

An explicit formula for the characters of the symmetric group

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Abstract

We give an explicit expression of the normalized characters of the symmetric group in terms of the “contents” of the partition labelling the representation.

1 Introduction

The characters of the irreducible representations of the symmetric group play an important role in many areas of mathematics. However, since the early work of Frobenius [5] in 1900, no explicit formula was found for them. The characters of the symmetric group were computed through various recursive algorithms, but explicit formulas were only known for about ten particular cases [5, 9]. The purpose of this paper is to give such an explicit expression in the general case.

The irreducible representations of the symmetric group S_n of n letters are labelled by partitions λ of n (i.e. weakly decreasing sequences of positive integers summing to n). Their characters χ^λ are evaluated at a conjugacy class of S_n , labelled by a partition μ giving the cycle-type of the class. Let χ_μ^λ be the value of the character χ^λ at a permutation of cycle-type μ . We shall give an explicit formula for the normalized character $\hat{\chi}_\mu^\lambda = \chi_\mu^\lambda / \dim \lambda$. This result was announced in [20].

It should be first emphasized that our formula gives the dependence of $\hat{\chi}_\mu^\lambda$ with respect to λ in terms of the “contents” of this partition. More precisely the normalized character $\hat{\chi}_\mu^\lambda$ is expressed as some (unique) symmetric function evaluated on the contents of λ .

This description of characters by content evaluation was proved in [13] and [3]. Previously the importance of contents had been apparent from the works of Jucys [11] and Murphy [24]. The fact had been noticed by Suzuki [29], Lascoux and Thibon [15] and Garsia [6].

Later tables and conjectures were independently given by Katriel [12] and by the author [16, Sections 8-11]. Actually the conjectures of [16] were formulated in the framework of Jack polynomials. But, as emphasized in Section 11 of [16], once specialized to $\alpha = 1$, they correspond to the characters of the symmetric group.

However the symmetric function expressing $\hat{\chi}_\mu^\lambda$ remained quite obscure, even in the very elementary situation of a partition μ having only one non-unary part. The purpose of this paper is to give an explicit expression.

It is a second remarkable fact that this symmetric function can only be written by using a new family of positive integers, which we have introduced in [18]. The connection of these integers with the symmetric group is still mysterious and certainly needs more investigation.

We emphasize that our method provides a very efficient algorithm, implemented on computer. Tables giving $\hat{\chi}_\mu^\lambda$ for $|\mu| - l(\mu) \leq 12$ will be available on a web page [21].

2 Notations

We briefly recall some basic notions about the characters of the symmetric group, referring the reader to [7] and to [22, Section 1.7] for an elementary introduction.

2.1 Characters

A partition $\lambda = (\lambda_1, \dots, \lambda_r)$ is a finite weakly decreasing sequence of nonnegative integers, called parts. The number $l(\lambda)$ of positive parts is called the length of λ , and $|\lambda| = \sum_{i=1}^r \lambda_i$ the weight of λ . For any integer $i \geq 1$, $m_i(\lambda) = \text{card}\{j : \lambda_j = i\}$ is the multiplicity of the part i in λ . Clearly $l(\lambda) = \sum_{i \geq 1} m_i(\lambda)$ and $|\lambda| = \sum_{i \geq 1} i m_i(\lambda)$. We shall also write $\lambda = (1^{m_1}, 2^{m_2}, 3^{m_3}, \dots)$. We set

$$z_\lambda = \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)!.$$

We identify λ with its Ferrers diagram $\{(i, j) : 1 \leq i \leq l(\lambda), 1 \leq j \leq \lambda_i\}$.

Let n be a fixed positive integer and S_n the group of permutations of n letters. Each permutation $\sigma \in S_n$ factorizes uniquely as a product of disjoint cycles, whose respective lengths are ordered such as to form a partition $\mu = (\mu_1, \dots, \mu_r)$ with weight n . This partition is called the cycle-type of σ and determines each permutation up to conjugacy in S_n . Conjugacy classes are thus labelled by partitions μ with $|\mu| = n$.

The irreducible representations of S_n and their corresponding characters are also labelled by partitions λ with weight $|\lambda| = n$. We write χ_μ^λ for the value of the character $\chi^\lambda(\sigma)$ at any permutation σ of cycle-type μ .

The dimension $\dim \lambda = \chi_{1^n}^\lambda$ of the representation λ is well known, see [22, Example 1.7.6, p. 116] or [7, p. 54]. We have

$$\dim \lambda = \frac{n!}{\prod_{i=1}^{l(\lambda)} (\lambda_i + l(\lambda) - i)!} \prod_{1 \leq i < j \leq l(\lambda)} (\lambda_i - \lambda_j + j - i).$$

We write $\hat{\chi}_\mu^\lambda = \chi_\mu^\lambda / \dim \lambda$ for the corresponding normalized character.

2.2 Symmetric functions

Let $A = \{a_1, a_2, a_3, \dots\}$ a (possibly infinite) set of independent indeterminates, called an alphabet. The generating functions

$$E_z(A) = \prod_{a \in A} (1 + za) = \sum_{k \geq 0} z^k e_k(A), \quad H_z(A) = \prod_{a \in A} (1 - za)^{-1} = \sum_{k \geq 0} z^k h_k(A)$$

define symmetric functions known as respectively elementary and complete. The power sum symmetric functions are defined by $p_k(A) = \sum_{i \geq 1} a_i^k$. For any partition μ , we define functions e_μ , h_μ or p_μ by

$$f_\mu = \prod_{i=1}^{l(\mu)} f_{\mu_i} = \prod_{k \geq 1} f_k^{m_k(\mu)},$$

where f_i stands for e_i , h_i or p_i .

When A is infinite, each of the three sets of functions e_i , h_i or p_i forms an algebraic basis of \mathcal{S} , the symmetric algebra with coefficients in \mathbb{R} . Each of the sets of functions e_μ , h_μ , p_μ is a linear basis of this algebra.

Another linear basis is formed by the Schur functions s_λ , which are defined by the Jacobi-Trudi formula

$$s_\lambda = \det_{1 \leq i, j \leq l} [h_{\lambda_i - i + j}],$$

with $h_i = 0$ for $i < 0$. This definition is usually written for a partition λ with length l . However it remains valid when λ is replaced by any sequence of integers $\lambda \in \mathbb{Z}^l$, not necessarily in descending order. Then using the obvious rule

$$s_{\dots, a, b, \dots} = -s_{\dots, b-1, a+1, \dots}$$

for $b > a$, it is easily seen that s_λ is either 0, either equal to $\pm s_\mu$, with μ a partition.

2.3 Shifted symmetric functions

Although the theory of symmetric functions goes back to the early 19th century, shifted symmetric functions are quite recent. They were introduced and studied in [13, 26].

Being given a finite alphabet $A = \{a_1, a_2, \dots, a_r\}$, a polynomial in A is “shifted symmetric” if it is symmetric in the shifted variables $a_i - i$. When $A = \{a_1, a_2, a_3, \dots\}$ is infinite, in analogy with symmetric functions, a “shifted symmetric function” f is a family $\{f_i, i \geq 1\}$ such that f_i is shifted symmetric in (a_1, a_2, \dots, a_i) , together with the stability property $f_{i+1}(a_1, a_2, \dots, a_i, 0) = f_i(a_1, a_2, \dots, a_i)$.

This defines \mathcal{S}^* , the shifted symmetric algebra with coefficients in \mathbb{R} , which is algebraically generated by the “shifted power sums”

$$p_k^*(A) = \sum_{i \geq 1} \left((a_i - i + 1)_k - (-i + 1)_k \right).$$

Here for an indeterminate z and any positive integer p , the *lowering factorial*

$$(z)_p = z(z-1)\dots(z-p+1) = \sum_{i=1}^p s(p,i) z^i,$$

is the generating function of the Stirling numbers of the first kind $s(p,i)$. Conversely

$$z^p = \sum_{i=1}^p S(p,i)(z)_i$$

is the generating function of the Stirling numbers of the second kind $S(p,i)$.

