Algebraic Combinatorics
A tale of two rings: SYM and QSYM

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¡Gracias por la oportunidad de estar aquí y de compartir este tema con Uds. en un lugar tan lindo!

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My Philosophy

“Combinatorics is the nanotechnology of mathematics”

Photo Credit: Austin Roberts
Outline

Introduction to SYM and QSYM

Permutations, partitions, compositions, graphs, RSK

Bases for Symmetric Functions

Bases for Quasisymmetric Functions
Tale of Two Rings

**Power Series Ring.**: \( \mathbb{Z}[[X]] \) over a finite or countably infinite alphabet \( X = \{x_1, x_2, \ldots, x_n\} \) or \( X = \{x_1, x_2, \ldots\} \).

**Two subrings.** of \( \mathbb{Z}[[X]] \):

- Symmetric Functions (SYM)
- Quasisymmetric Functions (QSYM)
Defn. $f(x_1, x_2, \ldots) \in \mathbb{Z}[[X]]$ is a symmetric function if for all $i$

$$f(\ldots, x_i, x_{i+1}, \ldots) = f(\ldots, x_{i+1}, x_i, \ldots).$$

Example. $x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + \ldots$

Defn. $SYM = \text{symmetric functions of bounded degree.}$
QSYM = Ring of Quasisymmetric Functions

**Defn.** \( f(x_1, x_2, \ldots) \in \mathbb{Z}[[X]] \) is a **symmetric function** if for all \( i \)

\[
f(\ldots, x_i, x_{i+1}, \ldots) = f(\ldots, x_{i+1}, x_i, \ldots).
\]

**Example.** \( x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + \ldots \)

**Defn.** \( f(x_1, x_2, \ldots) \in \mathbb{Z}[[X]] \) is a **quasisymmetric function** if

\[
\text{coef}(f; x_1^{\alpha_1}x_2^{\alpha_2} \ldots x_k^{\alpha_k}) = \text{coef}(f; x_a^{\alpha_1}x_b^{\alpha_2} \ldots x_c^{\alpha_k})
\]

for all \( 1 < a < b < \cdots < c \).

**Example.** \( f(X) = x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + \ldots \)
Why study SYM and QSYM?

- Symmetric Functions (SYM): Used in representation theory, combinatorics, algebraic geometry, over past 200+ years. More recently expanding applications in number theory, theoretical physics, economics, quantum computing!


- SYM and QSYM are easy to study with Sage!
High level goals

1. Develop intuition for some of the universal tools in algebraic combinatorics.

2. Build up vocabulary to introduce some important open problems and approaches to attack them.

3. Inspire you to learn more about quasisymmetric functions and find more applications.
Main tool: Permutations

**Defn.** A *permutation* $w$ in the symmetric group $S_n$ is a bijection on the set $[n] = \{1, 2, \ldots n\}$.

**Fact.** $S_n$ is a group under composition of bijections with the identity function as the identity for the group.

**Example.** $w : [4] \rightarrow [4]$ given by

$w(1) = 2, w(2) = 3, w(3) = 4, w(4) = 1$

$w^{-1}(1) = 4, w^{-1}(2) = 1, w^{-1}(3) = 2, w^{-1}(4) = 3$

$id(1) = 1, id(2) = 2, id(3) = 3, id(4) = 4$
Some Applications of Permutations

- Card shuffling and card tricks.
- The determinant of a $n \times n$ matrix $M = [m_{ij}]$ is by definition
  \[
  \det(M) = \sum_{w \in S_n} (-1)^{\text{inv}(w)} m_{1,w(1)} m_{2,w(2)} \cdots m_{n,w(n)}.
  \]
- Cryptography.
- Differentiating species by DNA strings and phylogenetic trees.
- Detecting near duplicate webpages for search engines (Broder Algorithm).
- Symmetric functions and symmetric polynomials.
Six more ways to represent a permutation

\[
\begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
= \begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 1
\end{bmatrix} = [2, 3, 4, 1]
\]

