cover of x, x is an MSP for w.

Now we prove the case shown in Figure 29(2) of three decreasing sequences for $\widetilde{x} = x_{k,l,m}$. Note that

$$l(w_{k,l,m}) = \binom{k}{2} + \binom{l}{2} + \binom{m}{2} + k + m + 2(l-2) + 1, \tag{9.9}$$

$$l(x_{k,l,m}) = \binom{k}{2} + \binom{l}{2} + \binom{m}{2}, \tag{9.10}$$

$$\#\mathcal{R}(x, w) = l(k + m) \text{ and} \tag{9.11}$$

$$l(w) - l(x) - \#\mathcal{R}(x, w) = (l-1)\left(1 + \frac{l-2}{l-1} - k - m\right). \tag{9.12}$$

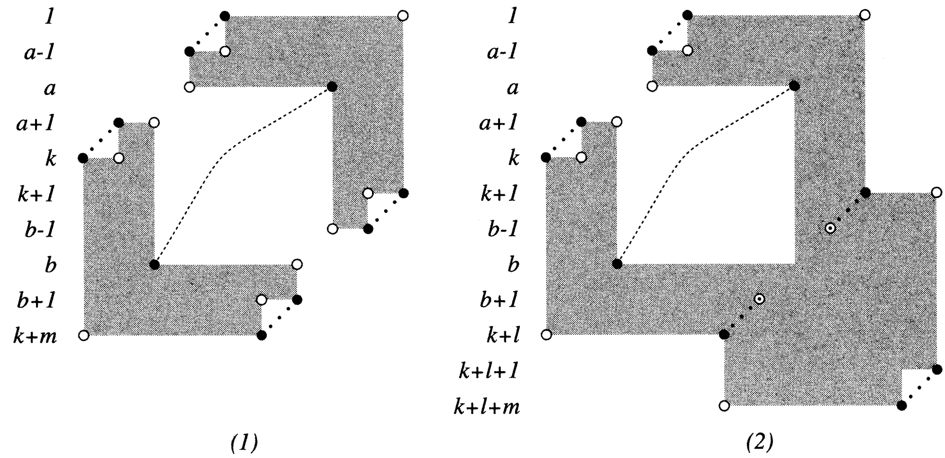


FIGURE 30

Since $k, m \geq 1$ and $l \geq 2$, (9.12) is negative. So by Theorem 11, x is a singular point of X_w. To prove it is an MSP for w, as above we consider some $t_{a,b} \in \mathcal{R}(x, w)$ and let $y = xt_{a,b}$. We have $l(w) - l(y) = k + m + 2(l - 2)$.

Viewing Figure 30(2), it is clear that $\#\mathcal{R}(y, w) = (k - 1) + (l - 1) + m(l - 1)$. If $l = 2$, then $\#\mathcal{R}(y, w) = k + m = l(w) - l(y)$. If $l > 2$, then by (2) of Proposition 35, we have that $k = m = 1$, and $\#\mathcal{R}(y, w) = 2(l - 1) = l(w) - l(y)$. So, in either case, y is a smooth point of X_w.

So in both cases, x is an MSP of X_w as claimed. □

This completes the proof of Theorem 37. It is easy to check that the above formulation is equivalent to Theorem 1. (Note, however, that the values of k, l, m in the statement of Theorem 37 differ from those used in Theorem 1.)

10. Lakshmibai-Sandhya Conjecture

Let $w = [w(1), \ldots, w(n)] \in \mathfrak{S}_n$. Define E_w to be the set of all $x = [x(1), \ldots, x(n)]$ satisfying the following conditions:

(1) There exist $i < j < k < l$ and $i' < j' < k' < l'$ such that (as sets)

$$\{w(i), w(j), w(k), w(l)\} = \{x(i'), x(j'), x(k'), x(l')\}.$$

(2) One of the following holds:
  (a) $\mathrm{fl}_{ijkl}(w) = 3412$ and $\mathrm{fl}_{i'j'k'l'}(x) = 1324$.
  (b) $\mathrm{fl}_{ijkl}(w) = 4231$ and $\mathrm{fl}_{i'j'k'l'}(x) = 2143$.