An element $f \in \mathcal{S}^*$ may be evaluated at any sequence (a_1, a_2, \dots) with finitely many non zero terms, hence at any partition λ . Moreover by analyticity, f is entirely determined by its restriction $f(\lambda)$ to partitions. This identification is usually performed and \mathcal{S}^* is considered as a function algebra on the set of partitions.

2.4 Contents

Given a partition λ , the “content” of any node $(i, j) \in \lambda$ is defined as $j - i$. Denote $A_\lambda = \{j - i, (i, j) \in \lambda\}$ the finite alphabet of the contents of λ . The symmetric algebra $\mathcal{S}[A_\lambda]$ is generated by the power sums

$$p_k(A_\lambda) = \sum_{(i,j) \in \lambda} (j-i)^k = \sum_{i=1}^{l(\lambda)} \sum_{j=1}^{\lambda_i} (j-i)^k.$$

It is well known [13, 26] that the quantities $p_k(A_\lambda)$ are shifted symmetric polynomials of λ . Indeed for any integer $k \geq 1$, applying the identity $r(z)_{r-1} = (z+1)_r - (z)_r$, we have

$$\begin{aligned} p_k(A_\lambda) &= \sum_{r=1}^k \sum_{(i,j) \in \lambda} S(k,r) (j-i)_r \\ &= \sum_{r=1}^k \frac{S(k,r)}{r+1} \sum_{i=1}^{l(\lambda)} \left((\lambda_i - i + 1)_{r+1} - (-i + 1)_{r+1} \right), \\ &= \sum_{r=1}^k \frac{S(k,r)}{r+1} p_{r+1}^*(\lambda). \end{aligned}$$

Hence the statement. For instance we have

$$p_1(A_\lambda) = \frac{1}{2} p_2^*(\lambda), \quad p_2(A_\lambda) = \frac{1}{3} p_3^*(\lambda) + \frac{1}{2} p_2^*(\lambda), \quad p_3(A_\lambda) = \frac{1}{4} p_4^*(\lambda) + p_3^*(\lambda) + \frac{1}{2} p_2^*(\lambda).$$

As a straightforward consequence, the shifted symmetric algebra \mathcal{S}^* is algebraically generated by the functions $p_k(A_\lambda)$, $k \geq 1$ together with $p_1^*(\lambda) = |\lambda|$. The latter corresponds to the cardinal of the alphabet A_λ .

Any shifted symmetric function may be written $f(A_\lambda)$, with $f \in \mathbb{R}[\text{card}, p_1, p_2, p_3, \dots]$. Moreover this expression is unique. From now on we shall abbreviate

$$p_k(\lambda) := p_k(A_\lambda), \quad p_\mu(\lambda) := p_\mu(A_\lambda).$$

This notation will not bring any confusion with the power sum $p_k(\lambda_1, \lambda_2, \dots) = \sum_{i \geq 1} \lambda_i^k$, which is never used in the sequel.

2.5 Murnaghan rule

The transition matrices between Schur functions and power sums are given by the Frobenius formulas

$$s_\lambda = \sum_{\mu} z_\mu^{-1} \chi_\mu^\lambda p_\mu, \quad p_\mu = \sum_{\lambda} \chi_\mu^\lambda s_\lambda.$$

Both formulas remain valid when the partition λ , with length l , is replaced by any sequence of integers $\lambda \in \mathbb{Z}^l$. This allows to define a generalized (or virtual) character χ^λ for any sequence of integers $\lambda \in \mathbb{Z}^l$.

For such λ , given some positive integer r , we have [22, Example 1.3.11, p. 48]

$$p_r s_\lambda = \sum_{i=1}^{l+1} s_{\lambda+r\epsilon_i},$$

with ϵ_i the sequence having 1 in the i -th place and 0 elsewhere.

The following recurrence property is a straightforward consequence. Given a partition λ with length $l(\lambda)$, and two integers k, p with $k \leq l(\lambda)$, let $\lambda - p\epsilon_k$ be the multi-integer $(\lambda_1, \dots, \lambda_{k-1}, \lambda_k - p, \lambda_{k+1}, \dots, \lambda_{l(\lambda)}) \in \mathbb{Z}^{l(\lambda)}$.

Murnaghan rule. *Let λ, μ be two partitions with weight n . Let p be some part of μ , and $\mu \setminus p$ the partition obtained by subtracting p from μ . Then we have*

$$\chi_\mu^\lambda = \sum_{k=1}^{l(\lambda)} \chi_{\mu \setminus p}^{\lambda - p\epsilon_k}.$$

In general the quantities $\chi^{\lambda - p\epsilon_k}$ appearing on the right-hand side are not characters, but virtual characters. In the usual formulation of the rule [22, Example 1.7.5, p. 117], these virtual characters are expressed in terms of characters. This is Nakayama's version [25] of Murnaghan's formula [23]. Our method has the advantage of making this operation totally unnecessary.

We shall need a formulation of the previous rule in terms of normalized characters. Given a partition λ with weight n and length $l(\lambda)$, and two integers k, p with $k \leq l(\lambda)$, we write

$$\begin{aligned} d_\lambda(k, p) &= \frac{n!}{(n-p)!} \frac{\dim(\lambda - p\epsilon_k)}{\dim \lambda} \\ &= \frac{(\lambda_k + l(\lambda) - k)!}{(\lambda_k + l(\lambda) - k - p)!} \prod_{\substack{i=1 \\ i \neq k}}^{l(\lambda)} \frac{\lambda_k - \lambda_i + i - k - p}{\lambda_k - \lambda_i + i - k}. \end{aligned}$$

The Murnaghan rule then writes

$$(n)_p \hat{\chi}_\mu^\lambda = \sum_{k=1}^{l(\lambda)} d_\lambda(k, p) \hat{\chi}_{\mu \setminus p}^{\lambda - p\epsilon_k}.$$

This recurrence relation is our first ingredient for the computation of $\hat{\chi}_\mu^\lambda$.

3 Lagrange interpolation

Our second ingredient is Lagrange interpolation, written under the following form [14]. For any two alphabets A and B , their difference $A - B$ (which is not their difference as sets) is defined by

$$H_z(A - B) = H_z(A) H_z(B)^{-1} = \frac{\prod_{b \in B} (1 - zb)}{\prod_{a \in A} (1 - za)}.$$

Alain Lascoux [14] mentions that when B is empty, the following result was already known to Euler.

Lagrange lemma. *Let A and B be two finite alphabets with respective cardinals m and n . For any integer $r \geq 0$ we have*

$$\sum_{a \in A} a^r \frac{\prod_{b \in B} (a - b)}{\prod_{c \in A, c \neq a} (a - c)} = h_{n-m+r+1}(A - B).$$

Given a partition λ , let r be the number of nodes in the main diagonal of its Ferrers diagram, and α_i (resp. β_i) be the number of nodes in the i -th row (resp. column) on the right of (resp. below) the node (i, i) . The couples (α_i, β_i) , $i = 1 \dots r$ are known as Frobenius coordinates.

The ‘‘Frobenius function’’ is defined by

$$F(z; \lambda) = \prod_{i=1}^r \frac{z - \alpha_i}{z + \beta_i + 1} = \prod_{i \geq 1} \frac{z - \lambda_i + i}{z + i}.$$

The equality goes back to Frobenius [5], see [22, Example 1.1.15, p. 17] or [10].

