Algebraic Combinatorics
A tale of two rings: SYM and QSYM

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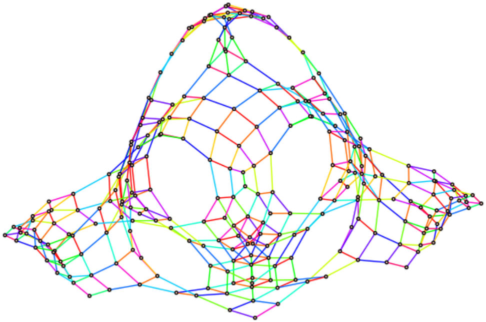
¡Gracias por la oportunidad de estar aquí y de compartir este tema con Uds. en un lugar tan lindo!

June 9, 2014

My Philosophy

"Combinatorics is the nanotechnology of mathematics"



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Outline

Tale of Two Rings

Power Series Ring.: $\mathbb{Z}[[X]]$ over a finite or countably infinite alphabet $X = \{x_1, x_2, \ldots, x_n\}$ or $X = \{x_1, x_2, \ldots\}$.

Two subrings. of $\mathbb{Z}[[X]]$:

- Symmetric Functions (SYM)
- Quasisymmetric Functions (QSYM)

SYM=Ring of Symmetric Functions

Defn. $f(x_1, x_2, \ldots) \in \mathbb{Z}[[X]]$ is a *symmetric function* if for all i

$$f(\ldots, x_i, x_{i+1}, \ldots) = f(\ldots, x_{i+1}, x_i, \ldots).$$

Example. $x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + \ldots$

Defn. SYM = symmetric functions of bounded degree.

QSYM=Ring of Quasisymmetric Functions

Defn. $f(x_1, x_2, \dots) \in \mathbb{Z}[[X]]$ is a *symmetric function* if for all i

$$f(\dots, x_i, x_{i+1}, \dots) = f(\dots, x_{i+1}, x_i, \dots).$$

Example. $x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + \dots$

Defn. $f(x_1, x_2, \dots) \in \mathbb{Z}[[X]]$ is a *quasisymmetric function* if

$$\mathrm{coef}(f; x_1^{\alpha_1} x_2^{\alpha_2} \dots x_k^{\alpha_k}) = \mathrm{coef}(f; x_a^{\alpha_1} x_b^{\alpha_2} \dots x_c^{\alpha_k})$$

for all $1 < a < b < \cdots < c$.

Example. $f(X) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + \dots$

Why study SYM and QSYM?

- Symmetric Functions (SYM): Used in representation theory, combinatorics, algebraic geometry, over past 200+ years. More recently expanding applications in number theory, theoretical physics, economics, quantum computing !
- Quasisymmetric Functions (QSYM): 0-Hecke algebra representation theory, Schubert calculus, enumeration of linear extensions of posets, Hopf dual of NSYM=non-commutative symmetric functions, terminal object in the category of combinatorial Hopf algebras.
- SYM and QSYM are easy to study with Sage!

High level goals

1. Develop intuition for some of the universal tools in algebraic combinatorics.
2. Build up vocabulary to introduce some important open problems and approaches to attack them.
3. Inspire you to learn more about quasisymmetric functions and find more applications.

Main tool: Permutations

Defn. A *permutation* w in the symmetric group S_n is a bijection on the set $[n] = \{1, 2, \ldots n\}$.

Fact. S_n is a group under composition of bijections with the identity function as the identity for the group.

Example. $w : [4] \longrightarrow [4]$ given by
$w(1) = 2, w(2) = 3, w(3) = 4, w(4) = 1$

$w^{-1}(1) = 4, w^{-1}(2) = 1, w^{-1}(3) = 2, w^{-1}(4) = 3$

$id(1) = 1, id(2) = 2, id(3) = 3, id(4) = 4$

Some Applications of Permutations

- Card shuffling and card tricks.
- The determinant of a $n \times n$ matrix $M = [m_{ij}]$ is by definition

$$\det(M) = \sum_{w \in S_n} (-1)^{inv(w)} m_{1,w(1)} m_{2,w(2)} \cdots m_{n,w(n)}.$$

- Cryptography.
- Differentiating species by DNA strings and phylogenetic trees.
- Detecting near duplicate webpages for search engines (Broder Algorithm).
- Symmetric functions and symmetric polynomials.