(3) Using the notation of Section 3, set

$$\underline{w} = \mathrm{unfl}^{w}_{ijkl}(\mathrm{fl}_{i'j'k'l'}(x)) \text{ and}$$
$$\widehat{x} = \mathrm{unfl}^{x}_{i'j'k'l'}(\mathrm{fl}_{ijkl}(w)).$$

Then

$$\underline{w} \leq x \leq \widehat{x} \leq w. \tag{10.1}$$

Theorem 38 (Conjecture in [21]). *For $w \in \mathfrak{S}_n$, the singular locus of X_w is equal to $\cup_x X_x$, where x runs over the maximal elements of E_w in Bruhat order.*

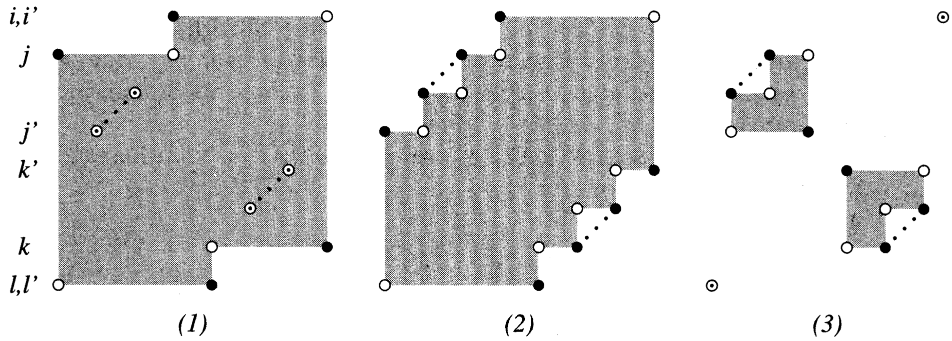


FIGURE 31. (1) $\underline{w}, w$; (2) x, w; (3) $\widehat{x}, w$.

Proof. That the points constructed by Lakshmibai and Sandhya are singular was first proved by Gasharov [17]. We refer the reader there for a proof of this direction of the conjecture. Independent proofs can also be found in Manivel [27] and Kassel-Lascoux-Reutenauer [19].

We only give the argument that $\operatorname{maxsing}(X_w) \subseteq \bigcup_{x \in E_w} X_x$ for singular points of the type 4231 (i.e., those described in case (1) of Theorem 1). The argument for singular points of type 3412 and 45312 is analogous.

Fix some $x \in \operatorname{maxsing}(X_w)$ (of type 4231). We will choose indices i, j, k, l and i', j', k', l' as described in the definition of E_w and show that (10.1) is satisfied for our choice of indices. So, using the notation of Theorem 1, let

$$\alpha_1 < \beta_1 < \beta_2 < \cdots < \beta_{k-1} < \alpha_2 < \alpha_3 < \cdots < \alpha_m < \beta_k \tag{10.2}$$

correspond to a type 4231 pattern in w. Then set

$$i = \alpha_1, \quad j = \beta_1, \quad k = \alpha_{m-1} \quad \text{and } l = \beta_k, \tag{10.3}$$

$$i' = \alpha_1, \quad j' = \beta_{k-1}, \quad k' = \alpha_2 \quad \text{and } l' = \beta_k. \tag{10.4}$$

Now, recall from Lemma 3 that $u \leq v$ if and only if $d_{u,v}$ is everywhere non-negative. But then (10.1) follows from Figure 31 along with the observation that $d_{u,v} \leq 1$ in each of these diagrams.

□

11. KAZHDAN-LUSZTIG POLYNOMIALS AT ELEMENTS OF $\operatorname{maxsing}(X_w)$

The Kazhdan-Lusztig polynomials associated to $SL(n)$ are indexed by two permutations x, w. Through work of Kazhdan and Lusztig [20], Beilinson-Bernstein [1] and Brylinski-Kashiwara [11], they have important interpretations in the context of Verma modules. In addition, these polynomials are related to the singular loci of Schubert varieties by a result of Kazhdan and Lusztig [20] that the Kazhdan-Lusztig polynomial $P_{x,w} = 1$ if and only if e_x is a smooth point of $X_w \subseteq SL(n)/B$. For further properties of these polynomials, see [18].

