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Bases for the Bruhat–Chevalley Order on All Finite Coxeter Groups

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1. INTRODUCTION

Let (W, S) be a finite Coxeter system; that is, W is a finite group with a presentation of the form

$$\langle s \in S \mid s^2 = 1, \ \underbrace{sts\cdots}_{m_s \text{ factors}} = \underbrace{tst\cdots}_{m_{st} \text{ factors}} \quad \text{for } s, t \in S, s \neq t \rangle,$$

where m_{st} (for $s, t \in S$) are positive integers with $m_{ss} = 1$ and $m_{st} > 1$ if $s \neq t$.

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Every element $w \in W$ can be written in the form $w = s_1 \cdots s_l$ with $l \geq 0$ and $s_1, \ldots, s_l \in S$. Here, the empty product is considered to be equal to the identity element in W. If l is minimal with this property we let $l(w) := l$ be the length of w and call the above expression "reduced." Let $\leq$ denote the Bruhat–Chevalley order on W. We have $v \leq w$ (for $v, w \in W$) if and only if there exists a reduced expression $w = s_1 \cdots s_l$ as above and a subsequence $1 \leq i_1 < \cdots < i_k \leq l$ such that $v = s_{i_1} \cdots s_{i_k}$. In particular, we write $v < w$ if $v \leq w$ and $v \neq w$. We refer to (2.2) below, where we recall basic properties about this order and to [2] for various other characterizations.

Lascoux and Schützenberger [7] have initiated the program of describing a so-called "base" for the Bruhat–Chevalley order. This is the unique subset $B := \mathrm{Base}(W) \subset W$ which is minimal with respect to set-theoretic inclusion such that if we let $(\mathcal{P}(B), \subseteq)$ be the partially ordered set of all subsets of B, then the map

$$W \to \mathcal{P}(B), \qquad w \mapsto \{b \in B \mid b \leq w\},$$

is an isomorphism of partially ordered sets onto its image; cf. [7, Proposition 2.4]. The existence of the subset B allows us to encode each element $w \in W$ by the boolean vector $\delta(W) := (\delta_b \mid b \in B)$, where $\delta_b = 1$ or 0 according to whether $b \leq w$ or not. For $v, w \in W$ the condition that $v \leq w$ is then equivalent to the purely boolean condition that $\delta(v)_b = 1$ implies $\delta(w)_b = 1$ for all $b \in B$.

In [7], the bases are determined for $(W, \leq)$ of type A_{n-1} and B_n. The construction in [7] essentially amounts to embedding $(W, \leq)$ into a lattice called the "enveloping lattice" of the group. In type A_{n-1} and B_n this lattice even turns out to be distributive. In this paper, we explicitly describe bases for all finite Coxeter groups.

The basic result [7, Théorème 3.6] states that the base B of $(W, \leq)$ is contained in the set of all bi-grassmannians of W. By definition, an element $w \in W$ is a bi-grassmannian if each of the sets

$$\mathcal{L}(w) := \{s \in S \mid \mathit{l}(sw) < \mathit{l}(w)\}, \qquad \mathcal{R}(w) := \{s \in S \mid \mathit{l}(ws) < \mathit{l}(w)\}$$

consists of precisely one element (which may be different for the two sets). In order to achieve this, Lascoux and Schützenberger use another characterization of the Bruhat–Chevalley order which appears in [2, Lemma 3.6] (and which goes back to Ehresmann [4] in the case of the symmetric groups): we have $v \leq w$ if and only if $p_s(v) \leq p_s(w)$ for each $s \in S$, where $p_s(w)$ is the unique element of minimal length in the coset $W_J w$ where W_J is the parabolic subgroup of W generated by $J = S \setminus \{s\}$. A tableau description of the Bruhat–Chevalley order for the classical types A_{n-1}, B_n, D_n has been introduced by Proctor in [9].

In this paper, we take a different approach. Instead of using the above definition of the base, we work with another characterization established in [7, Proposition 2.4], which seems to be more suitable for explicit computations: the base is the set of all elements $w \in W$ which cannot be obtained as the supremum of a subset of W not containing w. (We recall basic results concerning suprema in partially ordered sets in (2.4)). Our key tools for dealing with the problem of determining suprema are provided in Lemma 2.3 and Lemma 2.7. The first of these leads to a new and somewhat more direct proof of the above result that the base is contained in the set of all bi-grassmannians (see Theorem 2.5), while the second one leads to an efficient and practical criterion for determining which bi-grassmannians are base elements (see Corollary 2.8).

In the framework of the theory developed in Section 2, we determine the base for (W, S) of type A_{n-1}, B_n, and D_n in Theorems 3.4, 4.6, and 5.7, respectively. In order to state our results about bi-grassmannians and base elements in these cases, we use a coding of the elements of W which is particularly well suited for this purpose. This is given in terms of products of minimal right coset representatives along a naturally chosen chain of parabolic subgroups of W (see (3.1), (4.1), and (5.1)).

Modulo the general results about partially ordered sets in the appendix of [7], this paper is self-contained and independent of [7]. In particular, we obtain new proofs for the results on A_{n-1}, B_n. We also point out that we always work with the above definition of the Bruhat–Chevalley order in terms of subexpressions of reduced expressions for the elements of W.

Our methods yield a straightforward algorithm for the determination of the base of any given finite Coxeter group. We have implemented this algorithm in the computer algebra systems GAP [10] and CHEVIE [5], and we have used these programs to construct bases for the finite Coxeter groups of exceptional type. In order to give an idea of the complexity of these computations we just mention that it took about 28 h to calculate the base for type E_8 on a SUN Sparc station 5 computer. These GAP programs and explicit tables with reduced expressions for the base elements are available on request via e-mail to the authors. Using our programs we have found that the "enveloping lattice" is no more distributive in type D_n. In fact, the smallest example where this distributivity fails is type D_4 (see Example 5.8).

In all cases it turns out that the size of the base is rather small compared to the group order. Indeed, if W is of type A_{n-1}, B_n, or D_n, then the cardinality of W is $n!$, $2^n n!$, and $2^{n-1} n!$, respectively, while the number of base elements in each case is given by a polynomial in n of degree 3. See the end of Sections 3, 4, 5 for precise formulae giving the exact number of bi-grassmannians and base elements for the classical types, and Table I (2.10) for the exceptional types.

TABLE I
Base and Bi-grassmannians for Exceptional Types

(W, S)	$\#W$	$\#\mathrm{BiGr}(W)$	$\#\mathrm{Base}(W)$	'clivage'
$I_2(m)$	$2m$	$2(m-1)$	$2(m-1)$	yes
H_3	120	43	42	yes
H_4	14400	756	469	yes
G_2	12	10	10	yes
F_4	1152	108	96	no
E_6	51840	232	182	no
E_7	2903040	945	528	no
E_8	696729600	8460	2060	no

2. BI-GRASSMANNIANS AND BASE ELEMENTS

Let (W, S) be any finite Coxeter system and $\leq$ the Bruhat–Chevalley order on W. We denote by $\mathrm{Base}(W)$ and $\mathrm{BiGr}(W)$ the base and the set of all bi-grassmannians, respectively. For $s, t \in S$ we let ${}^sW^t$ be the subset of $\mathrm{BiGr}(W)$ consisting of all $w \in W$ such that $\mathcal{L}(w) = \{s\}$ and $\mathcal{R}(w) = \{t\}$. Thus, we have

$$\mathrm{BiGr}(W) = \bigcup_{s, t \in S} {}^sW^t.$$

Thus, an element $w \in W$ belongs to ${}^sW^t$ if and only if *each* reduced expression for w starts with s and ends with t.

Our first aim will be to show that the base of $(W, \leq)$ is contained in $\mathrm{BiGr}(W)$, a result originally due to Lascoux and Schützenberger.

We start with the following result which gives the exact conditions for the sets ${}^sW^t$ to be empty or not. It also shows that in order to study bi-grassmannians we can reduce to the case of irreducible finite Coxeter groups (see also the remarks at the end of this section).

Remark 2.1. Let $(S_i)_{i \in I}$ be the finest partition of S such that each element of S_i commutes with each elements of S_j for $i \neq j$. For each $i \in I$ let W_i be the subgroup of W generated by S_i so that W is the direct product of the groups $W_i (i \in I)$ and each (W_i, S_i) is an irreducible Coxeter system.

(a) If $s \in S_i$ and $t \in S_j$ with $i \neq j$, then ${}^sW^t$ is empty.

(b) If $s, t \in S_i$, then ${}^sW^t$ is non-empty.

In particular, we have $\mathrm{BiGr}(W) = \bigcup_{i \in I} \mathrm{BiGr}(W_i)$.

Proof. (a) Assume, if possible, that there exists some $g \in {}^sW^t$. We can write uniquely $g = \prod_{k \in I} g_k$, where $g_k \in W_k$ for all k. Since $l(sg) < l(g)$ and since $s \in S_i$ commutes with all generators in $S \setminus S_i$, we have $l(sg_i) < l(g_i)$ and, in particular, $g_i \neq 1$. But then there also exists some $t' \in S_i$ such that $l(g_i t') < l(g_i)$. Since t' also commutes with all generators in $S \setminus S_i$, we conclude that $t' \in R(g)$. But we also have $t \in R(g)$ and $i \neq j$, and hence $|R(g)| > 1$, a contradiction.

(b) If $s = t$, then $s \in {}^sW^t$. Now assume that $s \neq t$. Since (W_i, S_i) is irreducible, the Coxeter graph associated with (W_i, S_i) is a tree, by [1, Chap. V, Section 4, Proposition 8]. This means that there exists a unique sequence $(s_1, \ldots, s_q)$ of *different* elements in S_i such that $s_1 = s$, $s_q = t$, and s_k does not commute with s_{k+1} for $1 \leq k \leq q - 1$. Then $g := s_1 \cdots s_q$ is a Coxeter element in the parabolic subgroup of W generated by the subsystem $\{s_1, \ldots, s_q\} \subseteq S$. We have $L(g) = \{s_1\}$ and $R(g) = \{s_q\}$, since the Dynkin diagram of that subsystem does not have any branch points and, therefore, the above reduced expression for g is unique. So ${}^sW^t$ is nonempty, and the proof is complete. ∎

We now recall some basic facts about the Bruhat–Chevalley order $\leq$.

2.2. The following construction is taken from [3, Remark 5.2]. Let M be the monoid whose elements are the subsets of W and where the product is given by $A \cdot B := \{ab \mid a \in A, b \in B\}$ (for $A, B \subseteq W$). It is readily checked that the map $f: S \to M$, $s \mapsto \{1, s\}$, satisfies the assumptions of Matsumoto's theorem [1, Chap. IV, Section 1, Proposition 5]. So there exists a unique map $F: W \to M$ such that $F(w) = f(s_1) \cdots f(s_l)$ whenever $w \in W$ and $w = s_1 \cdots s_l$ ($s_i \in S$) is a reduced expression.

For each $w \in W$, the set $F(w)$ consists of all elements of W which are obtained by taking subexpressions of a given reduced expression of w, and this does not depend on the choice of a reduced expression. Thus, for any $v, w \in W$, we have $v \leq w$ if and only if $F(v) \subseteq F(w)$. In particular, this characterization now immediately shows that, indeed, $\leq$ is a partial order.

We will now collect some properties which will be useful for inductive arguments. Let $w \in W$ and $r \in S$. Then

$$\begin{aligned} r \in L(w) \quad &\Leftrightarrow \quad l(rw) = l(w) - 1 \quad \Leftrightarrow \quad rw < w, \\ r \notin L(w) \quad &\Leftrightarrow \quad l(rw) = l(w) + 1 \quad \Leftrightarrow \quad rw > w. \end{aligned}$$

There is a similar result for right multiplication with r, which immediately follows from the observation that $v \leq w$ if and only if $v^{-1} \leq w^{-1}$ for any $v, w \in W$.

Now let $v, w \in W$ and $r \in S$. Assume that $v < w$ and $rw < w$. Then we have:

(a) If $rv < v$, then $rv \leq rw$.

(b) if $rv > v$, then $rv \leq w$ and $v \leq rw$.

A proof can be found in [2] or [6, Lemma 7.4].

Of course, we also have symmetrical statements where we consider right multiplication by r throughout.

The following result is fundamental for characterizing the base of W.

LEMMA 2.3. *Let $x \in W$ and assume that $\mathsf{R}(x) \supseteq \{s, t\}$ with $s, t \in S$, $s \neq t$. Then we have the following implication for every $y \in W$:*

$$\textit{if } xs \leq y \textit{ and } xt \leq y, \textit{ then } x \leq y.$$

In other words, if $\mathsf{R}(x)$ contains two different generators s, t, then the set $\{xs, xt\}$ admits a supremum which is x. Symmetrically, a similar result also holds with $\mathsf{L}(x)$.

Proof. Let $y \in W$ with $xs \leq y$ and $xt \leq y$. Then we have $l(x) \leq l(y)$ (since $s \neq t$). We will proceed by induction on $l(y) + (l(y) - l(x))$. If this is zero, there is nothing to prove. Now let $y \neq 1$ and choose any $r \in S$ such that $ry < y$.

