INDEX AND OVERVIEW OF FEDERICO ARDILA'S LECTURES ON HOPF ALGEBRAS AND COMBINATORICS

OVERVIEW BY SARA BILLEY

This is an index for Federico Ardila's lectures on Hopf algebras available on Youtube. The study of Hopf algebras is a beautiful subject which is just coming into its prime in my opinion. It is much more developed now than it was 25 years ago when I first learned the subject with G.C. Rota. I enjoyed these lectures in particular because of the emphasis on examples. I found these video lectures easier to process than books on the subject, but they are much harder to peruse. Federico said on a website somewhere that an index was needed, so I started taking some notes. I was surprised to find that I learned something from summarizing each lecture and giving it a title. I encourage you to make your own index and give each lecture a title in your own words as a simple step toward digesting the subject.

One more tip, you can speed up the lectures to 2x speed using the settings button. It sounds very reasonable at that speed and not like a chipmunk lecturing on Hopf algebras. Must be some Fourier trick going on there.

Lecture 1: What is a Hopf Algebra?. A grand overview of Hopf algebras is given. Hopf algebras are vector spaces with a product, coproduct, an antipode, a unit and a counit. These maps have to satisfy certain compatibility conditions which can be defined in terms of commutative diagrams. There are a lot of diagrams show here! Don't be alarmed. It will take time to develop all the necessary vocabulary and intuition needed to digest these diagrams. The next few lectures will start with some examples of coproducts on familiar objects and some basics from algebra.

Example: K[G]: Group algebra (1:02:45) p. 4.

References:

Moss Sweedler, Hopf Algebras, 1969.

H. Figueroa and J. Gracia-Bondía, Hopf algebras in quantum field theory I. 2005

Lecture 2: Examples of Hopf Algebras. Examples are the key to understanding!

Topics:

(1) K-algebras, (1:04:59) p. 7

Date: May 17, 2014.

(2) Unit, (1:13:00) p. 8

Examples:

(1) K[G]: Group algebra (0:06:00) p. 4.
(2) K[x]: polynomials (0:09:00) p. 5.
(3) Hopf algebra on finite graphs (0:20:00) p. 6
(4) Shuffle/Cut Hopf algebra on Permutations, (0:38:00) p. 5

Lecture 3: Algebras and Tensor Products. This lecture covers background on rings, vector spaces, algebras, homomorphisms, ideals, subalgebras, quotients.

Topics:

(1) K-algebras, building up in more advanced ways (0:10:40) (0:30:00) p. 7
(2) Homomorphisms of algebras (0:16:39) p.8
(3) Tensor products (1:07:00) p. 3,10

Example: Matrix algebra (0:10:16) p. 3.

Lecture 4: Fun with Tensor Products. Tensor products are the natural setting for bilinear functions like multiplication. Learn to appreciate them. This point of view is developed as a Universal Property. Basic properties are developed.

Topics:

(1) Universal Property of tensor products, (0:5:00) p. 10
(2) Useful lemma when a sum of pure tensors is zero, (1:08:00) p.13

Example: Coalgebra of a set, p. 16

Lecture 5: Algebras, graded algebras and coalgebras.

Topics:

(1) Extension of scalars in an algebra using tensor product. (0:16:00) p. 14
(2) Graded algebras. (0:25:00) p. 15
(3) Coalgebras (0:50:00) p. 16
(4) Coassociativity (0:53:00) p. 16
(5) Counit (0:54:00) p. 16

Examples:

(1) Coalgebra of a set (1:01:17) p. 16
(2) Coalgebra of a poset (1:05:00) p. 17

Lecture 6: Duality between coalgebras and algebras. The vector space dual of a coalgebra is an algebra. The converse is true for finite dimensional algebras. These facts are proved and some examples discussed. Sweedler notation is introduced.

Topics:

(1) Duality (0:5:32) p. 18
(2) Sweedler notation (1:01:00) p. 21

Examples:

(1) Incidence coalgebra of a poset (0:41:34) p. 20
(2) Proving identities with Sweedler notation (1:14:20) p. 22

Lecture 7: Sweedler Notation: Feel the power. Sweedler notation is further developed. The power comes from proving identities in coalgebras proved in Sweedler notation. This will be critical for working with antipodes in a few lectures. Coalgebra homomorphisms, coideals and subcoalgebras are introduced. More on duality with regard to homomorphisms. Big theorem here is that coideals are dual to subalgebras done at the end of the lecture.