For any indeterminate z and positive integer p , we consider the function

$$(z)_p \frac{F(z-p; \lambda)}{F(z; \lambda)} = (z)_p \prod_{i=1}^{l(\lambda)} \frac{z - \lambda_i + i - p}{z + i - p} \frac{z + i}{z - \lambda_i + i},$$

and its expansion in descending powers of z , i.e. its Taylor series at infinity,

$$(z)_p \frac{F(z-p; \lambda)}{F(z; \lambda)} = \sum_{r \geq -p} C_r(\lambda; p) z^{-r}.$$

Theorem 1. *For any $r \geq 0$ we have*

$$C_{r+1}(\lambda; p) = -p \sum_{k=1}^{l(\lambda)} d_\lambda(k, p) (\lambda_k - k)^r.$$

Remark 1. For $r = 0$, this result is due to Frobenius [5], see [22, Example 1.7.7, p. 118]. The cases $r = 1, 2$ were investigated by Ingram [9].

Proof. We apply the Lagrange lemma for the two following alphabets

$$\begin{aligned} A &= \{a_i = \lambda_i - i, \quad i = 1, \dots, l(\lambda)\}, \\ B &= \{b_i = \lambda_i - i + p, \quad i = 1, \dots, l(\lambda) + p\} \end{aligned}$$

We have

$$\begin{aligned} H_z(A - B) &= \prod_{i=1}^{l(\lambda)} \frac{1 - z(\lambda_i - i + p)}{1 - z(\lambda_i - i)} \prod_{i=1}^p (1 + z(l(\lambda) - p + i)) \\ &= \frac{F(1/z - p; \lambda)}{F(1/z; \lambda)} \prod_{i=1}^{l(\lambda)} \frac{1 + z(i - p)}{1 + zi} \prod_{i=1}^p (1 + z(l(\lambda) - p + i)) \\ &= \frac{F(1/z - p; \lambda)}{F(1/z; \lambda)} \prod_{i=1}^p (1 + z(i - p)) \\ &= z^p (1/z)_p \frac{F(1/z - p; \lambda)}{F(1/z; \lambda)}. \end{aligned}$$

Therefore

$$(z)_p \frac{F(z-p; \lambda)}{F(z; \lambda)} = z^p H_{1/z}(A - B) = \sum_{k \geq 0} z^{p-k} h_k(A - B) = \sum_{r \geq -p} z^{-r} h_{r+p}(A - B).$$

On the other hand, it is obvious that

$$-p d_\lambda(k, p) = \frac{\prod_{b \in B} (a_k - b)}{\prod_{c \in A, c \neq a_k} (a_k - c)},$$

and for any $r \geq 0$ the Lagrange lemma writes

$$-p \sum_{k=1}^{l(\lambda)} d_\lambda(k, p) (\lambda_k - k)^r = h_{r+p+1}(A - B).$$

Hence the statement. □

4 Explicit series expansion

Our third ingredient is a formula giving explicitly the Taylor series at infinity of

$$(z)_p \frac{F(z - p; \lambda)}{F(z; \lambda)}.$$

Some preliminary results are necessary.

4.1 Positive integers

Let n, p, k be three integers with $0 \leq p \leq n$ and $k \geq 1$. Define

$$\binom{n}{p}_k = \frac{n}{k} \sum_{r \geq 0} \binom{p}{r} \binom{n-p}{r} \binom{n-r-1}{k-r-1}.$$

We have obviously

$$\binom{n}{p}_k = 0 \quad \text{for } k > n, \quad \binom{n}{p}_1 = n, \quad \binom{n}{p}_k = \binom{n}{n-p}_k.$$

These numbers generalize the classical binomial coefficients, since we have

$$\binom{n}{0}_k = \binom{n}{k}, \quad \binom{n}{1}_k = k \binom{n}{k}, \quad \binom{n}{p}_n = \binom{n}{p},$$

the last property being a direct consequence of the classical Chu-Vandermonde formula.

The numbers $\binom{n}{p}_k$ were studied in [18]. It was proved that they are *positive integers*, and their generating function

$$G_n(y, z) = \sum_{p=0}^n \sum_{k=1}^n \binom{n}{p}_k y^p z^k$$

was shown to be [31]

$$G_n(y, z) = 2^{-n} \left((1+y)(1+z) + \sqrt{(1+y)^2(1+z)^2 - 4y(1+z)} \right)^n \\ + 2^{-n} \left((1+y)(1+z) - \sqrt{(1+y)^2(1+z)^2 - 4y(1+z)} \right)^n - 1 - y^n.$$

4.2 Extension to partitions

For any integers $0 \leq p \leq |\lambda|$ and $k \geq 1$, we define

$$\left\langle \begin{matrix} \lambda \\ p \end{matrix} \right\rangle_k = \sum_{(p_i)} \sum_{(k_i)} \prod_{i=1}^{l(\lambda)} \binom{\lambda_i}{p_i}_{k_i},$$

the sum being taken over all decompositions $p = \sum_{i=1}^{l(\lambda)} p_i$, $k = \sum_{i=1}^{l(\lambda)} k_i$ with $0 \leq p_i \leq \lambda_i$ and $k_i \neq 0$ for any i . Observe that there is no such restriction for p_i .

This definition yields easily

$$\left\langle \begin{matrix} \lambda \\ p \end{matrix} \right\rangle_k = 0 \quad \text{except if } l(\lambda) \leq k \leq |\lambda|.$$

Indeed it is obvious that $\left\langle \begin{matrix} \lambda \\ p \end{matrix} \right\rangle_k = 0$ for $k < l(\lambda)$, and since $\binom{n}{p}_k = 0$ for $k > n$, we have also $\left\langle \begin{matrix} \lambda \\ p \end{matrix} \right\rangle_k = 0$ for $k > |\lambda|$.

For instance $\left\langle \begin{matrix} \lambda \\ p \end{matrix} \right\rangle_1 = 0$ except if λ is a row partition (n) , in which case $\left\langle \begin{matrix} (n) \\ p \end{matrix} \right\rangle_k = \binom{n}{p}_k$. We have easily

$$\left\langle \begin{matrix} \lambda \\ 1 \end{matrix} \right\rangle_k = k \left\langle \begin{matrix} \lambda \\ 0 \end{matrix} \right\rangle_k, \quad \left\langle \begin{matrix} \lambda \\ p \end{matrix} \right\rangle_k = \left\langle \begin{matrix} \lambda \\ |\lambda| - p \end{matrix} \right\rangle_k, \quad \left\langle \begin{matrix} \lambda \\ p \end{matrix} \right\rangle_{|\lambda|} = \binom{|\lambda|}{p}.$$

As a direct consequence of their definition, the generating function for the positive integers $\left\langle \begin{matrix} \lambda \\ p \end{matrix} \right\rangle_k$ is the following

$$\sum_{p=0}^{|\lambda|} \sum_{k=l(\lambda)}^{|\lambda|} \left\langle \begin{matrix} \lambda \\ p \end{matrix} \right\rangle_k y^p z^k = \prod_{i=1}^{l(\lambda)} G_{\lambda_i}(y, z) = \prod_{i=1}^{l(\lambda)} \left(G_i(y, z) \right)^{m_i(\lambda)}.$$

4.3 New symmetric functions

For any integers $n \geq 1$, $k \geq 1$ and $0 \leq p \leq n$, we define the symmetric function

$$F_{npk} = \sum_{|\mu|=n} \frac{\left\langle \begin{matrix} \mu \\ p \end{matrix} \right\rangle_k}{z_\mu} p_\mu.$$

Since $\left\langle \begin{matrix} \mu \\ p \end{matrix} \right\rangle_k = 0$ for $k < l(\mu)$, this sum is restricted to partitions with $l(\mu) \leq k$. Similarly since $\left\langle \begin{matrix} \mu \\ p \end{matrix} \right\rangle_k = 0$ for $k > |\mu|$, one has $F_{npk} = 0$ for $k > n$.