Case 1. $rx > x$. Since $xs < x$ this implies that $l(rxs) = l(x)$. (Indeed, $l(rx) > l(x)$ and so $l(rxs) \geq l(rx) - 1 = l(x)$; on the other hand, $l(xs) < l(x)$ and so $l(rxs) \leq l(xs) + 1 = l(x)$.) We can deduce from this that $r(xs) > xs$. Setting $v := xs$ and $w := y$ we see that all assumptions of (2.2)(b) are satisfied; hence $rxs = rv \leq w = y$. By a completely similar argument we also find that $rxt \leq y$. The equalities $l(rxs) = l(rxt) = l(x)$ also imply that $rxs < rx$ and $rxt < rx$. Hence all conditions of the assertion that we are trying to prove are satisfied for the pair (rx, y). Since $l(y) - l(rx) < l(y) - l(x)$ we can apply the induction and obtain that $x < rx \leq y$. Hence we are done in this case.

Case 2. $rx < x$. In this case, we find that $l(rxs) = l(xs) - 1$ or $l(rxs) = l(x)$, by a similar argument as above. Suppose we have $l(rxs) = l(x)$. Since $l(xs) = l(x) - 1 = l(rx)$ this can only happen if $x = rxs$, by [6, Lemma 7.2]. Thus since $x = rxs > xs$ we can apply (2.2)(b) with $v := xs$ and $w := y$ and obtain that $x = rxs = rv \leq w = y$, as desired. Similarly for t, in case $l(rxt) = l(x)$, we get $x \leq y$ as desired.

So it remains to consider the case where $rxs < xs$ and $rxt < xt$. Applying (2.2)(a) with $w := y$ and $v := xs$ (or $v := xt$, respectively) now yields that $rxs \leq ry$ and $rxt \leq ry$. Thus, all the assumptions of the result we are trying to prove are satisfied for the pair (rx, ry). Since $l(ry) < l(y)$ we can apply induction and obtain that $rx \leq ry$. Since $ry < y$ and $x = r(rx) > rx$ we get $r \leq y$ again from (2.2)(b) with $v := rx$ and $w := y$. So we are done, and the proof is complete. ∎

2.4. We now recall some basic facts about the base of partially ordered sets. Let $(X, \leq)$ be any partially ordered set with X finite (and nonempty). Given any nonempty subset $Y \subseteq X$ and an element $x \in X$ such that $y \leq x$ for all $y \in Y$, we say that x is the *supremum* of Y, writing $x = \sup(Y)$, if the following condition is satisfied:

$$\text{if } x' \in X \text{ and } y \leq x' \text{ for all } y \in Y, \text{ then } x \leq x'.$$

If the subset Y does not have a supremum we write $\sup(Y) = \emptyset$. When we replace $\leq$ by $\geq$ in the above condition we obtain the definition of the *infimum* of a subset Y, denoted $\inf(Y)$.

In order to simplify notation, we shall write $[\leq x] := \{y \in X \mid y \leq x\}$ for $x \in X$. By [7, Lemme 2.3], the following two sets of elements of X are equal:

(a) the subset of all elements $x \in X \setminus \inf(X)$ which cannot be obtained as the supremum of a subset $Y \subseteq X$ with $x \notin Y$;

(b) the subset of all elements $x \in X \setminus \inf(X)$ for which there exists an element $z \in X$ such that x is a minimal element in the complement of $[\leq z]$.

This subset of X is called the *base* of X, being denoted by $B = \mathrm{Base}(X)$.

This coincides with the definition given in the introduction for $(W, \leq)$. Namely, by [7, Proposition 2.4], the base B has the following property: if we let $(\mathcal{P}(B), \subseteq)$ be the partially ordered set of all subsets of B, then the map $X \to \mathcal{P}(B)$, $x \mapsto [\leq x] \cap B$, is an isomorphism of partially ordered sets onto its image, and moreover, any other subset of X with this property contains B. This characterization shows that

(c) $x = \sup([\leq x] \cap B)$ for each $x \in X$.

Thus, in order to check that the relation $x \leq y$ holds for any two elements $x, y \in X$ it is sufficient to check if $b \leq y$ for all $b \in \mathrm{Base}(W)$ such that $b \leq x$ (see the similar remark in the Introduction about coding elements in W by boolean vectors).

For practical purpose, the characterization of B in (b) seems to be the most efficient one; this will be used in our algorithmic description in (2.9) below. A simple application of the characterization in (b) will be given in Remark 2.6 below.

The following theorem is now a direct consequence of Lemma 2.3. Originally, it was proved by Lascoux and Schützenberger in [7, Théorème 3.6] by a different method.

THEOREM 2.5. *The base of* $(W, \leq)$ *is contained in the set of bi-grassmannians. Moreover, a bi-grassmannian* $g \in {}^sW^t$ (*for* $s, t \in S$) *which is not in the base can be obtained as the supremum of a subset* $Y \subseteq {}^sW^t$ *not containing* g.

Proof. Let $x \in W$ and assume that x is not a bi-grassmannian. Then $\mathcal{L}(x)$ or $\mathcal{R}(x)$ contains at least two generators $s, t \in S$, $s \neq t$, and Lemma 2.3 implies that $x = \sup(\{sx, tx\})$ or $x = \sup(\{xs, xt\})$, respectively. This yields that x is not in the base, by the characterization in (2.4)(a). This argument shows that $\mathrm{Base}(W) \subseteq \mathrm{BiGr}(W)$, as desired.

In order to prove the second assertion, we take any $g \in {}^sW^t$ which is not in the base. By (2.4)(a), there exists a subset $Y \subseteq \mathrm{Base}(W) \subseteq \mathrm{BiGr}(W)$ with $g \notin Y$ such that $g = \sup(Y)$. Suppose we have an element $y \in Y$ which is not in ${}^sW^t$. We will now describe a procedure which replaces y by a set Y_1 of base elements of strictly smaller length. Then, after a finite number of repetitions of this procedure we arrive at a subset $Y' \subseteq {}^sW^t$ such that $g \notin Y'$ and $g = \sup(Y')$.

So suppose, for example, that we have an element $y \in Y$ with $\mathcal{R}(y) = \{r\}$ and $r \neq t$. Let $Y_i := [\leq yr] \cap \mathrm{Base}(W)$; then $yr = \sup(Y_1)$ by (2.4)(c). Moreover, every element in Y_1 has length strictly smaller than y. We claim that Y_1 is the desired set, that is,

$$g = \sup\big((Y \setminus \{y\}) \cup Y_1\big).$$

To prove this, let $z \in W$ such that $y' \leq z$ for all $y' \in Y \setminus \{y\}$, and $y_1 \leq z$ for all $y_1 \in Y_1$. The latter conditions imply that $yr \leq z$.

If $zr > z$, then $yr < zr$ and we can apply (2.2)(b) with $v := yr$ and $w := zr$ to conclude that $y \leq zr$. Now we have $y' \leq zr$ for all $y' \in Y$ and hence $g \leq zr$. Since $gr > g$ we can apply once more (2.2)(b) with $v := g$ and $w := zr$ to conclude that $g \leq z$.

If $zr < z$, then (2.2)(b) with $v := yr$ and $w := z$ yields that $y \leq z$. Again, we now have $y' \leq z$ for all $y' \in Y$ and hence $g \leq z$. Thus, the above claim is established.

In the case when we have an element $y \in Y$ with $\mathcal{L}(y) = \{r\}$ and $r \neq s$ we can argue symmetrically. The proof is complete. ∎

The following results provide very powerful criteria to decide whether a bi-grassmannian is in the base or not. We will frequently make use of them in the following sections.

Remark 2.6. Let $J \subset S$ and $W_J \subset W$ be the corresponding parabolic subgroup of W. Then (W_J, J) itself is a finite Coxeter system, and the Bruhat–Chevalley order of (W_J, J) is obtained by restricting the ordering on W to W_J (see [6, Corollary 5.10]). We have $\mathrm{BiGr}(W_J) \subset \mathrm{BiGr}(W)$, by just noting that $sw' > w'$ and $w's > w'$ for all $w' \in W_J$ and $s \in S \setminus J$. Furthermore, we have

$$\mathrm{Base}(W_J) \subset \mathrm{Base}(W).$$

Indeed, suppose that $b \in \mathrm{Base}(W_J)$. Applying (2.4)(b) with respect to (W_J, J) we see that there exists some $z \in W_J$ such that b is a minimal element in $W_J \setminus [\leq z]$. If $b' \in W$ is such that $b' \leq b$, then b' can be obtained as a subexpression of b and, hence, $b' \in W_J$. From this, we see that b is also minimal in $W \setminus [\leq z]$. Hence applying (2.4)(b) with respect to (W, S), we get $b \in \mathrm{Base}(W)$.

LEMMA 2.7. *Let $s, t \in S$ and $g \in {}^sW^t$, $Y \subseteq {}^sW^t$ such that $y \leq g$ for all $y \in Y$. Assume that the following condition holds*:

$$\text{if } g' \in {}^sW^t \text{ and } y \leq g' \text{ for all } y \in Y, \text{ then } g \leq g'.$$

Then $g = \sup(Y)$.

Proof. Suppose the conclusion is false. Then there exists some $z \in W$ such that $y \leq z$ for all $y \in Y$ but $g \nleq z$. Choose z of minimal possible length with this property.

Step 1. We claim that z is a bi-grassmannian. Assume that this is not the case. Then $\mathcal{L}(z)$ or $\mathcal{R}(z)$ contains at least two different generators $r, r' \in S$, $r \neq r'$. Suppose first that $r, r' \in \mathcal{R}(z)$. We consider the parabolic subgroup $H \subseteq W$ generated by r, r' and write $z = xh$, where $h \in H$ and $x \in W$ is a minimal left coset representative with respect to H; then $l(z) = l(x) + l(h)$ (see [6, Section 1.10]). Since $r, r' \in \mathcal{R}(z)$ it follows that also $hr < h$ and $hr' < h$. (Indeed, if hr or hr' were reduced, then xhr or xhr' would also be reduced, a contradiction.) But H is a Coxeter group on the two generators r, r'; hence h is the unique longest element in H (see [6, Section 1.8]). So we can write $h = rr'r \cdots = r'rr' \cdots$, where the number of factors in both products is $m :=$ order of $rr' \in W$. Thus, a reduced expression for z is given by choosing any reduced expression for x and one of the two possible reduced expressions for h.

Let $y \in Y$. Since $y \leq z$, we can get a reduced expression for y by taking a suitable subexpression of the above reduced expression of $z = xh$. If we had $t \neq r, r'$, then this would already imply that $y \leq x$. Since this holds for all $y \in Y$ and since $x < z$, the minimality of z would imply that $g \leq x \leq z$, a contradiction. Hence we have $t = r$ or $t = r'$. We arrange notation so that $t = r$ and we choose the reduced expression for h which ends with r'. Let $z_1 := xhr' < xh = z$. A reduced expression for z_1 is obtained by just deleting the last factor r' from the reduced expression for z. Since any

$y \in Y$ is a subexpression of z and since $R(y) = \{r\}$, we conclude that y must already be a subexpression of z_1. Again, the minimality of z leads to a contradiction.

Thus, our assumption was wrong and hence $|R(z)| = 1$. If $L(z)$ contains at least two different generators, then using the symmetry of replacing an element by its inverse we again reach a contradiction using the previous argument. Hence the claim is proved.

Step 2. We know by Step 1 that z is a bi-grassmannian. Now we claim that $L(z) = \{s\}$ and $R(z) = \{t\}$. Suppose, for example, that $R(z) = \{r\}$ with $r \neq t$. Choose any reduced expression for $z' := zr < z$. If we add r to z', we obtain a reduced expression for z. Since any $y \in Y$ is given by a subexpression of this reduced expression for z, and since $r \notin R(y)$, we conclude that we must have $y \leq z'$. Again, the minimality of z leads to a contradiction. If $L(z) = \{r\}$ with $r \neq s$, then we use once more the symmetry given by inversion of elements to reach a contradiction by the previous argument.

This completes the proof. ∎

The following result is a key to the explicit computation of the base of W: it shows that it is sufficient to work only in the smaller sets ${}^sW^t$, for $s, t \in S$, instead of the whole of W.

COROLLARY 2.8. *Let $s, t \in S$. By restricting $\leq$ to ${}^sW^t$ we obtain a partially ordered set $({}^sW^t, \leq)$. Let* $\inf({}^sW^t)$ *be its infimum (if it exists) and* $\mathrm{Base}({}^sW^t)$ *its base. Then*

$$\mathrm{Base}(W) \cap {}^sW^t = \inf({}^sW^t) \cup \mathrm{Base}({}^sW^t).$$

Proof. First recall that $\inf(W) = 1 \notin \mathrm{BiGr}(W)$ and that an element $1 \neq x \in W$ lies in the base if and only if it cannot be obtained as the supremum of a subset $Y \subseteq W$ with $x \notin Y$.

Next observe that $g := \inf({}^sW^t)$, if it exists, always lies in $\mathrm{Base}(W)$. Indeed, if this were not the case, then g would have to be supremum of a subset $Y \subseteq {}^sW^t$ with $g \notin Y$ (see Theorem 2.5). But since $g = \inf({}^sW^t)$, this is impossible.

To have a separate notation for a supremum taken with respect to $({}^sW^t, \leq)$ we denote such a supremum by $\sup_{st}$.

