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On Schubert Varieties in the Flag Manifold of Sl$(n, \mathbb{C})$

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In this paper we study Schubert varieties in the flag manifold of Sl$(n, \mathbb{C})$. Two main results are obtained:

1. A combinatorial method is derived whereby one can determine the singular locus of a Schubert variety. In particular, this allows one to determine whether a given Schubert variety is singular or non-singular. These results are given in Theorems I and II, which are stated in Sect. II.

2. Using the method derived in Sect. II, a detailed geometric description of the non-singular Schubert varieties as repeated fibrations of Grassmannians is obtained. This result is given in Theorems III and IV of Sect. IV.

The sections of this paper are arranged as follows:

Section I presents the definitions and notations that will be used throughout the remainder of the paper. Section II develops and presents the combinatorial algorithm for determining the singular locus of a Schubert variety. Section III develops the combinatorics which are the basis for the main structure theorem about non-singular Schubert varieties. Section IV presents the structure theorem for non-singular Schubert varieties.

This paper is a reworking of the author's doctoral thesis [R]. The combinatorial algorithm for locating the singular locus of a Schubert variety was also discovered independently and more or less concurrently by Lakshmibai and Seshadri [LS]. (See also a closely related paper of Deodhar [D].) The result of [LS] is stated in a slightly different form here, in order to facilitate the calculations required in Sects. III and IV.

I. Definitions and Notation

1. *Flags and Schubert Varieties*

Let Sl$(n, \mathbb{C})$ denote the set of all $n \times n$ complex matrices with determinant $=1$, and let B be the Borel subgroup of Sl$(n, \mathbb{C})$ consisting of all the upper triangular matrices. We write Fl(n) for the flag manifold Sl$(n, \mathbb{C})/B$ associated to Sl$(n, \mathbb{C})$. The B-orbits in Fl(n) are called Bruhat cells and their closures are called *Schubert* *varieties.* For the purposes of this paper we find it convenient to regard Fl(n) and its Schubert varieties in the following more geometric manner.

Definition. A (complete) *flag* in $\mathbb{C}^n$ is a sequence of subspaces of $\mathbb{C}^n$

$$0 \subset V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset \mathbb{C}^n, \quad \text{with} \quad \dim(V_i) = i.$$

We write $\{(V_i)\}$ for such a sequence.

Fl(n) is then the collection of all complete flags in $\mathbb{C}^n$, endowed with the obvious topology. Fl(n) is a smooth complex algebraic variety of complex dimension $n(n-1)/2$. We similarly make the following

Definition. A *partial flag* of type $(i_1, i_2, \ldots, i_k)$ in $\mathbb{C}^n$ is a sequence of subspaces $0 \subset V_{i_1} \subset V_{i_2} \subset \ldots \subset V_{i_k} \subset \mathbb{C}^n$, with $\dim(V_{i_j}) = i_j$ and $i_1 < i_2 < \ldots < i_k$.

We write Fl$(i_1, i_2, \ldots, i_k; \mathbb{C}^n)$ to denote the collection of all partial flags of type $(i_1, \ldots, i_k)$ in $\mathbb{C}^n$. Fl$(i_1, i_2, \ldots, i_k; \mathbb{C}^n)$ is a smooth compact complex algebraic variety of complex dimension

$$i_1(n-i_1) + (i_2 - i_1)(n - i_2) + \ldots + (i_k - i_{k-1})(n - i_k).$$

We write Fl$(*; \mathbb{C}^n)$ to denote a partial flag manifold when the exact type of the flags of which it is composed is unimportant.

2. *The Correspondence Between Schubert Varieties and Permutations*

Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ denote the standard basis for $\mathbb{C}^n$ and let $\mathbb{C}^k$ denote the span of the first k of these basis vectors.

Let $s = (s(1), s(2), \ldots, s(n))$ be any permutation of $(1, 2, \ldots, n)$. Let $E(s)$ be the flag given by $V_k = \text{span}(\mathbf{e}_{s(1)}, \mathbf{e}_{s(2)}, \ldots, \mathbf{e}_{s(k)})$ for $k = 1, \ldots, n$. [In the case where s is the identity permutation $(1, 2, \ldots, n)$ we write E for the flag $E(s)$.] Observe that each Bruhat cell in Fl(n) is the B-orbit of exactly one $E(s)$; we write B_s for the Bruhat cell which contains $E(s)$ and write X_s for the Schubert variety which is the closure of B_s. B_s consists precisely of all those flags $F = \{(V_i)\}$ which satisfy

$$\dim(V_j \cap \mathbb{C}^k) = \#(\{s(1), \ldots, s(j)\} \cap \{1, \ldots, k\}) \quad \text{for all } j, k.$$

Given a Bruhat cell B_s we may compute the numbers $d_{jk}(s) = \dim(V_j \cap \mathbb{C}^k)$, $j = 1, \ldots, n-1$; $k = 1, \ldots, n$ for any flag F in B_s. We call each such equation a Schubert condition which B_s satisfies. B_s is uniquely determined by the set of all Schubert conditions which it satisfies.

Note too that the Bruhat cell B_s satisfies

$$\begin{aligned} \operatorname{codim} B_s &= \sum_k \#\{s(i) : i < k \text{ and } s(i) < s(k)\} \\ &= n(n-1)/2 - \text{length}(s), \end{aligned}$$

where length is the standard length function on the permutation group S_n.

Similarly, given a Schubert variety $X = X_s$ we see that X is uniquely determined by all of the inequalities $\dim(V_j \cap \mathbb{C}^k) \geq d_{jk}$, $j = 1, \ldots, n-1$; $k = 1, \ldots, n$ which are satisfied by every flag in X. We again call these the Schubert conditions for X. In the case $d_{jk} = \min(j, k)$ we write the corresponding Schubert condition as $V_j \subset \mathbb{C}^k$ (or

$\mathbb{C}^k \subset V_j$). We write $<$ to denote the Bruhat order on the set of all permutations; that is $t<s$ iff $X_t \subset X_s$. Observe that $t<s$ iff $d_{jk}(t) \geqq d_{jk}(s)$ for all j, k.

Finally, we remark that the notions of Bruhat cell, Schubert variety, and Schubert condition all extend mutatis mutandis to the case of a partial flag manifold $\mathrm{Fl}(i_1, \ldots, i_k; \mathbb{C}^n)$.

II. Locating Singularities in Schubert Varieties

In this section we provide a method whereby one can determine the singular locus of a Schubert variety in $\mathrm{Fl}(n)$, and in particular one can determine if a Schubert variety is non-singular. The results are contained in the following two theorems.

Theorem I. *Let $X = X_s$ be the Schubert variety in $\mathrm{Fl}(n)$ corresponding to the permutation s. Let $A = A(s)$ be the set of all pairs (i,j) which satisfy*

(i) $1 \leqq j < i \leqq n$ *and*

(ii) *either* $\{s(1), \ldots, s(j)\} \subset \{1, 2, \ldots, i-1\}$ *or* $\{1, \ldots, j\} \subset \{s(1), \ldots, s(i-1)\}$.

Then X is non-singular $\Leftrightarrow \# A = \operatorname{codim} X$ (in $\mathrm{Fl}(n)$).

Theorem I is restated in Sect. III after certain combinatorial diagrams are introduced; it is referred to there as Theorem Ia.

Theorem II. *Let $X = X_s$ as above and let $Y = Y_t$ be contained in X. For every pair (k, m), $1 \leqq k, m \leqq n$ let $B(k,m) =$ the empty set if*

$$\#(\{t(1), \ldots, t(k)\} \cap \{1, \ldots, m\}) \neq \#(\{s(1), \ldots, s(k)\} \cap \{1, \ldots, m\})$$

while if

$$\#(\{t(1), \ldots, t(k)\} \cap \{1, \ldots, m\}) = \#(\{s(1), \ldots, s(k)\} \cap \{1, \ldots, m\})$$

let $B(k,m) =$ the set of all pairs (i,j) which satisfy

(i) $i > m$ *and* $i \notin (\{t(1), \ldots, t(k)\} \cap \{m+1, \ldots, n\})$ *and*

(ii) $j \leqq k$ *and* $t(j) \leqq m$.

