Straightening for Standard Monomials on Schubert Varieties

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We give an elementary proof of the following known fact: any multihomogenous component in the homogeneous coordinate ring of a Schubert variety inside $GL_n/B$ has basis given by the standard monomials (V. Lakshmibai et al., 1986, *J. Algebra* 100, 462–557).

INTRODUCTION

Hodge and Pedoe [5] constructed bases of the homogeneous coordinate rings of the Schubert varieties in the Grassmannian, consisting of certain products of Plücker coordinates called standard monomials. Lakshmibai, Musili, and Seshadri [6] have generalized this theory extensively, giving standard monomial bases of the spaces of global sections of certain line bundles over unions of Schubert varieties in $G/B$ where $G$ is a classical group and $B$ is a Borel subgroup. The purpose of this article is to give an elementary proof that the standard monomials yield a basis of the coordinate ring of a single Schubert subvariety of the flag variety (that is, $G$ is of type $A$). The part of the original proof that is not elementary is the

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argument that the standard monomials are a spanning set. This is accomplished here using an explicit straightening algorithm based on a classical determinantal identity. As a byproduct of the proof, the Plücker coordinates that vanish on a given Schubert variety are shown to generate its prime ideal. Our methods are valid over the integers.

2. SCHUBERT VARIETIES

This section establishes notation and reviews standard facts regarding the Schubert varieties in the flag variety (see [3, 8]). For convenience of exposition the base field \( k \) is assumed to be algebraically closed.

Let \( E \) be an \( n \)-dimensional vector space over \( k \) with the fixed ordered basis \( \{ e_i : 1 \leq i \leq n \} \), \( G = GL(E) = GL_n(k) \) the general linear group, \( B \) the Borel subgroup of lower triangular matrices in \( G \), and \( \mathfrak{g} \) and \( \mathfrak{b} \) the Lie algebras of \( G \) and \( B \), respectively. Let \( \mathcal{F} \) be the set of complete flags in \( E \), whose typical element has the form \( F = (F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n) \), where \( F_i \) is a subspace of \( E \) of dimension \( i \). Composing the well-known Plücker and Segre maps gives an embedding of \( \mathcal{F} \) into the projective space of one-dimensional subspaces of \( \Lambda^n(E) \). It can be shown that the image of this map is irreducible and closed, giving \( \mathcal{F} \) the structure of a projective variety. There is an isomorphism given by the right \( G \)-equivariant map \( B \cdot \mathcal{F} \to \mathcal{F} \) such that \( Bg \to F \), where \( F \) is the span of the first \( r \) rows of the matrix \( g \).

The multihomogeneous coordinates of the embedding of \( \mathcal{F} \) in \( Z \) can be described as follows. The vectors \( e_u = e_{u_1} \wedge \cdots \wedge e_{u_r} \) form a basis of \( \Lambda^r(E) \), where \( u = u_1 u_2 \cdots u_r \) runs over the strictly increasing words of length \( r \) with letters in the set \( [n] \). Projective coordinates on \( \Lambda^r(E) \) are given by the basis \( \{ x_u \} \) of \( \Lambda^r(E)^* \) dual to \( \{ e_u \} \), so that the multihomogeneous coordinate ring \( k[Z] \) of \( Z \) is isomorphic to the polynomial ring \( k[x_u] \). The ring \( k[x_u] \) is \( \mathbb{N}^n \)-graded; every variable \( x_u \) indexed by a word of length \( r \) has degree given by the \( r \)th standard basis vector of \( \mathbb{N}^n \).

Let \( k[z_{ij}] \) be the coordinate ring of \( \mathfrak{g} \), where \( z_{ij} \) is the \((i,j)\)th matrix entry function. For an arbitrary word \( u = u_1 u_2 \cdots u_r \) with \( u_i \in [n] \) and \( 0 \leq r \leq n \), let \( p_u \) be the minor of the generic matrix \((z_{ij})\) with row indices \( 1, 2, \ldots, r \) and column indices \( u_1, u_2, \ldots, u_r \). By direct calculation, the multihomogeneous coordinates of the image of \( F \) in \( Z \) are given by \( Bg \mapsto p_u(g) \), where \( u \) is a strictly increasing word with letters in \( [n] \). Since \( G \) is Zariski dense in \( \mathfrak{g} \), the prime ideal of the image of \( F \) in \( Z \) is given by the kernel \( I \) of the map \( \theta : k[x_u] \to k[z_{ij}] \) given by \( x_u \mapsto p_u \).
The coordinate ring \( k[\mathfrak{F}] \) has the following well-known presentation. A proof is given in [5], where it is attributed to Mertens.