For $k = 0$ the previous definition is extended by the convention $F_{np0} = 0$ with the only exception $F_{000} = 1$. For $k = 1$ and any $p \leq n$ we have $F_{np1} = p_n$. For $k = n$ we obtain

$$F_{npn} = \binom{n}{p} \sum_{|\mu|=n} \frac{p_\mu}{z_\mu} = \binom{n}{p} F_{n0n} = \binom{n}{p} h_n,$$

where we have used [22, p. 25]. We have also $F_{n1k} = k F_{n0k}$ and $F_{npk} = F_{n,n-p,k}$.

From now on we abbreviate $F_{npk}(\lambda) := F_{npk}(A_\lambda)$, the value of the symmetric function F_{npk} on the alphabet A_λ . As a consequence of Section 2.4, it is a shifted symmetric function of λ .

4.4 Taylor expansion

For any partition λ the “content polynomial” of λ is defined by

$$C_\lambda(z) = \prod_{(i,j) \in \lambda} (z + j - i).$$

Since

$$\frac{C_\lambda(z)}{C_\lambda(z-1)} = \prod_{i=1}^{l(\lambda)} \prod_{j=1}^{\lambda_i} \frac{z + j - i}{z + j - i - 1} = \prod_{i=1}^{l(\lambda)} \frac{z + \lambda_i - i}{z - i},$$

we have

$$\frac{F(z-p; \lambda)}{F(z; \lambda)} = \frac{C_\lambda(-z+p)}{C_\lambda(-z+p-1)} \frac{C_\lambda(-z-1)}{C_\lambda(-z)}.$$

Given two indeterminates x, y , the expansion of

$$\frac{C_\lambda(x-y+1)}{C_\lambda(x-y)} \frac{C_\lambda(-y)}{C_\lambda(-y+1)} = -x \sum_{r \geq 0} c_r^\lambda(x) y^{-r}$$

in descending powers of y was explicitly obtained in [19, Corollary 5.2, p. 3464], in the more general context of “ α -contents”. The coefficients $c_r^\lambda(x)$ are given by

$$c_r^\lambda(x) = \sum_{\substack{i,j,m \geq 0 \\ 2m+i+j \leq r}} (-x)^{m-1} (x+1)^i \binom{m+i+j-1}{i} \left(\sum_{k=0}^{\min(m,r-2m-i)} \binom{|\lambda|+m-1}{m-k} F_{r-2m-i,j,k}(\lambda) \right).$$

We have $c_0^\lambda(x) = -1/x$ since $F_{000}(\lambda) = 1$, and $c_1^\lambda(x) = 0$ since $F_{1p0}(\lambda) = 0$. With $\hat{x} = x + 1$, first values are given by

$$\begin{aligned} c_2^\lambda(x) &= |\lambda|, & c_3^\lambda(x) &= 2p_1(\lambda) + \hat{x}|\lambda|, \\ c_4^\lambda(x) &= 3p_2(\lambda) + 3\hat{x}p_1(\lambda) - x \binom{|\lambda|+1}{2} + \hat{x}^2|\lambda|, \\ c_5^\lambda(x) &= 4p_3(\lambda) + 6\hat{x}p_2(\lambda) - x(|\lambda|+1) \left(2p_1(\lambda) + \hat{x}|\lambda| \right) + 4\hat{x}^2p_1(\lambda) + \hat{x}^3|\lambda|. \end{aligned}$$

From Section 2.4 it is obvious that any $c_r^\lambda(x)$ is a shifted symmetric function of λ . These auxiliary functions will be our main tool in the sequel.

Theorem 2. *We have the following Taylor series at infinity*

$$(z)_p \frac{F(z-p; \lambda)}{F(z; \lambda)} = -p \sum_{r,s,t \geq 0} (-1)^s \binom{r+s-1}{s} s(p, t) c_r^\lambda(p) z^{t-r-s}.$$

Proof. We have

$$\frac{F(z-p; \lambda)}{F(z; \lambda)} = -p \sum_{r \geq 0} c_r^\lambda(p) (1+z)^{-r} = -p \sum_{r, s \geq 0} (-1)^s \binom{r+s-1}{s} c_r^\lambda(p) z^{-r-s}.$$

□

The following result asserts that some *rational* function of λ (defined at the left-hand side) is actually a shifted symmetric *polynomial*.

Theorem 3. *For any $r \geq 0$ we have*

$$\sum_{k=1}^{l(\lambda)} d_\lambda(k, p) (\lambda_k - k)^r = (-1)^r \sum_{i, j \geq 0} (-1)^j \binom{r}{j} s(p+1, i-j) c_i^\lambda(p).$$

Proof. By Theorem 1 the left-hand side is the coefficient of z^{-r-1} in the Taylor series expansion of $(-1/p)(z)_p F(z-p; \lambda)/F(z; \lambda)$. By Theorem 2 this coefficient evaluates as

$$\sum_{i, j \geq 0} (-1)^j \binom{i+j-1}{j} s(p, i+j-r-1) c_i^\lambda(p)$$

But we have

$$\sum_{j \geq 0} (-1)^j \binom{i+j-1}{i-1} s(p, i+j-r-1) = (-1)^r \sum_{k \geq 0} (-1)^k \binom{r}{k} s(p+1, i-k).$$

This is a direct consequence of the Chu-Vandermonde identity

$$\binom{i+j-1}{i-1} = \sum_{k \geq 0} \binom{r}{k} \binom{i+j-r-1}{i-k-1},$$

together with the easy $r = 0$ case

$$\sum_{m \geq 0} \binom{m-1}{k-1} |s(p, m-1)| = |s(p+1, k)|.$$

□

5 One non-unary cycle

Let $|\lambda| = n$, ρ a partition with $|\rho| \leq n$ and $\mu = (\rho, 1^{n-|\rho|})$. A fundamental result [30, 13, 10] asserts that the normalized character $(n)_{|\rho|} \hat{\chi}_\mu^\lambda$ is a shifted symmetric function of λ . Our purpose is to express this character in terms of the auxiliary shifted symmetric functions $c_r^\lambda(p)$ previously introduced.

The proof is done by recurrence over the number of parts of μ distinct from 1, i.e. the number of non-unary cycles of permutations having cycle-type μ .

We begin with permutations which are p -cycles, i.e. having only one cycle of length p and all other cycles unary.

Theorem 4. For $\mu = (p, 1^{n-p})$ we have

$$(n)_p \hat{\chi}_\mu^\lambda = \sum_{i \geq 2} s(p+1, i) c_i^\lambda(p).$$

Proof. By the Murnaghan rule, we have

$$(n)_p \hat{\chi}_\mu^\lambda = \sum_{k=1}^{l(\lambda)} d_\lambda(k, p).$$

We apply Theorem 3 with $r = 0$. □

For $p = 1$ the sum is restricted to $i = 2$ and we recover $\hat{\chi}_{1^n}^\lambda = 1$. Frobenius [5] computed the three cases $2 \leq p \leq 4$, Suzuki [29] the cases $p = 2, 3$ and Ingram [9] the case $p = 5$. Their results may be recovered as follows :

$$\begin{aligned} (n)_2 \hat{\chi}_{2, 1^{n-2}}^\lambda &= c_3^\lambda(2) - 3c_2^\lambda(2) = 2p_1(\lambda) \\ (n)_3 \hat{\chi}_{3, 1^{n-3}}^\lambda &= c_4^\lambda(3) - 6c_3^\lambda(3) + 11c_2^\lambda(3) = 3p_2(\lambda) - 3 \binom{n}{2} \\ (n)_4 \hat{\chi}_{4, 1^{n-4}}^\lambda &= c_5^\lambda(4) - 10c_4^\lambda(4) + 35c_3^\lambda(4) - 50c_2^\lambda(4) = 4p_3(\lambda) - 4(2n-3)p_1(\lambda) \\ (n)_5 \hat{\chi}_{5, 1^{n-5}}^\lambda &= c_6^\lambda(5) - 15c_5^\lambda(5) + 85c_4^\lambda(5) - 225c_3^\lambda(5) + 274c_2^\lambda(5) \\ &= 5p_4(\lambda) - 5(3n-10)p_2(\lambda) - 10p_1^2(\lambda) + 25 \binom{n}{3} - 15 \binom{n}{2}. \end{aligned}$$

Remark 2. Since the left-hand side is a shifted symmetric *polynomial* of λ , Theorem 4 keeps true for virtual characters, i.e. when λ is replaced by *any sequence of integers*, not necessarily in descending order. This extension is obtained by analytic continuation.

