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Multiplying Schur Functions

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A new combinatorial rule for expanding the product of Schur functions as a sum of Schur functions is formulated. The rule has several advantages over the Littlewood–Richardson rule (D. E. Littlewood and A. R. Richardson, *Philos. Trans. Roy. Soc. London Ser. A* **233** (1934), 49–141). First this rule allows for direct computation of the expansion of the product of *any* number of Schur functions, not just the product of two Schur functions. Also, the rule is easily stated and is well suited to computer implementation. It is shown that the rule implies the Littlewood–Richardson rule and gives a combinatorial proof that the coefficient of S_λ in the product $S_\mu S_\nu$ equals the coefficient of S_ν in the expansion of the skew Schur function $S_{\lambda/\mu}$. The rule is derived from some results proved independently by A. P. Hillman and R. M. Grassl (*J. Combin. Comput. Sci. Systems* **5** (1980), 305–316) and by D. White (*J. Combin. Theory Ser. A* **30** (1981), 237–247) on the Robinson–Schensted–Knuth correspondence. © 1984 Academic Press, Inc.

The Littlewood–Richardson rule [4] gives a method for computing the coefficients in the expansion of the product of two Schur functions as a sum of Schur functions. In this paper, we formulate a new rule for expanding products of Schur functions as a sum of Schur functions which has several advantages over the Littlewood–Richardson rule. First our rule allows for direct computation of the expansion of the product of *any* number of Schur functions, not just the product of two Schur functions. Also, we feel our rule is simpler to apply in that it avoids the notions of lattice permutations or Yamanouchi words and our rule seems better suited to computer implementation. Indeed, computer programs have been developed using the rule and will appear in [1]. Our rule is an easy consequence of a lemma concerning the Robinson–Schensted–Knuth correspondence proved independently by Hillman and Grassl in [2] and White in [11]. However, it seems that both

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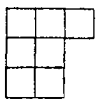
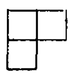
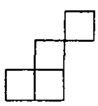
authors did not recognize that one can derive such a simple and powerful rule for the multiplication of Schur functions from this lemma.

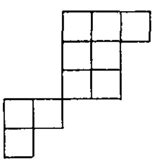
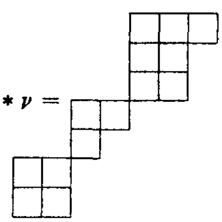
In Section 1, we shall describe our rule and give several illustrations of its application. In Section 2, we shall show how the rule follows from the Hillman–Grassl–White lemma and show how the Littlewood–Richardson rule follows easily from our rule. Also, we shall show how our rule is easily adaptable to calculate the expansion of skew Schur functions as a sum of Schur functions and give a combinatorial proof by combining our results with those of White [11] that the coefficient of S_λ in the expansion of the product $S_\mu S_\nu$, $g^{\lambda}_{\mu,\nu}$, is the same as the coefficient of S_ν in the expansion of the skew Schur function $S_{\lambda/\mu}$, $h^{\nu}_{\lambda/\mu}$.

1. The Schur Function Multiplication Rule

For λ, a partition of a natural number n into positive integers, we write $\lambda = (\lambda_1, \lambda_2, \cdots)$ where $\lambda_1 \geqslant \lambda_2 \geqslant \ldots$, and let $|\lambda| = n$. Given another partition $\mu = (\mu_1, \mu_2, \ldots)$, we write $\mu \leqslant \lambda$ if $\mu_i \leqslant \lambda_i$ for all i and let $|\lambda/\mu| = |\lambda| - |\mu|$. The partition λ will often be identified with the *Ferrers diagram* of shape λ which consists of left-justified rows of squares or cells with λ_1 cells in the first row, λ_2 cells in the second row, etc. If $\mu \leqslant \lambda$, then λ/μ denotes the cells of λ that remain after the cells of μ have been removed and we shall refer to the resulting diagram as the skew diagram of shape λ/μ. Thus we can regard the diagram λ as the skew diagram λ/ϕ where ϕ is the empty partition. For example, if $\lambda = (3,2,2)$ and $\mu = (2,1)$, then

$\lambda =$  , $\mu =$  , and $\lambda/\mu =$  .

Given any partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_l)$, we let $\lambda * \mu$ denote the skew diagram $(\lambda_1 + \mu_1, \lambda_2 + \mu_1, \ldots, \lambda_k + \mu_1, \mu_1, \ldots, \mu_l)/(\mu_1)^k$ where $(\mu_1)^k$ denotes the partition which consists of k parts of size μ_1. For partitions $\lambda^1, \ldots, \lambda^n$ with $n > 2$, we define $\lambda^1 * \cdots * \lambda^n$ by induction as $\lambda^1 * \cdots * \lambda^n = (\lambda^1 * \cdots * \lambda^{n-1}) * \lambda^n$. For example, if $\lambda = (3,2,2)$, $\mu = (2,1)$, and $\nu = (2,2)$, then

$\lambda * \mu =$  and $\lambda * \mu * \nu =$ 

A *skew tableau* T of shape λ/μ is a filling of the cells of the skew diagram of shape λ/μ with positive integers in such a way that the numbers weakly increase from left to right in each row and strictly increase from top to bottom in each column. A skew tableau T of shape λ/μ is called *standard* if T is filled with the numbers $1,\ldots,n = |\lambda/\mu|$. Skew tableaux of shape $\lambda/\phi = \lambda$ will be referred to as simply tableaux or standard tableaux of shape λ. For example the standard tableaux of shape $(2,1)$ are

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Given a skew tableau T of shape λ/μ, we define the *weight* of T, $W(T)$, to be the monomial $x_{i_1}x_{i_2}\cdots x_{i_n}$ where T is filled with the integers $i_1,\ldots,i_n$. Thus, for example,

$$W\left(\begin{array}{cccc} \cline{2-4} \multicolumn{1}{c|}{} & 2 & 2 & \multicolumn{1}{c|}{3} \\ \hline \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} & & \\ \cline{1-2} \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{5} & & \\ \cline{1-2} \end{array}\right) = x_2x_2x_3x_3x_4x_4x_5 = x_2^2x_3^2x_4^2x_5^1.$$

We shall refer to λ/μ as the shape of the tableau T and write $\rho(T) = \lambda/\mu$.

