Consequences of the Lakshmibai-Sandhya Theorem; the ubiquity of permutation patterns in Schubert calculus and related geometry

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1. Every Schubert variety $X(w) \subset GL_n/B$ is defined by determinantal equations coming from rank conditions.

2. $X(v) \subset X(w)$ if and only if $v \leq w$ in Bruhat order.

3. (Lakshmibai-Seshadri) The tangent space at $v$ to $X(w)$ has dimension $\# \{ t_{ij} : vt_{ij} \leq w \}$.

4. The Bruhat graph on $w$ has vertices indexed by $\{ v : v \leq w \}$ and edges between vertices which differ by a transposition.
Lakshmibai-Sandhya Theorem

**Fact.** There exists a simple criterion for characterizing smooth Schubert varieties using pattern avoidance.

**Theorem:** Lakshmibai-Sandhya 1990 (see also Haiman, Ryan, Wolper)

$X_w$ is non-singular $\iff$ $w$ has no subsequence with the same relative order as 3412 and 4231.

**Example:**

$w = 625431$ contains $6241 \sim 4231$ $\implies X_{625431}$ is singular

$w = 612543$ avoids $4231 \implies X_{612543}$ is non-singular
22 Years Later . . .

Consequences of the Lakshmibai-Sandhya Theorem.

Many geometrical properties of Schubert varieties are now characterized by pattern avoidance or a variation on this theme.

Let me tell you about 10 of them!
10 Pattern Properties

Property 1. (Carrell-Peterson, Deodhar, Gasharov) (ca 1994)
The following are equivalent

1. $X_w$ is smooth.

2. The Bruhat graph for $w$ is regular and every vertex has degree $\ell(w)$.

3. $\ell(w) = \# \{ t_{ij} \leq w \}$.

4. $w$ avoids 3412 and 4231.

5. The Poincare polynomial for $w$, $P_w(t) = \sum_{v \leq w} t^{l(v)}$ is palindromic.

6. The Poincare polynomial for $w$ factors nicely

$$P_w(t) = \prod_{i=1}^{k} (1 + t + t^2 + \cdots + t^{e_i})$$

Example. $P_{4321}(t) = (1 + t)(1 + t + t^2)(1 + t + t^2 + t^3)$
10 Pattern Properties

Prop 1 (Continued).
The following are also equivalent

1. $X_w$ is smooth.

2. $w$ avoids 3412 and 4231.

3. In the inversion hyperplane arrangement defined by $x_i - x_j = 0$ for all $i < j$ such that $w(i) > w(j)$, the generating function

$$R_w(t) = \sum_r q^{d(r)} = \sum_{v \leq w} t^{l(v)} = P_w(t)$$

4. The Kazhdan-Lusztig polynomial $P_{x,w}(t) = 1$ for all $x \leq w$.

5. The Kazhdan-Lusztig polynomial $P_{id,w}(t) = 1$.


Example. $P_{id,3412}(t) = (1 + t)$
Aside on KL-polys

“Everything you need to know to get started:”

• $S_n$ generated by adjacent transpositions $s_i = t_{i,i+1}$ for $1 \leq i < n$.

• $\mathcal{H} = \text{Hecke algebra}$ associated to $S_n$ generated by $\{T_1, T_2, \ldots, T_{n-1}\}$ with relations

  1. $(T_i)^2 = (q - 1)T_i + q$.
  2. $T_i T_j = T_j T_i$ if $|i - j| > 1$.
  3. $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ for all $1 \leq i < n$.

• $T_w = T_{i_1} T_{i_2} \cdots T_{i_p}$ if the reduced expression $s_{i_1} s_{i_2} \cdots s_{i_p} = w \in S_n$.

Easy Fact. $\{T_w : w \in W\}$ form a linear basis for $\mathcal{H}$. 
Observation. $T_w$'s are invertible over $\mathbb{Z}[q, q^{-1}]$.

- Recall the relation $(T_i)^2 = (q - 1)T_i + q$.
- $(T_i)^{-1} = q^{-1}T_i - (1 - q^{-1})$.
- $(T_{w-1})^{-1} = (T_{i_1})^{-1} \cdots (T_{i_p})^{-1}$ if $s_{i_1}s_{i_2} \cdots s_{i_p} = w$ (reduced).