Lascoux and Schützenberger [24], Zelevinskiĭ [31], Lascoux [23], Brenti [6, 7, 8] and others [5, 9, 28] all calculate explicit formulas for these polynomials in specific cases. In this section, we compute $P_{x,w}$ when $x \in \operatorname{maxsing}(X_w)$. We note that Theorem 42(3) is proved in [29] and Theorem 42(2) is proved in [24], but both are only proved in the case where $\widetilde{x} = x$ and $\widetilde{w} = w$.

A result of Polo, [28], states that every polynomial in $\mathbb{N}[q]$ with constant term 1 can be realized as a Kazhdan-Lusztig polynomial in $\mathfrak{S}_n$ for some n. However, as Theorem 42 shows, the Kazhdan-Lusztig polynomials at elements of $\mathrm{maxsing}(X_w)$ are of very limited forms.

For pairs of permutations $x, w \in \mathfrak{S}_n$, we can define the Kazhdan-Lusztig polynomials by the following properties:

(1) $P_{x,w} = 0$ if $x \not\leq w$.
(2) $P_{x,w} = 1$ if $x \leq w$ and $l(w) - l(x) \leq 2$.
(3) $\deg(P_{x,w}) \leq \frac{1}{2}(l(w) - l(x) - 1)$.
(4) If $s \in \mathcal{S}$ such that $ws < w$, then

$$P_{x,w} = q^c P_{x,ws} + q^{1-c} P_{xs,ws} - \sum_{x \leq z < ws, zs < z} \mu(z, ws) q^{(l(w)-l(z))/2} P_{x,z}, \tag{11.1}$$

where $\mu(z, ws)$ is the coefficient of $q^{(l(ws)-l(z)-1)/2}$ in $P_{z,ws}$ and $c = 1$ if $xs < x$, $c = 0$ if $xs > x$.

Lemma 39. *If $i \notin \Delta(x, w)$, then $P_{x,w} = P_{x^{\hat{i}}, w^{\hat{i}}}$*

Proof. Fix $x < w$ and pick some $i \notin \Delta(x, w)$. We know by Proposition 14 that $x(i) = w(i)$. By Corollary 21, this implies that if $x \leq z \leq w$ for some z, then $z(i) = x(i) = w(i)$.

With these facts, the result then follows easily by induction on $l(w)$ using (11.1). (Note that our base case of $l(w) = 1$ is trivial.) □

Corollary 40. $P_{\widetilde{x}, \widetilde{w}} = P_{x,w}$.

For reference we state the following fact [18, Cor. 7.14]:

Fact 41. For $s, s' \in \mathcal{S}$, $ws < w$, $s'w < w$, then $P_{x,w} = P_{xs,w} = P_{s'x,w}$.

We are now ready to calculate $P_{x,w}$ for $x \in \mathrm{maxsing}(X_w)$. By Theorem 37 and Corollary 40, it is enough to calculate $P_{x,w}$ for the pairs $x_{k,m}$, $w_{k,m}$ and $x_{k,l,m}$, $w_{k,l,m}$.

Theorem 42. *All Kazhdan-Lusztig polynomials at maximal singular points:*

(1) *4231-type: For $k, m \geq 2$,*
$$P_{x_{k,m}, w_{k,m}} = 1 + q + \cdots + q^{\min(k-1, m-1)}.$$
(2) *3412-type: For $k, m \geq 1$,*
$$P_{x_{k,2,m}, w_{k,2,m}} = 1 + q.$$
(3) *45312-type: For $l \geq 2$,*
$$P_{x_{1,l,1}, w_{1,l,1}} = 1 + q^{l-1}.$$

Proof. We refer the reader to Manivel [26] or Cortez [14] for independent proofs of this theorem. However, in order to illustrate the utility of our diagrams, we include a proof for the 45312-type polynomials here (the other types are analogous).

Let x and w be $x_{1,l,1}$ and $w_{1,l,1}$ respectively. We apply induction on l. The case of $l = 2$ is covered by the 3412-type case, so we assume $l \geq 3$.