Now let $g \in {}^sW^t$ and assume that $g \notin \mathrm{Base}(W)$. Then $g = \sup(Y)$ for a subset $Y \subseteq {}^sW^t$ with $g \notin Y$ (see Theorem 2.5). But then the defining condition for a supremum is also satisfied with respect to $({}^sW^t)$, and so $g = \sup_{st}(Y)$. Thus, we have the inclusion $\mathrm{Base}({}^sW^t) \subseteq \mathrm{Base}(W) \cap {}^sW^t$.

Conversely, let $b \in \mathrm{Base}(W) \cap {}^sW^t$ and assume that $b \neq \inf({}^sW^t)$, if the infimum exists. Suppose we have $b = \sup_{st}(Y)$ for some subset $Y \subseteq {}^sW^t$ with $b \notin Y$. Then Lemma 2.7 implies that $b = \sup(Y)$. But this would mean that $b \neq 1$ is not in the base of W, a contradiction. ∎

2.9. We shall now describe an algorithm to compute the sets $\mathrm{Base}(W)$ and $\mathrm{BiGr}(W)$, based on the above results. Let us fix $s, t \in S$ and consider the subset ${}^sW^t \subseteq \mathrm{BiGr}(W)$.

(a) In order to compute ${}^sW^t$ we proceed as follows. Let W^t be the set of distinguished right coset representatives with respect to the maximal parabolic subgroup of W generated by $S \setminus \{t\}$. This set can be constructed recursively as follows. For each $i \geq 0$ let W_i^t be the subset of W^t of elements of length i. We start with $W_0^t = \{1\}$. Let now $i > 0$ and assume that W_{i-1}^t has already been constructed. Then W_i^t is the set of all xr where $x \in W_{i-1}^t$ and $r \in S$ such that $\ell(r'xr) > \ell(xr)$ for all $r' \in S \setminus \{t\}$.

Having computed W^t in this recursive way, we obtain ${}^sW^t$ as the set of all $d \in W^t$ such that $\mathcal{L}(d) = \{s\}$.

(b) Now we want to determine $\mathrm{Base}(W) \cap {}^sW^t$. For this purpose we consider the partially ordered set $({}^sW^t, \leq)$. Using 2.4)(b) we determine its base by computing, for any $g \in {}^sW^t$, the minimal elements in ${}^sW^t \setminus [\leq g]$. By Lemma 2.7, $\mathrm{Base}(W) \cap {}^sW^t$ consists precisely of these minimal elements and $\inf({}^sW^t)$, if it exists.

In practice, this works even in large examples due to the fact that in (a) we can proceed by induction on the length of elements and that, in general, the cardinalities of the sets ${}^sW^t$ are rather small as compared to $|W|$. Thus, the algorithm works as long as we can afford to compute explicitly all minimal right coset representatives (with respect to a maximal parabolic subgroup) of a given length.

2.10. Let (W, S) be a Coxeter system of exceptional type. In Table I, we give for each of these types the number of bi-grassmannians and the number of elements in the base. These results were obtained by using an implementation of the algorithm (2.9) in GAP [10] and CHEVIE [5]. Explicit tables with reduced expressions for the elements in the base are available on request to the authors.

The column "clivage" refers to a notion introduced in [7, Définition 2.7]: the answer is "yes" if for every element $b \in \mathrm{Base}(W)$ there exists an element $\bar{b} \in W$ such that W is the disjoint union of $[\geq b]$ and $[\leq \bar{b}]$. Moreover, this condition holds if and only if the "enveloping lattice" of $(W, \leq)$ is distributive, according to [7, Théorème 2.8].

Since the Dynkin diagrams of type E_6, E_7, and E_8 contain D_4 as a subdiagram, the answer is "no" in these cases since the answer is already "no" for D_4 (see Example 5.8). For type F_4 with diagram $\overset{s_1}{\bullet}\!-\!\!-\overset{s_2}{\bullet}\!=\!=\overset{s_3}{\bullet}\!-\!\!-\overset{s_4}{\bullet}$

there are 16 base elements which do not admit a "clivage"; reduced expressions for them are given as follows: 2132132, 2132432, 2321432, 3213243, 3214323, 3234323, 23214323, 32132432, 213213243, 213234323, 321432132, 323432132, 2132132432, 2321432132, 3213234323, 3234321323 (where we write $21 \cdots$ instead of $s_2 s_1 \cdots$ to abbreviate the notation).

2.11. Let $(W_i, S_i)_{i \in I}$ be the irreducible components of (W, S). By Remark 2.1 we have $\mathrm{BiGr}(W) = \bigcup_{i \in I} \mathrm{BiGr}(W_i)$. We claim that we also have

$$\mathrm{Base}(W) = \bigcup_{i \in I} \mathrm{Base}(W_i).$$

To prove this, first note that the right-hand side is contained in the left-hand side by Remark 2.6. Conversely, let $b \in \mathrm{Base}(W)$. By (2.4)(b) there exists some $z \in W$ such that b is minimal in $W \setminus [\leq z]$. We can write uniquely $z = \prod_{k \in I} z_k$ with $z_k \in W_k$ for all k. Theorem 2.5 and Remark 2.1 imply 2.1 imply that $b \in W_j$ for some $j \in I$. But then b is also minimal in $W_j \setminus [\leq z_j]$ and, hence, $b \in \mathrm{Base}(W_j)$, as required.

Thus, we are reduced to the case of irreducible finite Coxeter groups, and we can proceed according to the known classification of these groups (see [1, Chap. VI, Section 4, Théorème 1]). The exceptional and noncrystallographic types have been considered in (2.10) above. In the following three sections we consider the classical types.

3. TYPE A_{n-1}

Let $n \geq 2$ and $\mathfrak{S}_n$ be the symmetric group of degree n. For $1 \leq i \leq n-1$ denote by s_i the basic transposition $(i, i+1)$. Then $(\mathfrak{S}_n, \{s_1, \dots, s_{n-1}\})$ is a Coxeter system of type A_{n-1} with the following Dynkin diagram.

A_{n-1},


, $n \geq 2$.

3.1. We will now describe a coding of the elements of $\mathfrak{S}_n$ in terms of certain sequences of nonnegative integers of length $n-1$ (cf. [7]).

We have a chain of parabolic subgroups $\mathfrak{G}_2 \subset \mathfrak{G}_3 \subset \cdots \subset \mathfrak{G}_n$, corresponding to the subsets of generators $\{s_1\} \subset \{s_1, s_2\} \subset \cdots \subset \{s_1, \dots, s_{n-1}\}$. For $1 \leq k \leq n-1$ we let

$$\mathfrak{R}_k^A := \{1, s_k, s_k s_{k-1}, \dots, s_k s_{k-1} \cdots s_1\}.$$

(Note that R_k^A contains $k+1$ elements.) Then we can see that $R_1^A = \mathfrak{S}_2 = \{1, s_1\}$, and R_k^A is the set of minimal right coset representatives of $\mathfrak{S}_k$ in $\mathfrak{S}_{k+1}$, for $2 \leq k \leq n-1$.

Every element $w \in \mathfrak{S}_n$ can therefore be written uniquely in the form $w = r_1, \dots, r_{n-1}$, where $r_k \in R_k^A$ for all k, and we have $l(w) = l(r_1) + \cdots + l(r_{n-1})$. We call this the *canonical form* of w, and the $r_k \neq 1$ the canonical factors of w. Since every element in R_k^A is uniquely determined by its length, we can therefore represent w by the sequence $[c_1, \dots, c_{n-1}]$, where $c_k = l(r_k)$ for all k. We will frequently identify an element w with its code $[c_1, \dots, c_{n-1}]$.

For example, the six elements in $\mathfrak{S}_3$ are coded as follows: $1 = [0,0]$, $s_1 = [1,0]$, $s_2 = [0,1]$, $s_1 s_2 = [1,1]$, $s_2 s_1 = [0,2]$, $s_1 s_2 s_1 = [1,2]$.

It will turn out that this coding is well adapted to characterizing which elements in $\mathfrak{S}_n$ are bi-grassmannians. We have the following basic relations. For $1 \leq c < k \leq n$ let $r_{k-1,c} := s_{k-1} s_{k-2} \cdots s_{k-c}$ (the unique element of length c in R_k^A). The generator s_i commutes with $r_{k-1,c}$ for $i > k$; the remaining products $r_{k-1,c} s_i$ and $s_i r_{k-1,c}$ are given by

$$\begin{array}{ll} s_{i-1} r_{k-1,c} & \text{if } k-c < i \leq k-1 \\ r_{k-1,c-1} & \text{if } i = k-c \\ r_{k-1,c+1} & \text{if } i = k-c-1 \\ s_i r_{k-1,c} & \text{if } i < k-c-1 \end{array}$$

and

$$\begin{array}{ll} r_{k,c+1} & \text{if } i = k \\ r_{k-2,c-1} & \text{if } i = k-1 \\ r_{k-1,c} s_{i+1} & \text{if } k-c \leq i < k-1 \\ r_{k-1,c} s_i & \text{if } i < k-c-1, \end{array}$$

respectively. Note that this covers all possibilities except for the two products $r_{k-1,c} s_{k+1}$ and $s_{k-c-1} r_{k-1,c}$ which are already in canonical form.

PROPOSITION 3.2. *An element $w \in \mathfrak{S}_n$ is a bi-grassmannian if and only if there exist integers $l, m \geq 0$ and $c > 0$ such that the code of w has the form*

$$b(l,m;c) := \Big[\underbrace{0,\dots,0}_{m}, \underbrace{c,\dots,c}_{l}, \underbrace{0,\dots,0}_{n-1-m-l}\Big], \qquad \textit{where } l > 0.$$

Proof. The fact that elements of the above form are bi-grassmannians is readily checked using the multiplication rules in (3.1). Conversely, let $w \in \mathfrak{S}_n$ be a bi-grassmannian. We want to show that w must have the above form. We do this by induction on n. If $n = 2$, then $\mathfrak{S}_2 = \{1, s_1\}$ and there is nothing to prove. So let $n > 2$ and assume that the assertion is already proved for $\mathfrak{S}_{n-1}$. We write $w = r_1 \cdots r_{n-1}$ with $r_k \in \mathbf{R}_k^A$ for all k.

If $r_{n-1} = 1$, then $w \in \mathfrak{S}_{n-1}$ and we are done by induction. So assume that $1 \neq r_{n-1} = r_{n-1,c}$ with $c \geq 1$; note that this means that $\mathbf{R}(w) = \{s_{n-c}\}$. Let $w' := r_1 \cdots r_{n-2} \in \mathfrak{S}_{n-1}$. If $w' = 1$, then $w = r_{n-1}$ has the required form. So assume that $w' \neq 1$.

We certainly have $\mathbf{L}(w') \subseteq \mathbf{L}(w)$, and hence $\mathbf{L}(w') = \mathbf{L}(w)$ because $\mathbf{L}(w)$ is a singleton set. This also implies that $r_{n-2} \neq 1$. (If we had $r_{n-2} = 1$, then w' would lie in $\mathfrak{S}_{n-2}$ and hence $s_{n-1} \in \mathbf{L}(w)$, which is impossible since $\mathbf{L}(w) = \mathbf{L}(w') \subseteq \mathfrak{S}_{n-1}$). We claim that w' is in fact a bi-grassmannian in $\mathfrak{S}_{n-1}$.

Since $\mathbf{L}(w') = \mathbf{L}(w)$ is a singleton set, we only need to consider the set $\mathbf{R}(w')$. Let $s_i \in \mathbf{R}(w')$ or, equivalently, $w's_i < w'$, for some $k < n - 1$. If $k < n - c - 1$, then $s_i r_{n-1} = r_{n-1} s_i$. Hence $s_i \in \mathbf{R}(w) = \{s_{n-c}\}$, which is impossible. So we must have $i \geq n - c - 1$. Assume that $i \geq n - c$. Then the above multiplication rules show that $s_i r_{n-1} = r_{n-1} s_{i+1}$, and hence, we would have $s_{i+1} \in \mathbf{R}(w)$. But $i + 1 \neq n - c$ for $i \geq n - c$, so again, this is impossible. We conclude that $\mathbf{R}(w') = \{s_{n-c-1}\}$, and our claim is established.

By induction, w' has a code of the desired form, where all nontrivial canonical factors have the same length. Since $\mathbf{R}(w') = \{s_{n-c-1}\}$, this length is c, and the proof is complete. ∎

Remark 3.3. The above proof shows that if $w = r_1 \cdots r_{n-1}$ is a bi-grassmannian in $\mathfrak{S}_n$ and $w' := r_1 \cdots r_{n-2} \neq 1$, then w' is a bi-grassmannian in $\mathfrak{S}_{n-1}$.

The following result can also be found in [7, Théorème 4.4], but note that the approach taken by Lascoux and Schützenberger is quite different from ours: they show that every $b \in \mathrm{BiGr}(\mathfrak{S}_n)$ admits a "clivage"; that is, there exists a (unique) $\bar{b} \in \mathfrak{S}_n$ such that $\mathfrak{S}_n$ is the disjoint union of $[\geq b]$ and $[\leq \bar{b}]$. This implies both the equality $\mathrm{Base}(\mathfrak{S}_n) = \mathrm{BiGr}(\mathfrak{S}_n)$ and the fact that the "enveloping lattice" for $(\mathfrak{S}_n, \leq)$ is distributive (cf. also the remarks in (2.10)).