Kazhdan-Lusztig Involution. Linear transformation interchanging

\[
T_w \overset{i}{\leftrightarrow} (T_{w-1})^{-1}
\]

\[
q \overset{i}{\leftrightarrow} q^{-1}
\]
Kazhdan-Lusztig Basis for $\mathcal{H}$

**Theorem.** (KL, 1979) There exists a unique basis \( \{ C'_w : w \in W \} \) for the Hecke algebra over \( \mathbb{Z}[q^{1/2}, q^{-1/2}] \) such that

1. \( i(C'_w) = C'_w \).

2. The change of basis matrix is upper triangular and determined by

\[
C'_w = q^{-1/2 \ell(w)} \sum_{x \leq w} P_{x,w}(q) T_x
\]

where \( P_{w,w} = 1 \) and for all \( x < w \), \( P_{x,w}(q) \in \mathbb{Z}[q] \) with degree at most

\[
\frac{\ell(w) - \ell(x) - 1}{2}.
\]

**Defn.** \( P_{x,w}(q) \) is the Kazhdan-Lusztig polynomial for \( x, w \).
Examples

\[ C'_{s_i} = q^{-\frac{1}{2}}(1 + T_i) = q^{\frac{1}{2}}(1 + T_i^{-1}) \]

\[ C'_{s_i} C'_{s_j} = q^{-1}(1 + T_i)(1 + T_j) \]
\[ = q^{-1}(1 + T_i + T_j + T_i T_j) \]
\[ = C'_{s_i s_j} \quad \text{for } i \neq j \]

\[ C'_{s_1} C'_{s_2} C'_{s_1} = q^{-\frac{3}{2}}(1 + T_1)(1 + T_2)(1 + T_1) \]
\[ = q^{-\frac{3}{2}}(1 + 2T_1 + T_2 + T_1 T_2 + T_2 T_1 + T_1^2 + T_1 T_2 T_1) \]
\[ = q^{-\frac{3}{2}}(1 + 2T_1 + T_2 + T_1 T_2 + T_2 T_1 + ((q - 1)T_1 + q) + T_1 T_2 T_1) \]

\[ C'_{s_1 s_2 s_1} = C'_{s_1} C'_{s_2} C'_{s_1} - C'_{s_1} \]
Deodhar Elements

**Defn.** $w \in S_n$ is *Deodhar* if $C'_w = C'_{s_{i_1}} C'_{s_{i_2}} \cdots C'_{s_{i_p}}$ for some reduced expression $s_{i_1} s_{i_2} \cdots s_{i_p} = w$.

- $C'_{s_1 s_2} = C'_{s_1} C'_{s_2}$ is *Deodhar*.
- $C'_{s_1 s_2 s_1} = C'_{s_1} C'_{s_2} C'_{s_1} - C'_{s_1}$ is *nonDeodhar*.
Kazhdan-Lusztig Polynomials

**Observation.** We have $P_{x, w} = P_{xs_i, w}$. If $ws < w$ and $xs < x < w$, 

$$P_{x, w}(q) = qP_{xs_i, ws_i}(q) + P_{x, ws_i}(q) - \sum_{zs_i < z} q^{\frac{\ell(w) - \ell(z)}{2}} \mu(z, ws_i) P_{x, z}(q).$$

where $\mu(x, w) = \text{coefficient of } q^{\frac{\ell(w) - \ell(x) - 1}{2}}$ in $P_{x, w}(q)$.

**Theorem.** (KL,1980)
If $W$ is a Weyl group or affine Weyl group then 

$$P_{x, w}(q) = \sum \dim \mathcal{H}^i_x(X_w) q^i.$$ 

**Corollary.**
The coefficients of $P_{x, w}(q)$ are non-negative integers with constant term 1.
### Examples of Kazhdan–Lusztig polynomials

Below are all $P_{x,w}(q)$ with $x = \text{id}$ and $w \in S_5$ which are different from 1:

<table>
<thead>
<tr>
<th>$w$</th>
<th>$P_{\text{id},w}$</th>
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<tr>
<td>(14523)</td>
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<td>(45312)</td>
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Interesting Properties

1. (Beilinson–Bernstein, Brylinski–Kashiwara, 1981) The multiplicities of irreducibles in the formal character of a Verma module are determined by the $P_{v,w}(1)$.