**Theorem 1.** The prime ideal \( I \) of the image \( \mathfrak{F} \) in \( Z \) is generated by the quadratic relations (see Section 5).

Let \( w \) be a permutation in the symmetric group \( S_n \) on the set \([n] = \{1, 2, \ldots, n\}\). We also denote by \( w \) the permutation matrix in \( G \) whose \((i, j)\)th entry is \( \delta_{w(i), j} \). The Schubert cell is defined by \( X_w = B \setminus BwB \). Every right \( B \)-orbit in \( B \setminus G \) is a Schubert cell. The Schubert variety \( X_w \) is defined to be the Zariski closure of the Schubert cell \( X_w \) in the projective variety \( B \setminus G \). To describe the right \( B \)-orbit decomposition of \( X_w \), recall the (strong) Bruhat order on \( S_n \). The Schubert variety is defined to be the Zariski closure of the Schubert cell \( X_w \) in the projective variety \( B \setminus G \). To describe the right \( B \)-orbit decomposition of \( X_w \), recall the strong Bruhat order on \( S_n \). The minimal length coset representatives \( s = s_1 s_2 \cdots s_r \) in \( S_n \) can be identified with the strictly increasing words of length \( r \) in the alphabet \( n \). If \( u \) and \( u' \) are two such words, write \( u \leq u' \) for all \( i \). Let \( \pi : S_n \to S_n/(S_n \times S_{n-1}) \) be the natural projection. Under the above identification, \( \pi(w) \) is the word obtained by sorting the values \( w(1), w(2), \ldots, w(r) \) into increasing order. Then let \( \nu \leq w \) if \( \pi(\nu) \leq \pi(w) \) for all \( 1 \leq r \leq n \). The Schubert variety has the well-known orbit decomposition

\[
X_w = \bigsqcup_{\nu \leq w} X_{\nu}^o.
\]

Consider the homomorphism of coordinate rings \( \eta_k : k[\mathfrak{g}] \to k[wb] \) given by a restriction of functions on \( \mathfrak{g} \) to the subvariety \( wb \). The following result appears as Exercise 10.5.11 in [3].

**Lemma 2.** The radical ideal \( I(X_w) \) of \( X_w \) is prime and is given by the kernel of the composite map \( \eta_k \circ \theta : k[x_w] \to k[wb] \).

**Proof.** Since \( k[wb] \) is a polynomial ring, the kernel of \( \eta_k \circ \theta \) is a prime ideal. Let \( f \in k[wb] \). Then the assertion of the lemma is deduced by the following chain of equivalences: \( f \in I(X_w) \) if and only if \( f \) vanishes on \( X_w \), if and only if \( \theta(f) \) vanishes on \( wb \), if and only if \( \theta(f) \) vanishes on \( wb \), if and only if \( \theta(f) \) vanishes on \( wb \). The first and second equivalences are trivial, and the fourth follows since \( wb \) is the closure of \( wb \) in \( \mathfrak{g} \). So it only remains to show the third equivalence. Without loss of generality assume that \( f \in k[x_w] \) is homogeneous of multidegree \( m = (m_1, m_2, \ldots, m_n) \). Let \( \lambda \) be the partition such that \( \lambda_r = m_1 + m_2 + \cdots + m_r \) for \( 1 \leq r \leq n \). A direct calculation shows that \( \theta(f)(bg) = (\prod_{r=1}^n b_r^{\lambda_r}) \theta(f)(g) \) for all \( b \in B \) and \( g \in G \).

Let \( J_w \) be the ideal in \( k[x_w] \) generated by \( I \) (the ideal of \( \mathfrak{F} \subset Z \)) and the variables \( x_u \) such that \( u \neq \pi_i(w) \), where \( r \) is the length of the word \( u \).
Lemma 3. The ideal $J_w$ defines $X_w$ set-theoretically.