THEOREM 3.4. *An element $w \in \mathfrak{S}_n$ is contained in the base of $(\mathfrak{S}_n, \leq)$ if and only if w is a bi-grassmannian.*

Proof. We proceed by induction on n. If $n = 2$, then $\mathfrak{S}_2 = \{1, s_1\}$ and there is nothing to prove. Now let $n > 2$ and $g \in \mathfrak{S}_n$ be a bi-grassmannian. By Theorem 2.5, there exists a subset $Y \subseteq \mathrm{Base}(\mathfrak{S}_n)$ such that $g = \sup(Y)$ and such that $\mathbf{L}(y) = \mathbf{L}(g)$, $\mathbf{R}(y) = \mathbf{R}(g)$ for all $y \in Y$.

We write $g = r_m \cdots r_{n-1}$ in canonical form with $1 \leq m \leq n - 1$ and $r_m \neq 1$. If $r_{n-1} = 1$, then g is a bi-grassmannian in $\mathfrak{G}_{n-1}$ and we are done by induction and Remark 2.6. Now suppose $r_{n-1} \neq 1$. If Y above were contained in $\mathfrak{S}_{n-1}$, then every $y \in Y$ would be smaller (with respect to $\leq$) than the longest element of $\mathfrak{S}_{n-1}$, thus so is $\sup(Y)$, and hence, $g = \sup(Y) \in \mathfrak{S}_{n-1}$. This is impossible since g contains the generator s_{n-1}. So there should be some $y \in Y$ which has a canonical form $y = r'_1 \cdots r'_{n-1}$ with $r'_{n-1} \neq 1$. Since $\mathcal{R}(y) = \mathcal{R}(g)$ the last factor r'_{n-1} must have the same length as the last factor of y, and so $r'_{n-1} = r_{n-1}$. By Proposition 3.2, every nontrivial factor in the decomposition of y has the same length. On the other hand, since $\mathcal{L}(y) = \mathcal{L}(g)$, any reduced expression for y must start with the same generator as any reduced expression for g. This forces that $r'_1 = \cdots = r'_{m-1} = 1$ and $r'_m = r_m \neq 1$. So we conclude that $g = y \in \mathrm{Base}(\mathfrak{S}_n)$, and the proof is complete. ∎

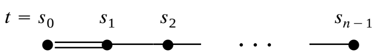
Let b_n denote the cardinality of the set of bi-grassmannians, which is the same as the set of base elements. We have $b_n = n(n^2 - 1)/6$, with the generating function given by

$$\sum_{n \geq 2} b_n z^{n-2} = \frac{1}{(1-z)^4} = 1 + 4z + 10z^2 + 20z^3 + 35z^4 + 56z^5 + \cdots.$$

4. TYPE B_n

Let $n \geq 1$ and $W_n \subset \mathrm{GL}_n(\mathbb{R})$ be the subgroup of all matrices which have exactly one nonzero entry in each row and each column and where this nonzero entry is ± 1. Let $t \in W_n$ be the diagonal matrix with diagonal entries $(-1, 1, \ldots, 1)$. And, for $1 \leq i \leq n-1$, let $s_i \in W_n$ the matrix obtained from the identity matrix by interchanging the ith and the $(i+1)$th row. Then $(W_n, \{t, s_1, \ldots, s_{n-1}\})$ is a Coxeter system of type B_n with the following Dynkin diagram.

B_n,



, $n \geq 1$

The group W_n is isomorphic to the wreath product of the cyclic group of order 2 with the symmetric group $\mathfrak{S}_n$. We let

$$t_0 := t, \qquad t_i := s_i t_{i-1} s_i \qquad \text{for } 1 \leq i \leq n-1.$$

Thus t_i is obtained from t by shifting the diagonal entry -1 from the first to the $(i+1)$th position. A normal subgroup in W_n of order 2^n is generated by $\{t_0, \ldots, t_{n-1}\}$, and a complementary subgroup isomorphic to $\mathfrak{S}_n$ is generated by $\{s_1, \ldots, s_{n-1}\}$.

4.1. There is again a coding of the elements of W_n, in a similar way as for type A_{n-1}. We have a chain of parabolic subgroups $W_1 \subset W_2 \subset \cdots \subset W_n$ corresponding to the subsets of generators $\{t\} \subset \{t, s_1\} \subset \cdots \subset$

$\{t, s_1, \dots, s_{n-1}\}$. Let $\mathsf{R}_0^B := W_1 = \{1, t\}$ and, for $1 \le k \le n-1$,

$$\mathsf{R}_k^B := \mathsf{R}_k^A \cup \mathsf{T}_k,$$

where

$$\mathsf{T}_k := \{t_k, s_k t_{k-1}, s_k s_{k-1} t_{k-2}, \dots, s_k s_{k-1} \cdots s_1 t_0\}.$$

(Note that R_k^B contains $2(k+1)$ elements.) Then R_k^B is the set of minimal right coset representatives of W_k in W_{k+1}, for $1 \le k \le n-1$.

Every element $w \in W_n$ can be written uniquely in the form $w = r_0 \cdots r_{n-1}$, where $r_k \in \mathsf{R}_k^B$ for all k, and we have $\mathsf{l}(w) = \mathsf{l}(r_0) + \cdots + \mathsf{l}(r_{n-1})$. Since every element in R_k^B is uniquely determined by its length, we can therefore represent w by the sequence $[c_0, \dots, c_{n-1}]$, where $c_k := \mathsf{l}(r_k)$ for all k. On the level of codings, the embeddings $\mathfrak{S}_n \subseteq W_n$ is given by the map $[c_1, \dots, c_{n-1}] \mapsto [0, c_1, \dots, c_{n-1}]$.

For example, the elements in W_2 (the dihedral group of order 8) are coded as follows: $1 = [0,0]$, $t = [1,0]$, $s_1 = [0,1]$, $ts_1 = [1,1]$, $s_1 t = [0,2]$, $ts_1 t = [1,2]$, $s_1 ts_1 = [0,3]$, $ts_1 ts_1 = [1,3]$.

Every element of W_n can be regarded as a "signed" permutation of the standard basis vectors of $\mathbb{R}^n$. The factors in R_k^A correspond to permutation matrices in W_n, where all nonzero entries are equal to 1. Therefore, we will call them *positive factors*. The factors in T_k are obtained from those in R_k^A by multiplying with a suitable factor t_{k-c}. This multiplication does not affect the induced permutation of the basis vectors but it does change the sign at exactly one basis vector. Therefore, we call these factors *negative factors*. Note also that a factor $r_k \in \mathsf{R}_k^B$ is positive or negative according to whether $c_k \le k$ or $c_k > k$, respectively (where $c_k = \mathsf{l}(r_k)$).

Now let $w \in W_n$ and write $w = r_0 \cdots r_{n-1}$ with $r_k \in \mathsf{R}_k^B$ for all k. Then we can define the *signed code* of w as the sequence obtained from the code $[c_0, \dots, c_{n-1}]$ of w by keeping all c_k with $c_k \le k$ and replacing all c_k with $c_k > k$ by $2k + 1 - c_k$. In order to distinguish between positive and negative factors, we attach a prime $'$ to the latter ones. Thus, for $c < n$, the factors $s_{n-1} \cdots s_{n-c}$ and $s_{n-1} \cdots s_{n-c} t_{n-c-1}$ have coding numbers c and c', respectively. We shall write $[c_0^{()}, \dots, c_{n-1}^{()}]$ if we do not want to specify which factors are positive or negative.

For example, the signed codes of the above elements of W_2 are: $1 = [0,0]$, $t = [0',0]$, $s_1 = [0,1]$, $ts_1 = [0',1]$, $s_1 t = [0,1']$, $ts_1 t = [0',1']$, $s_1 ts_1 = [0,0']$, $ts_1 ts_1 = [0',0']$.

We will see below that this modified coding is well-adapted to characterizing bi-grassmannians and base elements in W_n. Note also that the multiplication rules in (3.1) remain valid when we replace the element $r_{m-1,c} = s_{m-1} s_{m-2} \cdots s_{m-c}$ by $r'_{m-1,c} = s_{m-1} s_{m-2} \cdots s_{m-c} t_{m-c-1}$.

PROPOSITION 4.2. *An element $w \in W_n$ is a bi-grassmannian if and only if there exist integers $m, l, l_1 \geq 0$ and $c > 0$ such that the signed code of w has the form*

$$b(l, l_1, m; c)$$
$$:= \Big[\underbrace{0, \dots, 0}_{m}, \underbrace{(c-l)', \dots, (c-1)'}_{l}, \underbrace{c, \dots, c}_{l_1}. \underbrace{0, \dots, 0}_{n-m-l-l_1}\Big]$$

where $l + l_1 > 0$ and $l \leq c \leq l + m$.

Proof. The proof is similar to that of Proposition 3.2. First, it can be readily checked that every element with a signed code of the above form is a bi-grassmannian. Conversely, let $w \in W_n$ be a bi-grassmannian. We show by induction on n that its signed code has the above form. If $n = 1$, then $W_1 = \{1, t\}$ and there is nothing to prove. Now let $n > 1$ and write $w = r_1 \cdots r_{n-1}$ with $r_k \in \mathbf{R}_k^B$. If $r_{n-1} = 1$, then $w \in W_{n-1}$ and we are done by induction.

So let $r_{n-1} \neq 1$ and $w' := r_1 \cdots r_{n-2} \in W_{n-1}$. If $w' = 1$, then $w = r_{n-1}$ has the required form. So assume that $w' \neq 1$. As in the proof of Proposition 3.2 we see that $r_{n-2} \neq 1$. We claim that w' is in fact a bi-grassmannian in W_n and that the coding numbers of the two right-most factors r_{n-2}, r_{n-1} of w are arranged as claimed. We have $\mathbf{L}(w') = \mathbf{L}(w)$ and $\mathbf{R}(w) = \mathbf{R}(r_{n-1})$. For $\mathbf{R}(w')$, we consider the following three cases:

Case 1. $r_{n-1} = s_{n-1} \cdots s_{n-c}$ with coding number $c \geq 1$ and $\mathbf{R}(r_{n-1}) = \{s_{n-c}\}$. If $w's_i < w'$ for $1 \leq i \leq n-2$, then a similar reasoning as in the proof of Proposition 3.2 shows that $i = n - c - 1$; in particular, this implies $c < n - 1$. If $w't < w'$, then $c = n - 1$ for otherwise t would commute with r_{n-1} and then $t \in \mathbf{R}(w)$.

Hence, we have either $\mathbf{R}(w') = \{s_{n-c-1}\}$ (with $c < n - 1$) or $\mathbf{R}(w') = \{t\}$, and so w' is a bi-grassmannian. If $\mathbf{R}(w') = \{s_{n-c-1}\}$, then r_{n-2} is the positive factor $s_{n-2} \cdots s_{n-c-1}$ of length c, while if $\mathbf{R}(w') = \{t\}$, then r_{n-2} is the negative factor $s_{n-2} \cdots s_{n-c}t_{n-c-1}$ with coding number $(c-1)'$. In both cases, the coding numbers of r_{n-2} and r_{n-1} are arranged as claimed.

Case 2. $r_{n-1} = s_{n-1} \cdots s_1 t$ with coding number $(n-1)'$ and $\mathbf{R}(r_{n-1}) = \{t\}$. Assume, if possible, that $w's_i < w'$ for $1 \leq i \leq n-2$. Since then $s_i r_{n-1} = r_{n-1} s_{i+1}$, we would have $s_{i+1} \in \mathbf{R}(w)$, a contradiction. So $\mathbf{R}(w') = \{t\}$, $r_{n-2} = s_{n-2} \cdots s_1 t$, and w' is a bi-grassmannian whose right-most canonical factor is negative and has a coding number as claimed.

Case 3. $r_{n-1} = s_{n-1} \cdots s_{n-c} t_{n-c-1}$ with coding number c', $c < n - 1$, and $\mathbf{R}(r_{n-1}) = \{s_{n-c-1}\}$. In particular, this implies that t commutes with r_{n-1} and so $t \notin \mathbf{R}(w')$. Hence $w's_i < w'$ for some $1 \leq i \leq n-2$. The analogues for negative factors of the multiplication rules in (3.1) show that we must have $i = n - c - 1$. Thus, w' is a bi-grassmannian with $\mathsf{R}(w') = \{s_{n-c-1}\} = \mathsf{R}(w)$. Now we have two possibilities for r_{n-2}, namely either the positive factor $s_{n-2} \cdots s_{n-c-1}$ of length c, or the negative factor $s_{n-2} \cdots s_{n-c}t_{n-c-1}$ which has coding number $(c-1)'$. We claim that the first possibility cannot occur. To see this we show that if we had $r_{n-2} = s_{n-2} \cdots s_{n-c-1}$, then we would also have $s_{n-1} \in \mathsf{L}(w)$, which is impossible. We compute:

$$\begin{aligned} s_{n-1}r_{n-2}r_{n-1} &= s_{n-1}(s_{n-2} \cdots s_{n-c-1})(s_{n-1} \cdots s_{n-c}t_{n-c-1}) \\ &= (s_{n-1}s_{n-2}s_{n-1})(s_{n-3} \cdots s_{n-c-1})(s_{n-2} \cdots s_{n-c}t_{n-c-1}) \\ &= (s_{n-2}s_{n-1})s_{n-2}(s_{n-3} \cdots s_{n-c-1})(s_{n-2} \cdots s_{n-c}t_{n-c-1}) \\ &= \cdots \\ &= (s_{n-2}s_{n-1})(s_{n-3}s_{n-2}) \cdots (s_{n-c}s_{n-c-1})s_{n-c}t_{n-c-1} \\ &= (s_{n-2}s_{n-1})(s_{n-3}s_{n-2}) \cdots (s_{n-c-1}s_{n-c})s_{n-c-1}t_{n-c-1} \end{aligned}$$

which is not reduced since $s_{n-c-1}t_{n-c-1}$ is not reduced. Hence, $s_{n-1} \in \mathsf{L}(w)$, and we are done.