- **Matrix notation**
- **Two-line notation**
- **One-line notation**

\[
* * * * . \quad 1234
\]
\[
. . . . \quad = \quad \begin{array}{c}
0 \\
1 \\
2 \\
3 \\
4
\end{array}
\quad = \quad s_1 s_2 s_3
\]

- **Diagram of a permutation** $D^*(w)$
- **String diagram**
- **Product of $s_i$'s adjacent transpositions**
Permutation Statistics

- \( \text{inv}(w) = \#\{i < j \mid w(j) < w(i)\} = \ell(w) \) Inversions

- \( \text{des}(w) = \#\{i \mid w(i) > w(i + 1)\} \) Descents

- \( \text{peaks}(w) = \#\{i \mid w(i - 1) < w(i) > w(j)\} \) Peaks

Example.
\( w = [2, 5, 4, 3, 6, 1] \implies \text{inv}(w) = 8, \text{des}(w) = 3, \text{peaks}(w) = 2 \)
Permutation Statistics

- \( \text{inv}(w) = \# \{ i < j : w(j) < w(i) \} = \ell(w) \) Inversions

- \( \text{des}(w) = \# \{ i : w(i) > w(i + 1) \} \) Descents

- \( \text{peaks}(w) = \# \{ i : w(i - 1) < w(i) > w(j) \} \) Peaks

Example.
\( w = [2, 5, 4, 3, 6, 1] \implies \text{inv}(w) = 8, \text{des}(w) = 3, \text{peaks}(w) = 2 \)

\( \text{Inv}(w) = \{(1, 5), (2, 3), (2, 4), (2, 6), (3, 4), (3, 6), (4, 6), (5, 6)\} \)
\( \text{Des}(w) = \{2, 3, 5\} \)
\( \text{Peaks}(w) = \{2, 5\} \)
Generating Functions by Example

**Defn.** Let $A_n(x) = \sum_{w \in S_n} x^{1 + \text{des}(w)} = \sum A(d, n)x^d$. The $A(d, n)$ are called *Eulerian numbers*.

- $A_2(x) = x + x^2$
- $A_3(x) = x + 4x^2 + x^3$
- $A_4(x) = x + 11x^2 + 11x^3 + x^4$
Defn. Let $A_n(x) = \sum_{w \in S_n} x^{1+\text{des}(w)} = \sum A(d, n)x^d$. The $A(d, n)$ are called *Eulerian numbers*.

\begin{align*}
A_2(x) &= x + x^2 \\
A_3(x) &= x + 4x^2 + x^3 \\
A_4(x) &= x + 11x^2 + 11x^3 + x^4
\end{align*}

Thm. (Holte 1997, Diaconis-Fulman 2009) The probability of carrying $d$ on in the $j$th column when adding $n$ large numbers tend to $A(d, n)/n!$. 

Enumerative Results

**Thm.** (Gessel-Viennot 1985) The number of permutations in $S_n$ with a given descent set $S = \{s_1, \ldots, s_k\}$ is given by the binomial determinant

$$\det \left[ \begin{array}{c} n - s_i \\ s_{j+1} - s_i \end{array} \right]_{1 \leq i \leq j \leq k}$$

where $s_0 = 0$, $s_{k+1} = n$.

**Thm.** (Billey-Burdzy-Sagan 2013) The number of permutations with a given peak set $S = \{s_1 < \ldots < s_k\}$ for $n \geq s_k$ is determined by $2^{n-|S|-1} P_S(n)$ for the peak polynomial $P_S(n)$.

See also: “Properties of Peak Polynomials” by Fahrbach and Talmage (manuscript 2014).
Monomial Basis of SYM

**Defn.** A *partition* of a number $n$ is a weakly decreasing sequence of positive integers

$$
\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)
$$

such that $n = \sum \lambda_i = |\lambda|$.