6 Two non-unary cycles

The second step of our recurrence corresponds to a permutation having only two non-unary cycles with lengths $p \geq q$, whose cycle-type is the partition $\mu = (p, q, 1^{n-p-q})$. We shall need two auxiliary lemmas.

Lemma 1. Given four positive integers i, k, p, q with $k \leq l(\lambda)$, we have

$$c_i^{\lambda - p\epsilon_k}(q) = c_i^\lambda(q) + pq \sum_{\substack{r, s, t \geq 0 \\ r+s+t+2=i}} c_t^\lambda(q) (\lambda_k - k + q + 1)^r (\lambda_k - k - p + 1)^s.$$

Proof. By analytic continuation, both sides being shifted symmetric polynomials of λ , we may assume that $\lambda - p\epsilon_k$ is a partition. By definition we have

$$\begin{aligned} \frac{F(z - q - 1; \lambda - p\epsilon_k)}{F(z - 1; \lambda - p\epsilon_k)} \frac{F(z - 1; \lambda)}{F(z - q - 1; \lambda)} &= \frac{z - \lambda_k + k + p - q - 1}{z - \lambda_k + k - q - 1} \frac{z - \lambda_k + k - 1}{z - \lambda_k + k + p - 1} \\ &= 1 + \frac{pq}{(z - \lambda_k + k - q - 1)(z - \lambda_k + k + p - 1)} \\ &= 1 + \frac{pq}{z^2} \sum_{r, s \geq 0} (\lambda_k - k + q + 1)^r (\lambda_k - k - p + 1)^s z^{-r-s}. \end{aligned}$$

Expanding series and identifying coefficients, we can conclude. \square

Lemma 2. *Given two indeterminates x, y and two positive integers a, b , we have*

$$\sum_{\substack{r,s,u,v \geq 0 \\ r+s=a}} (-1)^{u+v} \binom{r}{u} \binom{s}{v} \binom{u+v}{a-b} (x+1)^{r-u} (y+1)^{s-v} = (-1)^{a+b} \binom{a+1}{b+1} \frac{x^{b+1} - y^{b+1}}{x-y}.$$

Proof. A direct consequence of the Chu-Vandermonde identity together with the two elementary properties

$$\begin{aligned} \sum_{u \geq 0} \binom{r}{u} \binom{u}{k} x^{r-u} &= \binom{r}{k} (1+x)^{r-k}, \\ \sum_{\substack{r+s=a \\ k+l=a-b}} \binom{r}{k} \binom{s}{l} x^{r-k} y^{s-l} &= \binom{a+1}{b+1} \frac{x^{b+1} - y^{b+1}}{x-y}. \end{aligned}$$

\square

Theorem 5. *For $\mu = (p, q, 1^{n-p-q})$ we have*

$$\begin{aligned} (n)_{p+q} \hat{\chi}_\mu^\lambda &= \sum_{i,j \geq 2} c_i^\lambda(p) c_j^\lambda(q) s(p+1, i) s(q+1, j) \\ &+ pq \sum_{i,j \geq 0} c_i^\lambda(p) c_j^\lambda(q) \left(\sum_{a,b \geq 0} \binom{a+1}{b+1} \frac{p(-p)^b + q^{b+1}}{p+q} s(p+1, i-a+b) s(q+1, j+a+2) \right). \end{aligned}$$

Proof. Both sums are obviously finite. The second is restricted to $i+j \leq p+q$ and $b \leq a \leq q-1$. The Murnaghan rule writes

$$(n)_{p+q} \hat{\chi}_{p,q,1^{n-p-q}}^\lambda = \sum_{k=1}^{l(\lambda)} d_\lambda(k, p) (n-p)_q \hat{\chi}_{q,1^{n-p-q}}^{\lambda - p\epsilon_k}.$$

In general $\lambda - p\epsilon_k$ is not a partition. However taking Remark 2 into account, Theorem 4 yields

$$(n)_{p+q} \hat{\chi}_{p,q,1^{n-p-q}}^\lambda = \sum_{k=1}^{l(\lambda)} d_\lambda(k, p) \sum_{i \geq 2} s(q+1, i) c_i^{\lambda - p\epsilon_k}(q).$$

By Lemma 1 we have

$$\begin{aligned} (n)_{p+q} \hat{\chi}_{p,q,1^{n-p-q}}^\lambda &= \sum_{i \geq 2} s(q+1, i) c_i^\lambda(q) \sum_{k=1}^{l(\lambda)} d_\lambda(k, p) \\ &+ pq \sum_{k=1}^{l(\lambda)} d_\lambda(k, p) \sum_{i \geq 2} s(q+1, i) \sum_{\substack{r,s,t \geq 0 \\ r+s+t+2=i}} c_t^\lambda(q) (\lambda_k - k + q + 1)^r (\lambda_k - k - p + 1)^s. \end{aligned}$$

Applying Theorem 3 with $r = 0$, the first term on the right-hand side is clearly

$$\sum_{i,j \geq 2} c_i^\lambda(p) c_j^\lambda(q) s(p+1, i) s(q+1, j).$$

The second term can be written

$$pq \sum_{r,s,t,u,v \geq 0} s(q+1, r+s+t+2) c_t^\lambda(q) \binom{r}{u} \binom{s}{v} (1+q)^{r-u} (1-p)^{s-v} \sum_{k=1}^{l(\lambda)} d_\lambda(k, p) (\lambda_k - k)^{u+v}.$$

Applying Theorem 3 we obtain

$$pq \sum_{j,r,s,u,v \geq 0} s(q+1, r+s+j+2) c_j^\lambda(q) \binom{r}{u} \binom{s}{v} (1+q)^{r-u} (1-p)^{s-v} \\ \times (-1)^{u+v} \sum_{i,k \geq 0} (-1)^k \binom{u+v}{k} s(p+1, i-k) c_i^\lambda(p).$$

We conclude by using Lemma 2. □

For $q = 1$ the sums are restricted to $j = 2$ (resp. $j = a = b = 0$), and we recover Theorem 4. Suzuki [29] computed the case $p = q = 2$, and Ingram [9] the four cases $2 \leq p \leq 4, q = 2$ and $p = q = 3$ (with many misprints).