Six more ways to represent a permutation

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix} = [2, 3, 4, 1]$$

matrix notation — two-line notation — one-line notation

$$\begin{array}{cccc} * & * & * & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{array} =$$

$$= \quad s_1 s_2 s_3$$

diagram of a permutation $D^*(w)$ — string diagram — product of s_i's adjacent transpositions

Permutation Statistics

- $inv(w) = \#\{i < j \;\; w(j) < w(i)\} = \ell(w)$ *Inversions*
- $des(w) = \#\{i \;\; : \;\; w(i) > w(i+1)\}$ *Descents*
- $peaks(w) = \#\{i \;\; : \;\; w(i-1) < w(i) > w(j)\}$ *Peaks*

Example.
$w = [2, 5, 4, 3, 6, 1] \implies inv(w) = 8, \; des(w) = 3, \; peaks(w) = 2$

Permutation Statistics

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Example.
$w = [2,5,4,3,6,1] \implies inv(w) = 8,\; des(w) = 3,\; peaks(w) = 2$

$Inv(w) = \{(1,5),(2,3),(2,4),(2,6),(3,4),(3,6),(4,6),(5,6)\}$
$Des(w) = \{2,3,5\}$
$Peaks(w) = \{2,5\}$

Generating Functions by Example

Defn. Let $A_n(x) = \sum_{w \in S_n} x^{1+des(w)} = \sum A(d,n)x^d$.
The $A(d,n)$ are called *Eulerian numbers*.

$A_2(x) = x + x^2$
$A_3(x) = x + 4x^2 + x^3$
$A_4(x) = x + 11x^2 + 11x^3 + x^4$

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Thm. (Holte 1997, Diaconis-Fulman 2009) The probability of carrying d on in the jth column when adding n large numbers tend to $A(d,n)/n!$.

Enumerative Results

Thm. (Gessel-Viennot 1985) The number of permutations in S_n with a given descent set $S = \{s_1, \ldots, s_k\}$ is given by the binomial determinant

$$\det \left[\binom{n - s_i}{s_{j+1} - s_i} \right]_{1 \leq i \leq j \leq k}$$

where $s_0 = 0, s_{k+1} = n$.

Thm.(Billey-Burdzy-Sagan 2013) The number of permutations with a given peak set $S = \{s_1 < \ldots < s_k\}$ for $n \geq s_k$ is determined by $2^{n-|S|-1} P_S(n)$ for the *peak polynomial* $P_S(n)$.

See also: "Properties of Peak Polynomials" by Fahrbach and Talmage (manuscript 2014).

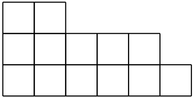
Monomial Basis of SYM

Defn. A *partition* of a number n is a weakly decreasing sequence of positive integers

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$$

such that $n = \sum \lambda_i = |\lambda|$.

Partitions can be visualized by their *Ferrers diagram*

$(6, 5, 2) \longrightarrow$ 

Defn/Thm. The *monomial symmetric functions*

$$m_\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k} + x_2^{\lambda_1} x_1^{\lambda_2} \cdots x_k^{\lambda_k} + \text{all other perms of vars}$$

form a basis for SYM_n = homogeneous symmetric functions of degree n.

Fact. $\dim SYM_n = p(n)$ = number of partitions of n.

More bases for SYM

Let $X = \{x_1, x_2, \ldots, x_m\}$ be the alphabet.

Defn. $e_k(X) = \displaystyle\sum_{1 \le i_1 < i_2 < \ldots < i_k \le m} x_{i_1} x_{i_2} \cdots x_{i_k}$.

For any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, set $e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}$.

These are *elementary symmetric functions*.

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These are *elementary symmetric functions*.

Application. If $F(x) = (x + r_1)(x + r_2) \cdots (x + r_m)$ is a polynomial with m roots, then

$$\begin{aligned} F(x) =& x^m + (r_1 + r_2 + \ldots + r_m)x^{m-1} + \ldots + (r_1 r_2 \cdots r_m) \\ =& x^m + e_1(r_1, \ldots, r_m)x^{m-1} + \ldots + e_m(r_1, \ldots, r_m). \end{aligned}$$

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Fundamental Theorem. $SYM(X) = \mathbb{Q}[e_1, e_2, ..., e_m]$ as a freely generated polynomial ring.

More bases for SYM

Defn. $h_k(X) = \sum_{1 \le i_1 \le i_2 < \ldots \le i_k \le m} x_{i_1} x_{i_2} \cdots x_{i_k}$. Set $h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_k}$. These are *homogeneous symmetric functions*.

Defn. $p_k(X) = x_1^k + x_2^k + x_3^k + \ldots + x_m^k$. Set $p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}$. These are *power symmetric functions*.