2. (Irving, 1988, Braden–MacPherson, 2001)
   \[ \text{If } x \leq y, \quad \text{coef}_{q^k} P_{x,w}(q) \geq \text{coef}_{q^k} P_{y,w}(q) \]

3. (Polo, 1999) Every polynomial with constant term 1 and nonnegative integer coefficients is the KL-poly of some pair of permutations.

4. (McLarnan-Warrington, 2003) Let $\mu(x, w) = \text{coefficient of } q^{\frac{\ell(w) - \ell(x) - 1}{2}}$. Then for $S_9$, $\mu(x, w) \in \{0, 1\}$. For $S_{10}$, $\mu(x, w) = 5$ is possible.

5. (Du Cloux 2003, Brenti 2004, Brenti-Caselli-Marietti 2006) For all Coxeter groups, there exists a formula for $P_{x,w}(q)$ which only depends on the abstract interval $[id, w]$ in Bruhat order.
10 Pattern Properties

Property 1 (Continued). Some localized smoothness tests are similar. The following are equivalent

1. $X_w$ is smooth at $v \in S_n$.

2. $\ell(w) = \{v t_{ij} \leq w\}$. (L-s)

3. The Kazhdan-Lusztig polynomial $P_{x,w}(t) = 1$ for all $v \leq x \leq w$. (K-L)

4. The Kazhdan-Lusztig polynomial $P_{v,w}(t) = 1$ (C-P,I,B-M).

Question. How do 3412 and 4231 patterns help to identify all singular points?
10 Pattern Properties

Property 2.
(Billey-Warrington, Manivel,Kassel-Lascoux-Reutenauer,Cortez ) (ca 2000)

\( X_v \) is an irreducible component of the singular locus of \( X_w \) \( \iff \)

\[ \nu = w \cdot \text{(1-cycle permutation)} \]

corresponding to a 4231 or 3412 or 45312 pattern of the following form

Here o’s denote 1’s in \( w \), ●’s denote 1’s in \( v \).
10 Pattern Properties

**Thm.** (Zariski) $X$ is a smooth variety iff the local ring at every point is regular.

**Def.** $X$ is *factorial* at a point $\iff$ the local ring at that point is a unique factorization domain.

**Property 3.** (Bousquet-Méloù+Butler, 2007, conj. by Woo-Yong) $X_w$ is factorial at every point $\iff w$ avoids $4231$ and $3412$. Here $3412$ means the 4 and 1 must be adjacent.

**Thm.** (Bousquet-Méloù+Butler) There is an explicit formula for counting the number $f_n$ of factorial Schubert varieties for $w \in S_n$:

$$F(t) = \frac{(1 - t)(1 - 4t - 2t^2) - (1 - 5t)\sqrt{1-4t}}{2(1 - 5t + 2t^2 - t^3)}$$

$$= x + 2x^2 + 6x^3 + 22x^4 + 89x^5 + 379x^6 + 1661x^7 + 7405x^8 + \ldots$$
**10 Pattern Properties**

**Property 4.** There exists a simple criterion for characterizing Gorenstein Schubert varieties using modified pattern avoidance.

**Def.** $X$ is *Gorenstein* if it is Cohen-Macaulay and its canonical sheaf is a line bundle.

**Theorem:** Woo-Yong (2004) $X_w$ is Gorenstein $\iff$

- $w$ avoids $31542$ and $24153$ with Bruhat restrictions $\{t_{15}, t_{23}\}$ and $\{t_{15}, t_{34}\}$

- for each descent $d$ in $w$, the associated partition $\lambda_d(w)$ has all of its inner corners on the same antidiagonal.
Gorenstein Schubert varieties

Sketch of proof.

• Step 1: Schubert varieties are all Cohen-Macaulay. (Ramanathan, 1985)

• Step 2: (Brion, Knutson, Kumar) Testing if $X_w$ is Gor. reduces to a comparison using the Weil divisor class group and the Cartier class group.

• Step 3: The Weil divisor class group is generated by the $[X_v] \in H^*(G/B)$ such that $w$ covers $v$ in Bruhat order ($v < \cdot w$).