Thus, in each case w' is a bi-grassmannian in W'_{n-1}. By induction, w' has a signed code as claimed. Moreover, we have found conditions on the coding numbers of the two right-most factors in w. These conditions show that the signed code of w is also as claimed. The proof is complete. ∎

Remark 4.3. The above proof shows that if $w = r_0 \cdots r_{n-1}$ is a bi-grassmannian in W_n and $w' := r_0 \cdots r_{n-2} \neq 1$, then w' is a bi-grassmannian in W_{n-1}; compare this with the analogous Remark 3.3 for $\mathfrak{S}_n$.

4.4. Let us fix an element $g \in \mathrm{BiGr}(W_n)$. We wish to describe a canonical procedure by which we can associate with g two elements $b_-, b_+ \in \mathrm{Base}(W_n)$ such that $g = \sup(\{b_-, b_+\})$. First we need to prepare some notation. Let $\mathsf{L}(g) = \{s_i\}$, $\mathsf{R}(g) = \{s_j\}$, where $0 \leq i, j \leq n-1$ (with $s_0 = t$). We consider the sets

$$\begin{aligned} {}^iW_n^j &:= \left\{ y \in \mathrm{BiGr}(W_n) \,\middle|\, \mathsf{L}(y) = \{s_i\},\ \mathsf{R}(y) = \{s_j\} \right\}, \\ Y_g &:= [\leq g] \cap {}^iW_n^j \cap \mathrm{Base}(W_n). \end{aligned}$$

(Note that ${}^iW^j$ is just a short notation for the set ${}^{s_i}W^{s_j}$ defined in Section 2.) By Theorem 2.5, we have $g = \sup(Y_g)$. We will find the desired elements b_-, b_+ in Y_g.

Let $\tau(g) \geq 0$ be the number of generators t in any reduced expression of g. (Note that this does not depend on the choice of a reduced expression.) Let $p := \max\{\tau(y) \mid y \in Y_g\}$. All elements in Y_g can be obtained as subexpressions of a given reduced expression of g. Thus, $p \leq \tau(g)$. Assume, if possible, that $p < \tau(g)$. We can write any $y \in {}^iW_n^j$ in the form $y = r_i \cdots r_{n-1}$ with $r_k \in \boldsymbol{R}_k^B$ for all k. Now note that $r_k \leq t_k$, if r_k is negative, and $r_k \leq s_k s_{k-1} \cdots s_1$, if r_k is positive. Thus, we would have $y \leq t_i \cdots t_{i+p-1}(s_{i+p} \cdots s_1)\cdots(s_{n-1} \cdots s_1) =: y_0$ for all $y \in Y_g$. But then also $g \leq y_0$ and so g would contain at most p generators t, a contradiction. Hence

(a) there exists some $b_- \in Y_g$ such that $\tau(b_-) = \tau(g)$.

Let $\nu(g) \geq 1$ be the biggest $k \geq 1$ such that the generator s_{k-1} occurs in a reduced expression for g. (Note again that this does not depend on the choice of a reduced expression.) Let $q := \max\{\nu(y) \mid y \in Y_g\}$. We certainly have $q \leq \nu(g)$. If $q < \nu(g)$, then all the elements in Y_g, and hence also g, would lie in the subgroup of W_n generated by $s_0, s_1, \ldots, s_{q-1}$. (More precisely, g would be smaller than or equal to the longest element in that subgroup.) This is a contradiction, and so we conclude that

(b) there exists some $b_+ \in Y_g$ such that $\nu(b_+) = \nu(g)$.

At some point we will have to show explicitly that certain bi-grassmannians are not in the base. The crucial result for this purpose is the following.

PROPOSITION 4.5. *Let* $g \in {}^iW_n^j$ *and consider any* $g_-, g_+ \in {}^iW_n^j$ *such that* $g_- \leq g, g_+ \leq g$ *and* $\tau(g_-) = \tau(g)$, $\nu(g_+) = \nu(g)$. *Then we have* $g = \sup(\{g_-, g_+\})$.

Proof. Let $\boldsymbol{L}(g) = \{s_i\}$ and $\boldsymbol{R}(g) = \{s_j\}$ as in (4.4). Assume that g has signed code $b(l, l_1, m; c)$, where $l + l_1 > 0, l \leq c \leq l + m$ as in Proposition 4.2. The condition $\boldsymbol{L}(g) = \{s_i\}$ is equivalent to $m = i$, and the condition $\boldsymbol{R}(g) = \{s_j\}$ is equivalent to $m + l + l_1 - c = j$. We have $\tau(g) = l$ and $\nu(g) = m + l + l_1$.

In order to prove that $g = \sup(\{g_-, g_+\})$ we use the criterion in Lemma 2.7. Let $g' \in {}^iW_n^j$ such that $g_- \leq g'$ and $g_+ \leq g'$. We must show that $g \leq g'$. Let g' have signed code $b(l', l'_1, m'; c')$. The conditions that $\boldsymbol{L}(g') = \{s_i\}$ and $\boldsymbol{R}(g') = \{s_j\}$ imply that

$$m' = i = m, \qquad l' + l'_1 - c' = j - i = l + l_1 - c. \tag{1}$$

Since g_- and g_+ can be obtained by taking subexpressions of g' we must have $l = \tau(g) = \tau(g_-) \leq \tau(g') = l'$ and $m + l + l_1 = \nu(g) = \nu(g_+) \leq \nu(g') = m' + l' + l'_1$. Combining this with the conditions in (1) we find that

$$l \leq l', \qquad c \leq c', \qquad l + l_1 \leq l' + l'_1. \tag{2}$$

In particular, g' contains at least as many negative factors as g, and each positive factor in g' is at least as long as each positive factor in g.

Let us write $g = r_m \cdots r_{\nu(g)-1}$ and $g' = r'_m \cdots r'_{\nu(g')-1}$ with $r_k, r'_k \in \mathbf{R}_k^B$ for all k. The first l or l' factors, respectively, are negative, while all other factors are positive.

Assume first that $c - l \leq c' - l'$. We claim that then $r_k \leq r'_k$ for all $k \geq m$. Indeed, if $k \leq m + l - 1$, then r_k and r'_k are negative (note that $l \leq l'$) and we certainly have $r_k \leq r'_k$, since $c - l \leq c' - l'$. If $k \geq m + l$, then r_k is positive and r'_k can be positive or negative; but since $c \leq c'$ we have again $r_k \leq r'_k$. Thus, in particular, we can conclude that $g \leq g'$, as desired.

Now assume that $c' - l' < c - l$. In this case we cannot argue factor by factor to conclude that $g \leq g'$. Instead, we rewrite the canonical form of g as a product of terms where all terms involving t are equal to the corresponding terms in g'. Let $d := (l - c) - (l' - c') \geq 1$. The conditions in (1) then also imply that $l'_1 = l_1 + d$. A straightforward computation shows that

$$r_m \cdots r_{m+l-1} = r'_m \cdots r'_{m+l-1} y_1 \cdots y_d,$$

where

$$y_k := s_{j-l'_1+k+l-1} \cdots s_{j-l'_1+k} \qquad \text{for } 1 \leq k \leq d.$$

So we have a new expression $g = r'_m \cdots r'_{m+l-1} y_1 \cdots y_d r_{m+l} \cdots r_{\nu(g)-1}$ which is also reduced. This new expression has been arranged so that the first l factors are exactly the same as those in g'. The product $y_1 \cdots y_d r_{m+l} \cdots r_{\nu(g)-1}$ consists of precisely $d + \nu(g) - m - l = d + l_1$ nontrivial factors which is exactly the number l'_1 of positive factors in g'. We will now prove that we can compare these last l'_1 positive factors term by term to conclude that $g \leq g'$.

Since $\mathbf{R}(g) = \mathbf{R}(g') = \{s_j\}$, the last $l_1 = l'_1 - d$ positive factors in g and in g' end with $s_{j-l'_1+d+1}, \ldots, s_j$ (where s_j is the last). Since $c \leq c'$, we can conclude that each of these factors in g is smaller than or equal to the corresponding factor in g'. On the other hand, both the preceding d factors $y_1, \ldots, y_d$ and the preceding d factors in g' end with $s_{j-l'_1+1}, \ldots, s_{j-l'_1+d}$. Again, each of the factors in the first product is smaller than or equal to the corresponding factor in the second product. This completes the proof. ∎

The following result is now a direct consequence of Proposition 4.5. It can also be found in [7, Théorème 7.4] but, again, Lascoux and Schützenberger's approach is quite different from ours: they use an order-preserving embedding $W_n \subseteq \mathfrak{S}_{2n}$ and thus reduce to the case of the symmetric groups already solved before. Again, their method also yields the stronger result that the "enveloping lattice" of $(W_n, \leq)$ is distributive, by showing that every base element admits a "clivage" (cf. the analogous remark preceding Theorem 3.4 for type A_{n-1}).

THEOREM 4.6. *Let $g \in W_n$ be a bi-grassmannian as in Proposition* 4.2, *with signed code*

$$\Big[\underbrace{0,\dots,0}_{m},\underbrace{(c-l)',\dots,(c-1)'}_{l},\underbrace{c,\dots,c}_{l_1},\underbrace{0,\dots,0}_{n-m-l-l_1}\Big]$$

$$(l + l_1 > 0, l \leq c \leq l + m).$$

Then g is in the base if and only if $l = 0$ (all factors are positive), or $l_1 = 0$ (all factors are negative), or $l = c$ (the first negative factor is $0'$), or $c = m + l$ (all negative factors ending in t).

Proof. Let $L(g) = \{s_i\}$, $R(g) = \{s_j\}$, and let $b(l, l_1, m; c)$ be the signed code of $g \in \mathrm{BiGr}(W_n)$. Let g_-, g_+ be any bi-grassmannians as in Proposition 4.5, so that $g = \sup(\{g_-, g_+\})$.

Case 1. First assume that $l = 0$, that is, all canonical factors in g are positive. Then g lies in the subgroup of W_n generated by $\{s_1, \dots, s_{n-1}\}$, and so g is a bi-grassmannian in $\mathfrak{S}_n$. By Theorem 3.4, we have $g \in \mathrm{Base}(\mathfrak{S}_n)$. So, using Remark 2.6, we conclude that $g \in \mathrm{Base}(W_n)$.

Case 2. Now assume that $l_1 = 0$ or that $l = c$. We claim that this forces $g = g_-$. Indeed, the coding numbers of the negative factors in g are $(c-l)', \dots, (c-1)'$. The reduced expression of the product of these factors ends with s_{m+l-c}, and the following l_1 nontrivial positive factors end with $s_{m+l-c+1}, \dots, s_{m+l-c+l_1}$ (note that the last index is just j and recall that $R(g) = \{s_j\}$). Now consider g_-. We have $\tau(g_-) = \tau(g) = l$ and so there exists some $k \geq l$ such that the negative factors in g_- have coding numbers $(k-l)', \dots, (k-1)'$. Again, the right-most of these factors ends with s_{m+l-k}, and the following positive factors end with $s_{m+l-k+1}, s_{m+l-k+2}, \dots$. Since $\tau(g_-) = \tau(g)$ and $g_- \leq g$ we have at most l_1 nontrivial positive factors in g_-. Hence the right-most of these factors ends with s_{m+l+l_1-k} or in a generator with a lower index. Since $R(g_-) = R(g)$ we can now deduce that $k \leq c$.

If $l = c$, then this also implies that $k = l$ and, hence, the negative factors in g_- and g are the same. The condition that $R(g_-) = R(g)$ then also forces the positive factors to be the same and, hence, $g_- = g$ as desired.

If $l_1 = 0$, then g_- cannot have any nontrivial positive canonical factors for otherwise we would have $\nu(g_-) > \nu(g)$. So $R(g_-)$ contains the generator at which the right-most negative factor ends. But this generator is s_{m+l-k}. Since $R(g_-) = R(g)$, we conclude that $k = c$ and hence that $g_- = g$ as desired.

We can now take g_- as b_- in (4.4)(a), and deduce that $g = b_- \in \mathrm{Base}(W_n)$.