In Figure 32, we depict the pairs x, w and x, ws_2 and xs_2, ws_2. We claim that the first two terms in (11.1) contribute $(1 + q)(1 + q^{l-2})$. First consider the pair x, ws_2. Since $ws_2 s_1 < ws_2$, by the induction hypothesis, Corollary 16

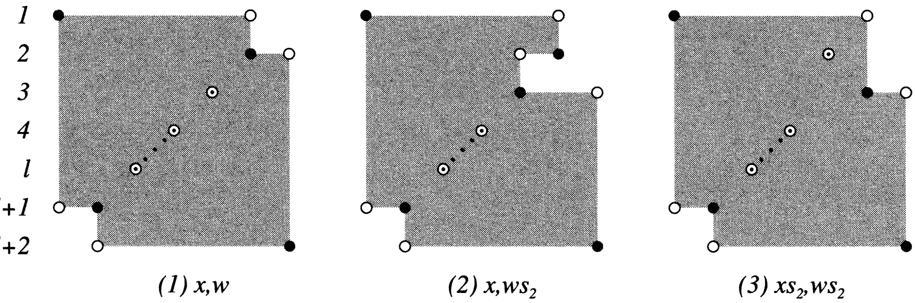


FIGURE 32

and Lemma 39, we see that $P_{x,ws_2} = P_{xs_1,ws_2} = 1 + q^{l-2}$. Now consider the pair xs_2, ws_2. Since $s_l ws_2 < ws_2$ and $ws_2 s_1 < ws_2$, it follows that $P_{xs_2,ws_2} = P_{s_l xs_2 s_1,ws_2}$. But since $s_l xs_2 s_1 = xs_1$, we get that $P_{xs_2,ws_2} = 1 + q^{l-2}$ also. Incorporating this information into (11.1), we can write

$$P_{x,w} = 1 + q^{l-2} + q + q^{l-1} - \sum_{\substack{x \leq z < ws \\ zs_2 < z}} \mu(z, ws_2) q^{\frac{l(w)-l(z)}{2}} P_{x,z}. \tag{11.2}$$

Now we check which z will appear in the sum in (11.2). First note that xs_1 is the unique MSP for ws_2. By induction, $P_{xs_1,ws_2} = 1 + q^{l-2}$. By Fact 41, $P_{e,ws_2} = P_{xs_1,ws_2}$. Hence, the only z such that $l(z) < l(ws_2)-1$ and $\deg(P_{z,ws_2})$ is maximized is $z = xs_1$. However, $xs_1s_2 > xs_1$, so xs_1 does not appear in the sum. So the only possible terms in the sum are those with $l(z) = l(ws_2) - 1$. From Figure 32(2), we see that $z = ws_2s_3$ is the only z satisfying both this length condition and $zs_2 < z$. Using Fact 41, Lemma 39 and the induction hypothesis, one can check that $P_{x,ws_2s_3} = 1 + q^{l-3}$. Hence, the sum in (11.2) contributes $-q - q^{l-2}$. Simplifying, we see that $P_{x,w} = 1 + q^{l-1}$ as claimed. □

Remark 43. In related work, Brion and Polo [10] compute the singular locus and Kazhdan-Lusztig polynomials for Schubert varieties associated to certain parabolic subgroups of connected semisimple algebraic groups.

12. EXAMPLES CALCULATING maxsing(X_w)

Example 44. Using Theorem 1, in Figure 33 we compute the singular locus

$$\operatorname{maxsing}(X_w) = X_{[48376512]} \cup X_{[64387512]} \cup X_{[46587312]} \cup X_{[68174325]} \tag{12.1}$$

of X_w where $w = [6, 8, 4, 7, 5, 3, 1, 2]$.

Remark 45. The cardinality of the set maxsing(X_w) may be $O(n^4)$. For example, $\#\operatorname{maxsing}(X_w) = \binom{n/2}{2}^2$ when $w = [n/2 + 1, \ldots, n, 1, \ldots, n/2]$ and n is even.