Given a skew tableau λ/μ, we index the rows from top to bottom and the columns from left to right. We let (i, j) denote the cell in the ith row of the jth column. Thus, for example, in the tableau pictured above the $(3,2)$-cell is filled with 5. We define the *reverse lexicographic filling* of λ/μ to be the filling of the skew diagram λ/μ with the numbers $1,\ldots,n = |\lambda/\mu|$ where $1,\ldots,\lambda_1 - \mu_1$ are placed in the first row in order from *right to left*, $\lambda_1 - \mu_1 + 1,\ldots,(\lambda_1 - \mu_1) + (\lambda_2 - \mu_2)$ are placed in the second row from right to left, etc. See Fig. 1 for the reverse lexicographic filling of $(5,3,2,2)/(2,1,1)$.

Next we shall define for any given skew diagram λ/μ, a set of tableaux $\mathcal{O}(\lambda/\mu)$ which is of fundamental importance for our Schur function multiplication rule. Given a cell (i, j) in a tableau T, we shall say a cell (i', j') is T is *weakly above and strictly to the right* of (i, j) if $i' \leqslant i$ and $j' > j$ and

Let $\lambda = (5,3,2,2)$ and $\mu = (2,1,1)$.

$$\begin{array}{ccccc} \cline{3-5} & \multicolumn{1}{c|}{} & 3 & 2 & \multicolumn{1}{c|}{1} \\ \cline{2-5} \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{4} & & \\ \cline{2-3} \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{6} & & & \\ \hline \multicolumn{1}{|c|}{8} & \multicolumn{1}{c|}{7} & & & \\ \cline{1-2} \end{array}$$

FIG. 1. The reverse lexicographic filling of λ/μ.

we shall say (i', j') is *weakly to the left and strictly below* (i, j) if $i' > i$ and $j' \leqslant j$. This given, we then define $\mathcal{O}(\lambda/\mu)$ to be the set of standard tableaux T of size $|\lambda/\mu|$ satisfying the following two conditions:

(i) If x and $x+1$ occur in the same row in the reverse lexicographic filling of λ/μ, then in T, $x+1$ must occur in a cell which is weakly above and strictly to the right of the cell in which x occurs in T.

(ii) If y occurs in the same column and in a row immediately below where x occurs in the reverse lexicographic filling of λ/μ, then in T, y must occur in a cell which is strictly below and weakly to the left of x.

Figure 2 gives a pictorial representation of these two conditions.

The Schur function S_λ for a partition λ is defined to be

$$S_\lambda(x_1, x_2, \ldots) = \sum_{T \in \mathfrak{I}_\lambda} W(T) \tag{1.1}$$

where $\mathfrak{I}_\lambda$ is the set of all tableaux of shape λ. It is well known that for any partitions, $\lambda^1, \lambda^2, \ldots, \lambda^k$,

$$\prod_{i=1}^{k} S_{\lambda^i} = \sum_{\substack{\lambda \\ |\lambda| = \sum_{i=1}^{n} |\lambda^i|}} g^{\lambda}_{\lambda^1, \ldots, \lambda^k} S_\lambda \tag{1.2}$$

where the sum on the right-hand side of (1.2) runs over all partitions of $|\lambda^1| + \cdots + |\lambda^k|$ and $g^{\lambda}_{\lambda^1, \ldots, \lambda^k}$ is a nonnegative integer. The main purpose of this section is to give a combinatorial rule to find the right-hand side of (1.2) given $\lambda^1, \ldots, \lambda^k$. We are now in a position to state our rule.

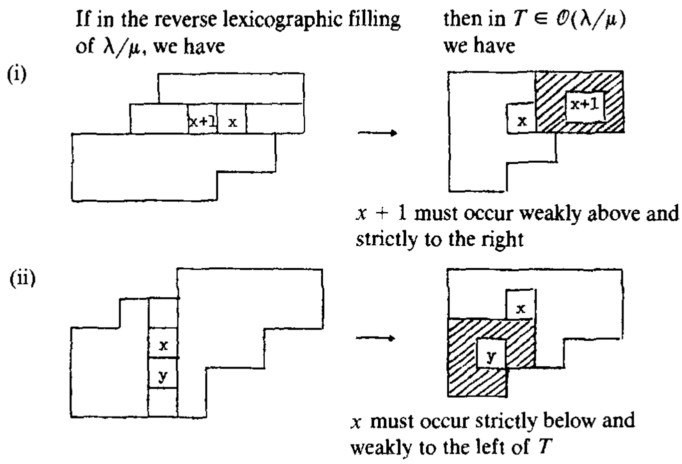


FIG. 2.

A Schur Function Multiplication Rule

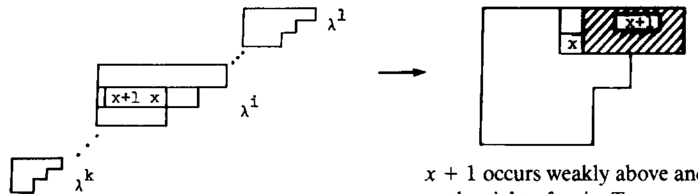
Given partitions $\lambda^1, \ldots, \lambda^k$, calculate

$$\Pi S_{\lambda^i} = \sum_{\substack{\lambda \\ |\lambda| = \sum_{i=1}^{k} |\lambda^i|}} g^{\lambda}_{\lambda^1, \ldots, \lambda^k} S_\lambda.$$

(1) Start with the reverse lexicographic filling L of the skew-diagram $\lambda^1 * \cdots * \lambda^k$.

