Small resolutions of minuscule Schubert varieties

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Abstract

Let $X$ be a minuscule Schubert variety. In this article, we use the combinatorics of quivers to define new quasi-resolutions of $X$. We describe in particular all relative minimal models $\tilde{\pi} : \tilde{X} \to X$ of $X$ and prove that all the morphisms $\tilde{\pi}$ are small (in the sense of intersection cohomology). In particular, all small resolutions of $X$ are given by the smooth relative minimal models $\tilde{X}$ and we describe all of them. As another application of this description of relative minimal models, we give a more intrinsic statement of the main result of [Pe1].

Introduction

Schubert varieties have been intensively studied and are of great importance in representation theory. There are several ways to understand the geometry and the singularities of Schubert varieties. One way is to describe the singular locus, its irreducible components and the generic singularity in each of these components. This has been completely achieved for $GL_n$ only recently (see [Man1], [Man2], [BW], [KLR] and [Cor2]). For the general case, there are only partial results (for an account, see [BL]). This description of singularities enables in particular to calculate the Kazhdan-Lusztig polynomials. For this study, combinatoric tools are useful but it is also useful to construct special resolutions of Schubert varieties (see for example [Cor1] and [Cor2]). Another way to study Schubert varieties is to calculate the cohomology of line bundles and especially to prove some vanishing theorems. This has been done in many ways, one of them thanks to the Bott-Samelson resolutions ([Ke], [MR], [RR] or [Ra]). In this article, we want to study the geometry of Schubert varieties thanks to the study of some particular resolutions of this variety.

A nice resolution of Schubert varieties is the Bott-Samelson resolution (see [De] or [Ha]). This resolution is useful to study the singularities and standard monomial theory of Schubert varieties (see [LLM]) and also to study the geometry of Schubert varieties (for curves on minuscule Schubert varieties, see for example [Pe1] and [Pe4]). However these resolutions are big in the sense that the fibers have big dimensions and there are many contracted subvarieties. Another
class of resolutions is of particular importance, the IH-small resolution (that is to say the small resolution in the sense of Intersection Cohomology, see definition [Ze]). These are well suited for the calculation of Kazhdan-Lusztig polynomials. In particular A. Zelevinsky in [Ze] constructed some IH-small resolutions for grassmannian Schubert varieties and gave a geometric interpretation of the combinatoric computation of Kazhdan-Lusztig polynomials by A. Lascoux and M.-P. Schützenberger in [LS]. Later P. Sankaran and P. Vanchinathan [SV1] and [SV2] constructed small resolution of some minuscule and cominuscule Schubert varities for $SO_{2n}$ and $Sp_{2n}$ and calculated the corresponding Kazhdan-Lusztig polynomials. In their article [SV1], they construct some minuscule Schubert varieties not admiting small resolutions. These examples are locally factorial Schubert variety (more precisely with singularities in codimension 2) for which the theorem of purity (see [Gr] theorem 21.12.12) says that there is no IH-small resolution.

In this article we study the IH-small resolutions of minuscule Schubert varities. We will generalise the constructions of A. Zelevinsky and P. Sankaran and P. Vanchinathan to any minuscule Schubert variety and describe all IH-small resolutions. We do not adress the problem of calculating Kazhdan-Lusztig polynomials. This has been done in a combinatoric way for all minuscule Schubert varities by B. D. Boe in [Bo] and we hope that our construction will lead to a geometric interpretation of these results.

In order to define our resolutions, we introduce a combinatorial objet: a quiver associated to a minuscule Schubert variety $X(w)$. This quiver is defined thanks to a reduced writing $\tilde{w}$ of $w$ which is unique for minuscule Schubert variety. These quivers seem to be the same as the quivers defined by S. Zelikson in [Zel] but we did not check this. To this quiver, we associate a configuration variety $\tilde{X}(\tilde{w})$ which is simply the Bott-Samelson resolution. The fact that the Bott-Samelson resolution can be seen as a configuration variety was already known by P. Magyar [MR] but the use of the quiver in the situation is very usefull. In the minuscule case, the quiver will have a very special and rigid geometry and in particular we will define the pics of the quiver and the height of a pic. As in the case of A. Zelevinsky’s construction [Ze], the choice of an order on the pics will lead to a partial resolution $\tilde{X}(\tilde{w}) \rightarrow X(w)$ of the Schubert variety. However, the variety $\tilde{X}(\tilde{w})$ will in general be locally factorial but not smooth.

These varieties are however interesting for the relative minimal model program. We study the relative minimal models of $X(w)$ and prove the following theorem:

**Theorem 0.1.** — The relative minimal models of $X(w)$ are the varieties $\hat{X}(\hat{w})$ obtained thanks to an order on the pics preserving the order on the heights of the pics (see construction [X] for more details).

For grassmannian Schubert varieties, there are always IH-small resolution [Ze]. As P. Sankaran and P. Vanchinathan proved this is not true for a general minuscule Schubert variety. The following theorem proves that we need to replace IH-small resolutions by relative minimal models to generalise the result of A. Zelevinsky to any minuscule Schubert variety (that is so say allow locally factorial singularities):
**Theorem 0.2.** — The morphism \( \hat{\pi} : \hat{X}(\hat{w}) \to X(w) \) from a relative minimal model to a minuscule Schubert variety \( X(w) \) is small.