Topics:

(1) Sweedler notation (0:01:00) p. 21
(2) Algebra homomorphisms revisited (0:16:46) p. 23
(3) Coalgebra homomorphisms (0:16:46) p. 24
(4) Subalgebra revisited (0:37:42) p. 23
(5) Ideal of an algebra revisited (0:38:00) p. 23
(6) Subcoalgebra (0:38:46) p. 24
(7) Coideal (0:39:30) p. 24
(8) Dual of coalgebra homomorphisms (0:44:12) p. 25

Examples:

(1) Proving identities with Sweedler notation (0:03:20) p. 22
(2) Coalgebra on a set maps to incidence coalgebra of the Boolean lattice (0:28:12) p. 27.5 (but labeled 24.5)

Lecture 8: A fundamental theorem of coalgebras. A key theorem is proved relating the image of a coalgebra homomorphism with the quotient of the domain mod the kernel. Note, F.A. says first isomorphism theorem in lectures and notes say fundamental theorem. Biaglebras are introduced toward the end of the lecture

Topics:

(1) Fundamental theorem of coalgebras (0:42:05) p. 31

(2) Bialgebras (0:58:00) p. 33
(3) Vandermonde convolution on binomial coefficients (1:23:16) p. 34

Examples:

(1) Group algebra is a bialgebra (1:15:58) p. 34
(2) Polynomial rings are bialgebras (1:18:24) p. 34

Lecture 9: Missing in action. This one didn't record properly. From the notes, it looks like F.A. covered the bialgebra obtained from the isomorphism classes of finite intervals in posets. He defined cocommutative coalgebras and Hopf algebras. Reading over the notes will fill this gap. You might try figuring out which of the coalgebra examples are cocommutative yourself before looking at the answer in the notes.

Topics: Cocommutative p. 36

Example: Bialgebras of intervals of finite posets p. 35

Lecture 10: Hopf algebras. The theory of Hopf algebras was developed from a need to put these examples into a common framework. The antipode is given for many of the examples we have done.

Topics:

(1) Hopf algebra (0:01:53) p. 37
(2) Hom(C,A) is an algebra (0:50:00) p. 39
(3) Convolution product (0:57:30) p. 39
(4) Antipode is inverse of identity under convolution (1:06:00) p. 40

Examples:

(1) Group algebra is a Hopf algebra (0:5:58) p. 38
(2) Polynomial ring is a Hopf algebra (0:43:46) p. 38
(3) Weird quotient example showing antipode not always an involution (0:13:00). From hw. Not in notes.
(4) Example of a bialgebra that is not a Hopf algebra (0:32:46) p. 38

Lecture 11: Properties of Antipodes. Several nice properties of antipodes are proved. A couple big theorems in the field are discussed at the end of the lecture by Taft and Larson Bradford 1988. Kaplansky conjecture says that if H is a finite dimensional Hopf algebra, then the antipode has finite order. Conjecture not in notes.

Topics:

(1) Antipodes anticommute (0:01:00) p. 41

(2) Antipodes are involutions for commutative or cocommutative Hopf algebras (0:54:00) p. 42
(3) Antipodes cannot have odd order larger than 1 (1:10:00) p. 45
(4) Kaplansky conjecture (1:12:00)

Lecture 12: Properties and constructions on Hopf algebras. A little was missing in the notes. Antipodes work well with Hopf algebra maps. Bialgebra/Hopf algebra quotients and ideals. Another version of the fundamental theorem on quotients of Hopf algebra. Then, on to Takeuchi theorem: Given a graded bialgebra then the antipode comes for free. Note, pages 48-49 scanned twice.

Topics:

(1) Quotients (0:7:35) p. 46
(2) Hopf ideal (0:09:30) p. 46
(3) Hopf dual (0:14:00) p. 46
(4) Graded Connected Hopf algebra (0:18:00) p. 47
(5) Takeuchi Theorem 1971 (0:47:45) p. 48. Note (-1) should be to power n in the formula.