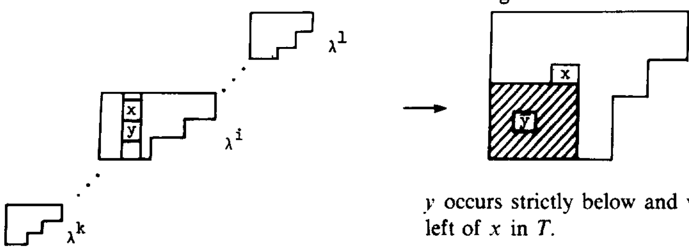
(2) Next generate $\mathcal{O}(\lambda^1 * \cdots * \lambda^k)$. That is, generate the set of standard tableaux T of size $|\lambda^1| + \cdots + |\lambda^k|$ satisfying the two conditions pictured below.

If in the reverse lexicographic filling L of $\lambda^1 * \cdots * \lambda^k$ we have, then in T we have



$x + 1$ occurs weakly above and strictly to the right of x in T

or (ii)



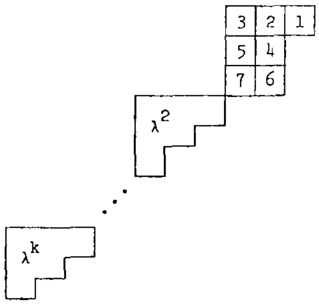
y occurs strictly below and weakly to left of x in T.

(3) Replace each tableau T in $\mathcal{O}(\lambda^1 * \cdots * \lambda^k)$ by the Schur function of the same shape and sum, i.e.,

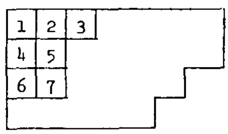
$$\prod_{i=1}^{k} S_{\lambda^i} = \sum_{T \in \mathcal{O}(\lambda^1 * \cdots * \lambda^k)} S_{\rho(T)}.$$

Before we give some examples of our rule, we shall make several remarks.

Remark 1. We note that the positions of the number $1, \ldots, |\lambda^1|$ will always be the same in any tableau $T \in \mathcal{O}(\lambda^1 * \cdots * \lambda^k)$. That is, if $\lambda^1 = (\lambda^1_1, \lambda^1_2, \ldots, \lambda^1_l)$, then in any $T \in \mathcal{O}(\lambda^1 * \cdots * \lambda^k)$ the numbers $1, \ldots, \lambda^1_1$ appear in order starting from the left in the first row of T, the numbers $1 + \lambda^1_1, \ldots, \lambda^1_1 + \lambda^1_2$ appear in order starting from the left in the second row of $T, \ldots,$ and the numbers $1 + \sum_{i=1}^{l-1} \lambda^1_i, \ldots, \sum_{i=1}^{l} \lambda^1_i$ appear in order starting from the left in the lth row of T. For example, if $\lambda^1 = (3, 2, 2)$ so that the reverse lexicographic filling of $\lambda^1 * \cdots * \lambda^k$ starts with



then it is easy to see that rules (i) and (ii) force that for every $T \in \mathcal{O}(\lambda^1 * \cdots * \lambda^k)$, the numbers $1, \dots, 7$ appear as pictured below.

$T =$


The well-known fact that $g^{\lambda}_{\lambda^1 * \cdots * \lambda^k} = 0$ unless $\lambda^i \leqslant \lambda$ for all $i = 1, \dots, k$ easily follows from the above observation. It also follows that in order to compute $\prod_{i=1}^{k} S_{\lambda^i}$ using our rule, it will be advantageous to pick λ^1 so that $|\lambda^1| \geqslant |\lambda^i|$ for all $i = 1, \dots, k$ since this will ensure that the largest number of possible elements will be forced to be in the same place in every tableau $T \in \mathcal{O}(\lambda^1 * \cdots * \lambda^k)$.

Remark 2. Note that our rule gives a combinatorial interpretation to the coefficients $g^{\lambda}_{\lambda^1 * \cdots * \lambda^k}$. That is, $g^{\lambda}_{\lambda^1 * \cdots * \lambda^k}$ equals the number of standard tableaux T in $\mathcal{O}(\lambda^1 * \cdots * \lambda^k)$ of shape λ.

Remark 3. Our rule is easily adapted for the calculation of a skew symmetric function as a sum of Schur functions,

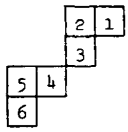
$$S_{\lambda/\mu} = \sum_{\substack{\nu \\ |\nu| = |\lambda/\mu|}} h^{\nu}_{\lambda/\mu} S_{\nu}.$$

That is, the rule is exactly the same except that we replace $\lambda^1 * \cdots * \lambda^k$ by λ/μ in steps 1 and 2. Although, it is not immediately obvious from this rule that $g^{\lambda}_{\mu,\nu} = h^{\nu}_{\lambda/\mu}$, we shall give a combinatorial proof of this fact in Section 2.