More bases for SYM

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Defn. $p_k(X) = x_1^k + x_2^k + x_3^k + \ldots + x_m^k$. Set $p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}$. These are *power symmetric functions*.

Theorem. $SYM = \mathbb{Q}[e_1, e_2, ...] = \mathbb{Q}[h_1, h_2, ...] = \mathbb{Q}[p_1, p_2, ...]$ over the alphabet $X = \{x_1, x_2, \ldots\}$.

Cor. SYM has three more bases $\{e_\lambda\}$, $\{h_\lambda\}$, $\{p_\lambda\}$ where the bases range over all partitions when X is infinite.

Schur basis for SYM

Let $X = \{x_1, x_2, \ldots, x_m\}$ be a finite alphabet.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ and $\lambda_p = 0$ for $p > k$.

Defn. The following are equivalent definitions for the Schur functions $S_\lambda(X)$:

1. $S_\lambda = \frac{\det(x_i^{\lambda_j+m-j})}{\det(x_i^j)}$ with indices $1 \leq i, j \leq m$.

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1. $S_\lambda = \dfrac{\det(x_i^{\lambda_j + m - j})}{\det(x_i^j)}$ with indices $1 \leq i, j \leq m$.
2. $S_\lambda = \sum x^T$ summed over all *column strict tableaux T of shape λ*.

Defn. T is *column strict* if entries strictly increase along columns and weakly increase along rows.

Example. A column strict tableau of shape $(5, 3, 1)$

$$T = \begin{array}{|c|c|c|c|c|} \hline 7 & & & & \\ \hline 4 & 7 & 7 & & \\ \hline 2 & 2 & 3 & 4 & 8 \\ \hline \end{array} \qquad x^T = x_2^2 x_3 x_4^2 x_7^3 x_8$$

Multiplying Schur Functions

Littlewood-Richardson Coefficients.

$$S_\lambda(X) \cdot S_\mu(X) = \sum_{|\nu|=|\lambda|+|\mu|} c^{\nu}_{\lambda,\mu} S_\nu(X)$$

$c^{\nu}_{\lambda,\mu} = \#$ skew tableaux of shape ν/λ such that $x^T = x^\mu$ and the reverse reading word is a lattice word.

Example. If $\nu = (4,3,2)$, $\lambda = (2,1)$, $\lambda = (3,2,1)$ then

$$\begin{array}{|c|c|c|c|} \hline 2 & 3 & & \\ \hline & 1 & 2 & \\ \hline & & 1 & 1 \\ \hline \end{array} \qquad \text{readingword} = 231211$$

Schur Functions/Schur Polynomials

Special properties.

1. The graded ring of representations of S_n for all $n > 0$ is isomorphic to SYM on an infinite alphabet. The irreducible representations are indexed by partitions. The map sends the irreducible V^{λ} to S_{λ}.

2. Schur polynomials are characters of irreducible GL_n representations.

3. Schur polynomials represent the Schubert basis in the Grassmannian manifolds.

Quasisymmetric Functions

Nice Algebraic Facts.

- There is an analog of the Frobenius characteristic from symmetric function theory giving an isomorphism the Grothendieck group of representations of 0-Hecke algebras to QSYM. It maps the irreducible 0-Hecke algebra representation L_α to F_α. (Duchamp-Krob-Leclerc-Thibon 1996)

- QSYM is Hopf dual to NSYM = non commutative symmetric functions. (Malvenuto-Reutenauer 1995, Gelfand-Krob-Lascoux-Leclerc-Retakh-Thibon 1995)

- QSYM is free over SYM on n variables and $\dim(\mathrm{QSY}(n)/\mathrm{SYM}(n)) = n!$ (Garsia-Wallach 2003)

- The quotient of $\mathbb{Z}[x_1, \ldots, x_n]$ mod quasisymmetric function with no constant term has Hilbert series $\sum C_n t^n$ where C_n is the n-th Catalan number (Aval-Bergeron-Bergeron 2004)

Monomial Basis of QSYM

Defn. A *composition* of a number n is a sequence of positive integers

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$$

such that $n = \sum \alpha_i = |\alpha|$.

Defn/Thm. The *monomial quasisymmetric functions*

$$M_\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} + x_2^{\alpha_1} x_3^{\alpha_2} \cdots x_{k+1}^{\alpha_k} + \text{all other shifts}$$

form a basis for $QSYM_n$ = homogeneous quasisymmetric functions of deg n.

Fact. $\dim QSYM_n$ = number of compositions of $n = 2^{n-1}$.