Proof. Suppose $u = u_1 u_2 \ldots u_r$ is a word such that $u \not\in \pi(w)$. Let $m$ be an index such that $u_m$ is greater than the $m$th largest letter in $w(1)w(2)\ldots w(r)$. Consider $\eta_p(p_u)$, which is the Plücker coordinate $p_u$ with the variables $z_i$ set to zero if $j > w(i)$. It vanishes, being the determinant of an $r \times r$ matrix that has a rectangular $(r+1-m) \times m$ submatrix of zeros. By Lemma 2, we have $J_w \subseteq I(x_w)$, so the zero set $Y$ of $J_w$ contains $X_w$. On the other hand, the ideal $J_w$ is $B$-stable under the action $(bf)(x) = f(bx)$. It follows that $Y$ is closed and right $B$-stable, hence a union of Schubert varieties. If $v \not\in w$, then there is an index $l$ such that $\pi_l(v) \not\in \pi(w)$. Setting $u = \pi_l(v)$, we have $x_u \in J_w$ and $p_u(v) \neq 0$, so $X_u \not\subseteq Y$. In light of (2.1) we have $Y = X_w$. 

We will obtain an elementary proof of the following result.

Theorem 4. The ideal $J_w$ is the prime ideal of the Schubert variety $X_w$.

3. Standard Monomials

Recall that the homogeneous coordinate ring $k[\mathcal{F}]$ of the flag variety $\mathcal{F}$ is isomorphic to the subring $k[p_u] \subseteq k[z_i]$ generated by the Plücker coordinates $p_u$. Let $k[p_u]_m$ denote the $m$th graded component $k[p_u]$, where $m = (m_1, m_2, \ldots, m_n) \in \mathbb{N}^n$. Let $\lambda$ be the partition defined by $\lambda_r = m_1 + m_2 + \cdots + m_r$ for $1 \leq r \leq n$. In characteristic zero, $k[p_u]_m$ affords the irreducible representation of $G$ of highest weight $\lambda$. This module was first constructed by Deruyts [2]. A tableau of shape $\lambda$ is a map from the Ferrers diagram $D(\lambda) = \{(i, j) : 1 \leq j \leq \lambda_i\}$ to the set $[n]$. It is clear that $k[p_u]_m$ is spanned by the elements $p_T = p_{u_1} p_{u_2} \ldots$, where $T$ is a tableau of shape $\lambda$ whose $i$th column $u_i$ is strictly increasing from top to bottom. A tableau $T$ is said to be standard if its rows are weakly increasing from left to right and its columns are strictly increasing from top to bottom. By abuse of language the monomial $p_T$ in the Plücker coordinates $p_u$ is said to be standard if the tableau $T$ is standard. The following basis was known to Young [10].

Theorem 5. The standard monomials form a basis of the homogeneous coordinate ring of $\mathcal{F}$.

The original definition of a standard monomial on a Schubert variety [6] can be translated into the following. Let $T$ be a standard tableau of shape $\lambda$. A defining chain for $T$ is a Bruhat-increasing chain of permutations $w^1 \leq w^2 \leq \cdots \leq w^{\lambda_1}$.
such that \( \pi_\lambda(w^k) \) is the \( k \)th column of \( T \), where \( \lambda_k \) is the number of cells in the \( k \)th column of the Ferrer diagram \( D(\lambda) \) of \( \lambda \). Say that \( T \) is standard on \( X_w \) (and by abuse of language that the monomial \( p_T \) is standard on \( X_w \)) if there is a defining chain \( \{w^i\} \) for \( T \) whose last permutation is less than or equal to \( w \), that is, \( w^{\lambda_i} \leq w \).

Example 6. A standard tableau \( T \) and defining chain for \( T \) are given below. This chain shows that \( T \) is standard on \( X_w \) for any \( w \in S_3 \) such that \( w \geq 596281347 \):

\[
\begin{array}{ccc}
1 & 2 & 5 \\
3 & 6 & 9 \\
\end{array}
\]

\[
T = 5 \quad 8 \\
7 \\
9
\]

\[
w^1 = 135792468 \\
w^2 = 268391457 \\
w^3 = 596281347.
\]

The following lemma about defining chains was proven by Deodhar.