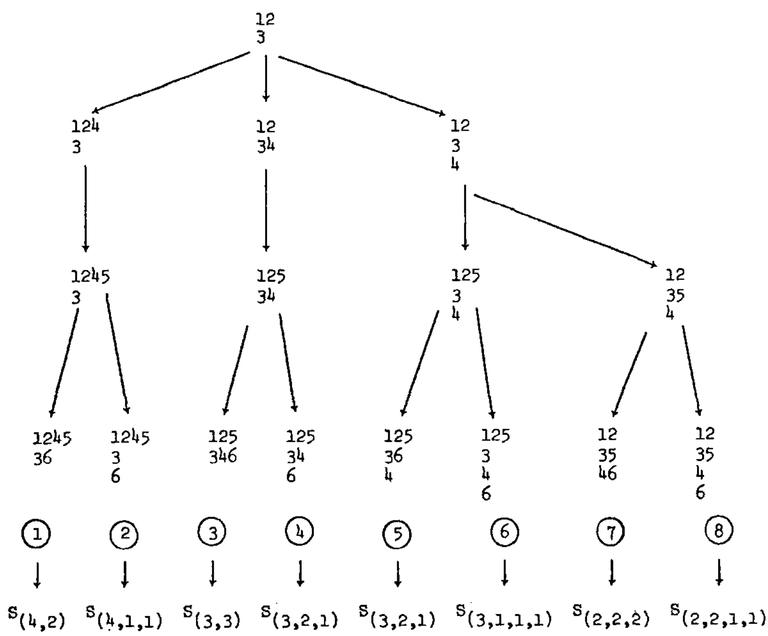
We now give some examples to illustrate our rule. In the first two examples, we calculate the expansion for the products of various Schur functions and in the third example, we calculate the expansion of a skew Schur function.

EXAMPLE 1. $S_{(2,1)} \cdot S_{(2,1)}$. We illustrate the three steps of our algorithm.

(1) *The reverse lexicographic filling of* $(2,1)*(2,1)$:



(2) *Construct* $\mathcal{O}((2,1)*(2,1))$. Note that by Remark 1 we know the positions of 1, 2, and 3 for all $T \in \mathcal{O}((2,1)*(2,1))$ so we can simply use the tree pictured below to generate the eight elements of $\mathcal{O}((2,1)*(2,1))$. The tree is constructed so that the successors of any node or tableau T in the tree represent all possible ways of adding the next number to T allowed by conditions (i) and (ii). For example, conditions (i) and (ii) impose no restrictions on the placement of 4 so it can be in any one of three positions consistent with the requirements to be a tableau. Similarly, there is only one successor of the tableau ${}^{124}_{3}$ since, by condition (i), 5 must be weakly above and strictly to the right of 4.



(3) *Replace each tableau* T by $S_{\rho(T)}$.

$$S_{(2,1)}S_{(2,1)} = S_{(4,2)} + S_{(4,1,1)} + S_{(3,3)} + 2S_{(3,2,1)} + S_{(3,1,1,1)} + S_{(2,2,2)} + S_{(2,2,1,1)}.$$

EXAMPLE 2. $S_{(2,1)} \cdot S_{(2,1)} \cdot S_{(2)}$

(1) *The reverse lexicographic filling*:

$$\begin{array}{cccccc} & & & & \boxed{2} & \boxed{1} \\ & & & & \boxed{3} & \\ & & \boxed{5} & \boxed{4} & & \\ & & \boxed{6} & & & \\ \boxed{8} & \boxed{7} & & & & \end{array}$$

(2) *Construct* $\mathcal{O}((2,1)*(2,1)*(2,1))$. It is clear that we need only continue each of the branches constructed in Example 1, i.e., we must add 7 and 8 onto each of the eight tableaux constructed in Example 1 according to rules (i) and (ii). We list below the tableaux that result from each of the eight tableaux of Example 1.

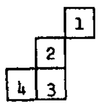
①	124578,	12458,	1245,	12458,	1245,	1245	
	36	367	3678	36	368	36	
				7	7	78	
②	124578,	12458,	1245,	12458,	1245		
	3	37	378	3	38		
	6	6	6	6	6		
				7	7		
③	12578,	1258,	125				
	346	346	346				
		7	78				
④	12578,	1258,	1258,	125,	1258,	125,	125
	34	347	34	348	34	348	34
	6	6	67	67	6	6	68
					7	7	7
⑤	12578,	1258,	1258,	125,	1258,	125,	125
	36	367	36	368	36	368	36
	4	4	47	47	4	4	48
					7	7	7
⑥	12578,	1258,	125,	1258,	125		
	3	37	378	3	38		
	4	4	4	4	4		
	6	6	6	6	6		
				7	7		
⑦	1278,	128,	12				
	35	35	35				
	46	46	46				
		7	78				
⑧	1278,	128,	128,	12			
	35	35	35	35			
	4	47	4	48			
	6	6	6	6			
			7	7			

(3) Now we just replace each tableau above by the Schur function of the same shape,

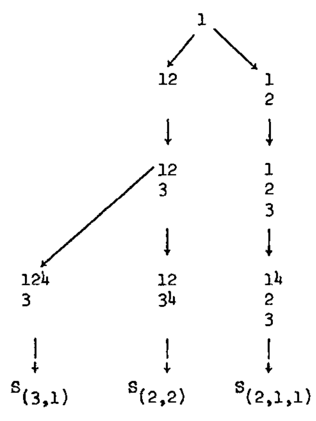
$$\begin{aligned} S_{(2,1)} \cdot S_{(2,1)} \cdot S_{(2)} = {} & S_{(6,2)} + S_{(6,1,1)} + 2S_{(5,3)} + 4S_{(5,2,1)} + 2S_{(5,1,1,1)} \\ & + S_{(4,4)} + 5S_{(4,3,1)} + 4S_{(4,2,2)} + 5S_{(4,2,1,1)} + S_{(4,1,1,1,1)} \\ & + 3S_{(3,3,2)} + 3S_{(3,3,1,1)} + 4S_{(3,2,2,1)} + 2S_{(3,2,1,1,1)} \\ & + S_{(2,2,2,2)} + S_{(2,2,2,1,1)}. \end{aligned}$$

EXAMPLE 3. $S_{(3,2,2)/(2,1)}$. Let $\mu = (2,1)$ and $\lambda = (3,2,2)$.