Let B be the union of all such $B(k,m)$, $1 \leqq k, m \leqq n$.

Then X is non-singular along $Y \Leftrightarrow \# B = \operatorname{codim} X$.

The remainder of this section is devoted to the proof of these theorems. As stated in the introduction, similar results are obtained in [LS] and [D].

The results of this section rely on the well-known Jacobian criterion for non-singularity of an algebraic variety.

1. *Coordinates on* $\mathrm{Fl}(n)$

We begin by defining a convenient set of coordinates on an open, dense affine subset of $\mathrm{Fl}(n)$.

Let U denote the collection of all $n \times n$ complex matrices $M = (x_{ij})$ which satisfy $x_{ii} = 1$ $(i = 1, \ldots, n)$ and $x_{ij} = 0$ for $j > i$. U is naturally identified with the affine space $\mathbb{C}^N$ where $N = n(n-1)/2$. We also regard U as a subset of $\mathrm{Fl}(n)$ by regarding each matrix M in U as representing the flag $F = \{(V_i)\}$ where V_i is the span of the first i columns of M. The matrix entries x_{ij} which are not required to be 0 or 1 provide the desired coordinates.

Observe that the origin of this coordinate system corresponds to the fixed flag E. Observe too that if X is singular, then it is necessarily singular at the point E and so we need only check at the origin of this coordinate system to determine if X is singular.

2. *Equations for Schubert Varieties*

Let X denote any Schubert variety in $\mathrm{Fl}(n)$. We know that X is determined by the Schubert conditions $\dim(V_k \cap \mathbb{C}^m) \geqq d_{km}$ which it satisfies. Each such Schubert condition may be expressed as polynomial equations in the coordinates x_{ij} as follows:

Let $\mathbf{c}_i$ denote the i^{th} column of the coordinate matrix M defined above, and let $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ be the standard basis for $\mathbb{C}^n$. (We regard the $\mathbf{e}_i$ as column vectors, each having exactly one non-zero entry.) Write d for d_{km}. We then have

$$\dim(V_k \cap \mathbb{C}^m) \geqq d \quad \text{iff} \quad \dim(V_k + \mathbb{C}^m) \leqq k + m - d$$

iff

$$\dim \operatorname{span}(\mathbf{c}_1, \ldots, \mathbf{c}_k, \mathbf{e}_1, \ldots, \mathbf{e}_m) \leqq k + m - d$$

iff

$$\operatorname{rank} \begin{bmatrix} | & & | & | & & | \\ e_1 & \cdots & e_m & c_1 & \cdots & c_k \\ | & & | & | & & | \end{bmatrix} \leqq k + m - d$$

iff every $(k+m-d+1) \times (k+m-d+1)$ minor of

$$\left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 & x_{21} & 1 & & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 & x_{m,1} & x_{m,2} & & x_{m,k} \\ \hline 0 & 0 & & 0 & x_{m+1,1} & x_{m+1,2} & & x_{m+1,k} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & x_{n,1} & x_{n,2} & \cdots & x_{n,k} \end{array}\right]$$

has determinant 0 iff every $(k-d+1) \times (k-d+1)$ minor of

$$\begin{bmatrix} x_{m+1,1} & x_{m+1,2} & \cdots & x_{m+1,k} \\ \vdots & \vdots & & \vdots \\ x_{n,1} & x_{n,2} & \cdots & x_{n,k} \end{bmatrix}$$

has determinant 0.

Equating all of the $(k-d+1) \times (k-d+1)$ minors in this block to 0 gives the required polynomials.

Given any Schubert variety X, we may list all of the Schubert conditions which X satisfies (and which define X) and following the above description, we obtain polynomial equations $f_k = 0$ in the coordinates x_{ij} corresponding to the vanishing of the minor determinants as above. These polynomials define X set-theoretically. These polynomials also form a basis for the ideal of X in the ring of polynomials in

$\{x_{ij}\}$, $1 \leqq j < i \leqq n$. (This is precisely Lemma 1 of [LS]. See also [MS, Theorem 5.5, Corollary 5.6, Theorem 6.1].)

3. *Calculation of Rank*

From II.1 above we have $U \cong \mathbb{C}^N$, where $N = n(n-1)/2$, and we have given $\mathbb{C}^N$ coordinates $\{x_{ij}\}$. Let $\{\mathbf{e}_{ij}\}$ be the basis of $\mathbb{C}^N$ which corresponds to the coordinates $\{x_{ij}\}$. We wish to calculate the rank of the Jacobian matrix $(\partial f_k/\partial x_{ij})$ evaluated at the origin. [We denote this matrix by J_X, and its evaluation at the origin by $J_X(0)$.] We regard rank $J_X(0)$ as being the dimension of the row space of $J_X(0)$. Let $\mathrm{C}(k,m,d)$ denote the Schubert condition $\dim(V_k \cap \mathbb{C}^m) \geqq d$. When no confusion will arise, we simply write C to denote a Schubert condition.

In general, a Schubert variety X will be defined by Schubert conditions $\mathrm{C}_1, \mathrm{C}_2, \ldots, \mathrm{C}_s$. Each Schubert condition C_i in turn corresponds to a series of determinantal conditions $f_{i,1}, \ldots, f_{i,t(i)}$. Finally, each f yields one row of the matrix $J_X(0)$. Let us write $R(\mathrm{C}_i)$ for the row space of $(\partial f_k/\partial x_{ij})|_0$ where the f's range over those determinantal equations corresponding to the condition C_i, and let us write $R(X)$ for the row space of $J_X(0)$. Note that if X is defined by Schubert conditions $\mathrm{C}_1, \ldots, \mathrm{C}_s$ then

$$R(X) = R(\mathrm{C}_1) + \ldots + R(\mathrm{C}_s).$$

We wish to compute $\dim R(X)$. We begin by computing $R(\mathrm{C})$ for a single Schubert condition $\mathrm{C} = \mathrm{C}(k,m,d)$. We divide Schubert conditions into 3 types:

I. $k \leqq m$ and $d = k$.

II. $k > m$ and $d = m$.

III. $d < \min(k,m)$.

Note that Type I corresponds to the Schubert condition $V_k \subset \mathbb{C}^m$, Type II corresponds to the condition $V_k \supset \mathbb{C}^m$, and Type III corresponds to a Schubert condition which cannot be expressed via a containment relation between a V_k and $\mathbb{C}^m$.

Let C denote any Schubert condition and let f denote any one of the minor determinants which must be set $=0$ according to C. Let M_f denote the minor matrix with $\det M_f = f = 0$.

Lemma 1. *If $x_{ij} \notin M_f$ then $\partial f/\partial x_{ij} = 0$. If $x_{ij} \in M_f$ then $\partial f/\partial x_{ij} = (-1^{i+j}) \det(K_{ij})$ where K_{ij} is the matrix obtained from M_f by deleting the row and column of x_{ij}.*

Proof. The first assertion is obvious. The second assertion is easily seen by expanding $\det M_f$ along the row (or column) of x_{ij}.

Corollary. $(\partial f/\partial x_{ij})|_0 = (-1)^{i+j} \det K_{ij}(0)$.

Remark. In case f is the equation $x_{ij} = 0$ (i.e., M_f is 1×1) then $\partial f/\partial x_{ij} = 1$. (In this case we define $\det K_{ij} = 1$.)

Let $\mathrm{C} = \mathrm{C}(k,m,d)$ as above. Recall that C yields the equations f_s which correspond to the vanishing of every $(k-d+1)$ by $(k-d+1)$ minor of the matrix

$$\begin{bmatrix} x_{m+1,1} & x_{m+1,2} & \cdots & x_{m+1,k} \\ \vdots & \vdots & & \vdots \\ x_{n,1} & x_{n,2} & \cdots & x_{n,k} \end{bmatrix}.$$

By Lemma 1, $\partial f_k/\partial x_{ij}|_0 = \det(K_{ij}(0))$ (at least up to sign). Observe that K_{ij} is a $(k-d)\times(k-d)$ minor of the matrix above (except in case $d=k\leqq m$, in which case $K_{ij}=1$), and that every entry of $K_{ij}(0)$ is either 0 or 1; the only non-zero entries of $K_{ij}(0)$ come from the 1's along the diagonal of the original coordinate matrix M.