Partitions can be visualized by their *Ferrers diagram*

$(6, 5, 2) \rightarrow$

```
  +---+---+---+
  |   |   |   |
  +---+---+---+
  |   |   |   |
  +---+---+---+
  |   |   |   |
  +---+---+
  |   |
  +---+
```

**Defn/Thm.** The *monomial symmetric functions*

$$
m_\lambda = x_1^{\lambda_1}x_2^{\lambda_2} \cdots x_k^{\lambda_k} + x_2^{\lambda_1}x_1^{\lambda_2} \cdots x_k^{\lambda_k} + \text{all other perms of vars}
$$

form a basis for $SYM_n = \text{homogeneous symmetric functions of degree } n$.

**Fact.** $\dim SYM_n = p(n) = \text{number of partitions of } n$. 
More bases for SYM

Let $X = \{x_1, x_2, \ldots, x_m\}$ be the alphabet.

**Defn.** $e_k(X) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq m} x_{i_1} x_{i_2} \cdots x_{i_k}$.

For any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, set $e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}$.

These are *elementary symmetric functions*. 
More bases for SYM

Let $X = \{x_1, x_2, \ldots, x_m\}$ be the alphabet.

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For any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, set $e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}$.

These are *elementary symmetric functions*.

**Application.** If $F(x) = (x + r_1)(x + r_2) \cdots (x + r_m)$ is a polynomial with $m$ roots, then

$$F(x) = x^m + (r_1 + r_2 + \ldots + r_m)x^{m-1} + \ldots + (r_1 r_2 \cdots r_m)$$

$$= x^m + e_1(r_1, \ldots, r_m)x^{m-1} + \ldots + e_m(r_1, \ldots, r_m).$$
More bases for SYM

Let \( X = \{ x_1, x_2, \ldots, x_m \} \) be the alphabet.

**Defn.** \( e_k(X) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq m} x_{i_1} x_{i_2} \cdots x_{i_k} \).

For any partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \), set \( e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k} \).

These are *elementary symmetric functions*.

**Application.** If \( F(x) = (x + r_1)(x + r_2) \cdots (x + r_m) \) is a polynomial with \( m \) roots, then

\[
F(x) = x^m + (r_1 + r_2 + \ldots + r_m)x^{m-1} + \ldots + (r_1 r_2 \cdots r_m) = x^m + e_1(r_1, \ldots, r_m)x^{m-1} + \ldots + e_m(r_1, \ldots, r_m).
\]

**Fundamental Theorem.** \( SYM(X) = \mathbb{Q}[e_1, e_2, \ldots, e_m] \) as a freely generated polynomial ring.
More bases for SYM

**Defn.** \( h_k(X) = \sum_{1 \leq i_1 \leq i_2 < \ldots < i_k \leq m} x_{i_1} x_{i_2} \cdots x_{i_k} \). Set \( h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_k} \). These are *homogeneous symmetric functions*.

**Defn.** \( p_k(X) = x_1^k + x_2^k + x_3^k + \ldots + x_m^k \). Set \( p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k} \). These are *power symmetric functions*.
More bases for SYM

**Defn.** \( h_k(X) = \sum_{1 \leq i_1 \leq i_2 < \ldots < i_k \leq m} x_{i_1}x_{i_2} \cdots x_{i_k} \). Set \( h_\lambda := h_{\lambda_1}h_{\lambda_2} \cdots h_{\lambda_k} \). These are **homogeneous symmetric functions**.

**Defn.** \( p_k(X) = x_1^k + x_2^k + x_3^k + \ldots + x_m^k \). Set \( p_\lambda := p_{\lambda_1}p_{\lambda_2} \cdots p_{\lambda_k} \). These are **power symmetric functions**.

**Theorem.** \( SYM = \mathbb{Q}[e_1, e_2, \ldots] = \mathbb{Q}[h_1, h_2, \ldots] = \mathbb{Q}[p_1, p_2, \ldots] \) over the alphabet \( X = \{x_1, x_2, \ldots\} \).