(1) *The reverse lexicographic filling*:



(2) *Construct* $\mathcal{O}(\lambda/\mu)$.



(3) $S_{(3,2,2)/(2,1)} = S_{(2,1)} + S_{(2,2)} + S_{(2,1,1)}$.

2. PROOFS

In this section, we shall give a brief outline of how the rules of Section 1 are proved, the connections between our Schur function multiplication rule and the Littlewood–Richardson rule, and a combinatorial proof that the coefficients $g^{\lambda}_{\mu,\nu}$ that arise in the expansion of $S_\mu S_\nu$ are the same as the coefficients $h^{\nu}_{\lambda/\mu}$ that arise in the expansion of $S_{\lambda/\mu}$.

We shall assume that the reader is familiar with the Robinson–Schensted–Knuth correspondence between words consisting of positive integers and pairs of tableaux (P, Q). This correspondence and its history may be found in Knuth's book [3]. We let U denote the set of words u of positive integers. If $u = u_1 \cdots u_n \in U$ we write $l(u) = n$ for the *length of* u and $W(u) = x_{u_1} \cdots x_{u_n}$ for the *weight of* u. For example if $u = 1\ 2\ 1\ 2\ 3\ 3\ 1$, then $W(u) = x_1^3 x_2^2 x_3^2$. In the course of constructing the pair of tableaux (P, Q) via the usual Schensted bumping algorithm, one can insert new elements in the first row and bump elements from row to row; this is usually called the *row insertion algorithm*. Or one can insert new elements in the first column and bump elements from column to column; this is called the *column insertion algorithm*. For technical reasons which facilitate the proof of Proposition 2 to follow, we shall assume that we process a word from left to right using the column insertion algorithm to construct a correspondence

$$\theta : U \to \left\{ (P, Q) \,\middle|\, \begin{matrix} P \text{ is a tableau and } Q \text{ is a standard tableau} \\ \text{of the same shape, i.e., } \rho(P) = \rho(Q). \end{matrix} \right\}$$

For example, if $u = 1\ 2\ 1\ 3\ 2$, the series of tableaux constructed by the column insertion algorithm are pictured below.

insert 1 $\quad \begin{array}{|c|}\hline 1 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline \end{array}$

insert 2 $\quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$

insert 1 $\quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$

insert 3 $\quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}$

insert 2 $\quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1} \end{array}$

$\qquad\quad \| \qquad\qquad \|$

$\qquad\quad P \qquad\qquad Q \qquad$ where $\theta(u) = (P, Q)$.

The basic properties of the Robinson–Schensted–Knuth correspondence are the following.

(2.1) If $u \in U$ and $\theta(u) = (P, Q)$,

(i) $\rho(P) = \rho(Q)$ and $|\rho(P)| = l(u)$,
(ii) P is a tableau with same weight as u, i.e., $W(u) = W(P)$, and
(iii) Q is a standard tableau.

Now given any skew tableau T of shape λ/μ, we let the *word of* T, $u(T)$, denote the word that results by reading the cells of T from top to bottom and from right to left, i.e., in the order of the reverse lexicographic filling of T. For example,

$$u\left(\begin{array}{|c|c|c|c|c|} \hline \multicolumn{3}{c|}{} & 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 2 & 2 & 3 & \multicolumn{1}{c}{} \\ \hline 1 & 3 & \multicolumn{3}{|c}{} \\ \cline{1-2} \end{array}\right) = 1\,1\,3\,2\,2\,3\,1.$$

The key lemma in proving the rules of Section 1 is to characterize when a standard tableau Q is such that there is a tableau T of shape λ/μ where $\theta(u(T)) = (P, Q)$. Such characterizations were developed independently by Hillman and Grassl in [2] and by White in [11]. However, the characterizations look different since the Hillman–Grassl version uses the row insertion algorithm while the White version uses the column insertion algorithm. The characterizations are equivalent, however, since for a given word u, the Q tableau that arises from row insertion can be mapped to the Q tableau that arises from the column insertion algorithm via the Schutzenberger evacuation operation (see [10]) and the evacuation operation transforms the Hillman–Grassl characterization to the White characterization and vice versa. We shall, however, state the White version since we are using column insertion. The following proposition, although not stated explicitly in this manner, is proved in [11].

PROPOSITION 1 (White [11]). *Fix a skew shape λ/μ. Then a standard tableau Q with $|\rho(Q)| = |\lambda/\mu|$ is in $\mathcal{O}(\lambda/\mu)$ if and only if there are tableaux T of shape λ/μ and P of shape $\rho(Q)$ such that $\theta(u(T)) = (P, Q)$ if and only if for all tableaux P of shape $\rho(Q)$, there is a tableaux T of shape λ/μ such that $\theta(u(T)) = (P, Q)$.*

Given Proposition 1, the proof of our Schur function multiplication rule is now almost immediate from the properties of the Robinson–

Schensted–Knuth correspondence θ. That is, it is clear that

$$S_{\lambda^1} \cdots S_{\lambda^k} = \Bigg(\sum_{T_1 \in \mathfrak{F}_{\lambda^1}} W(T_1) \Bigg) \cdots \Bigg(\sum_{T_k \in \mathfrak{F}_{\lambda^k}} W(T_k) \Bigg)$$
$$= \sum_{T \in \mathfrak{F}_{\lambda^1 * \cdots * \lambda^k}} W(T) \tag{2.2}$$

since every tableau T of shape $\lambda^1 * \cdots * \lambda^k$ can be decomposed into tableaux $T_i \in \mathfrak{F}_{\lambda^i}$. Now using the weight preserving properties of θ and Proposition 1, we have

$$\sum_{T \in \mathfrak{F}_{\lambda^1 * \cdots * \lambda^k}} W(T) = \sum_{\substack{T \in \mathfrak{F}_{\lambda^1 * \cdots * \lambda^k} \\ \theta(u(T)) = (P, Q)}} W(P)$$
$$= \sum_{Q \in \mathcal{O}(\lambda^1 * \cdots * \lambda^k)} \sum_{P \in \mathfrak{F}_{\rho(Q)}} W(P)$$
$$= \sum_{Q \in \mathcal{O}(\lambda^1 * \cdots * \lambda^k)} S_{\rho(Q)}. \tag{2.3}$$

Of course, exactly the same type of argument establishes the rule for the expansion of $S_{\lambda/\mu}$.