Example 46. Using a computer it is easy to calculate, for example, that $\#\operatorname{maxsing}(X_w) = 29$ for

$$w = [17, 6, 2, 15, 12, 11, 3, 8, 16, 7, 14, 5, 13, 9, 10, 1, 4]. \tag{12.2}$$

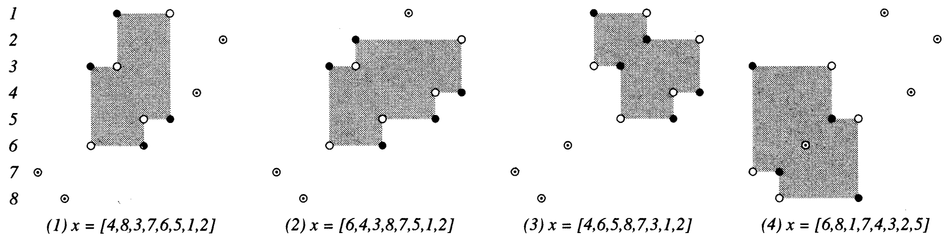


FIGURE 33

13. PATTERNS INDEXING maxsing(X_w)

Which 4231 or 3412 patterns lead to elements in maxsing(X_w)? We can describe these patterns by taking all 4231 and 3412 patterns in w and removing certain "useless patterns" contained in larger patterns of length 5 or 6. For example, if $w = [52341]$, the pattern 5241 will be useless since the shaded region it defines is not empty. We describe the useless patterns in the following way. For each pattern of length 5 or 6 in (13.1), remove the corresponding dotted pattern.

$$\begin{array}{lllll} (\dot{3}\dot{5}4\dot{1}\dot{2}) & (\dot{4}3\dot{5}\dot{1}\dot{2}) & (\dot{4}\dot{5}\dot{1}3\dot{2}) & (\dot{4}\dot{5}2\dot{1}\dot{3}) & \\ (\dot{5}\dot{2}3\dot{4}\dot{1}) & (\dot{5}\dot{2}\dot{4}3\dot{1}) & (\dot{5}3\dot{2}\dot{4}\dot{1}) & (\dot{5}\dot{3}\dot{4}2\dot{1}) & (\dot{5}4\dot{2}\dot{3}\dot{1}) \\ (\dot{6}\dot{3}52\dot{4}\dot{1}) & (\dot{5}\dot{6}34\dot{1}\dot{2}) & (\dot{5}2\dot{6}4\dot{1}\dot{3}) & (\dot{4}\dot{6}3\dot{1}5\dot{2}) & \end{array} \tag{13.1}$$

Each of the remaining "useful patterns" index a unique component in maxsing(X_w). For example, if $w = [7432651]$, then maxsing(X_w) has only one element — namely $x = [4321765]$ — and this element would be indexed by 7251. This example corresponds to the shape in Figure 1(1).

It would be interesting to know the distribution of the various sizes of maxsing(X_w) for all $w \in \mathfrak{S}_n$ for large n.

REFERENCES

1. A. Beilinson and J. Bernstein, *Localization of $\mathfrak{g}$-modules*, C. R. Acad. Sci. Paris Ser. I Math **292** (1981), 15–18. MR **82k:**14015
2. S. Billey, *Pattern avoidance and rational smoothness of Schubert varieties*, Adv. Math. **139** (1998), no. 1, 141–156. MR **99i:**14058
3. S. Billey and V. Lakshmibai, *Singular Loci of Schubert Varieties*, Progress in Mathematics, no. 182, Birkhäuser, 2000. MR **2001j:**14065
4. S. Billey and T. K. Lam, *Vexillary elements in the hyperoctahedral group*, J. Alg. Combin. **8** (1998), no. 2, 139–152. MR **2000d:**05124
5. S. Billey and G. Warrington, *Kazhdan-Lusztig polynomials for* 321*-hexagon-avoiding permutations*, J. Alg. Comb. **13** (2001), 111–136.
6. F. Brenti, *A combinatorial formula for Kazhdan-Lusztig polynomials*, Invent. Math. **118** (1994), no. 2, 371–394. MR **96c:**20074
7. ———, *Combinatorial expansions of Kazhdan-Lusztig polynomials*, J. London Math. Soc. **55** (1997), no. 2, 448–472. MR **99a:**05143
8. ———, *Kazhdan-Lusztig polynomials and R-polynomials from combinatorial point of view*, Discrete Math **193** (1998), no. 1–3, 93–116. MR **2000c:**05154
9. F. Brenti and R. Simion, *Explicit formulae for some Kazhdan-Lusztig polynomials*, J. Alg. Comb. **11** (2000), no. 3, 187–196. MR **2001e:**05137
10. M. Brion and P. Polo, *Generic singularities of certain Schubert varieties*, Math. Z. **231** (1999), no. 2, 301–324. MR **2000f:**14078