Examples:

(1) Polynomial rings are Hopf algebras (0:24:30) p. 47
(2) K-span of isomorphism classes of graphs is a Hopf algebra (0:28:00) p. 47
(3) Shuffle/Cut bialgebra of permutations is a Hopf algebra (0:33:47) p. 47
(4) Bialgebras of intervals of finite posets (0:44:20). Not in notes this time.

Lecture 13: Applying Takeuchi's formula for Antipodes. Initial comments on how to apply this infinite sum formula. In general, it is a research project to find a nice formula for an antipode. A nice formula has no cancellation or repeated terms. Then Bill Schmitt's work on incidence Hopf algebras is introduced. Antipode comes directly from Takeuchi's formula.

Topics:

(1) Conilpotent (0:10:45) p. 49
(2) Multinomial coefficients (0:26:58) p. 50
(3) Incidence Hopf algebras (1:06:05) p. 52
(4) Hereditary family (1:11:26) p. 53

Example: Takeuchi formula on polynomials (0:17:36) p. 50

References:

Joni and Rota, Coalgebra and Bialgebras in Combinatorics, 1979.

Schmitt, Incidence Hopf Algebras, 1994.

Lecture 14: Repertoire of Hopf Algebras. Hereditary families and incidence Hopf algebras are important tools for studying a lot of examples of Hopf algebras. See paper by Schmitt mentioned above for more details.

Topics: H(P) = Incidence Hopf algebra of a poset P mod reduced congruence (0:01:05) p. 52

Examples:

(1) All finite intervals mod isomorphism or not (0:24:12), not in notes.
(2) Binomial Hopf algebras = finite Boolean algebras (0:26:30) p. 54

Lecture 15: Incidence Hopf Algebras. The lecture starts with a careful description of reduced congruence for incidence Hopf algebras. Listen carefully because it is not in the notes. It is shown that the Hopf algebra of finite linear orders is isomorphic to the Hopf algebra of Symmetric Functions. The Hopf algebra of finite linear order without commutation is indexed by compositions and is isomorphic to Noncommutative Symmetric Functions.

Topics:

(1) Reduced congruence defined with 3 properties (0:01:00). Not in notes.
(2) H(P*) = free commutative incidence Hopf algebra (0:07:30) p. 54

Examples:

(1) Incidence Hopf algebra of finite linear orders (0:13:45) p. 55
(2) Hopf algebra of Symmetric Functions (0:27:48) p. 55
(3) Incidence Hopf algebra related to Finite Compositions = finite boxes (0:33:56) p. 56
(4) Noncommutative symmetric functions (0:48:51) p.57
(5) Incidence Hopf algebra of graphs (0:51:00) p. 58

References:

Brandon Humpert and Jeremy Martin, Incidence algebra of graphs, 2012

Lecture 16: Lattices and Distributive Latices. Two more examples of incidence Hopf algebras come from special posets called latices. Basics on lattices including meet, join, fundamental theorem of finite distributive lattices are covered.

Topics:

(1) Lattices (0:02:00) p. 62
(2) Distributive lattices (0:27:11) p.

Example: Another Hopf algebra on posets (1:05:00)

Lecture 17: The lattice of flats of a graph. The lecture starts out with a power series with no constant term y=F(x) then x = G(y) where G is the compositional inverse of F. This topic is not discussed any further until the next lecture, perhaps it is related to the homework. Instead, an incidence algebra of finite simple graphs is carefully spelled out which depends on taking contractions. This is equivalent to the incidence Hopf algebra on the lattice of flats of a graph.

Topics:

(1) Lattice of contractions of a graph = Lattice of flats of G (29:40) p. 66
(2) Definition of a flat of G (35:10) p. 66

Examples:

(1) Another Hopf algebra on graphs using deletion and contraction (0:05:00)
(2) Any family of graphs closed under taking deletions, contractions and disjoint union will give rise to another interesting hereditary family (1:14:00)

References:

Schmitt, Incidence Hopf Algebras, 1994.