Combining this with the above, we conclude: let $C=C(k,m,d)$ be as above and let f be any of the equations arising from C. Then $(\partial f/\partial x_{ij})|_0=0$ unless the block above contains a $(k-d)\times(k-d)$ identity matrix when evaluated at 0 and x_{ij} has this $I_{(k-d)}$ as cofactor in f, in which case $\partial f/\partial x_{ij}|_0=1$ (or -1). Now, fix k and m. We wish to determine those d for which the block above contains a $(k-d)\times(k-d)$ identity matrix. It is easy to check that the only condition under which an $I_{(k-d)}$ can occur in this block is when $d=m<k$, and then there is only one such $I_{(k-d)}$. Under these conditions, the x_{ij} which have this $I_{(k-d)}$ as cofactor are as shown in the block in the lower left hand corner of the following matrix:

$$\left[\begin{array}{cccc|ccc} x_{m+1,1} & x_{m+1,2} & \cdots & x_{m+1,m} & x_{m+1,m+1} & \cdots & x_{m+1,k} \\ \vdots & \vdots & & \vdots & \vdots & I_{k-d} & \vdots \\ x_{k,1} & x_{k,2} & \cdots & x_{k,m} & x_{k,m+1} & \cdots & x_{k,k} \\ \hline x_{k+1,1} & x_{k+1,2} & \cdots & x_{k+1,m} & x_{k+1,m+1} & \cdots & x_{k+1,k} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ x_{n,1} & x_{n,2} & \cdots & x_{n,m} & x_{n,m+1} & \cdots & x_{n,k} \end{array}\right].$$

The row in $J_X(0)$ corresponding to x_{ij} is the lower left block above with cofactor I_{k-d} is thus $\mathbf{e}_{ij}$ (or $-\mathbf{e}_{ij}$) $(i=k+1,\dots,n;\ j=1,\dots,m)$. Note that we may dispense with the minus signs above since we may choose $-f$ instead of f as a generator for the ideal of X. The only other circumstance which yields $\partial f/\partial x_{ij}|_0 \neq 0$ is the case mentioned above, when f is the condition $x_{ij}=0$, in which case the row of f in $J_X(0)$ is $\mathbf{e}_{ij}$. Combining all of this yields the following:

Proposition 1. *Suppose* $C=C(k,m,d)$ *is a single Schubert condition. Then* $R(C)=0$ *unless one of the following is satisfied:*

(i) $k=d$ *and* $m\geqq k$ *or*

(ii) $m=d$ *and* $k>m$.

In other words, $R(C)=0$ *unless C is of Type I or Type II. Furthermore, if C is the Type I condition* $V_k\subset\mathbb{C}^m$ *or the Type II condition* $\mathbb{C}^k\subset V_m$ *then* $R(C)$ *is spanned by the basis vectors in the following block:*

$$\begin{bmatrix} \mathbf{e}_{m+1,1} & \cdots & \mathbf{e}_{m+1,k} \\ \vdots & & \vdots \\ \mathbf{e}_{n,1} & \cdots & \mathbf{e}_{n,k} \end{bmatrix}.$$

Thus $R(X)=\sum R(C_i)$, where the C_i may now be taken to be only those Schubert conditions for X which express complete containments $V_k\subset\mathbb{C}^m$ or $V_k\supset\mathbb{C}^m$. [Note that since $R(C_i)$ is spanned by the basis vectors in the rectangular block shown above, $R(X)$ is spanned by the basis vectors in a Young diagram.]

Finally, observe that complete containment relations are easy to detect by inspection of the permutation of X. If $s=(s(1),\dots,s(n))$ is the permutation corresponding to X then

(i) X satisfies $V_k\subset\mathbb{C}^m$ iff $\{s(1),\dots,s(k)\}\subset\{1,\dots,m\}$.

(ii) X satisfies $V_k\supset\mathbb{C}^m$ iff $\{s(1),\dots,s(k)\}\supset\{1,\dots,m\}$.

Comparing this with the statement of Theorem I we see that the proof of Theorem I is complete.

4. *The Singular Locus of a Schubert Variety*

In this section we sketch the proof of Theorem II.

Let $X = X_s$ be any Schubert variety in $\mathrm{Fl}(n)$ and let $t < s$ in the Bruhat order. Let $\sum(X)$ denote the singular locus of X. We wish to determine if X_t is contained in $\sum(X)$. Since X is equi-singular at every point of B_t it suffices to determine whether the flag $E(t)$ is a singular point of X. We proceede by analogy with the previous section.

First, we choose coordinates $\{x_{ij}\}$ for an open dense affine subset U of X such that $E(t)$ is at the origin of this coordinate system. We do this by choosing as U the set of all flags $F = \{(V_i)\}$ which can be represented by a matrix

$$M = \begin{bmatrix} | & | & & | \\ c_1 & c_2 & \dots & c_n \\ | & | & & | \end{bmatrix}$$

whose columns $\mathbf{c}_i$ satisfy $V_i = \mathrm{span}(\mathbf{c}_1, \dots, \mathbf{c}_i)$ and whose entries (x_{ij}) satisfy

$$x_{t(i),i} = 1 \quad \text{for all } i \text{ and} \quad x_{t(i),j} = 0 \quad \text{whenever} \quad j > i\,.$$

We take as affine coordinates on U the matrix entries x_{ij} which are not required to be 0 or 1. (Note that this coordinate matrix is the same coordinate matrix as above except that the rows have been permuted by t, and the entries correspondingly renumbered.)

We now express $X \cap U$ as an algebraic subset of $U = \mathbb{C}^N$. As above, we list all of the Schubert conditions $C_1, \dots, C_r$ which X satisfies; each C_i in turn yields polynomials corresponding to the vanishing of the same minor determinants of the coordinate matrix as above. In this way we obtain polynomials f_k such that X is the vanishing set of these polynomials. As above, these polynomials generate the ideal of X (see [MS]).

Hence we may apply the Jacobian criterion to determine if X is singular at $E(t)$ (and hence along all of B_t). To do this we need only compare the codimension of X in $\mathrm{Fl}(n)$ with rank $J_{X,t}(0)$ where $J_{X,t}$ is the Jacobian matrix of X in the new coordinate system. As before, we consider rank $J_{X,t}$ to be the dimension of the row space of $J_{X,t}$.

Consider any one fixed row of $J_{X,t}(0)$ corresponding to some fixed $f = \det M_f$. As above, $\partial f/\partial x_{ij}|_0 = \det K_{ij}(0)$ where K_{ij} is the cofactor of x_{ij} in M_f. Clearly, $\det K_{ij}(0) = 0, 1$, or -1 since $K_{ij}(0)$ is a submatrix of the permutation matrix $M|_0$. It is easy to check that for any fixed f, at most one $K_{ij}(0)$ has non-zero determinant. Thus, each row of $J_{X,t}(0)$ is either 0 or else $\mathbf{e}_{ij}$ where $\mathbf{e}_{ij}$ denotes one of the standard basis vectors of $\mathbb{C}^N$. Therefore, to calculate the rank of $J_{X,t}(0)$ we need only determine which $\mathbf{e}_{ij}$ occur as rows of $J_{X,t}(0)$.

However, $\mathbf{e}_{ij}$ is a row of $J_{X,t}(0) \Leftrightarrow x_{ij}$ has a permutation matrix as cofactor in $M_f(0)$ for some f coming from a Schubert condition which X satisfies, or x_{ij} is the only entry in the matrix M_f (i.e., f is the equation $x_{ij} = 0$). Therefore, we need only count which x_{ij} satisfy one of these criteria. One should note that in this case, all Schubert conditions which define X must be considered, not just the ones which represent complete containments as was the case in the preceding section.

The following proposition makes precise which $\mathbf{e}_{ij}$ occur as rows of the matrix $J_{X,t}(0)$.