Monomial Basis of QSYM

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Bijection:

$$
\begin{aligned}
(\alpha_1, \alpha_2, \ldots, \alpha_k) \longrightarrow \{&\alpha_1, \\
&\alpha_1 + \alpha_2, \\
&\alpha_1 + \alpha_2 + \alpha_3, \\
&\ldots \\
&\alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}
\end{aligned}
$$

Counting Partitions

Asymptotic Formula:. (Hardy-Ramanujan) The number of partitions of n, denoted $p(n)$, grows like

$$p(n) \approx \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$$

Fundamental basis for QSYM

Defn. Let $D \subset [p-1] = \{1, 2, \ldots, p-1\}$.
The fundamental quasisymmetric function

$$F_D(X) = \sum x_{i_1} \cdots x_{i_p}$$

summed over all $1 \leq i_1 \leq \ldots \leq i_p$ such that $i_j < i_{j+1}$ whenever $j \in D$.

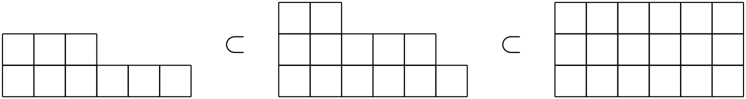
Example. $F_{\{3\}} = x_1x_1x_1x_2x_2 + x_1x_2x_2x_3x_3 + x_1x_2x_3x_4x_5 + \ldots$

Other bases of QSYM. dual immaculate basis (Berg-Bergeron-Saliola-Serrano-Zabrocki), quasi Schur basis (Haglund-Luoto-Mason-vanWilligenburg), matroid friendly basis (Luoto)

A Poset on Partitions

Defn. A *partial order* or a *poset* is a reflexive, anti-symmetric, and transitive relation on a set.

Defn. *Young's Lattice* on all partitions is the poset defined by the relation $\lambda \subset \mu$ if the Ferrers diagram for λ fits inside the Ferrers diagram for μ.



Defn. A standard Young tableau T of shape λ is a saturated chain in Young's lattice from $\emptyset$ to λ.

Example. $T =$

7				
4	5	9		
1	2	3	6	8

SYT=Standard Young Tableaux

Thm.(Frame-Robinson-Thrall 1954) The number of standard tableaux of shape $\lambda \vdash n$, denoted f^λ, is given by the *hook length formula*:

$$f^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h(i,j)}$$

where $h(i,j)$ is the *hook length* of the cell c in the Ferrers diagram for λ found by counting the number of cells above c plus the number to the right of c including itself.

See also the proof by Greene-Nijenhuis-Wilf (1979).

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Example. $f^{(3,2)} = \frac{5!}{1\cdot 1\cdot 2\cdot 3\cdot 4} = 5$

2	1	
4	3	1

RSK=Robinson-Schensted-Knuth Bijection

Rep Theory Facts. (see Sagan's book) The dimension of the S_n irreducible representation V^λ is f^λ. Hence

$$n! = \sum_{\lambda \vdash n} (f^\lambda)^2$$

because the the regular representation of S_n decomposes as the direct sum of $dim(V^\lambda)$ copies of V^λ.

Alternative Proof. RSK gives a bijection from S_n to $\cup_{\lambda \vdash n} SYT_\lambda^2$.

RSK=Robinson-Schensted-Knuth Bijection

Input: $w \in S_n$

Output: $(P, Q) \in SYT_\lambda$ for some shape $\lambda \vdash n$.

Start: Set $P = Q = \emptyset$.

Step i: Insert $w(i)$ into the first row of P by "bumping" the smallest value $b > i$ from the first row if it exists or adding i to the end of the first row otherwise. If b exists, bump it into the second row, continuing until nothing is bumped. The result is the new tableau P. Add a new cell containing i to Q in the same position as the new cell added to P.

Ex. $w = [1, 2, 6, 3, 5, 4] \mapsto P = \begin{array}{|c|c|c|c|} \hline 6 \\ \cline{1-1} 5 \\ \hline 1 & 2 & 3 & 4 \\ \hline \end{array}, \quad Q = \begin{array}{|c|c|c|c|} \hline 6 \\ \cline{1-1} 4 \\ \hline 1 & 2 & 3 & 5 \\ \hline \end{array}$

Gessel's formula for Schur functions

Thm.(Gessel,1984) For all partitions λ,

$$S_\lambda(X) = \sum F_{D(T)}(X)$$

summed over all standard tableaux T of shape λ.

Defn. The descent set of T, denoted $D(T)$, is the set of numbers i such that $i+1$ appears northwest of i in T. Equivalently, i is a descent if $i+1$ appears to the left of i in the reading word of T.