Case 3. Finally assume that $l, l_1 > 0$ and $l < c = l + m$, that is, there exist nontrivial positive and negative factors but all negative factors end with t. Since $\nu(g_+) = \nu(g)$, the total number of nontrivial factors in g_+ is the same as the corresponding number for g. Since $\tau(g_+) \leq \tau(g)$ and since $l_1 > 0$ we conclude that the last nontrivial factor in g_+ is positive. Hence, since also $R(g_+) = R(g)$, the last nontrivial canonical factors in g_+ and g must be equal. Suppose that g_+ contains exactly $p \geq 0$ negative factors. Since $g_- \leq g$ we certainly must have $p = \tau(g_+) \leq \tau(g) = l$. We claim that we have in fact equality. Indeed, since $\nu(g_+) = \nu(g)$ the number of nontrivial positive factors in g_+ is $l + l_1 - p$. Then we can compute that $l(g_+) \geq 1 + 2 + \cdots + p + c(l + l_1 - p)$ where we used the fact that the kth negative factor has at least length k. On the other hand, we have $l(g) = 1 + 2 + \cdots + l + cl_1$. Since $l(g_+) \leq l(g)$, the asserted equality $p = l$ now follows by just comparing these two formulae for the length. Thus, g_+ and g have the same number of positive and negative factors, respectively, and since the positive factors have the same length, this forces that $g = g_+$. We can take g_+ as b_+ in (4.4)(b) and deduce that $g = b_+ \in \mathrm{Base}(W_n)$.

Thus, every bi-grassmannian which is covered by one of the above three cases is in the base of W_n. Now assume that $l, l_1 > 0$ and $l < c < l + m$. We want to prove that $g \notin \mathrm{Base}(W_n)$. For this purpose we just have to find a particular choice for g_-, g_+ such that $g_- < g, g_+ < g$. This can be done as follows.

Let g_- be the bi-grassmannian in ${}^iW_n^j$ with signed code $b(l, l_1 - 1, m; c - 1)$. Note that the conditions on the parameters in Proposition 4.2 are satisfied since $l_1 > 0$ and $c > l > 0$; moreover, we have $\tau(g_-) = \tau(g)$. Let g_+ be the bi-grassmannian in ${}^iW_n^j$ with signed code $b(l - 1, l_1 + 1, m; c)$. Note again that the conditions on the parameters are satisfied since $l > 0$ and $c < m + l$. It is readily checked that $g_- < g$ and $g_+ < g$. Hence we have found the desired elements. Since $g = \sup(\{g_-, g_+\})$, the proof is complete. ∎

EXAMPLE 4.7. Using Theorem 4.6 we easily see that every bi-grassmannian is in the base for types B_1, B_2, and B_3. If $n = 4$ there are 45 bi-grassmannians and 44 base elements; the missing element is $[0, 0, 1', 2] = s_2 s_1 t s_1 s_3 s_2$. We have, in fact,

$$[0, 0, 1', 2] = \sup([0, 0, 2, 2], [0, 0, 0', 0]).$$

Let us consider the example B_5: there are 90 bi-grassmannians and 85 base elements. The five missing elements, together with expressions as suprema of base elements, are given as

$$\begin{aligned}
[0,0,1',2,0] &= \sup([0,0,0',0,0],[0,0,2,2,0]),\\
[0,0,1',2',3] &= \sup([0,0,0',1',0],[0,0,2',3,3]),\\
[0,0,1',2,2] &= \sup([0,0,0',1,0],[0,0,2,2,2]),\\
[0,0,0,2',3] &= \sup([0,0,0,1',0],[0,0,0,3,3]),\\
[0,0,0,1',2] &= \sup([0,0,0,0',0],[0,0,0,2,2]).
\end{aligned}$$

Note that the first element in this list is the missing element we had for B_4 before.

Let g_n denote the cardinality of the set of bi-grassmannians. Then we have $g_n = (n^4 + 10n^3 + 11n^2 + 2n)/24$, with generating function

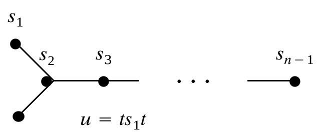
$$\sum_{n\geq 1} g_n z^{n-1} = \frac{1 - z + z^2}{(z-1)^5} = 1 + 6z + 19z^2 + 45z^3 + 90z^4 + 161z^5 + \cdots.$$

Let b_n be the cardinality of the set of base elements. Then we have $b_n = (2n^3 + n)/3$ and its generation function is given by

$$\sum_{n\geq 1} b_n z^{n-1} = \frac{(z+1)^2}{(1-z)^4} = 1 + 6z + 19z^2 + 44z^3 + 85z^4 + 146z^5 + \cdots.$$

5. TYPE D_n

Let $W_n \subset \mathrm{GL}_n(\mathbb{R})$ as in the previous section. For $n \geq 2$ consider the subgroup $\tilde{W}_n \subset W_n$ consisting of all elements $w \in W_n$ such that $\tau(w)$ is even (where, as in (4.4), $\tau(w)$ is the number of generators t in a reduced expression for w). In particular, $\tilde{W}_n$ has index 2 in W_n. If we set $u := ts_1t$, then $(\tilde{W}_n, \{u, s_1, s_2, \ldots, s_{n-1}\})$ is a Coxeter system of type D_n with the following Dynkin diagram:

D_n 

, $n \geq 2$.

We still have a semidirect product decomposition with an elementary abelian normal subgroup of order 2^{n-1} generated by $u_1, \ldots, u_{n-1}$, where

$$u_1 := us_1, \qquad u_i := s_i u_{i-1} s_i \qquad \text{for } 2 \leq i \leq n-1.$$

Note that, now both $\{s_1, s_2, \ldots, s_{n-1}\}$ and $\{u, s_2, \ldots, s_{n-1}\}$ generate complementary subgroups isomorphic to $\mathfrak{S}_n$.

The above Dynkin diagram admits a symmetry of order 2. This gives rise to a group automorphism $\sigma\colon \tilde{W}_n \to \tilde{W}_n$, $w \mapsto w^\sigma$, interchanging u and s_1 and fixing all s_i with $i \geq 2$. This automorphism is in fact given by conjugation with t inside W_n.

5.1. Again, we have a chain of parabolic subgroups $\tilde{W}_2 \subset \tilde{W}_3 \subset \cdots \subset \tilde{W}_n$ corresponding to the subsets of generators $\{u, s_1\} \subset \{u, s_1, s_2\} \subset \cdots \subset \{u, s_1, \ldots, s_{n-1}\}$. Let $\mathbf{R}_1^D := \tilde{W}_2 = \{1, s_1, u, us_1\}$ and, for $2 \leq k \leq n-1$,

$$\mathbf{R}_k^D := \mathbf{R}_k^A \cup \mathbf{U}_k,$$

where

$$\mathbf{U}_k := \{u_k, s_k u_{k-1}, \ldots, s_k s_{k-1} \cdots s_2 u_1\} \cup \{s_k s_{k-1} \cdots s_2 u\}.$$

(Note that $\mathbf{R}_k^D$ contains $2(k+1)$ elements.) Then $\mathbf{R}_k^D$ is the set of minimal right coset representatives of $\tilde{W}_k$ in $\tilde{W}_{k+1}$, for $2 \leq k \leq n-1$.

Every element $w \in \tilde{W}_n$ can be written uniquely in the form $w = r_1 \cdots r_{n-1}$, where $r_k \in \mathbf{R}_k^D$ for all k and we have $\mathit{l}(w) = \mathit{l}(r_1) + \cdots + \mathit{l}(r_{n-1})$. But now the elements in $\mathbf{R}_k^D$ are not determined by their length. So, in order to obtain a unique coding, we represent w by $[c_1, \ldots, c_{n-1}]$, where $c_k = \mathit{l}(r_k) + 1$ if $r_k \in \mathbf{U}_k$, and $c_k = \mathit{l}(r_k)$ otherwise. As we did before, we will often identify an element $w \in \tilde{W}_n$ with its code.

For example, we consider the elements of length 2 in type D_4; their codes are given as follows: $s_1 s_3 = [1,0,1]$, $s_2 u = [0,3,0]$, $s_3 s_2 = [0,0,2]$, $s_1 s_2 = [1,1,0]$, $s_2 s_1 = [0,2,0]$, $s_2 s_3 = [0,1,1]$, $us_3 = [2,0,1]$, $s_1 u = us_1 = [3,0,0]$, $us_2 = [2,1,0]$.

Again, we call a factor $r_k \in \mathbf{R}_k^A$ positive, and a factor $r_k \in \mathbf{U}_k$ negative. In a similar way as for type B_n we can also define a signed code for $w \in \tilde{W}_n$: this is obtained from the above code $[c_1, \ldots, c_{n-1}]$ of w by keeping each $c_k \leq k$ and replacing each $c_k > k$ by $2k + 1 - c_k$. In order to distinguish between positive and negative factors we attach a prime $'$ to the latter ones. Thus, the factors $s_{n-1} \cdots s_{n-c}$ and $s_{n-1} \cdots s_{n-c} u_{n-c-1}$ have coding numbers c and c', respectively. Note that this only works for $c < n-1$. The two factors with coding numbers $(n-1)$ and $(n-1)'$ are $s_{n-1} \cdots s_2 s_1$ and $s_{n-1} \cdots s_2 u$, respectively. In particular, the coding numbers for s_1, u, us_1 are 1, $1'$, $0'$, respectively. Again, we shall write $[c_0^0, \ldots, c_{n-1}^0]$ if we do not want to specify which factors are positive or negative.

For example, the signed codes for the above elements in $\tilde{W}_4$ are given by: $[1,0,1]$, $[0,2',0]$, $[0,0,2]$, $[1,1,0]$, $[0,2,0]$, $[0,1,1]$, $[1',0,1]$, $[0',0,0]$, $[1',1,0]$, respectively.

5.2. Our aim is to find a description of all bi-grassmannians in $\tilde{W}_n$ similar to that for type B_n as in Proposition 4.2. Recall that the proof of the latter result was achieved by an inductive argument, based on the observation that if $w = w'r_{n-1} \in \mathrm{BiGr}(W_n)$, where $1 \neq w' \in W_{n-1}$ and $r_{n-1} \in \boldsymbol{R}_{n-1}$, then we also have $w' \in \mathrm{BiGr}(W_{n-1})$.

This need no longer be true for bi-grassmannian in $\tilde{W}_n$. Indeed, for $n \geq 3$ we have a counterexample given by $w := s_{n-2} \cdots s_2 u s_1 s_{n-1} \cdots s_2 \in \mathrm{BiGr}(\tilde{W}_n)$ for which $w' := s_{n-2} \cdots s_2 u s_1 \notin \mathrm{BiGr}(\tilde{W}_{n-1})$, since $\boldsymbol{R}(w') = \{u, s_1\}$ is not a singleton set.

It will turn out, however, that we still can obtain a uniform description if we consider all the elements $w \in \tilde{W}_n$ such that each of $\boldsymbol{L}(w)$ and $\boldsymbol{R}(w)$ either consists of just one generator or else equals $\{u, s_1\}$. We call these elements pseudo bi-grassmannians.

PROPOSITION 5.3. *An element $w \in \tilde{W}_n$ is a pseudo bi-grassmannian if and only if there exist integers $m, l, l_1 \geq 0$ and $c > 0$ such that one of the following conditions is satisfied. Either $w^\sigma = w$ and the signed code of w has the form*

$$b_I(m, l, l_1; c) := \Big[\underbrace{0, \dots, 0}_{m}, \underbrace{(c-l)', \dots, (c-1)'}_{l}, \underbrace{c, \dots, c}_{l_1}, \underbrace{0, \dots, 0}_{n-1-m-l-l_1}\Big], \tag{I}$$

where $l + l_1 > 0$ and $l \leq c \leq l + m$ (cf. the similar conditions in Proposition 4.2 *for type B_n); or $w^\sigma \neq w$ and the signed code of one of w, w^σ has the form*

$$b_{II}(m, l, l_1; c) := \Big[\underbrace{0, \dots, 0}_{m}, \underbrace{(c-l)^0, \dots, (c-3)', c-2, (c-1)'}_{l}, \underbrace{c, \dots, c}_{l_1}, \underbrace{0, \dots, 0}_{n-1-m-l-l_1}\Big], \tag{II}$$

where $l > 0$ and $c = l + m + 1$.

The only pseudo bi-grassmannians which are not bi-grassmannians are those of form (I) with $c = l$, $m = 0$ (the first negative factor is us_1) or with $c = l + m$, $l_1 = 0$ (all factors are negative, and they are ending in us_1).

Proof. It is easily checked that all elements $w \in \tilde{W}_n$ such that the signed code of one of w, w^σ has the form (I) or (II) are pseudo bi-grassmannians. To prove the converse, we can again proceed by induction on n. If $n = 2$, then all nonidentity elements are pseudo bi-grassmannians and there is nothing to prove. So, now let $n > 2$ and $w = r_1 \cdots r_{n-1} \in \tilde{W}_n$ be a pseudo bi-grassmannian with $r_k \in \boldsymbol{R}_k^D$ for all k. If $r_{n-1} = 1$ we are done by induction. So, first we can assume that $r_{n-1} \neq 1$. If $w' = 1$, then $w = r_{n-1}$ has the required form and we are done. Hence, we can also assume that $w' := r_1 \cdots r_{n-2} \neq 1$. By a similar argument as in the proof of

Proposition 3.2 we must have $r_{n-2} \neq 1$. We now check that w' is a pseudo bi-grassmannian in $\tilde{W}_{n-1}$, by considering the following three cases.