Lecture 18: More examples of incidence Hopf algebras Any family of graphs closed under taking restriction to flats, contraction of flats and disjoint union will give rise to another interesting hereditary family. For example, the Faà di Bruno Hopf algebra is related to complete graphs. Faà di Bruno was a saint. Rota gave the Hopf algebra that name. This topic is closely related to Lagrange inversion via formulas using the partial Bell polynomials. The Doubilet-Rota-Stanley Thm says the characters of the Faà di Bruno Hopf algebra are isomorphic to the group of divided power series with multiplication given by composition. This implies the antipode is determined by the Lagrange inversion formula.

Topics:

(1) Faà di Bruno Hopf algebra (0:6:27) p. 69
(2) Partial Bell polynomial (0:32:00) p. 69
(3) Faà di Bruno formula for iterated nth derivative of a composition of power series(0:42:07) p. 70
(4) Lagrange inversion formula (0:54:30) p. 70
(5) Crash course on characters of Hopf algebras (1:02:10) p. 72
(6) Group of divided power series with multiplication given by composition (1:11:16) p. 71

Lecture 19: Characters of Hopf algebras. This lecture spells out many of the beautiful connections outlined in Lecture 18. In particular, the characters of a Hopf algebra are shown to be a group under convolution. The group of divided power series is shown to be a group under composition and expressed using the Lagrange inversion formula in terms of the partial Bell polynomials. After proving the Doubilet-Rota-Stanley theorem, it is an easy corollary to get the antipode for the Faà di Bruno Hopf algebra. This formula is a special case of how Möbius functions relate to antipodes which is the next topic.

Topics:

(1) Group of characters of a Hopf algebra (0:16:45) p. 72
(2) Group of divided power series (0:33:30)
(3) Proof of Doubilet-Rota-Stanley thm (0:41:30) p. 73)
(4) Antipode for Faà di Bruno Hopf algebra (0:53:00) p. 74
(5) Möbius functions (1:07:00) p. 75

References:

M. Haiman and W. Schmitt. Incidence Algebra Antipodes and Lagrange inversion in one and several variables, (1989).

Lecture 20: Antipodes to Möbius functions. Incidence Hopf algebras can be a useful tool to compute Möbius functions. It was shown in Lecture 19 that the algebra homomorphisms $F : H \longrightarrow K$ on a Hopf algebra form a group under convolution. To get the inverse of such a map f, just take f(S(x)). The Möbius function is an algebra map defined as the inverse of the zeta function which is 1 on every interval. The characteristic polynomial of a ranked interval is a generating function for the Möbius function. The characteristic polynomials often factor which implies a factorization of posets. Many examples are done. Joke is hard to hear: What did 0 say to 8? Answer: "I spell" ?? I didn't get it.

Topics:

(1) Möbius functions of posets (0:04:00) p. 77
(2) Characteristic polynomial of a ranked poset (0:16:50) p. 77
(3) The Möbius inversion formula (0:56:30) p. 80

Examples:

(1) Boolean lattice B(3) (0:18:30) p. 77
(2) Partition lattice p. 76
(3) Chains (0:41:00) p. 78
(4) Divisors of n ordered by divisibility (0:48:00) p. 79
(5) Inclusion-Exclusion formula (1:07:00) p. 80

(6) Partition lattice p. 76 but not in lecture

Lecture 21: Fun with Möbius functions. The goal of this lecture is to show some nice examples of Möbius functions and how they come up in different areas of math. The classic example is computing Euler's totient function from number theory. Apparently, the word totient comes from Latin and one source says it means "so many, that many". Another well known example is the partition lattice Π_n which is isomorphic to the lattice of flats of the complete graph with n vertices. So, one can compute its Möbius function using the interpretation of characters of the Faà di Bruno Hopf algebra as divided power series. This is a good example to show the power of Hopf algebra machinery to proving combinatorial theorems. A third connection is between Möbius functions of posets and Euler characteristics of certain simplicial complexes built from posets. The proof uses the famous Philip Hall theorem for Möbius functions. Note, there are two copies of p. 80-81. I don't see much difference besides the label of where Lecture 21 starts.