Lemma 7 [8, Chap. 4, Sect. 6]. For any standard tableau \( T \), there is a minimal defining chain \( w^1 \leq w^2 \leq \cdots \leq w^{\lambda_i} \) for \( T \) in the sense that if \( \{v^j\} \) is another defining chain for \( T \), then \( v^j \leq w^j \) for all \( j \).

In the previous example the sequence \((w^1, w^2, w^3)\) is the minimal defining chain for \( T \).

As a consequence of this lemma, checking standardness of \( p_T \) on \( X_w \) is easy if one knows the minimal defining chain \( w^1 \leq w^2 \leq \cdots \leq w^{\lambda_i} \) for \( T \), since one only needs to check if \( w^{\lambda_i} \leq w \). There is a nice way to compute this minimal defining chain, involving the notion of the right key of a tableau [7]. Given a standard tableau \( T \) of shape \( \lambda \), the right key \( K_+(T) \) of \( T \) is the tableau of shape \( \lambda \) given as follows. Let \( u^{i,j} \leq u^{i,j+1} \leq \cdots \leq u^{i,\lambda_i} \) be the unique chain of strictly increasing words (each of length \( \lambda_j \)) defined by the following properties:

- \( u^{i,j} \) is the \( j \)th column of \( T \), and
- \( u^{i,k} \) is minimal among words with \( u^{i,k} \geq u^{i,k-1} \) which contain the \( k \)th column of \( T \) as a subword (see paragraph below Lemma 8).

Then the \( j \)th column of \( K_+(T) \) is defined to the last word \( u^{i,\lambda_i} \).

Define the canonical lift \( w(T) \) of \( T \) to be the shortest permutation such that \( \pi_\lambda(w(T)) \) equals the \( j \)th column of \( K_+(T) \). We then have the following relationship.
Lemma 8. [7] For any standard tableau \( T \), the minimal defining chain \( w^1 \leq w^2 \leq \cdots \leq w^h \) for \( T \) has \( w^j \) equal to the canonical lift of the tableau consisting of the first \( j \) columns of \( T \).

Therefore, \( T \) is standard on \( X_w \) if and only if \( w(T) \leq w \).

The process involved in passing from \( u^{i,k-1} \) to \( u^{i,k} \) is equivalent to a jeu de taquin on a two-column tableau or a greedy partial matching. Here is an explicit algorithm which starts with the word \( u = u^{i,k-1} \) and produces the word \( v = u^{i,k} \). By definition the letters \( T(k,1) < T(k,2) < \cdots < T(k,q) \) (for \( q = \lambda'_i \)) must occur as a subword of \( v \), say in positions \( 1 \leq r_1 < r_2 < \cdots < r_q \leq \lambda'_i \) that are determined as follows. Let \( r_q \) be maximal such that \( u_{r_q} \leq T(k,q) \). Given \( r_m+1 \), let \( r_m \) be maximal such that \( r_m < r_{m+1} \) and \( u_{r_m} \leq T(k,m) \). It follows from the standardness of \( T \) that at each stage the index \( r_m \) exists. Define \( v \) to be the word such that

\[
v_r = \begin{cases} 
T(k,m) & \text{if } r = r_m \\
u_r & \text{if } r \notin \{r_1,r_2,\ldots,r_q\}.
\end{cases}
\]

Example 9. Three tableaux, \( T, T', K_+(T) \), are depicted below. The \( k \)th column of the tableau \( T' \) is the word \( u^{i,k} \) in the definition of \( K_+(T) \), where \( j = 1 \). A letter in the \( k \)th column of \( T' \) is starred if and only if it does not appear in the \( k \)th column of \( T \). The last column of \( T' \) yields the first column (since \( j = 1 \)) of the right key \( K_+(T) \) of \( T \). Performing similar calculations for \( j > 1 \), one obtains \( K_+(T) \), also shown below. From this we see that the canonical lift of \( T \) is \( w(T) = 596281347 \):

\[
T = 5 \ 8 \quad T' = 5 \ 6 \ 6* \quad K_+(T) = 6 \ 9 \\
1 \ 2 \ 5 \quad 1 \ 2 \ 2* \quad 2 \ 5 \ 5 \\
3 \ 6 \ 9 \quad 3 \ 3* \ 5 \quad 5 \ 6 \ 9 \\
7 \quad 7 \ 8 \ 8* \quad 8 \\
9 \quad 9 \ 9* \ 9 \quad 9
\]

Let \( Y = \bigcup_{i=1}^r X_{w^i} \) be a union of Schubert varieties. Say that a tableau \( T \) is standard on \( Y \) if it is standard on \( X_{w^i} \) for some \( i \), and again by abuse of language say that \( p_T \) is a standard monomial on \( Y \). Our main goal is to give an elementary proof of the following result in the special case that \( Y \) is a single Schubert variety.