**Cor.** \( SYM \) has three more bases \( \{e_\lambda\}, \{h_\lambda\}, \{p_\lambda\} \) where the bases range over all partitions when \( X \) is infinite.
Schur basis for SYM

Let $X = \{x_1, x_2, \ldots, x_m\}$ be a finite alphabet.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ and $\lambda_p = 0$ for $p > k$.

**Defn.** The following are equivalent definitions for the Schur functions $S_\lambda(X)$:

1. $S_\lambda = \frac{\det(x_i^{\lambda_j + m - j})}{\det(x_i^j)}$ with indices $1 \leq i,j \leq m$. 
   
**Example.** A column strict tableau of shape $(5,3,1)$

$$T = \begin{array}{ccc}
n & o & o \\
o & o & o \\
n & o & o \\
o & o & o \\
o & o & o
\end{array}$$

$$x_T = x_2x_4x_7x_2x_4x_3x_8$$
Schur basis for SYM

Let \( X = \{x_1, x_2, \ldots, x_m\} \) be a finite alphabet.

Let \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0) \) and \( \lambda_p = 0 \) for \( p > k \).

**Defn.** The following are equivalent definitions for the Schur functions \( S_\lambda(X) \):

1. \( S_\lambda = \frac{\det(x_i^{\lambda_j + m-j})}{\det(x_i^j)} \) with indices \( 1 \leq i, j \leq m \).
2. \( S_\lambda = \sum x^T \) summed over all column strict tableaux \( T \) of shape \( \lambda \).

**Defn.** \( T \) is *column strict* if entries strictly increase along columns and weakly increase along rows.

**Example.** A column strict tableau of shape \( (5, 3, 1) \)

\[
T = \begin{pmatrix}
  7 \\
  4 & 7 & 7 \\
  2 & 2 & 3 & 4 & 8 \\
\end{pmatrix}
\]

\( x^T = x_2^2 x_3 x_4^2 x_7^3 x_8 \)
Multiplying Schur Functions

**Littlewood-Richardson Coefficients.**

\[
S_\lambda(X) \cdot S_\mu(X) = \sum_{|\nu|=|\lambda|+|\mu|} c_{\lambda,\mu}^\nu S_\nu(X)
\]

\(c_{\lambda,\mu}^\nu = \# \text{ skew tableaux of shape } \nu/\lambda \text{ such that } x^T = x^\mu \text{ and the reverse reading word is a lattice word.}

**Example.** If \(\nu = (4, 3, 2)\), \(\lambda = (2, 1)\), \(\lambda = (3, 2, 1)\) then

\[
\begin{array}{ccc}
2 & 3 \\
1 & 2 \\
1 & 1 \\
\end{array}
\]

readingword = 231211
Schur Functions/Schur Polynomials

Special properties.

1. The graded ring of representations of $S_n$ for all $n > 0$ is isomorphic to $SYM$ on an infinite alphabet. The irreducible representations are indexed by partitions. The map sends the irreducible $V^\lambda$ to $S_\lambda$.

2. Schur polynomials are characters of irreducible $GL_n$ representations.

3. Schur polynomials represent the Schubert basis in the Grassmannian manifolds.
Quasisymmetric Functions

Nice Algebraic Facts.

▶ There is an analog of the Frobenius characteristic from symmetric function theory giving an isomorphism the Grothendieck group of representations of 0-Hecke algebras to QSYM. It maps the irreducible 0-Hecke algebra representation \( L_\alpha \) to \( F_\alpha \). (Duchamp-Krob-Leclerc-Thibon 1996)

▶ QSYM is Hopf dual to \( \text{NSYM} = \text{non commutative symmetric functions} \). (Malvenuto-Reutenauer 1995, Gelfand-Krob-Lascoux-Leclerc-Retakh-Thibon 1995)