We give the examples

$$(n)_6 \hat{\chi}_{3,3,1^{n-6}}^\lambda = (c_4^\lambda(3) - 6c_3^\lambda(3) + 11c_2^\lambda(3)) (c_4^\lambda(3) - 6c_3^\lambda(3) + 20c_2^\lambda(3)) \\ - 9c_6^\lambda(3) + 90c_5^\lambda(3) - 375c_4^\lambda(3) + 810c_3^\lambda(3) - 876c_2^\lambda(3)$$

$$(n)_7 \hat{\chi}_{4,3,1^{n-7}}^\lambda = (c_5^\lambda(4) - 10c_4^\lambda(4) + 35c_3^\lambda(4) - 50c_2^\lambda(4)) (c_4^\lambda(3) - 6c_3^\lambda(3) + 23c_2^\lambda(3)) \\ - 12 (c_7^\lambda(4) - 15c_6^\lambda(4) + 95c_5^\lambda(4) - 325c_4^\lambda(4) + 624c_3^\lambda(4) - 620c_2^\lambda(4)).$$

Remark 3. The expression given by Theorem 5 is not symmetrical with respect to (p, q) , though actually $\hat{\chi}_\mu^\lambda$ is. Thus the equivalence of formulas written for (p, q) and for (q, p) yields identities between the c_j^λ 's. The simplest case of such an identity is obtained for $\mu = (2, 1^{n-2})$. Then writing Theorem 5 for $(2, 1)$ and for $(1, 2)$, we have

$$(n)_3 \hat{\chi}_{2,1^{n-2}}^\lambda = (c_3^\lambda(2) - 3c_2^\lambda(2))(c_2^\lambda(1) - 2) = -2c_3^\lambda(1) + c_2^\lambda(1)(c_3^\lambda(2) - 3c_2^\lambda(2) + 4),$$

which gives

$$3c_2^\lambda(2) - 2c_2^\lambda(1) = c_3^\lambda(2) - c_3^\lambda(1) = n.$$

7 The general case

Theorem 5 may be written in a more compact form, by using the following notations. Let $\varepsilon \in \{0, 2\}$. Define $\theta = 1$ if $\varepsilon = 0$ and $\theta = pq$ otherwise. Then Theorem 5 reads

$${}^{(n)}_{p+q} \hat{\chi}_{p,q,1^{n-p-q}}^\lambda = \sum_{\varepsilon \in \{0,2\}} \sum_{i,j \geq 0} A_{ij}^{(\varepsilon)}(p, q) c_i^\lambda(p) c_j^\lambda(q),$$

with

$$A_{ij}^{(\varepsilon)}(p, q) = \sum_{a,b \geq 0} \theta \binom{a+1}{b+1} \frac{p(-p)^b + q^{b+1}}{p+q} s(p+1, i-a+b) s(q+1, j+a+\varepsilon),$$

and the convention that the sum on a, b is restricted to $a = b = 0$ when $\varepsilon = 0$.

A similar notation will be useful in the general case. Let $\rho = (\rho_1, \dots, \rho_r)$ be a partition with weight $|\rho| \leq n$. Let $\mathbf{M}^{(r)}$ denote the set of upper triangular $r \times r$ matrices with nonnegative integers, and 0 on the diagonal. For any $1 \leq i < j \leq r$ let $\varepsilon_{ij} \in \{0, 2\}$, and define $\theta_{ij} = 1$ if $\varepsilon_{ij} = 0$ and $\theta_{ij} = \rho_i \rho_j$ otherwise.

Theorem 6. For $\mu = (\rho_1, \dots, \rho_r, 1^{n-|\rho|})$ we have

$${}^{(n)}_{|\rho|} \hat{\chi}_\mu^\lambda = \sum_{\varepsilon \in \{0,2\}^{r(r-1)/2}} \sum_{(i_1, \dots, i_r) \in \mathbf{N}^r} A_{i_1, \dots, i_r}^{(\varepsilon)}(\rho_1, \dots, \rho_r) \prod_{k=1}^r c_{i_k}^\lambda(\rho_k),$$

with

$$A_{i_1, \dots, i_r}^{(\varepsilon)}(\rho_1, \dots, \rho_r) = \sum_{a, b \in \mathbf{M}^{(r)}} \left(\prod_{1 \leq i < j \leq r} \theta_{ij} \binom{a_{ij}+1}{b_{ij}+1} \frac{\rho_i (-\rho_i)^{b_{ij}} + \rho_j^{b_{ij}+1}}{\rho_i + \rho_j} \right) \\ \times \prod_{k=1}^r s\left(\rho_k + 1, i_k + \sum_{l < k} (a_{lk} + \varepsilon_{lk}) - \sum_{l > k} (a_{kl} - b_{kl})\right),$$

and the convention that the sum on a_{ij}, b_{ij} is restricted to $a_{ij} = b_{ij} = 0$ when $\varepsilon_{ij} = 0$.

Remark 4. The right-hand side is a finite sum. Indeed for any $s \geq 1$, if we sum up the $r - s + 1$ conditions

$$i_k + \sum_{l < k} (a_{lk} + \varepsilon_{lk}) - \sum_{l > k} (a_{kl} - b_{kl}) \leq \rho_k + 1$$

from $k = s$ to $k = r$, we obtain

$$\sum_{k=s}^r \left(i_k + \sum_{l < k} \varepsilon_{lk} + \sum_{l > k} b_{kl} + \sum_{l < s} a_{lk} \right) \leq \sum_{k=s}^r (\rho_k + 1).$$

Hence any summation quantity remains bounded.

Proof. Assuming the property true for $r - 1$, we shall apply the Murnaghan rule under the form

$$(n)_{|\rho|} \hat{\chi}_\mu^\lambda = \sum_{l=1}^{l(\lambda)} d_\lambda(l, \rho_1) (n - \rho_1)_{|\rho| - \rho_1} \hat{\chi}_{\mu \setminus \rho_1}^{\lambda - \rho_1 \epsilon_l}.$$

In general $\lambda - \rho_1 \epsilon_l$ is not a partition. However by analytic continuation of a shifted symmetric polynomial, the recurrence assumption still writes

$$(n - \rho_1)_{|\rho| - \rho_1} \hat{\chi}_{\mu \setminus \rho_1}^{\lambda - \rho_1 \epsilon_l} = \sum_{\varepsilon' \in \{0, 2\}^{(r-1)(r-2)/2}} \sum_{(i_2, \dots, i_r) \in \mathbb{N}^{r-1}} A_{i_2, i_3, \dots, i_r}^{(\varepsilon')}(\rho_2, \dots, \rho_r) \prod_{k=2}^r c_{i_k}^{\lambda - \rho_1 \epsilon_l}(\rho_k),$$

with ε' standing for $\{\varepsilon_{ij}, 2 \leq i < j \leq r\}$. By Lemma 1 each $c_{i_k}^{\lambda - \rho_1 \epsilon_l}(\rho_k)$ can be written

$$\begin{aligned} c_{i_k}^{\lambda - \rho_1 \epsilon_l}(\rho_k) &= c_{i_k}^\lambda(\rho_k) + \rho_1 \rho_k \sum_{\substack{r, s, t \geq 0 \\ r+s+t+2=i_k}} c_t^\lambda(\rho_k) (\lambda_l - l + \rho_k + 1)^r (\lambda_l - l - \rho_1 + 1)^s \\ &= c_{i_k}^\lambda(\rho_k) + \rho_1 \rho_k \sum_{\substack{r, s, t, u, v \geq 0 \\ r+s+t+2=i_k}} c_t^\lambda(\rho_k) \binom{r}{u} \binom{s}{v} (1 + \rho_k)^{r-u} (1 - \rho_1)^{s-v} (\lambda_l - l)^{u+v}, \end{aligned}$$

Defining

$$B_k(r, s, u, v) = (-1)^{u+v} \binom{r}{u} \binom{s}{v} (1 + \rho_k)^{r-u} (1 - \rho_1)^{s-v},$$

we obtain

$$c_{i_k}^{\lambda - \rho_1 \epsilon_l}(\rho_k) = \sum_{\varepsilon_{1k} \in \{0, 2\}} \theta_{1k} \sum_{\substack{r_k, s_k, t_k \geq 0 \\ r_k + s_k + t_k + \varepsilon_{1k} = i_k}} c_{t_k}^\lambda(\rho_k) \sum_{u_k, v_k \geq 0} B_k(r_k, s_k, u_k, v_k) (-\lambda_l + l)^{u_k + v_k},$$

with $\theta_{1k} = 1$ if $\varepsilon_{1k} = 0$, $\theta_{1k} = \rho_1 \rho_k$ if $\varepsilon_{1k} = 2$, and the convention that the sum on r_k, s_k is restricted to $r_k = s_k = 0$ when $\varepsilon_{1k} = 0$. Inserting this expression in the recurrence assumption, we get

$$\begin{aligned} (n - \rho_1)_{|\rho| - \rho_1} \hat{\chi}_{\mu \setminus \rho_1}^{\lambda - \rho_1 \epsilon_l} &= \sum_{\varepsilon \in \{0, 2\}^{r(r-1)/2}} \sum_{(t_2, \dots, t_r) \in \mathbb{N}^{r-1}} \prod_{k=2}^r \theta_{1k} c_{t_k}^\lambda(\rho_k) \\ &\quad \times \sum_{\substack{r_k, s_k \geq 0 \\ u_k, v_k \geq 0}} B_k(r_k, s_k, u_k, v_k) (-\lambda_l + l)^{u_k + v_k} A_{z_2, z_3, \dots, z_r}^{(\varepsilon')}(\rho_2, \dots, \rho_r), \end{aligned}$$

where for clarity of display, the notation z_j ($2 \leq j \leq r$) stands for $z_j = r_j + s_j + t_j + \varepsilon_{1j}$.