Theorem 10. [6, 8] The standard monomials on \( Y \) form a basis of the homogeneous coordinate ring \( k[Y] \).
This result has the following well-known consequence for Demazure modules [1]. Let $\lambda$ be a partition with at most $n$ parts and $m_i$ the number of columns of $\lambda$ of length $i$. The partition $\lambda$ defines a $B$-equivariant line bundle $L_\lambda = \bigoplus_{i=1}^{n} L_i^{\otimes m_i}$ on $Y$, where $L_i$ is the restriction from $F = B \setminus G$ to $Y$ of the pullback of $O(1)$ under the composite morphism

$$B \setminus G \to P_i \setminus G \xrightarrow{\text{Plücker}} \mathbb{P}(\wedge^i(E))$$

and $P_i$ is the maximal parabolic subgroup generated by the subgroups $GL_i \times GL_{n-i}$ and $B$ of $GL_n$. The Demazure module is defined to be the space of global sections $H^0(Y, L_\lambda)$ and can be shown to be isomorphic to the $m = (m_1, \ldots, m_n)$th multihomogeneous component of $k[Y]$. Therefore Theorem 10 implies that the Demazure module has a basis given by the elements $p_T$ where $T$ is a tableau of shape $\lambda$ that is standard on $Y$.

In the case $Y = X_w$, Theorems 4 and 10 follow immediately from Lemma 3 and the following two results.

**Theorem 11.** The monomials that are standard on a union $Y$ of Schubert varieties are linearly independent on $Y$.

**Theorem 12.** The monomials that are standard on $X_w$ span $k[x_w]/J_w$.

Theorems 11 and 12 are proved in Sections 4 and 5, respectively.

### 4. Independence

For the sake of completeness we repeat the elementary proof of Theorem 11 furnished by standard monomial theory [8]. We work in the ring $k[p_u]$. Suppose there is a linear dependence relation among a collection $T$ of standard monomials $p_T$ on $Y$. Without loss of generality we may assume that all the tableaux in $T$ have the same shape $\lambda$, and that $\lambda$ has a minimal number of columns. Let

$$\sum_{T \in \mathcal{T}} c_T p_T \big|_Y = 0$$

be such a relation with as few terms as possible, so that $0 \neq c_T \in k$, where $f \big|_Y$ denotes the restriction of the function $f$ to the subvariety $Y$. Among all the strictly increasing words that appear as the last column of a tableau $T \in \mathcal{T}$, let $u$ be any minimal element. Let $T_{\neq u}$ (resp. $T_{= u}$) be the set of $T \in \mathcal{T}$ whose last column is equal (resp. not equal) to $u$. Let $Y' = \bigcup_{T \in \mathcal{T}} X_{w(T)}$, which is contained in $Y$ since each $T \in \mathcal{T}$ is assumed to be standard on $Y$. 


Suppose first that $T_\neq u$ is empty, that is, every tableau $T \in T$ has last column $u$. For $T \in T$, let $T'$ be the tableau obtained by removing the last column of $T$. The relation (4.1) still holds when restricted further to the subvariety $Y' \subseteq Y$, and factors into the relation

$$p_u \sum_{T \in T} c_T P_T^{+} \mid_{Y'} = 0. \quad (4.2)$$

Let $T \in T$. By Lemma 8 and the definitions, $w(T)$ contains $u$ as an initial subword. Thus, $p_u$ does not vanish at any point of the Schubert cell $X_{w(T)}$. It follows that the sum in (4.2) vanishes on $\bigcup_{T \in T} X_{w(T)}$ and hence on its closure $Y'$. This gives a linear dependence relation among monomials that are standard on $Y'$, which correspond to tableaux with one less column than $\lambda$, which is a contradiction.