▶ QSYM is free over \( \text{SYM} \) on \( n \) variables and \( \dim(\text{QSYM}(n)/\text{SYM}(n)) = n! \) (Garsia-Wallach 2003)

▶ The quotient of \( \mathbb{Z}[x_1, \ldots, x_n] \) mod quasisymmetric function with no constant term has Hilbert series \( \sum C_n t^n \) where \( C_n \) is the \( n \)-th Catalan number (Aval-Bergeron-Bergeron 2004)
Monomial Basis of QSYM

**Defn.** A *composition* of a number $n$ is a sequence of positive integers

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$$

such that $n = \sum \alpha_i = |\alpha|$.

**Defn/Thm.** The *monomial quasisymmetric functions*

$$M_\alpha = x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_k^{\alpha_k} + x_2^{\alpha_1}x_3^{\alpha_2} \cdots x_k^{\alpha_k} + \text{all other shifts}$$

form a basis for $QSYM_n = \text{homogeneous quasisymmetric functions of deg } n$.

**Fact.** $\dim QSYM_n = \text{number of compositions of } n = 2^{n-1}.$
Monomial Basis of QSYM

**Fact.** \(\dim QSYM_n = \text{number of compositions of } n = 2^{n-1}.\)

Bijection:

\[
(\alpha_1, \alpha_2, \ldots, \alpha_k) \mapsto \{\alpha_1, \\
\alpha_1 + \alpha_2, \\
\alpha_1 + \alpha_2 + \alpha_3, \\
\ldots \\
\alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}
\]
Asymptotic Formula: (Hardy-Ramanujan) The number of partitions of $n$, denoted $p(n)$, grows like

$$p(n) \approx \frac{1}{4n\sqrt{3}} e^{\pi \sqrt{\frac{2n}{3}}}$$
Fundamental basis for QSYM

**Defn.** Let $D \subset [p - 1] = \{1, 2, \ldots, p - 1\}$.

The fundamental quasisymmetric function

$$F_D(X) = \sum x_{i_1} \cdots x_{i_p}$$

summed over all $1 \leq i_1 \leq \ldots \leq i_p$ such that $i_j < i_{j+1}$ whenever $j \in D$.

**Example.** $F_{\{3\}} = x_1x_1x_1x_2x_2 + x_1x_2x_2x_3x_3 + x_1x_2x_3x_4x_5 + \ldots$

**Other bases of QSYM.** dual immaculate basis (Berg-Bergeron-Saliola-Serrano-Zabrocki), quasi Schur basis (Haglund-Luoto-Mason-vanWilligenburg), matroid friendly basis (Luoto)
A Poset on Partitions

**Defn.** A *partial order* or a *poset* is a reflexive, anti-symmetric, and transitive relation on a set.

**Defn.** *Young’s Lattice* on all partitions is the poset defined by the relation $\lambda \subset \mu$ if the Ferrers diagram for $\lambda$ fits inside the Ferrers diagram for $\mu$.

**Defn.** A *standard Young tableau* $T$ of shape $\lambda$ is a saturated chain in Young’s lattice from $\emptyset$ to $\lambda$.

**Example.** $T = \begin{array}{cccc} 7 \\ 4 & 5 & 9 \\ 1 & 2 & 3 & 6 & 8 \end{array}$
Thm. (Frame-Robinson-Thrall 1954) The number of standard tableaux of shape $\lambda \vdash n$, denoted $f^\lambda$, is given by the hook length formula:

$$f^\lambda = \frac{n!}{\prod_{(i,j) \in \lambda} h(i,j)}$$

where $h(i,j)$ is the hook length of the cell $c$ in the Ferrers diagram for $\lambda$ found by counting the number of cells above $c$ plus the number to the right of $c$ including itself.

See also the proof by Greene-Nijenhuis-Wilf (1979).
Thm. (Frame-Robinson-Thrall 1954) The number of standard tableaux of shape $\lambda \vdash n$, denoted $f^\lambda$, is given by the hook length formula:

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See also the proof by Greene-Nijenhuis-Wilf (1979).