Proposition 2. *Let $X=X_s$ be given and let $t<s$. Let $C=C(k,m,d)$ be any Schubert condition satisfied by X. The row space of $J_{X,t}(0)$ is spanned by a subset of the $\mathbf{e}_{ij}$, some contributed by each $C(k,m,d)$ which X satisfies. $C(k,m,d)$ contributes 0 except in the case*

$$d=\#(\{t(1),\ldots,t(k)\}\cap\{1,\ldots,m\})$$
$$=\#(\{s(1),\ldots,s(k)\}\cap\{1,\ldots,m\}).$$

In this case, suppose that $i_1,\ldots,i_{k-d}$ are $\geqq m+1$. Then $C(k,m,d)$ contributes all those $\mathbf{e}_{ij}$ which satisfy:

(i) $i>m$,

(ii) $j\leqq k$,

(iii) $i\neq t(i_q)$ *for any* $q=1,\ldots,k-d$ *(i.e.,* $i\in(\{t(1),\ldots,t(k)\}\cap\{1,\ldots,m\})$, *and*

(iv) $j\neq i_q$ *for any* $q=1,\ldots,k-d$ *(i.e.,* $t(j)\leqq m$*).*

The verification of this proposition is trivial.

Theorem II follows immediately from this proposition.

III. Combinatorics

The remainder of this paper is devoted to studying the structure of the non-singular Schubert varieties in $\mathrm{Fl}(n)$. We begin by introducing some diagrams which will facilitate the combinatorics required for our investigation. Using these diagrams, we restate Theorem I more compactly as Theorem Ia. The remainder of this section is given over to investigating combinatorial properties satisfied by the non-singular Schubert varieties.

1. Combinatorial Diagrams for Schubert Varieties

Let s be any permutation of $(1,2,\ldots,n)$. On an $n\times n$ grid [with the squares numbered so that (1, 1) is in the lower left corner] we begin by placing a dot in each of the squares $(i,s(i))$, $i=1,\ldots,n$. We call this the *basic diagram* of s. On this diagram, we now mark off two regions, called R_1 and R_2, be declaring

(i) $(i,j)\in R_1$ iff $j>s(k)$ for all $k\leqq i$.

(ii) $(i,j)\in R_2$ iff $s(k)>j$ for all $k\geqq i$.

For $s=(3,2,5,1,4)$, R_1 and R_2 are shown in Fig. 1.

We call this the R-diagram associated to s (and X_s).

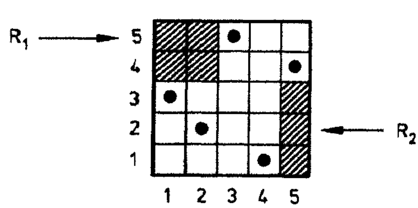


Fig. 1

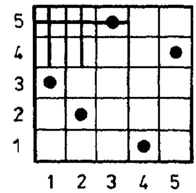


Fig. 2

Remark. R_1 is the largest Young diagram which can be drawn in the upper left corner of the basic diagram of s without containing any of the squares $(i, s(i))$. R_2 is similarly the largest Young diagram which can be drawn in the lower right corner.

Observe that $(i,j) \in R_1 \Leftrightarrow$ all of $s(1), \ldots, s(i)$ are $<j$

$$\Leftrightarrow V_i \subset \mathbb{C}^{j-1} \text{ is satisfied for every flag } F \in X(s).$$

Similarly, $(i,j) \in R_2 \Leftrightarrow \mathbb{C}^j \subset V_{i-1}$ is satisfied for every flag $F \in X(s)$. Thus, R_1 and R_2 completely describe all Type I and Type II Schubert conditions which X satisfies.

We now define a new region (on the same diagram) denoted by ${}^tR_2$ and defined by ${}^tR_2 = \{(i,j) : (j,i) \in R_2\}$. We draw a new diagram (Fig. 2) which shows R_1 and ${}^tR_2$ (but not R_2 itself). Here, R_1 consists of those squares which are marked by a vertical line; ${}^tR_2$ consists of those squares which are marked by a horizontal line.

We refer to this last diagram as the R^*-diagram associated to s (and to X_s). Observe that $(i,j) \in R_1 \cup {}^tR_2 \Leftrightarrow$ either $V_i \subset \mathbb{C}^{j-1}$ or $\mathbb{C}^j \subset V_{i-1}$. Comparing this with the results of Sect. II we have the following:

Proposition 3. *The rank of the matrix* $J_X(0)$ *is equal to* $\#(R_1 \cup {}^tR_2)$. *Consequently,* X *is non-singular iff* $\#(R_1 \cup {}^tR_2) = \operatorname{codim} X$.

Beginning again with the basic diagram for s we now define two more regions:

$$Q_1 = \{(i,j) : j > s(i) \text{ and } j = s(k) \text{ for some } k > i\}$$
$$Q_2 = \{(i,j) : j < s(i) \text{ and } j = s(k) \text{ for some } k < i\}.$$

These are shown in Fig. 3

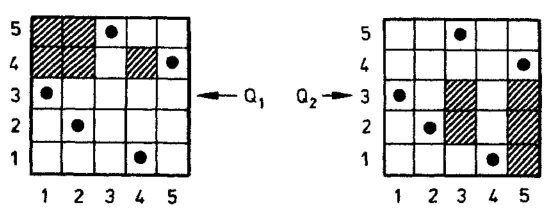


Fig. 3

Observe that $R_1 \subset Q_1$, $R_2 \subset Q_2$, and $\operatorname{codim} X = \# Q_1 = \# Q_2$. Note too that there is a natural pairing of Q_1 with Q_2, namely if (i,j) is in Q_1 then $(i,j) = (i, s(k))$ for some $k > i$. This is paired with $(k, s(i)) \in Q_2$. We shall refer to such a pair as a q-pair.

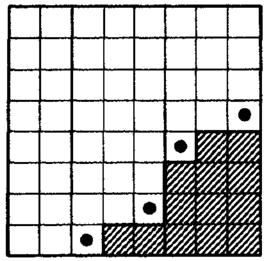


Fig. 4

Definition. Let $X = X_s$ be any Schubert variety in Fl(n). Suppose that for every q-pair $(i, s(k)) \leftrightarrow (k, s(i))$, $i < k$, we have either $(i, s(k)) \in R_1$ or $(k, s(i)) \in R_2$ (or both). We then say that X is *defined by containment conditions* and write X is *dcc*.

Proposition 4. *Suppose X is dcc. Suppose that X has associated to it the regions $R_1(X)$ and $R_2(X)$. Let $Z \neq X$ be another Schubert variety which has associated to it the exact same regions R_1 and R_2. Then* $\mathrm{codim}(Z) > \mathrm{codim}(X)$.

Proof. Since $Q_1(Z) \supset R_1(Z) = R_1(X)$ and $Q_2(Z) \supset R_2(Z) = R_2(X)$, one can easily see that every q-pair for X is also a q-pair for Z. Thus, $Q_1(Z) \supset Q_1(X)$, whence $\mathrm{codim}\, Z > \mathrm{codim}\, X$. (Note that $\mathrm{codim}\, X$ cannot equal $\mathrm{codim}\, Z$, otherwise X and Z would have the same Q_1 and Q_2 and consequently would be the same Schubert variety.) QED

As a result of Proposition 4, if X is *dcc* then X has minimal codimension among all Schubert varieties which satisfy precisely the same containment conditions as X.

Let us now make some observations about the R-diagram of X. First, observe that those squares of the grid which are not contained in either R_1 or R_2 but which are bounded on 2 sides by one of the R_i must contain dots. We refer to these dots as sitting in the recessed corners formed by the R_i. See Fig. 4.

Note that these $(i, s(i))$ completely determine R_1 and R_2, and conversely that if R_1 and R_2 are given then these squares must contain dots. For X *dcc*, the rest of the dots in the squares $(i, s(i))$ are then determined by the property that they define the cell of largest possible dimension.