Otherwise suppose the set $T_\neq u$ is nonempty. Restricting the relation (4.1) to $Y' \subseteq Y$, one has $p_T \mid_{Y'} = 0$ for $T \in T_\neq u$, since the Plücker coordinate corresponding to the last column of such a tableau $T$ vanishes on $Y'$. This means that there is a linear dependency among the monomials $p_T$ for $T \in F_{w'u}$, which are standard on $Y'$. This new relation has fewer turns, which is a contradiction.

**Remark 13.** Notice that the inductive nature of this proof naturally leads one to consider spaces $Y$ which are unions of Schubert varieties, rather than just single Schubert varieties $X_w$.

## 5. SPANNING

Two kinds of straightening relations are required here, the famous quadratic relations (see Theorem 1) and another relation that is also given by a classical determinantal identity.

Both straightening relations use the following total order on tableaux. Given a tableau $T$ of shape $\lambda$, let $|T|$ be the tableau obtained by sorting the columns of $T$ into increasing order from top to bottom (if there is a repeated entry in some column of $T$ then $p_T = 0$). Define the word of the tableau $T$ (written word$(T)$) to be the concatenation $u^1 u^2 \ldots u^\lambda$, where $u^i$ is the word given by reading the $j$th column of $T$ from bottom to top. Let $S < T$ if word$(S)$ precedes word$(T)$ lexicographically.

Theorem 12 is an immediate corollary to the following straightening law.

**Proposition 14.** Let $T$ be a tableau of shape $\lambda$ that is not standard on $X_w$. Then there is an expression of the form

$$p_T = \sum_{S < T} c_S p_S(\text{mod } J_w), \quad (5.1)$$

where $c_S \in \mathbb{Z}$.
It may be assumed that the columns of $T$ are strictly increasing from top to bottom. There are two cases:

1. $T$ is not standard, that is, one of its rows is not weakly increasing from left to right.
2. $T$ is standard but not standard on $X_w$.

In the first case a quadratic relation may be employed to produce a relation of the form (5.1). For the sake of completeness we describe the quadratic relations and repeat the classical argument.

Consider two adjacent columns $j$ and $j+1$ of a tableau $T$ of shape $\lambda$, and let $1 \leq i \leq \lambda'_j + 1$. Then one has the relation

$$p_T = \sum_S p_S,$$  \hspace{1cm} (5.2)

where $S$ runs over all the tableaux obtained from $T$ by exchanging any $i$ of the letters in the $j$th column with the first $i$ letters of the $(j + 1)$st.

So if $T$ is not standard, $T(i, j) > T(i, j + 1)$ for some cell $(i, j)$. Note that the first $i$ letters of the $(j + 1)$st column of $T$ are all strictly smaller than the last $\lambda'_j - i$ letters in the $j$th column:

$$T(\lambda'_j, j) > \cdots > T(i + 1, j) > T(i, j) > T(i, j + 1)$$

$$> \cdots > T(2, j + 1) > T(1, j + 1).$$

It is not hard to verify that on the right hand side of (5.2), each tableau $S$ satisfies $S < T$, for in passing from $T$ to $S$, at least one of the largest $\lambda'_j - i$ letters in the $j$th column of $T$ is replaced by a strictly smaller letter. This yields a relation of the form (5.1).

For the second case, a little more notation is required. Let $S_A$ be the symmetric group on the set $A$ and $\text{Col}(\lambda)$ the Young subgroup of $S_{D(\lambda)}$ that stabilizes each column of $D(\lambda)$. The group $S_{D(\lambda)}$ has a natural right action on tableaux of shape $\lambda$ by $(T \sigma)(i, j) = T(\sigma(i, j))$.

We now describe the second kind of straightening relation. Let $A$ be a subset of $D(\lambda)$ that has at most one cell in each row, $j$ the minimum index of a column that contains a cell of $A$, and $A'$ be the portion of $A$ that is strictly to the right of the $j$th column. Let $f$ be any injection from $A'$ into the $j$th column of $D(\lambda)$ whose image $C'$ is disjoint from $A$. Finally, let $C$ consist of the cells in the $j$th column of $D(\lambda)$ that are not in $A$ nor in $C'$. 