Now we may insert this expression in the Murnaghan rule and apply Theorem 3. We obtain

$$\begin{aligned}
(n)_{|\rho|} \hat{\chi}_\mu^\lambda &= \sum_{\varepsilon \in \{0,2\}^{r(r-1)/2}} \sum_{(t_2, \dots, t_r) \in \mathbf{N}^{r-1}} \prod_{k=2}^r \theta_{1k} c_{t_k}^\lambda(\rho_k) \\
&\quad \times \sum_{\substack{r_k, s_k \geq 0 \\ u_k, v_k \geq 0}} B_k(r_k, s_k, u_k, v_k) A_{z_2, z_3, \dots, z_r}^{(\varepsilon')}(\rho_2, \dots, \rho_r) \\
&\quad \times \sum_{t_1, j \geq 0} (-1)^j \binom{\sum_{l \geq 2} (u_l + v_l)}{j} s(\rho_1 + 1, t_1 - j) c_{t_1}^\lambda(\rho_1).
\end{aligned}$$

Writing $j = \sum_{l \geq 2} j_l$ and applying the Chu-Vandermonde formula, this can be rewritten

$$\begin{aligned}
(n)_{|\rho|} \hat{\chi}_\mu^\lambda &= \sum_{\varepsilon \in \{0,2\}^{r(r-1)/2}} \sum_{(t_1, \dots, t_r) \in \mathbf{N}^r} c_{t_1}^\lambda(\rho_1) \prod_{k=2}^r c_{t_k}^\lambda(\rho_k) \theta_{1k} \\
&\quad \times \sum_{\substack{r_k, s_k, j_k \geq 0 \\ u_k, v_k \geq 0}} (-1)^{j_k} B_k(r_k, s_k, u_k, v_k) \binom{u_k + v_k}{j_k} \\
&\quad \times s\left(\rho_1 + 1, t_1 - \sum_{l > 1} j_l\right) A_{z_2, z_3, \dots, z_r}^{(\varepsilon')}(\rho_2, \dots, \rho_r).
\end{aligned}$$

But by Lemma 2 we have

$$\sum_{\substack{r_k, s_k, u_k, v_k \geq 0 \\ r_k + s_k = a_{1k}}} B_k(r_k, s_k, u_k, v_k) \binom{u_k + v_k}{a_{1k} - b_{1k}} = (-1)^{a_{1k} - b_{1k}} \binom{a_{1k} + 1}{b_{1k} + 1} \frac{\rho_k^{b_{1k} + 1} - (-\rho_1)^{b_{1k} + 1}}{\rho_k + \rho_1}.$$

Writing $r_k + s_k = a_{1k}$ and $j_k = a_{1k} - b_{1k}$, ($2 \leq k \leq r$), we obtain

$$\begin{aligned}
(n)_{|\rho|} \hat{\chi}_\mu^\lambda &= \sum_{\varepsilon \in \{0,2\}^{r(r-1)/2}} \sum_{(t_1, \dots, t_r) \in \mathbf{N}^r} \sum_{\substack{(a_{12} \dots a_{1r}) \\ (b_{12} \dots b_{1r})}} c_{t_1}^\lambda(\rho_1) \prod_{k=2}^r c_{t_k}^\lambda(\rho_k) A_{z_2, z_3, \dots, z_r}^{(\varepsilon')}(\rho_2, \dots, \rho_r) \\
&\quad \times \theta_{1k} \binom{a_{1k} + 1}{b_{1k} + 1} \frac{\rho_1 (-\rho_1)^{b_{1k}} + \rho_k^{b_{1k} + 1}}{\rho_1 + \rho_k} s\left(\rho_1 + 1, t_1 - \sum_{l > 1} (a_{1l} - b_{1l})\right),
\end{aligned}$$

where z_j stands now for $z_j = t_j + a_{1j} + \varepsilon_{1j}$. Hence the result. \square

For $\rho_r = 1$ either all ε_{lr} , $1 \leq l \leq r-1$, are equal to 0, and the sum is restricted to $i_r = 2$. Either only one ε_{lr} is equal to 2, and the sum is restricted to $i_r = 0$. Or two or more ε_{lr} are equal to 2, such a contribution being zero. Summing up the r non zero contributions brings a factor $(n - \sum_{i=1}^{r-1} \rho_i)$, and we recover the formula for $r-1$.

Ingram [9] had only computed the case $\rho = (2, 2, 2)$ (with many misprints). It writes

$$(n)_6 \hat{\chi}_{2,2,2,1^{n-6}}^\lambda = (c_3^\lambda(2) - 3c_2^\lambda(2))^3 + 2(c_3^\lambda(2) - 3c_2^\lambda(2))(27c_3^\lambda(2) - 47c_2^\lambda(2) - 6c_4^\lambda(2)) \\ + 40c_5^\lambda(2) - 240c_4^\lambda(2) + 560c_3^\lambda(2) - 600c_2^\lambda(2).$$

For $\rho = (3, 2, 2)$ a new example is

$$(n)_7 \hat{\chi}_{3,2,2,1^{n-7}}^\lambda = \\ \left((c_3^\lambda(2) - 3c_2^\lambda(2))^2 - 4c_4^\lambda(2) + 18c_3^\lambda(2) - 50c_2^\lambda(2) \right) (c_4^\lambda(3) - 6c_3^\lambda(3) + 11c_2^\lambda(3)) \\ - 12(c_3^\lambda(2) - 3c_2^\lambda(2))(c_5^\lambda(3) - 8c_4^\lambda(3) + 23c_3^\lambda(3) - 28c_2^\lambda(3)) \\ + 72(c_6^\lambda(3) - 10c_5^\lambda(3) + 40c_4^\lambda(3) - 80c_3^\lambda(3) + 79c_2^\lambda(3)).$$

Remark 5. As mentioned in Remark 3, since the order of the ρ_k is irrelevant, there are many ways of writing the right-hand side in terms of the c_j^λ 's. Their equivalence yields identities between the c_j^λ 's.

8 Jucys-Murphy elements

We briefly recall the connection of our results with the structure of the center \mathcal{Z}_n of the group algebra \mathcal{CS}_n of S_n . These facts are not new, and may be found in [3, Section 4].

Given a partition μ , denote \mathcal{C}_μ the conjugacy class of permutations having cycle-type μ , and identify this class with the formal sum of its elements. Then it is well known that we obtain a basis of \mathcal{Z}_n .

For $1 \leq i \leq n$ the Jucys-Murphy elements J_i are defined by $J_i = \sum_{j < i} (ji)$, where (ji) is a transposition. These elements were introduced independently in [11] and [24]. They generate a maximal commutative subalgebra of \mathcal{CS}_n .