Next we shall sketch the connection of our Schur function multiplication rule with the Littlewood–Richardson rule. To state the Littlewood–Richardson rule we need the notion of a lattice permutation. Given a word $u \in U$, we let $|u|_i$ denote the number of occurrences of i in u. A word $u = u_1 \cdots u_n \in U$ is a *lattice permutation* if for all i and all $1 \leqslant k \leqslant n$, $|u_1 \cdots u_k|_i \geqslant |u_1 \cdots u_k|_{i+1}$, i.e., if in any initial segment of u, there are at least as many occurrences of i as occurrences of $i+1$. We can now state the Littlewood–Richardson rule as follows.

The Littlewood–Richardson Rule

Suppose

$$S_\mu S_\nu = \sum_{\substack{\lambda \\ |\lambda| = |\mu| + |\nu|}} g^\lambda_{\mu,\nu} S_\lambda.$$

$g^\lambda_{\mu,\nu}$ = number of tableau T of shape λ/μ such that the word of T, $u(T)$, is a lattice permutation and $W(T) = x_1^{\nu_1} \ldots x_n^{\nu_n}$ where $\nu = (\nu_1, \ldots, \nu_n)$. (2.4)

To see the equivalence of our multiplication rule and the Littlewood–Richardson rule, we need only show the following.

PROPOSITION 2. *Given partitions* $\mu = (\mu_1, \ldots, \mu_m)$, $\nu = (\nu_1, \ldots, \nu_n)$, *and* $\lambda = (\lambda_1, \ldots, \lambda_p)$ *with* $|\lambda| = |\mu| + |\nu|$,

$$\begin{aligned} g^{\lambda}_{\mu,\nu} &= \#\text{ of } Q \in \mathcal{O}(\mu * \nu) \text{ of shape } \lambda \\ &= \#\text{ of } T \in \mathfrak{T}_{\lambda/\mu} \text{ such that } u(T) \text{ is a lattice permutation} \\ &\qquad \text{and } W(T) = x_1^{\nu_1} \cdots x_n^{\nu_n}. \end{aligned} \tag{2.5}$$

Proof. Let $\mathcal{O}_\lambda(\mu * \nu) = \{Q \in \mathcal{O}(\mu * \nu) | \rho(Q) = \lambda\}$ and $LR_\nu(\lambda/\mu) = \{T \in \mathfrak{T}_{\lambda/\mu} | u(T)$ is a lattice permutation and $W(T) = x_1^{\nu_1} \cdots x_n^{\nu_n}\}$. We shall show there is a simple bijection R: $\mathcal{O}_\lambda(\mu * \nu) \to LR_\nu(\lambda/\mu)$.

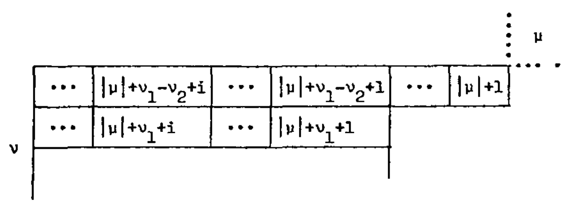
We can assume $\lambda \geqslant \mu$ since otherwise both $\mathcal{O}_\lambda(\mu * \nu)$ and $LR_\nu(\lambda/\mu)$ are empty. By our first remark following the statement of the multiplication rule, we know that every tableau $Q \in \mathcal{O}_\lambda(\mu * \nu)$ has a subtableau T_μ of shape μ where in T_μ, the numbers $1, \ldots, \mu_1$ appear in the first row, the numbers $\mu_1 + 1, \ldots, \mu_1 + \mu_2$ appear in the second row, etc. Thus in Q, the numbers $|\mu| + 1, \ldots, |\mu| + |\nu|$ lie in the skew diagram λ/μ. The map R, called the replacement map, simply takes the elements of Q in the skew diagram λ/μ and replaces those elements that correspond to the numbers of the reverse lexicographic filling of $\mu * \nu$ that appear in the first row of the diagram ν by 1's, those that correspond to numbers of the reverse lexicographic filling that appear in the second row of the diagram ν by 2's, etc. That is, the elements $|\mu| + 1, \ldots, |\mu| + \nu_1$ are replaced by 1's, the elements $|\mu| + \nu_1 + 1, \ldots, |\mu| + \nu_1 + \nu_2$ are replaced by 2's, etc. For example, using Example 1 of Section 1 for the expansion of $S_{(2,1)}S_{(2,1)}$ and $\lambda = (3,2,1)$, we see that there are two elements of $\mathcal{O}_{(3,2,1)}\{(2,1) * (2,1)\}$ and

$$R\begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & \\ 6 & & \end{pmatrix} = \begin{array}{ccc} & & \boxed{1} \\ & \boxed{1} & \\ \boxed{2} & & \end{array}$$

and

$$R\begin{pmatrix} 1 & 2 & 5 \\ 3 & 6 & \\ 4 & & \end{pmatrix} = \begin{array}{ccc} & & \boxed{1} \\ & \boxed{2} & \\ \boxed{1} & & \end{array}.$$