Case 1. $r_{n-1} = s_{n-1} \cdots s_{n-c}$ with $1 \leq c \leq n-1$. Then r_{n-1} has coding number c, and we have $\mathcal{R}(w) = \{s_{n-c}\}$. Note that if $c = n-1$, then $r_{n-1}u \in \mathcal{R}^D_{n-1}$ is reduced, and so we cannot have $wu < w$. As in the proof of Proposition 4.2 we see that if $w's_i < w'$, then $i = n-c-1 > 0$. On the other hand, if $w'u < w'$, then $c \geq n-2$ (note that u commutes with $s_{n-1}, \ldots, s_3, s_1$). Combining these two conditions we see that w' is a pseudo bi-grassmannian, with

(a) either $c \leq n-2$ and $\mathcal{R}(w') = \{s_{n-c-1}\}$,

(b) or $c = n-2$ and $\mathcal{R}(w') = \{u, s_1\}$,

(c) or $c \geq n-2$ and $\mathcal{R}(w') = \{u\}$.

If (a) holds, then r_{n-2} could either be the positive factor $s_{n-2} \cdots s_{n-c-1}$ with coding number c or the negative factor $s_{n-2} \cdots s_{n-c}u_{n-c-1}$ with coding number $(c-1)'$. If (b) holds, then r_{n-2} is the negative factor $s_{n-2} \cdots s_2 u s_1$ with coding number $(n-3)' = (c-1)'$. Finally, if (c) holds, then r_{n-2} is the negative factor $s_{n-2} \cdots s_2 u$ with coding number $(n-2)'$.

Case 2. $r_{n-1} = s_{n-1} \cdots s_2 u$ with coding number $(n-1)'$ and $\mathcal{R}(w) = \{u\}$. (A similar argument as in Case 1 shows that we cannot have $ws_1 < w$.) Since $s_i r_{n-1} = r_{n-1} s_{i+1}$ for $i \geq 2$, we cannot have $w's_i < w'$ in this case. Nor can we have $w'u < w'$, since $ur_{n-1} = s_{n-1} \cdots s_3 u s_2 u = s_{n-1} \cdots s_2 s_2 u s_2$. Thus we should have $\mathcal{R}(w') = \{s_1\}$, and hence, r_{n-2} is the positive factor $s_{n-2} \cdots s_2 s_1$ with coding number $n-2$.

Case 3. $r_{n-1} = s_{n-1} \cdots s_{n-c} u_{n-c-1}$ with $1 \leq c \leq n-2$. Then r_{n-1} has coding number c' and we have $\mathcal{R}(w) = \{s_{n-c-1}\}$ (for $c \leq n-3$) or $\mathcal{R}(w) = \{s_1, u\}$ (for $c = n-2$). As before, we can check that if $w's_i < w'$, then we must have $i = n-c-1$. Furthermore, if $c \leq n-3$ we cannot have $w'u < w'$, since $ur_{n-1} = r_{n-1}s_1$. Thus, there are only the following possibilities:

(a) either $r_{n-2} = s_{n-2} \cdots s_{n-c-1}$ with coding number c,

(b) or $c > 1$ and $r_{n-2} = s_{n-2} \cdots s_{n-c} u_{n-c-1}$ with coding number $(c-1)'$,

(c) or $c = n-2$ and $r_{n-2} = s_{n-2} \cdots s_2 u$ with coding number c'.

By a similar computation as in Case 3 of the proof of Proposition 4.2, we can check that we would have $s_{n-1} \in \mathcal{L}(w)$ if (a) or (c) holds. For example, in case (c) we would find that

$$\begin{aligned} s_{n-1} r_{n-2} r_{n-1} &= s_{n-1}(s_{n-2} \cdots s_2)(s_{n-1} \cdots s_3) u s_2 u s_1 \\ &\vdots \\ &= (s_{n-2}s_{n-1})(s_{n-3}s_{n-2}) \cdots (s_3 s_2) s_3 u s_2 u s_1 \\ &= (s_{n-2}s_{n-1})(s_{n-3}s_{n-2}) \cdots (s_3 s_2) s_3 s_2 u s_2 s_1 \end{aligned}$$

which is not reduced since $s_3 s_2 s_3 s_2$ is not reduced. Hence, these two cases can in fact not occur.

Summarizing, we see that in each case above w' is a pseudo bi-grassmannian. Hence, by induction, w' has the desired form. Let $[c_1^0, \dots, c_{n-1}^0]$ be the signed code of w. From the above cases we also have found that one of the following five conditions must hold for the two right-most coding numbers r_{n-2}, r_{n-1}:

(I) or (II) $\begin{cases} \text{both } r_{n-2} \text{ and } r_{n-1} \text{ are positive and } c_{n-2} = c_{n-1}; \\ r_{n-2} \text{ is negative, } r_{n-1} \text{ is positive, and } c_{n-2} = c_{n-1} - 1; \end{cases}$

only (I) $\{ \text{both } r_{n-2} \text{ and } r_{n-1} \text{ are negative and } c_{n-2} = c_{n-1} - 1;$

only (II) $\begin{cases} r_{n-2} \text{ is positive, } r_{n-1} \text{ is negative, } n-2 = c_{n-2} = c_{n-1} - 1; \\ r_{n-2} \text{ is negative, } r_{n-1} \text{ is positive, and } n-2 = c_{n-2} = c_{n-1}. \end{cases}$

If the first, second, or third of the above conditions is satisfied, we can apply induction and conclude that our element has a signed code of form (I).

If the fourth condition is satisfied, we apply induction again and conclude that our element has a signed code of form (II).

If the fifth condition is satisfied, we first apply σ and then we see that, by induction, our element has a signed code of form (II).

This completes the proof of the assertions about the signed code of pseudo bi-grassmannians.

The assertions about bi-grassmannians now easily follow: we just need to exclude the cases when $\mathsf{L}(w)$ or $\mathsf{R}(w)$ equals $\{u, s_1\}$. From the above discussion, we see that the only case where we have $\mathsf{R}(w) = \{s_1, u\}$ is when r_{n-1} has coding number $(n-2)'$. But then r_{n-2} should have coding number $(n-3)'$ by Case 3(b). Thus, by induction, the signed code of w has the form $[0, \dots, 0, (n-l-1)', \dots, (n-2)']$. Similarly, we can have $\mathsf{L}(w) = \{s_1, u\}$ only if the first negative factor is us_1, that is, only if the signed code has the form $[0', 1', \dots, (l-1)', l, \dots, l]$. This completes the proof. ∎

5.4. Consider the embedding $\tilde{W}_n \hookrightarrow W_n$. Given any $w \in \tilde{W}_n$, we can rewrite a reduced expression for w in the generators of $\tilde{W}_n$ as a reduced expression in the generators of W_n, by using the rules $s_i \mapsto s_i$ (for all i) and $u \mapsto ts_1t$.

Now let $g \in \tilde{W}_n$ be a bi-grassmannian with signed code $b_I(m, l, l_1; c)$ or $b_{II}(m, l, l_1; c)$ as in Proposition 5.3, and consider its image in W_n. We claim that this image satisfies one of the following conditions:

(a) If l is even, then g is also a bi-grassmannian in W_n, with signed code $b(m+1, l, l_1; c)$;

(b) If l is odd, then tg is a bi-grassmannian in W_n, with signed code $b(m+1, l, l_1; c)$.

To prove this, assume first that g is of the form (I). We write $g = r_1 \cdots r_{n-1}$ with $r_k \in \boldsymbol{R}_k^D$ for all k. If r_k is a positive factor, then the reduced expressions for r_k in $\tilde{W}_n$ and in W_n are identical. If $r_k = s_k \cdots s_{k-c+1} u_{k-c}$ (for $c < k$) or $r_k = s_k \cdots s_2 u$, then, using the rule $u_i = tt_i = t_i t$ (for all i) and the fact that t commutes with all s_i, $i \geq 2$, we obtain $r_k = ts_k \cdots s_{k-c+1} t_{k-c}$ or $r_k = ts_k \cdots s_2 s_1 t$, respectively. Thus, each of the l negative factors produces a negative factor in W_n and an extra factor t. Since t commutes with all s_i for $i \geq 2$, these extra factors cancel out in pairs. Hence we obtain a sequence of l negative factors in W_n, and an extra factor t at the beginning if l is odd. If g has form (II), then the l alternating positive and negative factors produce a sequence of l negative factors in W_n ending with t, and again an extra factor t at the beginning if l is odd. In both cases we see that the resulting element in W_n has the form as claimed.

However, if $g \in \tilde{W}_n$ is a bi-grassmannian such that $g \neq g^\sigma$ and g^σ has a signed code of form (II), then neither g nor tg is a bi-grassmannian in W_n. (Take, for example, the element $g = us_2 \in \mathrm{BiGr}(\tilde{W}_3)$ with signed code $[1', 1]$.) The signed code of g is then given by

(c) $b_{II}^\sigma(m, l, l_1; c)$

$$:= \Big[\underbrace{0, \ldots, 0}_{m}, \underbrace{(c-l)^0, \ldots, (c-2)', c-1}_{l}, \underbrace{c', c, \ldots, c}_{l_1}, \underbrace{0, \ldots, 0}_{n-1-m-l-l_1}\Big],$$

where $c = m + l + 1$ as before.

5.5. In order to find the base for $\tilde{W}_n$, our strategy will be mainly the same as the one for type B_n (see the steps in the proof of Theorem 4.6).

First note that the Bruhat–Chevalley order on $\tilde{W}_n$ is not the restriction of the Bruhat–Chevalley order on W_n. (Take, for example, the elements $s_1, u = ts_1 t$ which are comparable in W_n but not in $\tilde{W}_n$.) For this reason, in order to distinguish the ordering on $\tilde{W}_n$ from the one on W_n, we will denote it from now on by $\preccurlyeq$. Since $\tilde{W}_n$ is a subgroup of W_n generated by the reflections $u = ts_1 t, s_1, \ldots, s_{n-1}$ of W_n, we have the implications (see [8, Lemma 1.9]):

(a) if $x, y \in \tilde{W}_n$ with $x \preccurlyeq y$, then $x \leq y$ and $tx \leq ty$.

Furthermore, we will need an analogue of the function τ in (4.4). For any $w \in \tilde{W}_n$ we define $\tilde{\tau}(w) \geq 0$ as follows. Write $w = r_1 \cdots r_{n-1}$ in canonical form with $r_k \in \mathbf{R}_k^D$ for all k. If all r_k are positive let $\tilde{\tau}(w) := 0$. Otherwise, there exists at least one negative factor (in particular, $w \neq 1$) and we let

$$\tilde{\tau}(w) := \max\{k \geq 1 \mid r_k \in \mathbf{U}_k\} - \min\{m \geq 1 \mid r_m \neq 1\} + 1.$$

Thus, $\tilde{\tau}(w)$ is the length of that part of the canonical form of w which starts with the left-most nontrivial factor and ends with the right-most negative factor.

Now consider $g \in \mathrm{BiGr}(\tilde{W}_n)$. Then one of g, g^σ has a signed code of the form (I) or (II) as in Proposition 5.3. We embed g into W_n and compare the values $\tilde{\tau}(g), \tau(g)$. Using the embedding rules in (5.4) we find that

$$\text{(b)} \quad \begin{cases} tg > g \text{ and } \tau(g) = \tilde{\tau}(g) & \text{if } \tilde{\tau}(g) \text{ is even,} \\ tg < g \text{ and } \tau(g) = \tilde{\tau}(g) + 1 & \text{if } \tilde{\tau}(g) \text{ is odd.} \end{cases}$$

With these notations we can now state:

LEMMA 5.6. *Let $g \in \tilde{W}_n$ be a bi-grassmannian.*

(*a*) *There exists some $b_- \in \mathrm{Base}(\tilde{W}_n)$ with $b_- \preccurlyeq g$ such that $\mathbf{L}(b_-) = \mathbf{L}(g)$, $\mathbf{R}(b_-) = \mathbf{R}(g)$, and $\tilde{\tau}(b_-) = \tilde{\tau}(g)$.*

(b) *If the signed code of g has the form $b_I(m, l, l_1; c)$ with $l = c$ or $l_1 = 0$, then $g = b_- \in \mathrm{Base}(\tilde{W}_n)$.*

Proof. Let $Y_g \subseteq \mathrm{Base}(\tilde{W}_n)$ be similarly defined as in (4.4), so that $g = \sup(Y_g)$. Let $e := \max\{\tilde{\tau}(y) \mid y \in Y_g\}$. We want to show that $e = \tilde{\tau}(g)$. Take any $z \in Y_g$ with $\tilde{\tau}(z) = e$.

Assume, if possible, that $e > \tilde{\tau}(g)$. Since $z \preccurlyeq g$ we also have $z \leq g$ by the first inequality in (5.5)(a), and so $\tau(z) \leq \tau(g)$. On the other hand, by (5.5)(b), we have $e = \tilde{\tau}(z) \leq \tau(z)$ and so $e \leq \tau(g)$. Our assumption $e > \tilde{\tau}(g)$ now yields $\tau(g) \geq e > \tilde{\tau}(g)$. Using again (5.5)(b) we conclude that e must be even and $\tilde{\tau}(g) = \tau(g) - 1 = e - 1$ is odd. But in this case we also have $tg < g$ and $tz > z$. This implies that $\tau(tg) < \tau(g) = \tilde{\tau}(g) + 1 = e$ and $\tau(tz) > e$. The second inequality in (5.5)(a) yields that $tz \leq tg$ and so $e < \tau(tz) \leq \tau(tg) < e$, a contradiction. Hence the assumption was wrong and we must have $e \leq \tilde{\tau}(g)$.