Jucys proved the two following fundamental properties.

- (i) We have $\mathcal{S}[J_1, \dots, J_n] = \mathcal{Z}_n$. More precisely the elementary symmetric functions $e_k(J_1, \dots, J_n)$ are given by

$$e_k(J_1, \dots, J_n) = \sum_{|\mu| - l(\mu) = k} \mathcal{C}_\mu.$$

- (ii) Viewing any central function χ as the formal sum $\sum_\sigma \chi(\sigma)\sigma$, for any symmetric function f , we have

$$f(J_1, \dots, J_n) \chi^\lambda = f(A_\lambda) \chi^\lambda.$$

Now for any $\mu = (\rho, 1^{n-|\rho|})$ with $\rho = (\rho_1, \dots, \rho_r)$ having no part 1, define $\mathbf{z}_\mu = \mathbf{z}_\rho$. As in Section 2.4, we write

$$(n)_{|\rho|} \hat{\chi}_\mu^\lambda = f_\mu(A_\lambda),$$

for a unique $f_\mu \in \mathbb{R}[\text{card}, p_1, p_2, p_3, \dots]$. Therefore the central character of λ at μ is

$$\omega_\mu^\lambda = n! z_\mu^{-1} \hat{\chi}_\mu^\lambda = z_\mu^{-1} f_\mu(A_\lambda).$$

Thus we have

$$\mathcal{C}_\mu \chi^\lambda = \omega_\mu^\lambda \chi^\lambda = z_\mu^{-1} f_\mu(A_\lambda) \chi^\lambda = z_\mu^{-1} f_\mu(J_1, \dots, J_n) \chi^\lambda,$$

which yields immediately

$$\mathcal{C}_\mu = z_\mu^{-1} f_\mu(J_1, \dots, J_n).$$

Since f_μ is unique, this expression of \mathcal{C}_μ as a symmetric function in the Jucys-Murphy elements is unique. Similarly from

$$e_k(A_\lambda) = \sum_{|\mu| - l(\mu) = k} z_\mu^{-1} f_\mu(A_\lambda),$$

we obtain the decomposition

$$e_k = \sum_{|\mu| - l(\mu) = k} z_\mu^{-1} f_\mu.$$

For instance we have

$$e_2 = \frac{1}{3} f_{31^{n-3}} + \frac{1}{8} f_{221^{n-4}} = p_2 - \binom{n}{2} + \frac{1}{2} p_1^2 - \frac{3}{2} p_2 + \binom{n}{2}.$$

The expression of the power sums of the Jucys-Murphy elements $p_k(J_1, \dots, J_n)$ in terms of the conjugacy classes \mathcal{C}_μ was studied in [15]. To give a similar expression of $F_{npk}(J_1, \dots, J_n)$ is an interesting open problem.

9 Application to Hecke algebras

The Hecke algebra $H_n(q_1, q_2)$ is a deformation of the group algebra $\mathcal{C}S_n$. More precisely $H_n(q_1, q_2)$ is the algebra over $\mathbb{C}(q_1, q_2)$, the field of rational functions in two indeterminates (q_1, q_2) , generated by T_1, T_2, \dots, T_{n-1} with relations

$$(T_i - q_1)(T_i - q_2) = 0, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i, \quad \text{if } |i - j| > 1.$$

The case of S_n corresponds to $q_1 = -q_2 = 1$, and $T_i = s_i$, the simple transposition switching i and $i + 1$. The algebra $H_n(q_1, q_2)$ has a linear basis $\{T_\sigma, \sigma \in S_n\}$ defined by $T_\sigma = T_{s_{i_1}} T_{s_{i_2}} \dots T_{s_{i_k}}$ where $s_{i_1} s_{i_2} \dots s_{i_k}$ is any reduced decomposition of σ .

Like those of S_n , the irreducible representations of $H_n(q_1, q_2)$ are indexed by partitions $|\lambda| = n$ and have dimension $\dim \lambda$. We shall denote χ_H^λ (resp. $\hat{\chi}_H^\lambda$) the corresponding character (resp. normalized character).

For any permutation $\sigma \in S_n$ with cycle-type $\mu = (\mu_1, \mu_2, \dots, \mu_l)$, we write $T_\mu = T_{\gamma_{\mu_1} \times \gamma_{\mu_2} \times \dots \times \gamma_{\mu_l}}$, with γ_k the k -cycle $s_{k-1} s_{k-2} \dots s_1 \in S_k$. Then it is known [27, Corollary 5.2, p. 477] that the characters χ_H^λ are uniquely determined by their values $\chi_H^\lambda(T_\mu), |\mu| = n$.

In the following we adopt “ λ -ring” notations (see [27, p. 471] or [17, p. 220] for a short survey). In other words, given any symmetric function f taken on an alphabet A , we shall write $f[(q_1 + q_2)A]$ for the image of f under the ring homomorphism uniquely determined by

$$p_k[(q_1 + q_2)A] = (q_1^k - (-q_2)^k)p_k(A) = (q_1^k - (-q_2)^k) \sum_{a \in A} a^k.$$

Then we have the generalized Frobenius formulas

$$\begin{aligned} (q_1 + q_2)^{-l(\mu)} h_\mu[(q_1 + q_2)A] &= \sum_{\lambda} \chi_H^\lambda(T_\mu) s_\lambda(A), \\ s_\lambda[(q_1 + q_2)A] &= \sum_{\mu} (q_1 + q_2)^{l(\mu)} \chi_H^\lambda(T_\mu) m_\mu(A), \end{aligned}$$

with $m_\mu(A)$ the monomial symmetric function of A , i.e. the sum of distinct monomials $\prod_i a_i^{m_i}$ such that (m_i) is a permutation of μ . The first of these formulas was proved in [27, Theorem 4.14, p. 475], see also [28]. For their equivalence, see [4].

By a classical result (see for instance [17, p. 238]), for any positive integer r we have

$$h_r[(q_1 + q_2)A] = \sum_{|\rho|=r} z_\rho^{-1} \prod_{i=1}^{l(\rho)} (q_1^{\rho_i} - (-q_2)^{\rho_i}) p_\rho(A).$$

As a direct consequence, for $\mu = (\mu_1, \dots, \mu_l)$ we have

$$\hat{\chi}_H^\lambda(T_\mu) = (q_1 + q_2)^{-l} \sum_{|\rho^{(1)}|=\mu_1, \dots, |\rho^{(l)}|=\mu_l} \prod_{i=1}^l z_{\rho^{(i)}}^{-1} \prod_{j=1}^{l(\rho^{(i)})} (q_1^{\rho_j^{(i)}} - (-q_2)^{\rho_j^{(i)}}) \hat{\chi}_{\cup_i \rho^{(i)}}^\lambda,$$

with $\hat{\chi}_{\cup_i \rho^{(i)}}^\lambda$ given by Theorem 6. Here the partition $\mu \cup \nu$ is formed by parts of μ and ν .

10 Final remark

In this paper the normalized characters of the symmetric group have been written in terms of the contents of the partition λ . A different approach has been recently devoted to the same problem, using “Kerov polynomials”.

In this alternative framework the “free cumulants” of λ are considered. These probabilistic quantities $R_i(\lambda)$, $i \geq 2$ arise in the asymptotic study of the representations of symmetric groups [1].

In the simplest case of a p -cycle $\mu = (p, 1^{n-p})$, Kerov set the problem of writing the normalized character $(n)_p \hat{\chi}_{p, 1^{n-p}}^\lambda$ as a polynomial in the free cumulants $R_i(\lambda)$. He conjectured that the coefficients of this polynomial are positive integers.

In [2, 8] this normalized character was obtained as a particular coefficient in some Taylor series. Special cases were also computed. However Kerov’s positivity conjecture is still open. It would be interesting to study the connection between both descriptions (contents vs free cumulants).

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