Example. $f^{(3,2)} = \frac{5!}{1 \cdot 1 \cdot 2 \cdot 3 \cdot 4} = 5$  \[
\begin{array}{c}
\begin{array}{ccc}
2 & 1 \\
4 & 3 & 1
\end{array}
\end{array}
\]
Rep Theory Facts. (see Sagan’s book) The dimension of the $S_n$ irreducible representation $V^\lambda$ is $f^\lambda$. Hence

$$n! = \sum_{\lambda \vdash n} (f^\lambda)^2$$

because the the regular representation of $S_n$ decomposes as the direct sum of $\dim(V^\lambda)$ copies of $V^\lambda$.

Alternative Proof. RSK gives a bijection from $S_n$ to $\bigcup_{\lambda \vdash n} SYT^2_\lambda$. 
RSK = Robinson-Schensted-Knuth Bijection

**Input:** \( w \in S_n \)

**Output:** \((P, Q) \in \text{SYT}_\lambda\) for some shape \( \lambda \vdash n \).

**Start:** Set \( P = Q = \emptyset \).

**Step \( i \):** Insert \( w(i) \) into the first row of \( P \) by “bumping” the smallest value \( b > i \) from the first row if it exists or adding \( i \) to the end of the first row otherwise. If \( b \) exists, bump it into the second row, continuing until nothing is bumped. The result is the new tableau \( P \). Add a new cell containing \( i \) to \( Q \) in the same position as the new cell added to \( P \).

**Ex.** \( w = [1, 2, 6, 3, 5, 4] \mapsto P = \begin{array}{c}
6 \\
5 \\
1 \ 2 \ 3 \ 4
\end{array}, \quad Q = \begin{array}{c}
6 \\
4 \\
1 \ 2 \ 3 \ 5
\end{array} \)
Gessel’s formula for Schur functions

**Thm.** (Gessel, 1984) For all partitions $\lambda$,

$$S_\lambda(X) = \sum F_{D(T)}(X)$$

summed over all standard tableaux $T$ of shape $\lambda$.

**Defn.** The descent set of $T$, denoted $D(T)$, is the set of numbers $i$ such that $i + 1$ appears northwest of $i$ in $T$. Equivalently, $i$ is a descent if $i + 1$ appears to the left of $i$ in the reading word of $T$.

**Example.** Expand $S_{(3,2)}$ in the fundamental basis

$$S_{(3,2)}(X) = F_{\{3\}}(X) + F_{\{2,4\}}(X) + F_{\{2\}}(X) + F_{\{1,4\}}(X) + F_{\{1,3\}}(X)$$

\[
\begin{array}{cccc}
4 & 5 & & \\
1 & 2 & 3 & \\
3 & 5 & & \\
1 & 2 & 4 & \\
3 & 4 & & \\
1 & 2 & 5 & \\
2 & 5 & & \\
1 & 3 & 4 & \\
2 & 4 & & \\
1 & 3 & 5 & \\
\end{array}
\]
Gessel’s formula for Schur functions

**Thm.** (Gessel, 1984) For all partitions $\lambda$,

$$S_\lambda(X) = \sum F_{D(T)}(X)$$

summed over all standard tableaux $T$ of shape $\lambda$.

**Proof.** Partition the set of column strict tableaux of shape $\lambda$ according to their standardization. Given $T$, replace the 1’s from left to right bijectively by $1, 2, \ldots, a$. Then replace the 2’s by $a+1, a+2, \ldots, b$ from left to right. Then the 3’s, etc. The result is a standard tableau $std(T)$ with the same shape and $x^T$ is compatible with the descent set of $T$. 
Macdonald Polynomials

**Defn/Thm.** (Macdonald 1988, Haiman-Haglund-Loehr, 2005)

\[ \tilde{H}_\mu(X; q, t) = \sum_{w \in S_n} q^{inv_\mu(w)} t^{maj_\mu(w)} F_{Des(w^{-1})} \]

where \( Des(w) \) is the descent set of \( w \) in one-line notation.