We remark in passing that not all pairs of Young diagrams can actually arise as R-regions of some Schubert variety in Fl(n); we do, however, have the following

Proposition 5. *Suppose that $X = X_s$ has associated to it the regions R_1 and R_2. Then there is a unique permutation t such that $Y = Y_t$ is dcc and Y has associated to it the same regions R_1 and R_2.*

Proof. The uniqueness of such a t follows from Proposition 4. Thus, we need only establish the existence of a permutation t with Y_t *dcc* and with $R_1(Y) = R_1(X)$ and $R_2(Y) = R_2(X)$.

We construct t by modifying the original permutation s. If s itself is *dcc* then we are done. If not, then there is some q-pair for s, say $(i, s(k)) \leftrightarrow (k, s(i))$, $i < k$, which has neither member in an R-region. Form the permutation s_1 from s by interchanging $s(i)$ and $s(k)$. Observe that this has the effect of eliminating the original q-pair. (It may eliminate some other q-pairs as well.) If s_1 is not *dcc* we may repeat the above procedure to obtain s_2, etc. This procedure must terminate since codimension cannot be lowered indefinitely. Thus, we eventually reach a permutation which is *dcc*. Since the procedure above does not change the original R-regions at any stage, the proposition is proven. QED

Proposition 6. *If X is non-singular then X is dcc.*

Proof. For X non-singular we have $\operatorname{codim} X = \operatorname{rank} J_X(0)$, while in general one has $\operatorname{codim} X \geqq \operatorname{rank} J_X(0)$. Thus equality can only hold if $\operatorname{codim} X$ is minimal for the given R_1 and R_2. But such X must be *dcc*. QED

Let us now define a partial order on the squares of R_2 by declaring that $(i,j) < (k, l)$ if $k < i$ and $j > l$. We similarly partially order R_1 by declaring $(i,j) < (k, l)$ if $i < k$ and $j > l$. Let C_1 denote the set of all maximal elements of R_1 and C_2 the set of all maximal elements of R_2. (In other words, the C_i are the squares which are at the protruding corners of the R_i.) These are indicated in Fig. 5.

Note that each C_i completely determines the corresponding R_i and consequently, for X *dcc*, the both C_i taken together determine X (since X is *dcc*, hence the largest Schubert variety having the given R_1 and R_2). Furthermore, the elements of C_1 completely describe the Type I Schubert conditions which X satisfies, since (i,j) is in C_1 iff every flag in X satisfies $V_i \subset \mathbb{C}^{j-1}$. We shall henceforth say that R_1 implies the Schubert condition $V_i \subset \mathbb{C}^{j-1}$; we denote this condition by simply writing $(i, j-1)_1$.

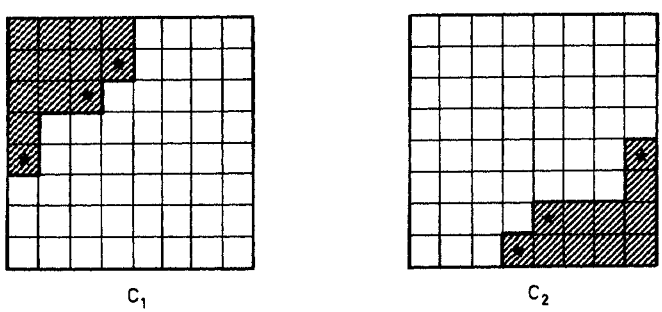


Fig. 5

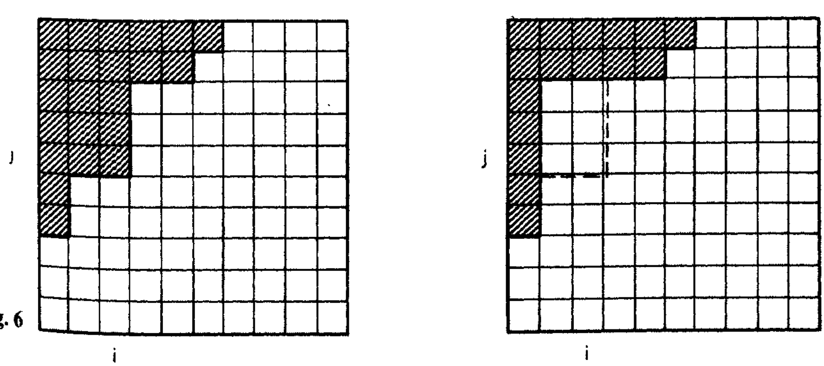


Fig. 6

Similarly, the elements of C_2 completely describe the Type II Schubert conditions which X satisfies, since (i,j) is in C_2 iff every flag in X satisfies $\mathbb{C}^j \subset V_{i-1}$. We say that R_2 implies the Schubert condition $\mathbb{C}^j \subset V_{i-1}$; we denote this condition by $(j, i-1)_2$.

Definition. We say that X^* is obtained from X by *removing the condition* (i,j) if $C_1(X^*)$ and $C_2(X^*)$ satisfy the property that they may be obtained from $C_1(X)$ and $C_2(X)$ by deleting exactly the one element (i,j) from either $C_1(X)$ or $C_2(X)$. Figure 6 shows the result on R_1 of removing the condition (i,j) from X.

Proposition 7. *Suppose that X has associated to it the regions R_1 and R_2. Let R_1^* be obtained from R_1 by removing all those squares $(i,j) \in R_1$ which are $\leqq$ exactly one fixed $(i_0, j_0) \in C_1$, and let $R_2^* = R_2$. Then there is a unique dcc Schubert variety X^* such that the R-regions associated to X^* are precisely R_1^* and R_2^*. Furthermore,* $\operatorname{codim} X^* < \operatorname{codim} X$.

Proof. By virtue of Proposition 5, we need only verify that the resulting R-diagram, composed of R_1^* and R_2^* is the R-diagram of some Schubert variety. But this is obvious: we need only take the permutation obtained as indicated in Fig. 7 to obtain the desired R-diagram. The assertion that $\operatorname{codim} X^* < \operatorname{codim} X$ is clear. QED

Proposition 8. *Let X be given, with regions R_1 and R_2 associated to it. Let C_1 and C_2 be as above. Suppose that for some $(i_0, j_0) \in C_1$ we have $(j_0, i_0) \in R_2$. (That is, suppose that on forming $R_1 \cup {}^tR_2$ one of the protruding corners (i_0, j_0) of R_1 disappears.) Then X is singular.*

Proof. Let Y be the Schubert variety formed by removing the condition (i_0, j_0) from X. By hypothesis, $R_1(Y) \cup {}^tR_2(Y) = R_1(X) \cup {}^tR_2(X)$, whence $\operatorname{rank} J_X = \operatorname{rank} J_Y$. But then $\operatorname{rank} J_X = \operatorname{rank} J_Y \leqq \operatorname{codim} Y < \operatorname{codim} X$. Therefore, X is singular, as claimed. QED

A similar statement applies if the roles of R_1 and R_2 are reversed.

As a result of this proposition, we see that, for X non-singular,

(i) all Type I and Type II Schubert conditions which X satisfies are implied by the protruding corners in the Young diagram $R_1 \cup {}^tR_2$.

(ii) By virtue of (i) we may totally order the containment conditions which X satisfies. This is done by taking them in the order in which they naturally occur along the edge of $R_1 \cup {}^tR_2$.

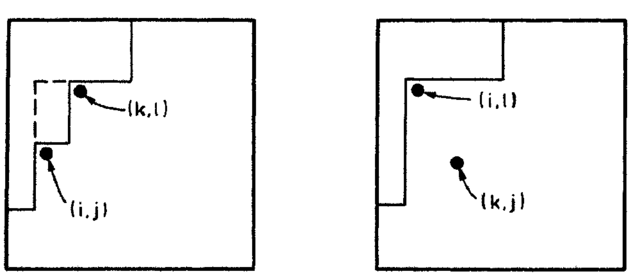
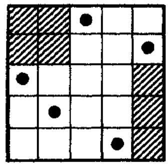
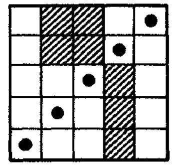


Fig. 7

Fig. 8





We now construct yet another diagram which is associated to a Schubert variety X, which we shall call the S-diagram of X. Beginning with the R-diagram for X, permute the columns of this diagram so that all of the dots lie along the diagonal Δ of the diagram. Let S_1 be the image of R_1 and let S_2 be the image of R_2. These are shown in Fig. 8.