Let $\beta$ be the involution in $S_{D(\lambda)}$ that exchanges each cell in $A'$ with its image under $f$. Then for any tableau $T$ of shape $\lambda$, the Plücker coordinates satisfy the relation

$$\sum_{\sigma \in S_d/(S_d \cap \text{Col}(\lambda))} (-1)^{\sigma} p_{T^\sigma} = \sum_{\tau \in S_d \cup C/(S_d \cap C \cap \text{Col}(\lambda))} (-1)^{\tau} p_{T^\tau}. \tag{5.3}$$

This follows from the determinantal identity [9, III.11.11].

So assume that $T$ is standard but not standard on $X_w$. This means that $w \neq w$, or in other words, for some $j$, the $j$th column $u$ of $K_w(T)$ satisfies $u \neq \pi_j(w)$. Let $j$ be maximal with this property, so that all columns to the right of the $j$th are strictly shorter than the $j$th. Let $r$ be an index such that $u_r$ exceeds the $r$th letter in $\pi_j(w)$.

We now determine a set $A$ as in (5.3) by selecting a suitable strictly decreasing subword of word $T$. This is most easily done by examining the tableau $T'$, whose columns consist of the words $u^{j/j}$, $u^{j+1/j}$, $\ldots$, $u^{\lambda_1/j}$ that occur in the calculation of (the $j$th column of) the right key of $T$. Let us index the columns of $T'$ from left to right by $j$ through $\lambda_1$. By the definition of right key, the rightmost column of $T'$ is equal to $u$.

Let $u'_r$ be the letter in the $r$th row and last column of $T'$. It follows from the right key algorithm that given $u'_r$ through $u'_{r-1}$, there is an unstarred letter $u'_s$ in the $s$th row of $T'$ that is in a weakly earlier column that $u'_{s-1}$ with $u'_s \geq u'_{s-1}$. Repeating this process, one eventually has that the letter $u'_s$ must be given by the bottom letter of the $j$th column of $T'$. Since the unstarred letters in the $p$th column of $T'$ correspond to letters in the $p$th column of $T$, the word $u' = u'_s \ldots u'_{s+1} u'_s$ may be regarded as a strictly decreasing subword of word $T$. Let $A$ be the set of cells in $D(\lambda)$ that correspond to the letters of $u'$. It is enough to show that the desired relation (5.1) is obtained by applying (5.3) to the tableau $T$ and the set of cells $A$. The reader is urged to keep the following example in mind while reading the remainder of the proof.

**Example 15.** Let $T$ be as in the running example, $w = 64281357$, and $j = 1$, so that $T'$ is as before. Here $u = 25689$ and $\pi_j(w) = 24689$, so $r = 2$. One choice for $u'$ is given in bold in $T'$ and in $T$:

$$
\begin{array}{c}
1 & 2 & 2* & 1 & 2 & 5 \\
3 & 3* & 5 & 3 & 6 & 9 \\
7 & 8 & 8* & 7 \\
9 & 9* & 9 & 9 \\
\end{array}
\quad
\begin{array}{c}
T' = 5 & 6 & 6* & T = 5 & 8 \\
7 & 8 & 8* & 7 \\
9 & 9* & 9 & 9 \\
\end{array}
$$

Let $f$ be any injection from $A'$ into the $j$th row of $D(\lambda)$ whose image $C$ is disjoint from $A$. Consider any tableau $T \beta T$ appearing on the right hand side of (5.3). Its $j$th column (call it $x$) contains the letters of $u'$. Since $u'$
has $\lambda_j^r - r + 1$ letters, the smallest of which is $u_j$, there at least $\lambda_j^r - r + 1$ letters in $x$ that are greater than or equal to $u_j$. Since $x$ has $\lambda_j^r$ letters, its $r$th largest letter must be greater than or equal to $u_j$. It follows that $x \notin \pi_i(w)$, so that $p_s$ and $p_T$ vanish on $X_{u_j}$. Consider the left hand side of (5.3). Since $u_j'$ appears as a strictly decreasing subword of word$(T)$, it follows that $p_T$ appears once and the rest of the summands have the form $\pm p_S$ where $S < T$. Thus one has a relation of the form (5.1).

**Problem 16.** Devise a straightening algorithm that proves spanning for standard monomials on unions of Schubert varieties.

**REFERENCES**

3. W. Fulton, Young tableaux with applications to representation theory and geometry, June 1995.