Assume now, if possible, that $e < \tilde{\tau}(g)$. By a similar argument as in (4.4)(a) this implies that $g \preccurlyeq u_m \cdots u_{m+e-1} y_0$, where y_0 is a certain product of generators s_j. Using the first inequality in (5.5)(a) we obtain $g \leq t^e t_m \cdots t_{m+e-1} y_0$ and so $\tau(g) \leq e$ (for e even) or $\tau(g) \leq e + 1$ (for e odd). Since $\tilde{\tau}(g) \leq \tau(g)$ we conclude, using our assumption $e < \tilde{\tau}(g)$, that e must be odd and $\tilde{\tau}(g) = \tau(g) = e + 1$ is even. The second inequality in (5.5)(a) yields that $tg \leq t_m \cdots t_{m+e-1} y_0$, and so $\tau(tg) \leq e$. Since $\tilde{\tau}(g)$ is even we have $tg > g$ (see (5.5)(b)). Hence we conclude that $\tilde{\tau}(g) = \tau(g) < \tau(tg) \leq e$, again a contradiction. Thus, (a) is proved.

Now consider (b). Let $g \in \mathrm{BiGr}(\tilde{W}_n)$ be of form (I), with signed code $b_I(m, l, l_1; c)$ such that $l = c$ or $l_1 = 0$. By (a) we can find an element $z \in Y_g$ with $\tilde{\tau}(z) = \tilde{\tau}(g)$.

If z has form (I) or (II), then we embed g, z into W_n, use the rules in (5.4)(a), (b), and see that we are in a completely similar situation as in Case 2 of the proof of Theorem 4.6. In a similar way as in that proof, we can conclude that $g = z \in \mathrm{Base}(\tilde{W}_n)$, and we are done.

It remains to consider the case where our element z is a bi-grassmannian such that $z \neq z^\sigma$, where z^σ has form (II). Then the signed code of z has a form as in (5.4)(c). Since $\tilde{\tau}(z) = \tilde{\tau}(g)$ the right-most negative factor in z appears at the same position in the canonical form as the right-most negative factor in g does. Now all negative factors in g have the form $s_k \cdots s_{k-c+1} u_{k-c}$ for $k > c$. So we can just insert suitable factors u or s_1 into the reduced expression for z so that z becomes a bi-grassmannian of form (I); call this new element z_1. (Note that now the lengths of the right-most negative and of the left-most positive factor in the new element are arranged correctly so as to give a bi-grassmannian of form (I).)

We still have $z_1 \preccurlyeq g$. Again, we embed these elements into W_n, use the rules in (5.4), and conclude that $z_1 = g$ in the same way as above. But note that every negative factor of z_1 ends with us_1, and hence the same holds for g. The assumption that $l = c$ or $l_1 = 0$ would therefore imply that g starts or ends with us_1, and so g would be a pseudo bi-grassmannian but not a bi-grassmannian (see the excluded parameters in Proposition 5.3). This contradiction completes the proof of (b). ∎

THEOREM 5.7. *If $g \in \tilde{W}_n$ is a bi-grassmannian such that the signed code of g or g^σ has form (II), then g lies in* $\mathrm{Base}(\tilde{W}_n)$. *A bi-grassmannian $g \in \tilde{W}_n$ with $g^\sigma = g$ and signed code $b_I(m, l, l, l_1; c)$ of form (I) lies in* $\mathrm{Base}(\tilde{W}_n)$ *if and only if*

$$l = 0, \qquad l_1 = 0, \qquad \textit{or} \qquad c = l.$$

Furthermore, the map $[c_1^0, \ldots, c_{n-1}^0] \mapsto [0, c_1^0, \ldots, c_{n-1}^0]$ defines a bijection between $\mathrm{BiGr}(\tilde{W}_n) \setminus \mathrm{Base}(\tilde{W}_n)$ *and the corresponding set for W_n.*

Proof. First let $g \in \tilde{W}_n$ be a bi-grassmannian such that g or g^σ has form (II). Since σ preserves the Bruhat–Chevalley order we deduce that $g \in \mathrm{Base}(\tilde{W}_n)$ if and only if $g^\sigma \in \mathrm{Base}(\tilde{W}_n)$. So we can assume that g has form (II) with signed code $b_{II}(m, l, l_1; c)$. By Theorem 2.5, there exists a subset $Y \subseteq \mathrm{Base}(\tilde{W}_n)$ such that $g = \sup(Y)$ and such that $\mathsf{L}(y) = \mathsf{L}(g)$, $\mathsf{R}(y) = \mathsf{R}(g)$ for all $y \in Y$. By an analogous argument as in (4.4)(b) there exists some $b_+ \in Y$ such that $\tilde{\nu}(b_+) = \tilde{\nu}(g)$. Here $\tilde{\nu}(g)$ is defined as the biggest $k \geq 2$ such that the generator s_{k-1} occurs in a reduced expression for g (cf. the similar definition in (4.4) for type B_n).

We claim that we have in fact $g = b_+ \in \mathrm{Base}(\tilde{W}_n)$. In order to prove this, suppose first that b_+ has form (I), with signed code $b_I(m', l', l_1'; c')$. Since $\mathcal{L}(g) = \mathcal{L}(b_+)$ we have $m = m'$. On the other hand, we have $\mathcal{R}(b_+) = \{s_{m'+l'+l_1'-c'+1}\}$ and $\mathcal{R}(g) = \{s_{l_1}\}$ (for $l_1 > 0$) or $\mathcal{R}(g) = \{u\}$ (for $l_1 = 0$). Since $\mathcal{R}(b_+) = \mathcal{R}(g)$ and b_+ has the form (I) we conclude that $0 < l_1 = m + l' + l_1' - c' + 1$. The equality $l + l_1 + 1 = \tilde{\nu}(g) = \tilde{\nu}(b_+) = l' + l_1' + 1$ implies that $c' = m + l + 1 = c$. But $c' \leq m + l'$ and so $l + 1 \leq l'$. Now we can estimate the length of b_+ and find a contradiction by a similar argument as in Case 3 of the proof of Theorem 4.6. Hence b_+ or b_+^σ must be of form (II). But in this case, the conditions that $b_+ \preccurlyeq g$, $\mathcal{L}(b_+) = \mathcal{L}(g)$, $\mathcal{R}(b_+) = \mathcal{R}(g)$, and $\tilde{\nu}(b_+) = \tilde{\nu}(g)$ imply that $g = b_+$. Hence we have $g \in \mathrm{Base}(\tilde{W}_n)$, and we are done.

From now on, assume that $g^\sigma = g$ and $g \in \mathrm{BiGr}(\tilde{W}_n)$ has form (I). Let us first consider the case where g satisfies one of the conditions $l = 0$, $l_1 = 0$, or $c = l$. If $l = 0$, then $g \in \mathfrak{S}_n$ and we are done, using (2.6) and Theorem 3.4. The cases for $l_1 = 0$ or $l = c$ have already been considered in Lemma 5.6(b). So now it remains to show that a bi-grassmannian $g \in \tilde{W}_n$ of form (I) is not a base element if it does not satisfy any of the above conditions.

We use a similar strategy as in the proof of Theorem 4.6. We construct bi-grassmannians $g_-, g_+ \in \tilde{W}_n$ of form (I) as follows: g_- is obtained from g by keeping the number of negative factors and decreasing the number of positive factors by 1; g_+ is obtained from g by keeping the total number of nontrivial factors but increasing the number of positive factors by 1. Moreover, this can be done in such a way that we still have $\mathcal{L}(g_-) = \mathcal{L}(g_+) = \mathcal{L}(g)$ and $\mathcal{R}(g_-) = \mathcal{R}(g_+) = \mathcal{R}(g)$, and these conditions uniquely determine g_- and g_+.

It can be readily checked that $g_+ \prec g$ and $g_+ \prec g$, and now our aim is to show that $g = \sup(\{g_-, g_+\})$. By Lemma 2.7, it is sufficient to take any bi-grassmannian $g' \in \tilde{W}_n$ with $g_- \preccurlyeq g', g_+ \preccurlyeq g'$ and $\mathcal{L}(g') = \mathcal{L}(g)$, $\mathcal{R}(g') = \mathcal{R}(g)$. Then next, by showing that $g \preccurlyeq g'$ we complete the proof by a completely similar argument as in the proof of Proposition 4.5. We omit further details. ∎

EXAMPLE 5.8. Consider the example type D_4. There are 30 bi-grassmannians; their signed codes are given as follows:

$$[1',0,0],[1',2,3'],[1',2,0],[1',1,0],[1',2,2],[1',1,1],[1,2',0],$$
$$[1,0,0],[1,2',3],[1,1,0],[1,2',2],[1,1,1],[0,2',0],[0,2,3'],$$
$$[0,2,0],[0,2',3],[0,1,0],[0,0',0],[0,2',2],[0,2,2],[0,1',2],$$
$$[0,0',1'],[0,1,1],[0,0',1],[0,0,3'],[0,0,3],[0,0,2],[0,0,1'],$$
$$[0,0,1],[0,0,0'].$$

There is one bi-grassmannian which is not in the base: $[0,1',2] = s_2us_1s_3s_2$. This element can in fact be obtained as the $\sup([0,0',0], [0,2,2],[0,2',2])$.

The following three base elements do not admit a "clivage": $[0,0',0]$, $[0,2',2],[0,2,2]$. Hence, by [7, Théorème 2.8], we see that the "enveloping lattice" of $(W, \leq)$ is not distributive.

Consider now type D_5. There are 69 bi-grassmannians and 64 base elements. The signed codes of the five missing elements and expressions as suprema of base elements are given as follows:

$$\begin{aligned}
[0,1',2,0] &= \sup([0,0',0,0],[0,2,2,0],[0,2',2,0]),\\
[0,1',2',3] &= \sup([0,0',1',0],[0,2',3,3],[0,2,3',3]),\\
[0,1',2,2] &= \sup([0,0',1,0],[0,2,2,2],[0,2',2,2]),\\
[0,0,2',3] &= \sup([0,0,1',0],[0,0,3,3],[0,0,3',3]),\\
[0,0,1',2] &= \sup([0,0,0',0],[0,0,2,2]).
\end{aligned}$$

Note that, indeed, we have a bijection between the set of these elements and the corresponding set of five missing elements for type B_5 in Example 4.7. Note also that the elements with signed codes $[0,2',2,0]$, $[0,2,3',3]$, $[0,2',2,2]$, and $[0,0,3',3]$ do not have form (II), but only their images under σ.

Let g_n denote the cardinality of the set of bi-grassmannians. Then we have $g_n = (n^4 + 14n^3 - 37n^2 + 46n - 24)/24$, with generating function given by

$$\sum_{n\geq 2} g_n z^{n-2} = \frac{2 - z^3}{(1-z)^5} = 2 + 10z + 30z^2 + 69z^3 + 135z^4 + 237z^5 + \cdots.$$

Let b_n denote the cardinality of the set of base elements. Then we have $b_n = (5n^3 - 12n^2 + 13n - 6)/6$ and its generating function is given by

$$\sum_{n\geq 2} b_n z^{n-2} = \frac{2 + 2z + z^2}{(1-z)^4} = 2 + 10z + 29z^2 + 64z^3 + 120z^4 + 202z^5 + \cdots.$$

These formulae follow from Proposition 5.3 and Theorem 5.7.

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REFERENCES

1. N. Bourbaki, "Groupes et algèbres de Lie," Chaps. IV–VI, Hermann, Paris, 1968.
2. V. V. Deodhar, Some characterizations of Bruhat ordering on a Coxeter group and determination of the relative Möbius function, *Invent. Math.* **39** (1977), 187–198.
3. M. Dyer, "Hecke Algebras and Reflections in Coxeter Groups," Ph.D. thesis, University of Sydney, 1987.
4. C. Ehresmann, Sur la topologie de certains espaces homogènes, *Ann. of Math.* **35** (1934), 396–443.
5. M. Geck, G. Hiss, F. Lübeck, G. Malle, and G. Pfeiffer, CHEVIE-A system for computing and processing generic character tables for finite groups of Lie type, Weyl groups and Hecke algebras, *Appl. Algebra Engrg. Comm. Comput.* **7** (1996), 175–210.
6. J. E. Humphreys, "Reflection Groups and Coxeter Groups," Cambridge Stud. Adv. Math., Vol. 29, Cambridge Univ. Press, Cambridge, 1990.
7. A. Lascoux and M. P. Schützenberger, Treillis et bases des groupes de Coxeter, *Electron. J. Combin.* **3** (1995).
8. G. Lusztig, "Characters of Reductive Groups over a Finite Field," Ann. Math. Studies, Vol. 107, Princeton Univ. Press, Princeton, NJ, 1984.
9. R. A. Proctor, Classical Bruhat orders and lexicographic shellability, *J. Algebra* **77** (1982), 104–126.
10. M. Schönert *et al.*, "GAP—Groups, Algorithms, and Programming," Lehrstuhl D für Mathematik, fourth ed., RWTH Aachen, Germany, 1994.

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