**Thm.** (Haiman) Expanding \( \tilde{H}_\mu(X; q, t) \) into Schur functions

\[ \tilde{H}_\mu(X; q, t) = \sum_i \sum_j \sum_{|\lambda|=|\mu|} c_{i,j,\lambda} q^i t^j S_\lambda, \]

the coefficients \( c_{i,j,\lambda} \) are all non-negative integers.

\[ \Rightarrow \] Macdonald polynomials are **Schur positive**,

**Open I.** Find a “nice” combinatorial algorithm to compute \( c_{i,j,\lambda} \) showing these are non-negative integers.
Lascoux-Leclerc-Thibon Polynomials

**Defn.** Let \( \bar{\mu} = (\mu^{(1)}, \mu^{(1)}, \ldots, \mu^{(k)}) \) be a list of partitions.

\[
LLT_{\bar{\mu}}(X; q) = \sum q^{inv_{\mu}(T)} F_{Des(w^{-1})}
\]

summed over all bijective fillings \( w \) of \( \bar{\mu} \) where each \( \mu^{(i)} \) filled with rows and columns increasing. Each \( w \) is recorded as the permutation given by the content reading word of the filling.

**Thm.** For all \( \bar{\mu} = (\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(k)}) \)

1. \( LLT_{\bar{\mu}}(X; q) \) is symmetric. (Lascoux-Leclerc-Thibon)
Lascoux-Leclerc-Thibon Polynomials

**Open II.** Find a “nice” combinatorial algorithm to compute the expansion coefficients for $LLT$’s to Schurs.

**Known.** Each $\tilde{H}_\mu(X; q, t)$ expands as a positive sum of LLT’s so Open II implies Open I. (Haiman-Haglund-Loehr)
**k-Schur Functions**

**Defn.** (Lam-Lapointe-Morse-Shimozono + Lascoux, 2003-2010)

\[ S^{(k)}_{\lambda}(X; q) = \sum_{S^* \in SST(\mu, k)} q^{\text{spin}(S^*)} F_{\text{Des}}(S^*). \]

**Nice Properties.** Consider \( \{ S^{(k)}_{\lambda}(X; q = 1) \} \)

1. These are a Schubert basis for the homology ring of the affine Grassmannian of type \( A_k \). (Lam)

2. Structure constants are related to Gromov-Witten invariants of flag manifolds (Lapointe-Morse, Peterson, Lam-Shimozono).

3. There exists a \( k \)-Schur analog the Murnaghan-Nakayama rule. (Bandlow-Schilling-Zabrocki)
$k$-Schur Functions

**Defn.** (Lam-Lapointe-Morse-Shimozono + Lascoux, 2003-2010)

$$S^{(k)}_{\lambda}(X; q) = \sum_{S^* \in \text{SST}(\mu, k)} q^{\text{spin}(S^*)} F_D(S^*).$$

**Nice Conjectures.** Consider \( \{S^{(k)}_{\lambda}(X; q)\} \) with \( q \) an indeterminate

1. Macdonald polynomials expand as a positive sum of \( k \)-Schurs. (LLLMS)
2. LLT’s expand as a positive sum of \( k \)-Schurs (Assaf-Haiman)
Schur Positivity of $k$-Schurs

**Theorem.** (Lam-Lapointe-Morse-Shimozono, 2011) At $q = 1$, \( \{ S^{(k)}_\lambda(X; 1) \} \) is Schur positive. In fact, each $k$-Schur expands as a positive sum of $k + 1$-Schurs.

Partial progress toward a positivity proof for indeterminate $q$ in (Assaf-Billey 2012) and (Benedetti-Bergeron 2012)

**Open III.** Find a “nice” combinatorial algorithm to compute the expansion coefficients for $k$-Schurs to Schurs.