• Step 4: The Cartier class group is generated by $[X_{w_0 s_i}][X_w]$ and

$$[X_{w_0 s_i}][X_w] = \sum [X_{w t_{ab}}]$$

summed over all $w t_{ab} : a \leq i < b, \ell(v) = \ell(w) - 1$.

• The Schubert variety $X_w$ is Gorenstein if and only if there exists an integral solution $(\alpha_1, \ldots, \alpha_{n-1})$ to

$$\sum_{i=1}^{n-1} \alpha_i \left( \sum_{v = w t_{ab} : a \leq i < b, \ell(v) = \ell(w) - 1} [X_{w t_{ab}}] \right) = \sum_{v < \cdot w} [X_v]$$
10 Pattern Properties

Property 5. (Gasharov-Reiner, 2002)

Def. $X(w)$ is *defined by inclusions* if it can be described as the set of all flags $F_i$ where $F_i \subset E_j$ or $E_i \subset F_j$ form some collection of pairs $i, j$.

Theorem. (Gasharov-Reiner, 2002) $X(w)$ is *defined by inclusions* iff $w$ avoids 4231, 35142, 42513, 351624.

Theorem. (Hultman-Linusson-Shareshian-Sjöstrand, ca 2007) The number of regions in the inversion arrangement for $w$ is at most the number of elements below $w$ in Bruhat order iff $w$ avoids 4231, 35142, 42513, 351624.
Cohomology of Schubert varieties

Gasharov-Reiner show that Schubert varieties defined by inclusions have a nice presentation for their cohomology ring.

**Thm.** (Carrell, 1992) \( H^*(X_w) \cong H^*(G/B)/I_w \) where \( I_w \) is generated by all \( S_v = [X_{w_0}v] \) such that \( v \not\leq w \).

**Thm.** (Akyldiz-Lacoux-Pragacz, 1992) \( I_w \) is generated by \( S_v \) such that \( v \not\leq w \) and \( v \) is Grassmannian (at most 1 descent).

**Def.** \( x \) is *bigrassmannian* if both \( x \) and \( x^{-1} \) are Grassmannian.

**Thm.** (Reiner-Woo-Yong, 2010) \( I_w \) is generated by \( S_v \) such that \( v \not\leq w \), \( v \) is Grassmannian and there exists some bigrassmannian \( x \in E(w) \) such \( x \leq v \) and \( \text{Des}(x) = \text{Des}(v) \).
Essential sets

**Def.** Let $E(w)$ be the set of permutations which are minimal elements in Bruhat order in the complement of the interval $[id, w]$.

**Thm.** (Lascoux- Schützenberger 1992, Geck-Kim 1997) The elements in $E(w)$ are bigrassmannian.

**Thm.** (Reiner-Woo-Yong, 2010) There exists a bijection between $E(w)$ and Fulton’s essential set.

**Open.** What is the relationship between $E(w)$ and defining equations for Schubert varieties in other types?

**Open.** Find a minimal set of generators for $I_w$ for all $w \in S_n$. (See Reiner-Woo-Yong conjecture).
10 Pattern Properties

Property 6. (Deodhar, Billey-Warrington, 1998)
The following are equivalent

1. \( C'_w = C'_{s_{i_1}} C'_{s_{i_2}} \cdots C'_{s_{i_p}} \) for some/any \( s_{i_1} s_{i_2} \cdots s_{i_p} = w \) (reduced).

2. The Bott-Samelson resolution of \( X_w \) is small.

3. \( \sum_{v \leq w} t^{l(v)} P_{v,w}(t) = (1 + t)^{l(w)}. \)

4. For each \( v \leq w \), the Kazhdan-Lusztig polynomial
   \[
   P_{v,w}(t) = \sum_{\sigma \in E(v,w)} t^{\text{defect}(\sigma)}.
   \]

5. \( w \) is 321-hexagon avoiding, i.e. avoids
   \[
   321, 56781234, 56718234, 46781235, 46718235
   \]
10 Pattern Properties

Property 7. Boolean permutations

Theorem. (Tenner, 2006) The principle order ideal below $w$ in Bruhat order is a Boolean lattice $\iff w$ is $321$ and $3412$ avoiding.