Note that (i,j) is in R_1 (resp. R_2) iff $(s(i),j)$ is in S_1 (resp. S_2). Note too that if we think of each R-region as being composed of vertical rods of various lengths $r_1, \ldots, r_k$ then in the S-diagram, one rod of each length in each region must about the diagonal. This follows immediately from the fact that each recessed corner of R_1 and R_2 must contain a dot.

Remark. Suppose that $(i, s(j)) \leftrightarrow (j, s(i))$ is a q-pair for X. After permuting the columns of the R-diagram as described these are sent to $(s(i), s(j))$ and $(s(j), s(i))$, resp.

Now define ${}^tS_2 = \{(i,j) : (j,i) \in S_2\}$.

Proposition 9. *For X dcc we have* $\operatorname{codim} X = \#(S_1 \cup {}^tS_2)$.

Proof. We know that $\operatorname{codim} X =$ the number of q-pairs which X has. We need only observe that each such q-pair is counted exactly once in $S_1 \cup {}^tS_2$ owing to the remark above. QED

We now have:

Theorem Ia. *X is non-singular $\Leftrightarrow$ X is dcc and*

$$\#(R_1 \cup {}^tR_2) = \#(S_1 \cup {}^tS_2).$$

We denote these two sets by R and S, resp.

2. *Combinatorial Properties of Non-Singular Schubert Varieties*

We now examine under what conditions we have $\# R = \# S$.

By construction, R_1 and R_2 are both Young diagrams, whence so is $R = (R_1 \cup {}^tR_2)$. Furthermore, $R_1 \cap {}^tR_2$ is a sub-Young diagram of $R_1 \cup {}^tR_2$. Since we permute columns of R_1 and R_2 to obtain the respective S regions we shall think of R_1 and R_2 as being composed of vertical rods, whence ${}^tR_2$ is composed of horizontal rods (which are the transposes of the vertical rods). Likewise, S_1 and ${}^tS_2$ are thought of as being composed of vertical and horizontal rods, respectively. Furthermore, each individual rod in S_1 corresponds exactly to a rod in R_1 and similarly for ${}^tS_2$. $R_1 \cap {}^tR_2$ (resp., $S_1 \cap {}^tS_2$) consists of those squares (i,j) which lie in both a horizontal and a vertical rod in the respective diagram.

Now, for any Schubert variety X which is *dcc* we have

$$\#(S_1 \cup {}^tS_2) = \operatorname{codim} X \geqq \operatorname{rank} J_X(0) = \#(R_1 \cup {}^tR_2)$$

whence

$$\#(S_1 \cap {}^tS_2) \leqq \#(R_1 \cap {}^tR_2).$$

We now wish to define an inclusion

$$j: S_1 \cap {}^tS_2 \hookrightarrow R_1 \cap {}^tR_2 .$$

Recall that both $R_1 \cup {}^tR_2$ and $S_1 \cup {}^tS_2$ are thought of as the union of a collection of horizontal rods and a collection of vertical rods. The rods themselves are the same in each case; the only difference is that in R they are arranged in order of length to form a Young diagram, while in S they are not so arranged. Consider the respective intersections as being made up of vertical columns. The columns in the R-intersection are also arranged by length to form a Young diagram. The columns of the S-intersection may be rearranged by order of length (preserving the existing left-to-right order in case of two columns of the same length). It is not difficult to verify that after rearranging the columns of $S_1 \cap {}^tS_2$ by order of length, the i^{th} column is not longer than the i^{th} column of $R_1 \cap {}^tR_2$.

We now define the inclusion $j: S_1 \cap {}^tS_2 \hookrightarrow R_1 \cap {}^tR_2$ by sending the i^{th} longest column of $S_1 \cap {}^tS_2$ to the i^{th} longest column of $R_1 \cap {}^tR_2$ with the squares within each column remaining in order. (In case two columns have the same length, their left-to-right order is preserved by j.) Observe that X is non-singular if and only if j is a surjection.

Let X be a non-singular Schubert variety in $\mathrm{Fl}(n)$. Consider the R^*-diagram for X. Since none of the corners in C_1 or C_2 are hidden in $R_1 \cup {}^tR_2$, it follows that all squares (i,j) on the boundary of the Young diagram $R_1 \cup {}^tR_2$ are not contained in $R_1 \cap {}^tR_2$ and consequently can all be unambiguously labeled I or II. Furthermore, each protruding corner of $R_1 \cap {}^tR_2$ has as its 2 exterior edges one square labeled I and one square labeled II.

Proceeding from lower-left to upper-right along the boundary of $R_1 \cup {}^tR_2$, let (x, y) be any protruding corner of $R_1 \cap {}^tR_2$ at which the boundary squares change from I to II [that is (x, y) should have a I below it and a II to its right]. Observe that this square is the right-most and lowest square in any column of length $(n-y+1)$ in $R_1 \cap {}^tR_2$. Also, this square occurs as the intersection of the right-most vertical rod of length $>(n-y+1)$ (say it has length $n-y+1+b$) with the lowest horizontal rod of length $>x$ (say it has length $x+a$). Since X is non-singular, (x, y) must be in the image of the inclusion j. By the construction of j, $j^{-1}(x, y)$ is the lowest square in the right-most column of length $(n-y+1)$ in $S_1 \cap {}^tS_2$. It is easy to verify that both the row and the column which contain $j^{-1}(x,y)$ must abut the diagonal $\Delta = \{(i,i)\}$.

Consequently, the square $j^{-1}(x, y)$ must be in the intersection of a vertical rod of length $n-y+1+b$, which is located at distance $y-b$ from the left edge, and a horizontal rod of length $x+a$ which is located at height $x+a+1$. In order for these two rods to intersect, we must have $y-b \leqq x+a$.

Observe now that in order for (x, y) to be a protruding corner of the sort assumed, we must have that $(x, y-b+1) \in C_1$ and $(x+a, y) \in {}^tC_2$. Therefore,

$(x, y-b)_1$ and $(x+a, y-1)_2$ are implied Schubert conditions which x must satisfy. Since $y-b \leqq x+a$ we see that at a point along the boundary of $R_1 \cap {}^tR_2$ at which the labeling of the boundary squares changes from I to II we have that the implied conditions $(x, y-b)_1$ and $(x+a, y-1)_2$ satisfy $y-b \leqq x+a$.

We continue to suppose that $X = X_s$ is a non-singular Schubert variety in $\mathrm{Fl}(n)$. Let R_1 and R_2 be the R-regions for X, with corners $C_1 = \{(a_i, b_i+1)\}$ and $C_2 = \{(c_i+1, d_i)\}$. Then, the implied Schubert conditions which X satisfies are $(a_i, b_i)_1$ (i.e. $V_{a_i} \subset \mathbb{C}^{b_i}$) and $(d_i, c_i)_2$ (i.e. $\mathbb{C}^{d_i} \subset V_{c_i}$). Observe that for each implied Type I condition $(a_i, b_i)_1$ we have $a_i < b_i$ and for each implied Type II condition $(c_i, d_i)_2$ we have $d_i < c_i$.

We can arrange the elements of $C_1 \cup {}^tC_2$ into a totally ordered list by taking the squares in the natural order in which they occur along the boundary of $R_1 \cup {}^tR_2$ (say from lower-left to upper-right). Since all squares of $C_1 \cup {}^tC_2$ occur along the boundary of $R_1 \cup {}^tR_2$ this gives a total ordering. Observe that this is equivalent to ordering the implied Schubert conditions $(a_i, b_i)_1$ and $(d_i, c_i)_2$ according to their first (and smaller) element. Since this ordering will be preserved if we instead look at ${}^tR_1 \cup R_2$, we see that this is the same as ordering the (a_i, b_i) and (d_i, c_i) according to their second element.