Topics:

(1) Euler's Phi function (totient formula) (0:5:30) (0:18:00) p. 81
(2) Möbius function of the lattice of the partition lattice (20:38) p. 76
(3) Philip Hall Theorem (0:53:00) p. 82
(4) Order complex (1:00:00) p. 83
(5) Reduced Euler characteristic of a simplicial complex (1:06:00) p. 83
(6) Reduced Euler characteristic of an order complex of P is the Möbius function of P by Hall thm (1:06:00) p. 83

References:

R. Stanley, Enumerative Combinatorics Volume 1 Chapter 3.7, online version dated July 2011

Lecture 22: Face Lattice of a Polytopes. The Möbius function of the face lattice of a polytope L_P is shown to have a particularly simple formula. Plus, along the way it is shown that the face lattices of polytopes is a hereditary family of finite posets so there is an associated incidence Hopf algebra for polytopes. Then f-vectors, h-vectors and the Dehn-Sommerville relations for simplicial polytopes are proved. The video doesn't show the example computing Stanley's trick for h-vectors. But, its easy. Just write the f-numbers along a row and a diagonal of 1's moving northeast from the first entry in the row. Then reverse the process of Pascal's triangle. See the example in the notes.

Topics:

(1) Face lattice of a polytope and its Möbius function (0:02:00) p. 84

(2) f-vector of a polytope (0:40:16) p. 86
(3) h-vector of a polytope (0:44:00) p. 86
(4) Dehn-Sommerville relations (0:48:30) p. 86
(5) Stanley's trick for computing h-vectors (0:53:00) p. 86

Lecture 23: Matroids. Matroids are a combinatorial model generalizing the notion of independent sets of vectors in a vector space. Generalizing the notion of deletion and contraction one gets a Hopf algebra on matroids. Ends with two good questions. When do two graphs have the same matroids? Determined by Whitney's 2-isomorphism theorem. Is there a Hopf algebra of transversal matroids? No, use gamoids.

Topics:

(1) Definition of a matroid in terms of independent sets (0:03:18) p. 88
(2) Basis (0:18:30) p. 88
(3) Linear matroids (0:11:11) p. 89
(4) Graphical matroids (0:17:00) p. 89
(5) Algebraic matroids (0:23:00) p. 89
(6) Transversal matroids (0:28:50) not in notes
(7) Hopf algebra of a matroid (0:41:00) thru (1:02:00) p. 90-91

Lecture 24: Generalized Permutahedron. A generalized permutahedron is any polytope defined by inequalities which are given by submodular functions. Examples of submodular functions and polytopes are inspired by graphs, matroids, and posets. Then a recipe is given to make a Hopf algebra from generalized permutahedra which contains the Hopf algebras on graphs, matroids, and posets already discussed. A formula for the antipode is given. It is proved by Takeuchi's formula.

Topics:

(1) Permutahedron (0:04:00) p. 92
(2) Generalized permutahedron (0:16:50) p. 93
(3) Submodular functions (0:20:50) p. 93
(4) Graphs give rise to graphical zonotopes (0:30:21) p. 94
(5) Matroid polytope (0:32:30) .94
(6) Posets to poset polyhedron (0:45:00) p.95
(7) Hopf algebra of generalized permutahedron (0:49:18) p. 95
(8) Antipode results (1:10:50) p. 97

References:

M. Aguiar and F. Ardila, The Hopf monoid of generalized permutahedra. Still not out yet as of 5/13/2014.

Lecture 25: Symmetric functions. Symmetric function have several important bases including monomial symmetric functions, elementary symmetric functions, homogeneous symmetric functions.

Topics: coproduct p. 102

Lecture 26: More on the Hopf algebra of Symmetric Functions. The lecture builds up the Hopf structure on symmetric functions. It ends by showing the beautiful fact that the graded Hopf dual of SYM is isomorphic to itself. Something is missing at the end of the lecture.

Topics:

(1) antipode symmetric functions (0:16:00)-(0:26:00) p. 104 -105
(2) Dual Hopf algebra for symmetric functions (0:46:00) p. 107
(3) Graded dual of a graded Hopf algebra (0:55:00) p. 107

Lecture 27: Three variations of symmetric functions. Starts with a review of the Hopf algebra of symmetric functions on commuting variables. Then the lecture covers some variations of that theme. NCSYM versus NSYM. Graded dimensions for NCSYM are the Bell numbers. Bases are indexed by set partitions. For NSYM the graded dimension is 2^{n-1} and the bases are indexed by compositions. The Hopf algebra structure on quasisymmetric functions is introduced. It is just mentioned at the end that NSYM and QSYM are Hopf dual. There is a little more in the notes. See the Wikipedia page on quasisymmetric functions for more pointers to the literature.