Equivalently, the Bott-Samelson resolution of $X(w)$ isomorphic to $X(w)$.

Note: Boolean permutations are $321$-hexagon avoiding.

Theorem. (Fan ’98, West ’96). The number of Boolean permutations in $S_n$ is the Fibonacci number $F_{2n-1}$, e.g. $F_1 = 1, F_3 = 2, F_5 = 5$. 
10 Pattern Properties

Property 8. (Woo, 2009)
The Kazhdan-Lusztig polynomial $P_{id,w}(1) = 2 \iff w$ avoids 653421, 632541, 463152, 526413, 546213, and 465132 and the singular locus of $X_w$ has exactly 1 component.

Def. $KL_m = \{ w \in S_\infty \mid P_{id,w}(1) \leq m \}$.

Example. $KL_1$ are the permutations indexing smooth Schubert varieties.

Extension (Billey-Weed): $KL_2$ is characterized by 66 permutation patterns on 5,6,7 or 8 entries.

Open. $KL_m$ is closed under taking patterns. Can it always be described by a finite set of patterns?
KL$_2$ Patterns

Extension (Billey-Weed): KL$_2$ is characterized by 66 permutation patterns on 5, 6, 7 or 8 entries.

';; 44 patterns in S$_{5,6,7}$ ;;; 22 more in S$_8$

```
'(4 5 1 2 3) (3 4 5 1 2) (5 3 4 1 2) (5 2 3 4 1) (4 5 2 3 1)
(3 5 1 6 2 4) (5 2 3 6 1 4) (5 2 6 3 1 4) (6 2 4 1 5 3) (5 2 4 6 1 3)
(4 6 2 5 1 3) (5 2 6 4 1 3) (5 4 6 2 1 3) (3 6 1 4 5 2) (4 6 1 3 5 2)
(3 6 4 1 5 2) (4 6 3 1 5 2) (5 3 6 1 4 2) (4 6 5 1 3 2) (4 2 6 3 5 1)
(6 3 2 5 4 1) (6 3 5 2 4 1) (6 4 2 5 3 1) (6 5 3 4 2 1)
(3 6 1 2 7 4 5) (6 2 3 1 7 4 5) (6 2 4 1 7 3 5) (3 4 1 6 7 2 5)
(4 2 3 6 7 1 5) (4 2 6 3 7 1 5) (4 2 6 7 3 1 5) (3 7 1 2 5 6 4)
(7 2 3 1 5 6 4) (3 7 1 5 2 6 4) (3 7 5 1 2 6 4) (7 5 2 3 1 6 4)
(6 2 5 1 7 3 4) (7 2 6 1 4 5 3) (3 4 1 7 5 6 2) (3 5 1 7 4 6 2)
(4 5 1 7 3 6 2) (4 2 3 7 5 6 1) (5 3 4 7 2 6 1) (4 2 7 5 6 3 1)
(3 4 1 2 7 8 5 6) (4 2 3 1 7 8 5 6) (3 4 1 7 2 8 5 6)
(4 2 3 7 1 8 5 6) (4 2 7 3 1 8 5 6) (3 5 1 2 7 8 4 6)
(5 2 3 1 7 8 4 6) (5 2 4 1 7 8 3 6) (3 4 1 2 8 6 7 5)
(4 2 3 1 8 6 7 5) (3 4 1 8 2 6 7 5) (4 2 3 8 1 6 7 5)
(4 2 8 3 1 6 7 5) (3 4 1 8 6 2 7 5) (4 2 3 8 6 1 7 5)
(4 2 8 6 3 1 7 5) (3 5 1 2 8 6 7 4) (5 2 3 1 8 6 7 4)
(3 6 1 2 8 5 7 4) (6 2 3 1 8 5 7 4) (5 2 4 1 8 6 7 3)
```


10 Pattern Properties

Property 9. (LCI permutations)

Def. A local ring $R$ is a *local complete intersection (lci)* if it is the quotient of some regular local ring by an ideal generated by a regular sequence. A variety is lci if every local ring is lci.

Thm. (Úlfarsson-Woo, 2011) The Schubert variety $X(w)$ is lci if and only if $w$ avoids 53241, 52341, 52431, 35142, 42513, and 426153.
10 Pattern Properties

Property 10. A permutation is *vexillary* if it avoids 2143.