Now, let us regard the elements of this list as being grouped in blocks, a block being all those implied Schubert conditions of the same type which occur consecutively in this list. Given any such block B, say B is composed of the pairs $(x_i, y_i), \ldots, (x_j, y_j)$ all of the same type, let us define the extent of B to be

$$\mathrm{extent}(B) = \{x_i, \ldots, y_j\}.$$

Suppose that B_1 and B_2 are two consecutive blocks in the above list. Suppose that

$$\mathrm{extent}(B_1) = \{x_i, \ldots, y_j\}$$

and

$$\mathrm{extent}(B_2) = \{w_k, \ldots, z_m\}.$$

Claim. *w_k is greater than or equal to y_j.*

Proof. We consider two cases:

(i) B_1 is composed of elements of Type I and B_2 is composed of elements of Type II.

(ii) B_1 is composed of elements of Type II and B_2 is composed of elements of Type I.

In case (i), the last implied condition listed in B_1 is $(x_j, y_j)_1$ while the first implied condition listed in B_2 is $(w_k, z_k)_2$. Thus this corresponds exactly to the type of change in the boundary of $R_1 \cup {}^tR_2$ considered in Sect. 2 above, so we know that $y_j \leqq w_k$, as claimed.

To prove case (ii), we consider the permutation s^* whose basic diagram is obtained by transposing the basic diagram of s about the diagonal Δ. [Note: s^* is the inverse permutation to s, that is $s^*(i) = j \Leftrightarrow s(j) = i$.] Clearly, $\mathrm{codim}\, X_s = \mathrm{codim}\, X_s^*$. It is also clear that $R_1(s^*) = {}^tR_2(s)$ and ${}^tR_2(s^*) = R_1(s)$, whence X_s^* is non-singular precisely when X_s is.

Finally, observe that the implied Schubert conditions for s^* are gotten from those of s precisely by changing the type of every one. Therefore, in order to prove the desired case (ii) result for the permutation s, it is sufficient to prove the corresponding case (i) result for s^*. Since we already know case (i) for all non-singular Schubert varieties, the claim is proven.

IV. The Structure of Non-Singular Schubert Varieties

In this section we provide a geometric description of the non-singular Schubert varieties in $\mathrm{Fl}(n)$. In IV.1 a certain class of non-singular Schubert varieties is constructed; in IV.2 we show that every non-singular Schubert variety belongs to this class.

1. Obviously Non-Singular Schubert Varieties

Theorem III. *Suppose we are given pairs of numbers* (k_i, m_i), $i=1, \ldots, 2$ *and* (q_j, h_j), $j=1, \ldots, t$ *which satisfy*

$$0<h_t<h_{t-1}<\ldots<h_1 \leqq k_1<\ldots<k_s<n,$$

$$0<q_t<q_{t-1}<\ldots<q_1<n,$$

$$0<m_1<m_2<\ldots<m_s<n,$$

$$q_j<h_j \quad \text{for all } j \text{ and } k_i<m_i \text{ for all } i.$$

Let $M=\{$*all (partial) flags in* $\mathrm{Fl}(h_t, \ldots, h_1, k_1, \ldots, k_s; \mathbb{C}^n): V_{k_i} \subset \mathbb{C}^{m_i}$ *for all i and* $\mathbb{C}^{q_j} \subset V_{h_j}$ *for all j*$\}$. *Then M is a non-singular Schubert variety in* $\mathrm{Fl}(h_t, \ldots, k_s; \mathbb{C}^n)$.

Proof. Since M is defined strictly in terms of Schubert conditions, M is a union of Bruhat cells. Since M is closed, it is a union of Schubert varieties. In order to show that M is in fact a smooth Schubert variety, it suffices to show that M is non-singular (and consequently irreducible).

We proceed by induction on s and t. Suppose first that $s=t=1$.
Case 1: $k_1=h_1$. (We omit subscripts in this case.) Then M is given by

$$M=\{V_k : \mathbb{C}^q \subset V_k \subset \mathbb{C}^m\} \subset G_k(\mathbb{C}^n),$$

whence M is precisely the orbit in $G_k(\mathbb{C}^n)$ of $\mathbb{C}^k$ under the action of the group of all $n \times n$ matrices of the form .

$$\left[\begin{array}{c|c|c} I_q & 0 & 0 \\ \hline 0 & A & 0 \\ \hline 0 & 0 & I_{n-m} \end{array}\right],$$

where A is a $(m-q) \times (m-q)$ matrix of determinant 1. Therefore, M is non-singular. Note that in fact $M \cong G_{k-q}(\mathbb{C}^{m-q})$.
Case 2: $h<k$ (we still omit subscripts for simplicity). In this case

$$M=\{\text{flags in } \mathrm{Fl}(h, k; \mathbb{C}^n): \mathbb{C}^q \subset V_h \subset V_k \subset \mathbb{C}^m\}.$$

Let

$$N=\{\text{flags in } \mathrm{Fl}(h, k; \mathbb{C}^m): \mathbb{C}^q \subset V_h \subset V_k\}$$

and let

$$Z=\{V_h\in G_h(\mathbb{C}^m):\mathbb{C}^q\subset V_h\}.$$

Note that Z is smooth by case 1. We have the diagram

$$\begin{array}{c}\mathrm{Fl}(h,k;\mathbb{C}^m)\\ \downarrow\pi\\ Z\xrightarrow{i}G_h(\mathbb{C}^m),\end{array}$$

where π is the natural projection [which is a fibration with fibre $\cong G_{k-h}(\mathbb{C}^{m-h})$] and i is the natural inclusion.

We form the pull-back of the fibration above by this inclusion. This yields a fibre bundle with base Z and fibre $\cong G_{k-h}(\mathbb{C}^{m-h})$. The total space of this bundle is therefore smooth. Furthermore, this total space consists precisely of those flags in $\mathrm{Fl}(h,k;\mathbb{C}^m)$ which satisfy $\mathbb{C}^q\subset V_h\subset V_k$, that is the total space of this bundle is N, whence N is non-singular. Finally, M is precisely the image of N under the smooth inclusion

$$\mathrm{Fl}(h,k;\mathbb{C}^m)\hookrightarrow\mathrm{Fl}(h,k;\mathbb{C}^n).$$

Therefore, M is a smooth Schubert variety in $\mathrm{Fl}(h,k;\mathbb{C}^n)$.

Now suppose that we have shown the result for some s and t. That is suppose that given (k_i,m_i), $i=1,\dots,s$ and (q_j,h_j), $j=1,\dots,t$ satisfying the hypotheses of the theorem, we know that the corresponding Schubert variety is non-singular. We now show that we may increase either s or t by 1.

First, we show that s may be increased by 1.

Let (k_i,m_i), $i=1,\dots,s+1$ and (q_j,h_j), $j=1,\dots,t$ be given satisfying the hypotheses of the theorem. Consider the natural projection

$$\begin{array}{c}\mathrm{Fl}(h_t,\dots,h_1,k_1,\dots,k_s,k_{s+1};\mathbb{C}^{m_{s+1}})\\ \downarrow\\ \mathrm{Fl}(h_t,\dots,h_1,k_1,k_s;\mathbb{C}^{m_{s+1}}).\end{array}$$

This maps is a fibration with fibre $\cong G_{k_{s+1}-k_s}(\mathbb{C}^{m_{s+1}-k_s})$.

Let $Z=\{$all flags in $\mathrm{Fl}(h_t,\dots,h_1,k_1,\dots,k_s;\mathbb{C}^{m_{s+1}})$: $\mathbb{C}^{q_j}\subset V_{h_j}$ and $V_{k_i}\subset\mathbb{C}^{m_i}$ for all i and $j\}$. By the inductive hypothesis, Z is a non-singular Schubert variety. The inclusion of Z into the partial flag manifold is a smooth map. Let Z^* denote the pull-back of the fibration above via this inclusion. Then Z^* is a fibre bundle with base space Z and fibre $G_{k_{s+1}-k_s}(\mathbb{C}^{m_{s+1}-k_s})$.