Example. Expand $S_{(3,2)}$ in the fundamental basis

4	5	
1	2	3

3	5	
1	2	4

3	4	
1	2	5

2	5	
1	3	4

2	4	
1	3	5

$$S_{(3,2)}(X) = F_{\{3\}}(X) + F_{\{2,4\}}(X) + F_{\{2\}}(X) + F_{\{1,4\}}(X) + F_{\{1,3\}}(X)$$

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summed over all standard tableaux T of shape λ.

Proof. Partition the set of column strict tableaux of shape λ according to their standardization. Given T, replace the 1's from left to right bijectively by $1, 2, \ldots, a$. Then replace the 2's by $a+1, a+2, \ldots, b$ from left to right. Then the 3's, etc. The result is a standard tableau $std(T)$ with the same shape and x^T is compatible with the descent set of T.

Macdonald Polynomials

Defn/Thm. (Macdonald 1988, Haiman-Haglund-Loehr, 2005)

$$\widetilde{H}_\mu(X;q,t) = \sum_{w \in S_n} q^{inv_\mu(w)} t^{maj_\mu(w)} F_{Des(w^{-1})}$$

where $Des(w)$ is the descent set of w in one-line notation.

Thm. (Haiman) Expanding $\widetilde{H}_\mu(X;q,t)$ into Schur functions

$$\widetilde{H}_\mu(X;q,t) = \sum_i \sum_j \sum_{|\lambda|=|\mu|} c_{i,j,\lambda} q^i t^j S_\lambda,$$

the coefficients $c_{i,j,\lambda}$ are all non-negative integers.

$\Longrightarrow$ Macdonald polynomials are *Schur positive*,

Open I. Find a "nice" combinatorial algorithm to compute $c_{i,j,\lambda}$ showing these are non-negative integers.

Lascoux-Leclerc-Thibon Polynomials

Defn. Let $\bar{\mu} = (\mu^{(1)}, \mu^{(1)}, \ldots, \mu^{(k)})$ be a list of partitions.

$$LLT_{\bar{\mu}}(X; q) = \sum q^{inv_{\mu}(T)} F_{Des(w^{-1})}$$

summed over all bijective fillings w of $\bar{\mu}$ where each $\mu^{(i)}$ filled with rows and columns increasing. Each w is recorded as the permutation given by the content reading word of the filling.

Thm. For all $\bar{\mu} = (\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(k)})$

1. $LLT_{\bar{\mu}}(X; q)$ is symmetric. (Lascoux-Leclerc-Thibon)

Lascoux-Leclerc-Thibon Polynomials

Open II. Find a "nice" combinatorial algorithm to compute the expansion coefficients for LLT's to Schurs.

Known. Each $\widetilde{H}_{\mu}(X; q, t)$ expands as a positive sum of LLT's so Open II implies Open I. (Haiman-Haglund-Loehr)

k-Schur Functions

Defn. (Lam-Lapointe-Morse-Shimozono + Lascoux, 2003-2010)

$$S_\lambda^{(k)}(X;q) = \sum_{S^* \in SST(\mu,k)} q^{\mathrm{spin}(S^*)} F_{Des(S^*)}.$$

Nice Properties.: Consider $\{S_\lambda^{(k)}(X;q=1)\}$

1. These are a Schubert basis for the homology ring of the affine Grassmannian of type A_k. (Lam)
2. Structure constants are related to Gromov-Witten invariants of flag manifolds (Lapointe-Morse,Peterson, Lam-Shimozono).
3. There exists a k-Schur analog the Murnaghan-Nakayma rule. (Bandlow-Schilling-Zabrocki)

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$$S_\lambda^{(k)}(X;q) = \sum_{S^* \in SST(\mu,k)} q^{\mathrm{spin}(S^*)} F_{D(S^*)}.$$

Nice Conjectures.: Consider $\{S_\lambda^{(k)}(X;q)\}$ with q an indeterminate

1. Macdonald polynomials expand as a positive sum of k-Schurs. (LLLMS)
2. LLT's expand as a positive sum of k-Schurs (Assaf-Haiman)

Schur Positivity of k-Schurs

Theorem. (Lam-Lapointe-Morse-Shimozono, 2011) At $q = 1$, $\{S_\lambda^{(k)}(X;1)\}$ is Schur positive. In fact, each k-Schur expands as a positive sum of $k+1$-Schurs.

Partial progress toward a positivity proof for indeterminate q in (Assaf-Billey 2012) and (Benedetti-Bergeron 2012)

Open III. Find a "nice" combinatorial algorithm to compute the expansion coefficients for k-Schurs to Schurs.