11. J.-L. Brylinski and M. Kashiwara, *Kazhdan-Lusztig conjectures and holonomic systems*, Invent. Math. **64** (1981), 387–410. MR **83e:**22020
12. J. B. Carrell and J. Kuttler, *On the smooth points of T-stable varieties in G/B and the Peterson map*, Invent. Math. **151** (2003), no. 2, 353–379.
13. C. Chevalley, *Sur les décompositions cellulaires des espaces G/B*, Proceedings of Symposia in Pure Mathematics **56** (1994), no. Part I, 1–25. MR **95e:**14041
14. A. Cortez, *Singularites generiques et quasi-resolutions des variétés de Schubert pour le groupe lineaire*, C. R. Acad. Sci. Paris Ser. I Math. **33** (2001), 561–566.
15. V. Deodhar, *Local Poincaré duality and non-singularity of Schubert varieties*, Comm. Algebra **13** (1985), 1379–1388. MR **86i:**14015
16. W. Fulton, *Young tableaux; with applications to representation theory and geometry*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, New York, 1997. MR **99f:**05119
17. V. Gasharov, *Sufficiency of Lakshmibai-Sandhya singularity conditions for Schubert varieties*, Compositio Math. **126** (2001), no. 1, 47–56. MR **2002d:**14078
18. J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press, 1990. MR **92h:**20002
19. C. Kassel, A. Lascoux, and C. Reutenauer, *The singular locus of a Schubert variety*, IRMA Prepublication (2001); to appear in J. Algebra.
20. D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165–184. MR **81j:**20066
21. V. Lakshmibai and B. Sandhya, *Criterion for smoothness of Schubert varieties in $SL(n)/B$*, Proc. Indian Acad. Sci. (Math Sci.) **100** (1990), no. 1, 45–52. MR **91c:**14061
22. V. Lakshmibai and C. S. Seshadri, *Singular locus of a Schubert variety*, Bull. Amer. Math. Soc. **11** (1984), no. 2, 363–366. MR **85j:**14095
23. A. Lascoux, *Polynômes de Kazhdan-Lusztig pour les variétés de Schubert vexillaires*, C. R. Acad. Sci. Paris Sér. I Math. **321** (1995), no. 6, 667–670. MR **96g:**05144
24. A. Lascoux and M.-P. Schützenberger, *Polynômes de Kazhdan & Lusztig pour les grassmanniennes*, Young tableaux and Schur functors in algebra and geometry (Toruń, 1980), Soc. Math. France, Paris, 1981, pp. 249–266. MR **83i:**14045
25. A. Lascoux and M.-P. Schützenberger, *Schubert polynomials and the Littlewood-Richardson rule*, Letters in Math. Physics **10** (1985), 111–124. MR **87g:**20021
26. L. Manivel, *Generic singularities of Schubert varieties*, arXiv:math.AG/0105239 (2001).
27. L. Manivel, *Le lieu singulier des variétés de Schubert*, Internat. Math. Res. Notices (2001), no. 16, 849–871.
28. P. Polo, *Construction of arbitrary Kazhdan-Lusztig polynomials in symmetric groups*, Represent. Theory **3** (1999), 90–104, (electronic). MR **2000j:**14079
29. B. Shapiro, M. Shapiro, and A. Vainshtein, *Kazhdan-Lusztig polynomials for certain varieties of incomplete flags*, Discrete Math. **180** (1998), 345–355. MR **99b:**14054
30. J. Stembridge, *On the fully commutative elements of Coxeter groups*, J. Alg. Combin. **5** (1996), no. 4, 353–385. MR **97g:**20046
31. A. V. Zelevinskiĭ, *Small resolutions of singularities of Schubert varieties*, Functional Anal. Appl. **17** (1983), no. 2, 142–144. MR **85b:**14069

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