Combinatorial Properties.
1. (Edelman-Greene, Macdonald) The number of reduced words for a vexillary permutation $v$ is equal to the number of standard tableaux of shape determined by sorting the lengths of the rows of the diagram of $v$.

2. The Stanley symmetric function $F_v$ is a Schur function iff $v$ is vexillary.

$$F_v = \sum_{a=a_1a_2\ldots a_k \in R(v)} \sum_{i_1 \leq \cdots \leq i_k \in C(a)} x_{i_1}x_{i_2}\cdots x_{i_k}$$

where $R(v)$ are the reduced words for $v$ and $C(a)$ are the weakly increasing sequences of positive integers such that $i_j < i_{j+1}$ if $a_j < a_{j+1}$.

3. (Tenner, 2006) The permutation $v$ is vexillary iff for every permutation $w$ containing $v$, there exists a reduced decomposition $a \in R(w)$ containing a shift of some $b \in R(v)$ as a factor.
10 Pattern Properties

**Property 10.** A permutation is *vexillary* if it avoids 2143. (Lascoux-Schützenberger, 1984)

**Geometric Properties.**

1. (Fulton, 1992) The *essential set* for $w$, the cells in the diagram of $w$ with no neighbor directly east or north, lie on a decreasing piecewise linear curve.

2. (Lascoux, 1995) There exists a combinatorial approach to computing the Kazhdan-Lusztig polynomials $P_{v,w}$ when $w$ is *covexillary*, i.e. 3412-avoiding.

3. (Li-Yong, ca 2011) There exists a combinatorial rule for computing multiplicities for $X(w)$ when $w$ is 3412-avoiding.
10 Pattern Properties

Property 10. (continued) $k$-vexillary permutations.

Def. A permutation $w$ is $k$-vexillary if its Stanley symmetric function $F_w$ has at most $k$ terms in its expansion.

Example: $F_{2143} = s(2) + s(1,1)$, so $2143$ is 2-vexillary.

Thm. (Billey-Pawlowski) Let $w$ be a permutation.

1. $w$ is 2-vexillary iff $w$ avoids 35 patterns in $S_5, S_6, S_7, S_8$.

2. $w$ is 3-vexillary iff $w$ avoids 91 patterns in $S_5, S_6, S_7, S_8$.

Conjecture. The $k$-vexillary permutations form an order ideal in the poset on all permutations ordered by pattern containment.
10 Pattern Properties

List of 2-vexillar patterns:

`\texttt{(setf *2-vex* \\
'((3 2 1 5 4) (2 1 5 4 3)  \\
(2 1 4 3 6 5) (2 4 1 3 6 5) (3 1 4 2 6 5) (3 1 2 6 4 5)  \\
(2 1 4 6 3 5) (2 4 1 6 3 5) (2 3 1 5 6 4) (2 1 5 3 6 4)  \\
(3 1 5 2 6 4) (4 2 6 1 5 3) (5 2 7 1 4 3 6) (5 1 7 3 2 6 4)  \\
(4 2 6 5 1 7 3) (2 5 4 7 1 6 3) (5 4 7 2 1 6 3) (5 2 7 6 1 4 3)  \\
(6 1 8 3 2 5 4 7) (2 6 4 8 1 5 3 7) (6 4 8 2 1 5 3 7) (2 6 5 8 1 4 3 7)  \\
(6 5 8 2 1 4 3 7) (5 1 7 3 6 2 8 4) (5 1 7 6 3 2 8 4) (6 1 8 3 7 2 5 4)  \\
(6 1 8 7 3 2 5 4) (2 5 4 7 6 1 8 3) (5 4 7 2 6 1 8 3) (5 4 7 6 2 1 8 3)  \\
(2 6 4 8 7 1 5 3) (6 4 8 7 2 1 5 3) (2 6 5 8 7 1 4 3) (6 5 8 2 7 1 4 3)  \\
(6 5 8 7 2 1 4 3))))\texttt{)}`
List of 3-vexillary patterns:

```lisp
(setf *3-vex*
  '((2 1 4 3 6 5) (4 3 2 1 6 5) (3 2 1 6 4 5) (4 2 1 6 3 5) (3 2 1 5 6 4)
    (3 2 5 1 6 4) (3 1 2 6 5 4) (2 3 1 6 5 4) (3 2 1 6 5 4) (3 1 6 2 5 4)
    (3 2 6 1 5 4) (2 4 1 6 5 3) (4 2 1 6 5 3) (4 2 6 1 5 3) (2 1 6 5 4 3)
    (3 5 2 1 4 7 6) (4 2 5 1 3 7 6) (2 5 4 1 3 7 6) (5 2 4 1 3 7 6)
    (4 3 1 5 2 7 6) (3 5 1 4 2 7 6) (5 3 1 4 2 7 6) (4 1 5 3 2 7 6)
    (3 5 2 4 1 7 6) (4 2 5 3 1 7 6) (2 4 1 3 7 5 6) (3 1 4 2 7 5 6)
    (3 1 2 5 7 4 6) (2 3 1 5 7 4 6) (2 5 1 3 7 4 6) (3 1 5 2 7 4 6)
    (3 5 1 2 7 4 6) (2 4 1 5 7 3 6) (2 5 1 4 7 3 6) (2 5 4 1 7 3 6)
    (2 1 5 7 4 3 6) (2 5 1 7 4 3 6) (5 2 7 1 4 3 6) (2 4 1 3 6 7 5)
    (3 1 4 2 6 7 5) (3 1 2 6 4 7 5) (2 3 1 6 4 7 5) (3 1 6 2 4 7 5)
    (2 4 1 6 3 7 5) (3 1 4 6 2 7 5) (3 4 1 6 2 7 5) (3 1 6 4 2 7 5)
    (4 1 6 3 2 7 5) (3 1 6 2 7 4 5) (2 4 1 6 7 3 5) (2 1 6 4 7 3 5)
    (2 1 4 7 6 3 5) (2 1 7 4 6 3 5) (2 1 6 5 3 7 4) (3 1 6 5 2 7 4)
    (2 1 5 7 3 6 4) (2 1 7 5 3 6 4) (5 1 7 3 2 6 4) (2 1 6 3 7 5 4)
    (4 2 6 5 1 7 3) (2 1 5 7 4 6 3) (2 5 4 7 1 6 3) (5 4 7 2 1 6 3)
    (2 1 6 4 7 5 3) (5 2 7 6 1 4 3)

(2 4 6 1 3 5 8 7) (4 1 5 2 6 3 8 7) (4 1 2 3 8 5 6 7) (2 1 4 6 8 3 5 7)
(2 4 6 1 8 3 5 7) (6 1 8 3 2 5 4 7) (2 6 4 8 1 5 3 7) (6 4 8 2 1 5 3 7)
(2 6 5 8 1 4 3 7) (6 5 8 2 1 4 3 7) (3 4 1 2 7 8 5 6)
(2 3 4 1 6 7 8 5) (2 1 6 3 7 4 8 5) (4 1 6 2 7 3 8 5)
(5 1 7 3 6 2 8 4) (5 1 7 6 3 2 8 4) (6 1 8 3 7 2 5 4)
(6 1 8 7 3 2 5 4) (2 5 4 7 6 1 8 3) (5 4 7 2 6 1 8 3)
(5 4 7 6 2 1 8 3) (2 6 4 8 7 1 5 3) (6 4 8 7 2 1 5 3)
(2 6 5 8 7 1 4 3) (6 5 8 2 7 1 4 3) (6 5 8 7 2 1 4 3))
)
Future Work

Open Problems.

1. Characterize the Gorenstein, lci and factorial locus of $X(w)$ using patterns. (Woo-Yong)

2. (From Ùlfarsson) Is there a nice generating function to count the number of Gorenstein/lci permutations or Schubs defined by inclusions, etc.

3. Find a geometric explanation why a finite number of patterns suffice in all cases above.

4. What nice properties does the inversion arrangement have for other pattern avoiding families?

5. $KL_m$ is closed under taking patterns. Can it always be described by a finite set of patterns?

6. Conjecture (Woo): The Schubert varieties with multiplicity $\leq 2$ can be characterized by pattern avoidance.

7. What other filtrations on the set of all permutations can be characterized by (generalized) patterns?