It is immediate that if $Q \in \mathcal{O}_\lambda(\mu * \nu)$ and $T = R(Q)$, then $W(T) = x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n}$. The fact that the word of $T, u(T)$, is a lattice permutation easily follows from conditions (i) and (ii) of the definition of $\mathcal{O}(\mu * \nu)$. For example, consider the 1's and 2's in $u(T) = u_1 \cdots u_l$ where $l = |\lambda/\mu|$. Suppose we have an initial segment $u_1 \cdots u_k$ of $u(T)$ where $u_k = 2$ and u_k is the result of replacing in Q, the number $|\mu| + |\nu_1| + i$ where $1 \leqslant i \leqslant \nu_2$. Thus in the reverse lexicographic filling of $\mu * \nu$, we have the following picture:



Now by condition (i) of the definition of $\mathcal{O}(\mu * \nu)$, we know that the numbers $|\mu| + \nu_1 + i + 1, \ldots, |\mu| + |\nu_1| + |\nu_2|$ must occur weakly above and strictly to right of $|\mu| + \nu_1 + i$ in Q so that all of the images of $|\mu| + \nu_1 + i + 1, \ldots, |\mu| + \nu_1 + \nu_2$ under R lie in the word $u_1 \cdots u_k$. Similarly, condition (i) forces $|\mu| + \nu_1 + 1, \ldots, |\mu| + |\nu_1| + i - 1$ to lie weakly below and strictly to the left of $|\mu| + \nu_1 + i$ in Q so that none of their images under R are in $u_1 \cdots u_k$. Thus there are precisely $\nu_2 - i + 1$ 2's in $\mu_1 \cdots \mu_k$. But by condition (ii) of the definition of $\mathcal{O}(\mu * \nu)$, we know that for each $\nu_2 \geqslant j \geqslant i$, the number $|\mu| + \nu_1 + j$ must be strictly below and weakly to the left of the $|\mu| + \nu_1 - \nu_2 + j$ in Q so that the images under R of $|\mu| + \nu_1 - \nu_2 + i, \ldots, |\mu| + \nu_1 - \nu_2$ also lie in the word $u_1 \cdots u_k$ and hence there are at least $\nu_2 - i + 1$ 1's in $u_1 \cdots u_k$. The other conditions for $u(T)$ being a lattice permutation are checked similarly. Since Q is a tableau and under our replacement map we ensure that whenever $k \leqslant l$, their replacements $R(k)$ and $R(l)$ satisfy $R(k) \leqslant R(l)$, it immediately follows that the rows and columns of T are weakly increasing. The fact that the columns of T are strictly increasing follows from condition (i) of the definition of $\mathcal{O}(\mu * \nu)$; namely, if $k \leqslant l$ and $R(k) = R(l)$ so that k and l occur in the same row of the reverse lexicographic filling of $\mu * \nu$, then condition (i) forces that l occurs weakly above and strictly to the right of k in Q so that the same number cannot occur twice in any column of T.

To define the inverse of R, start with a tableau $T \in LR_\nu(\lambda/\mu)$ and replace the 1's in T in order from bottom to top and left to right by $|\mu| + 1, \ldots, |\mu| + \nu_1$, respectively, the 2's in the same order by $|\mu| + \nu_1 + 1, \ldots, |\mu| + \nu_1 + \nu_2$, etc. This will result in a tableau Q' of shape λ/μ and we then attach the tableau T_μ described above to form a standard tableau $Q = R^{-1}(T)$. For example, if $\lambda = (3,3,2)$, $\mu = (2,1)$, and $\nu = (2,2,1)$, then

$$R^{-1}\left(\begin{array}{cccc} & & & 1 \\ & & 1 & 2 \\ 2 & 3 & & \end{array}\right) = \begin{array}{ccc} 1 & 2 & 5 \\ 3 & 4 & 7 \\ 6 & 8 & \end{array}.$$

It is now not difficult to check that the order in which we replaced the 1's, 2's, etc., and the fact that T is a tableau will ensure that $Q = R^{-1}(T)$ satisfies condition (i) of the definition of $\mathcal{O}(\mu * \nu)$ and the fact that $u(T)$ is a lattice permutation will ensure that Q satisfies condition (ii) of the definition of $\mathcal{O}(\mu * \nu)$. It is clear, however, that $R(R^{-1}(T)) = T$ so that R is indeed a bijection from $\mathcal{O}_\lambda(\mu * \nu)$ and $LR_\nu(\lambda/\mu)$. □

Combining Propositions 1 and 2 gives another combinatorial proof of the Littlewood–Richardson rule. We note, however, that our approach differs from the previous proofs in the literature in that we deal directly with the product $S_\mu S_\nu = \sum_\lambda g^\lambda_{\mu,\nu} S_\lambda$. That is, Robinson [6] gave the first proof of the Littlewood–Richardson rule but, while it can be pushed through, his proof lacked a great number of details and is rather incomplete. In recent years, a number of combinatorial proofs of the Littlewood–Richardson rule have been given; see Schutzenberger [8], Thomas [9], and White [11]. However, all these recent proofs deal directly with the expansion $S_{\lambda/\mu} = \sum_\nu h^\nu_{\lambda/\mu} S_\nu$. That is, all the above proofs show that the set $LR_\nu(\lambda/\mu)$ defined in Proposition 1 counts the multiplicity of S_ν in the expansion of $S_{\lambda/\mu}$ and then use the fact that $h^\nu_{\lambda/\mu} = g^\lambda_{\mu,\nu}$. In particular, White used this approach and Proposition 1 to give his proof of the Littlewood–Richardson rule in [11]. That is, there is a natural map between lattice permutations and standard tableaux which we shall call $\mathcal{L}$. We start with a lattice permutation $u = u_1 \cdots u_n$ of weight $x_1^{\nu_1} \cdots x_n^{\nu_n}$ and we associate a standard tableau T of shape ν to u by letting the position of the i's in u determine the elements in the elements of the ith row in T; i.e., if $u_{j1} = u_{j2} = \cdots = u_{jk}$ represent all occurrences of i in u, then we place $j_1, \ldots, j_k$ in order in the ith row of T. For example,

$$\mathcal{L}(1\,1\,2\,1\,3\,2\,2\,3\,1) = \begin{array}{cccc} 1 & 2 & 4 & 9 \\ 3 & 6 & 7 & \\ 5 & 8 & & \end{array}.$$