Topics:

(1) NCSYM = Hopf algebra of symmetric functions in noncommuting variables (0:15:45) p. 108
(2) NSYM = Noncommutative symmetric functions (0:40:50)
(3) QSYM = Quasisymmetric functions (1:02:00) p. 111

References:

M. Rosas and B. Sagan, Symmetric functions in noncommuting variables (2006).

N. Bergeron, C. Reutenauer, M. Rosas, and M. Zabrocki, Invariants and coinvariants of the symmetric group in noncommuting variables (2005).

Gelfand, Krob, Lascoux, Leclerc, Retakh, Thibon, Noncommutative symmetric functions, (1995).

Lecture 28: Many Hopf algebras and how they are related. SYM, NSYM, NCSYM, QSYM, Shuffle/Cut Hopf algebra on permutations are all compared. The last one is also called the Malvenuto-Reutenauer Hopf algebra of permutations or

S-Sym in the lectures where the S is in fraktur font. The antipode of Shuffle/Cut has infinite order. These all fit together in a big commutative diagram as Hopf algebras. Note, at (0:25:42) he is talking about the fundamental quasisymmetric functions in QSYM which are F_a. At (0:28:00) he is talking about the multiplication and coproduct on the basis in the shuffle/cut Hopf algebra and these are denoted by F_{perm}. It is a little hard to differentiate in the lecture because the camera resolution is not perfect. In fact, the map from S-Sym to QSYM induces a similar multiplication and coproduct on the fundamental quasisymmetric functions.

Topics:

(1) Fundamental quasisymmetric functions F_a indexed by compositions in QSYM (0:25:40)p. 112
(2) Multiplication and coproduct on basis F_{perm} indexed by permutations for Shuffle/Cut (0:28:00)p. 114
(3) Hopf alg map from NCSYM to SYM is surjective (0:40:00) p. 115
(4) Hopf alg map from NSYM to SYM is surjective(0:45:00) p. 115
(5) Hopf alg map from NSYM to NCSYM is injective (0:57:00) p. 116
(6) Hopf alg map from SYM to QSYM is injective (1:00:00) p. 117
(7) Hopf alg map from S-SYM to QSYM is surjective (1:09:00) p. 117
(8) Hopf alg map from NSYM to S-SYM is injective (1:14:00) p. 118

Lecture 29: Combinatorial Hopf Algebras. Combinatorial Hopf algebras are graded connected Hopf algebras along with a character function which is linear and respects the product structure. They often have a basis indexed combinatorial objects like in the examples discussed in these lectures. The very important theorem here is that every combinatorial Hopf algebra maps in a unique way into QSYM with the evaluation character. Thus, QSYM is the terminal object in the category of combinatorial Hopf algebras. This is a theorem of Aguiar-Bergeron-Sottile.

Topics:

(1) Definition of a combinatorial Hopf algebra (0:02:00) p. 120
(2) Morphisms of CHA's (0:23:54) p. 121 cut off at top of page, more clear in lecture.
(3) Aguiar-Bergeron-Sottile Theorem p. 121
(4) Chromatic polynomials for graphs and symmetric functions p.124

Examples:

(1) Hopf algebra on graphs with character given by the counit(0:05:00) p. 120
(2) Incidence algebra of posets with 0 and 1 with character given by the Möbius function (0:07:30) p. 120

(3) Hopf algebra of generalized permutahedron with the Ehrhart polynomial as character(0:11:00)
(4) QSYM with character given by the evaluation at $x_1 = 1, x_i = 0$ all other i (0:14:30) p. 120
(5) SYM with same character given by evaluation (0:21:30)

References:

M. Aguiar, N. Bergeron and F. Sottile, Combinatorial Hopf algebras and generalized Dehn-Sommerville relations, 2006

Lecture 30: Polynomials associated to combinatorial Hopf algebras. There is a natural morphism from QSYM to polynomials in one variable. This has nice implications for all combinatorial Hopf algebras. Examples are chromatic polynomials, order polynomials, and counting order preserving maps in posets.

Examples:

(1) Chromatic polynomials p. 129
(2) Order polynomials p. 129
(3) Posets p. 130

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