Z^* consists precisely of those flags in $\mathrm{Fl}(h_t,\dots,h_1,k_1,\dots,k_s,k_{s+1};\mathbb{C}^m)$ which satisfy $\mathbb{C}^{q_i}\subset V_{h_i}$ $(i=1,\dots,t)$ and $V_{k_j}\subset\mathbb{C}^{m_j}$ $(j=1,\dots,s)$. Since Z^* is the pull-back of a smooth bundle via a smooth map, Z^* is non-singular. Since M is the image of Z^* under the inclusion of

$$\mathrm{Fl}(h_t,\dots,h_1,k_1,\dots,k_s,k_{s+1};\mathbb{C}^{m_{s+1}})$$

into

$$\mathrm{Fl}(h_t,\dots,h_1,k_1,\dots,k_s,k_{s+1};\mathbb{C}^n),$$

we conclude that M is non-singular.

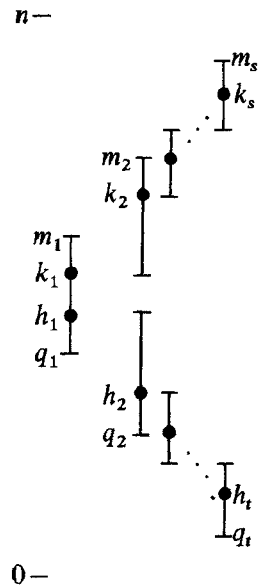
A similar argument shows that t may be increased by 1. This proves Theorem III.

Note that the construction above proceeds inductively by, at each stage, creating a new fibre bundle whose fibre is a Grassmannian. We shall refer to the non-singular Schubert varieties constructed in this way as repeated fibrations of Grassmannians.

The Schubert variety constructed above, namely

$$\{\text{flags in } \mathrm{Fl}(h_t, \ldots, k_s; \mathbb{C}^n)\colon V_{k_i} \subset \mathbb{C}^{m_i} \text{ and } \mathbb{C}^{q_j} \subset V_{h_j}\}$$

may be conveniently represented by the following diagram:



Here, each dot indicates one of the subspaces in the flag; the vertical line indicates the containment conditions which that subspace satisfies. The entire variety is constructed by fibering the Grassmannians in the order in which they appear in the diagram (from left to right).

Definition. We shall say that the Schubert variety in the diagram above has *extreme conditions* (q_t, m_s).

Now let X and Y be two Schubert varieties of the type constructed above, say

$$X \subset \mathrm{Fl}(k_1, \ldots, k_s; \mathbb{C}^n) \quad \text{and} \quad Y \subset \mathrm{Fl}(m_1, \ldots, m_t; \mathbb{C}^n).$$

Suppose further that X has extreme conditions (r_1, r_2), Y has extreme conditions (r_3, r_4) and $r_2 < r_3$. Observe that given any two flags $F_1 = \{(V_k)\} \in X$ and $F_2 = \{(V_m)\} \in Y$ we may form a new partial flag in $\mathrm{Fl}(k_1, \ldots, m_t; \mathbb{C}^n)$ by setting $F = \{(V_k, \ldots, V_k, V_m, \ldots, V_m)\}$. Note that this really is a flag since the assumption on the extreme conditions guarantees the requisite nesting of the V_i. The collection of all such flags forms a non-singular Schubert variety in $\mathrm{Fl}(k_1, \ldots, m_t; \mathbb{C}^n)$ which we denote by $X \times Y$.

Finally, consider the projection

$$\begin{array}{c}\mathrm{Fl}(n)\\ \downarrow\\ \mathrm{Fl}(*;\mathbb{C}^n).\end{array}$$

This map is a fibration whose fibre is a product of partial flag manifolds (and consequently a repeated fibration of Grassmannians). Given one of the non-singular Schubert varieties constructed above in $\mathrm{Fl}(*;\mathbb{C}^n)$, we may take the complete pre-image of this variety in $\mathrm{Fl}(n)$. Since the above projection respects the Schubert cell structures of the respective flag manifolds, we know that this pre-image is itself a Schubert variety. Since the map is a fibration with smooth fibre, this pre-image is in fact a non-singular Schubert variety. In the next section we shall prove that every non-singular Schubert variety in $\mathrm{Fl}(n)$ arises in this way.

2. *The Structure of Non-Singular Schubert Varieties*

Theorem IV. *Let $X = X_s$ be a non-singular Schubert variety in* $\mathrm{Fl}(n)$. *Then X coincides with a Schubert variety of the type constructed in IV.1, and consequently is a repeated fibration of Grassmannians.*

Proof. Let R_1 and R_2 be the R-regions for X, with corners C_1 and C_2. Let $\{(a_i, b_i)_1\}$ and $\{(d_i, c_i)_2\}$ be the implied conditions for X. As in Sect. III.3 we arrange these implied conditions in a single list as they naturally occur along the boundary of $R_1 \cup {}^tR_2$ (i.e., according to increasing a_i or d_i). We again regard this list as being composed of blocks B_i, a block consisting of all those implied conditions of the same type which occur consecutively in the list above. Suppose that X has associated to it q such blocks. Recall from Sect. III that we have defined the extent of such a block B_i and that these extents are disjoint.

The blocks B_i alternate by type; let us group together every pair of blocks which occur as a Type II block followed by a Type I block. (We may assume that each group consists of a Type II block followed by a Type I block, provided we allow that the Type II block may be empty in the case of the first group and the Type I block may be empty in the case of the last group.)

Consider now any such group of two blocks. Within this group we have a series of implied Type II conditions $(d_i, c_i)_2$ followed by a series of implied Type I conditions $(a_i, b_i)_1$.

Observe that the pairs (a_i, b_i) and (d_i, c_i) satisfy the inequalities of Theorem III of the preceding section and consequently we may construct the non-singular Schubert variety $\{F : V_{a_i} \subset \mathbb{C}^{b_i} \text{ and } \mathbb{C}^{d_i} \subset V_{c_i}\}$.

This construction may be repeated for each group of two blocks which occurs in the list of implied Schubert conditions for X. Now observe that the Schubert varieties so constructed all have extreme conditions which do not overlap; this is an immediate consequence of the disjointness of the extents of the blocks themselves. (Recall Sect. III.) Consequently, we may construct a single Schubert variety which is the product (in the sense of the preceding section) of these individual Schubert varieties. Call the resulting Schubert variety Y. Let Y^* denote the complete pre-image of Y in $\mathrm{Fl}(n)$ [under the natural projection of $\mathrm{Fl}(n)$ to the appropriate partial flag manifold].

Claim. $Y^* = X_s$.

Proof. Observe that the flag $E(s)$ satisfies all of the implied conditions of X [since every flag in X satisfies these conditions and $E(s) \in X_s$], and consequently $E(s) \in Y^*$. Since Y^* is a Schubert variety, Y^* must contain the entire closure of the orbit of $E(s)$, and consequently $X \subset Y^*$. Since Y^* is smooth (hence irreducible) the proposition will be proven if we can show that $\dim X = \dim Y^*$, or equivalently $\operatorname{codim} X = \operatorname{codim} Y^*$. The computation is routine; one uses the facts:

(i) $\operatorname{codim} X = \#(R_1 \cup {}^tR_2)$ since X is non-singular,

(ii) $\operatorname{codim} Y^* = \operatorname{codim} Y$ (in $\mathrm{Fl}(*; \mathbb{C}^n)$),

(iii) $\operatorname{codim} Y = \dim \mathrm{Fl}(*; \mathbb{C}^n) - \dim Y$.

This proves the claim and hence Theorem IV.

Observe finally that the construction above is reversible; that is given any diagram of the sort we have been using to represent a non-singular Schubert variety, one can recover the permutation corresponding to the pre-image of this Schubert variety in $\mathrm{Fl}(n)$ as follows:

(i) One can recover all the implied Schubert conditions for this Schubert variety by inspection of the diagram.

(ii) One can now draw regions R_1 and R_2 which yield these implied Schubert conditions (as corners C_1 and C_2).

(iii) One now places dots in all of the recessed corners of the R-regions and then fills in all remaining dots so as to minimize codimension. The desired permutation may now be read from this basic diagram.

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