Now if $\mathcal{O}_\nu(\lambda/\mu) = \{Q \in \mathcal{O}(\lambda/\mu) | Q \text{ is of shape } \nu\}$, then White showed in [11] that $\mathcal{L}$ is a bijection between $LR_\nu(\lambda/\mu)$ and $\mathcal{O}_\nu(\lambda/\mu)$. Thus White proves the following since $h^\nu_{\lambda/\mu} = |\mathcal{O}_\nu(\lambda/\mu)|$ by Proposition 1.

PROPOSITION 3 (White [11]). $|LR_\nu(\lambda/\mu)| = h^\nu_{\lambda/\mu}$.

Thus combining Propositions 1, 2, and 3, we have a completely combinatorial proof of the following.

PROPOSITION 4. *For any partitions* λ, μ, *and* ν *where* $|\lambda| = |\mu| + |\nu|$, $h^{\nu}_{\lambda/\mu} = g^{\lambda}_{\mu,\nu}$ *where* $S_\mu S_\nu = \Sigma_\lambda g^{\lambda}_{\mu,\nu} S_\nu$ *and* $S_{\lambda/\mu} = \Sigma_\lambda h^{\nu}_{\lambda/\mu} S_\nu$.

CONCLUSION

We end the paper with a brief discussion of the complexity of our Schur function multiplication rule. We shall restrict ourselves to the case of multiplying just two Schur functions although it will be clear that our discussion applies to arbitrary products.

To expand the product $S_\lambda \times S_\mu$ as the sum of Schur functions, we need to construct the set of tableaux $\mathcal{O}(\lambda * \mu)$. Assume that $|\lambda| \geqslant |\mu|$. The set $\mathcal{O}(\lambda * \mu)$ is constructed by constructing the tree of tableaux $\mathfrak{T}$ just as we did in Example 1. Thus $\mathfrak{T}$ is the tree whose root is the tableau T of shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ where the elements $1, \ldots, |\lambda_1|$ are in the first row of T, the elements $\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2$ are in the second row, etc. Then the sons of T are those tableaux T' which result by adding $|\lambda| + 1$ to T in all possible ways according to the conditions (i) and (ii) pictured in Fig. 2 and the sons of such a T' are those tableaux T'' which result by adding $|\lambda| + 2$ to T' in all possible ways according to conditions (i) and (ii), etc. The tree $\mathfrak{T}$ as given in Example 1 can be constructed in a depth first manner as follows. First we place the numbers $|\lambda| + 1, |\lambda| + 2, |\lambda| + 3, \ldots,$ in the tableau in order so that $|\lambda| + i$ is placed in the highest row allowed by conditions (i) and (ii). Finally when the tableau contains all the numbers $1, \ldots, |\lambda| + |\mu|$, we move the largest number in the tableau down to the next possible row allowed. If it is not possible to move the largest number down, then we remove the largest number and try to move the largest number that remains down to the next possible row allowed. We continue in this way to remove the largest number of the tableau until we find one that can be moved down in which case we move that number down to the next row allowed by conditions (i) and (ii) and then attempt to re-add those numbers which were removed to their highest possible allowed rows. A moment's thought will convince one that it is always possible to start from any tableau U where the number $|\lambda| + 1, \ldots, |\lambda| + k$ have been added to T according to conditions (i) and (ii) and add $|\lambda| + k + 1, \ldots, |\lambda| + |\mu|$ in order so that each number is added to its highest possible row to complete a tableau in $\mathcal{O}(\lambda * \mu)$. Thus, the process stops when we reach a point where we are trying to move $|\lambda| + 1$ down but cannot because we will no longer end up with a legitimate shape. The reader can check that the process described above does traverse the tree of Example 1 in depth first order.

It is not too difficult to concoct a data structure which will allow one to move from one node to another node in this process in constant time. There are $|\mathcal{O}(\lambda * \mu)|$ leaves in the tree $\mathfrak{T}$ and for each leaf there are $|\mu|$ edges from the leaf back to the root of $\mathfrak{T}$. Since each edge in the tree is traversed twice and each edge is traversed in constant time, it follows that to traverse the tree requires no more than $K|\mu||\mathcal{O}(\lambda * \mu)|$ steps for some constant K. If we allow $K(|\lambda| + |\mu|)$ steps to set up the tableaux T and the reverse lexicographic filling of $\lambda * \mu$, we get an estimate of $K(|\lambda| + |\mu|) + K|\mu||\mathcal{O}(\lambda * \mu)|$ for the complexity of the algorithm where we assume $|\lambda| \geqslant |\mu|$. One might quibble that the estimate $K|\mu||\mathcal{O}(\lambda * \mu)|$ is too high for traversing the tree, but that is not the main problem to determine the complexity of this algorithm. The first step for either a worst case or average complexity result for this algorithm is to find some reasonable estimates for $|\mathcal{O}(\lambda * \mu)|$. However, we do not know of any work done on estimates for $|\mathcal{O}(\lambda * \mu)|$.

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