In other words, the relative minimal models play the role for general minuscule Schubert varieties of \( IH \)-small resolution for grassmannian Schubert varieties. However they do not share the same nice properties and in particular are not as well fitted as \( IH \)-small resolutions for the computation of Kazhdan-Lusztig polynomials. We also describe all relative canonical models of \( X(w) \). Furthermore, the following result of B. Totaro [To] using a key result of J. Wisniewski [Wi] tells us to look for \( IH \)-small resolutions in the class of relative minimal models:

**Theorem 0.3.** — [Totaro-Wisniewski] Any \( IH \)-small resolution of a normal variety \( X \) is a relative minimal model for \( X \).

In particular, in our situation, all \( IH \)-small resolutions of \( X(w) \) are given by the smooth relative minimal models. We then give a combinatorial criterion on the quiver for the relative minimal model \( \hat{X}(\hat{w}) \) to be smooth. We get the following:

**Theorem 0.4.** — The variety \( \hat{X}(\hat{w}) \) is an \( IH \)-small resolution if and only if the order on the pics preserves the order on the heights of the pics and if at step \( i \) the pic \( p_i \) is minuscule for the quiver (see definition [7.17]).

In particular, we are able to say which minuscule Schubert variety admits an \( IH \)-small resolution. At the end of the article we sketch another way the prove this: an \( IH \)-small resolution has to factor through the relative canonical minimal model \( X_{can}(w) \) of \( X(w) \) and is a crepant resolution of \( X_{can}(w) \). In particular, the stringy Euler number \( e_{st} \) defined by V. Batyrev [Ba] of \( X_{can}(w) \) has to be an integer. We give a formula for \( e_{st}(X_{can}(w)) \) and it is an easy verification that we recover in this way all minuscule Schubert varieties not admitting an \( IH \)-small resolution.

Another motivation for the study of resolutions and partial resolutions of Schubert varieties (and in fact our motivation at the beginning of the study) is the following reinterpretation of our result in [Pe1]. Let \( X \) be a minuscule Schubert variety and \( \pi : \hat{X} \to X \) any relative minimal model. For a 1-cycle class \( \alpha \in \mathbb{A}_1(X) \) define the set

\[
ne(\alpha) = \{ \beta \in NE(\hat{X}) / \pi_\ast \beta = \alpha \}
\]

where \( NE(\hat{X}) \) is the cone of effective 1-cycles in \( \hat{X} \).

**Theorem 0.5.** — The irreducible components of \( Hom_{\alpha}(\mathbb{P}^1, X) \) of the scheme of morphisms from \( \mathbb{P}^1 \) to \( X \) of class \( \alpha \) are indexed by \( ne(\alpha) \).

The same kind of results are true for other special Schubert varieties (for example for cominuscule Schubert varieties, this will be studied in [Pe2]). It is also the case for cones over homogeneous varieties (see [Pe3]).

Let us give an overview of the article. In paragraphs 1 and 2 we recall some basic notations, definitions and results on elements of the Weyl group and minuscule Schubert varieties. In the
third paragraph, we define the quiver associated to a reduced writing $\tilde{w}$ and the corresponding configuration variety. We study basic properties of these varieties (Weil, Cartier and canonical divisors, 1-cycles and intersection formulae), and link them with the geometry of the quiver. In the fourth paragraph, we describe the particular geometry of a quiver associated to a minuscule Schubert variety and study the link with the geometry of the Schubert variety (Weil, Cartier and canonical divisors, 1-cycles and intersection formulae). In the fifth paragraph, we construct and study a generalisation of Bott-Samelson resolution which is between the Schubert variety and the Bott-Samelson resolution. In the sixth paragraph, we describe the geometry of this generalisation (one more time Weil, Cartier and canonical divisors, 1-cycles and intersection formulae) and describe all the relative Mori theory for a minuscule Schubert variety. In the last paragraph, we prove that all relative minimal models are $IH$-small and describe all $IH$-small resolutions of minuscule Schubert varieties. In this paragraph, in contrast with the rest of the paper, we use a case by case analysis. A general proof could be possible but we think it would be more complicated and would lead to much more combinatorics. In an appendix, we describe the quivers of minuscule Schubert variety. We use this description intensively in the last paragraph.
1 Notations

Let $G$ be a semi-simple algebraic group, fix $T$ a maximal torus and $B$ a Borel subgroup containing $T$. Let us denote by $\Delta$ the set of all roots, by $\Delta^+$ (resp. $\Delta^-$) the set of positive (resp. negative) roots, by $S$ the set of simple roots associated to the data $(G, T, B)$ and by $W$ the associated Weyl group. If $P$ is a parabolic subgroup containing $B$ we note $W_P$ the subgroup of $W$ corresponding to $P$. We will also denote by $\Sigma(P)$ the set of simple roots $\beta$ such that $U_\beta \not\subset P$.

**Definition 1.1.** — Let $w \in W$, let us denote by $P^w$ the largest parabolic subgroup of $G$ such that the morphism $BwB/B \to BwP^w/P^w$ is a $P^w/B$ fibration.

Let us also denote by $P_w$ the stabiliser of $X(w) = BwP^w/P^w$ in $G/P^w$.

**Definition 1.2.** — (i) Let $w \in W$, we define the support of $W$ denoted by $\text{Supp}(w)$ to be the set of simple roots $\beta$ such that $s_\beta$ appears in a reduced writing of $w$. This set is independent of the reduced writing and only depends on $w$.

(ii) We will denote by $G_w$ the smallest reductive subgroup of $G$ containing all the groups $U_\beta$ for $\beta \in \text{Supp}(w)$. It is easy to see that there is an isomorphism

$$X(w) = P_wwP^w/P^w \simeq (P_w \cap G_w)w(P_w \cap G_w)/(P^w \cap G_w).$$

(iii) Let us also define the boundary of $G_w$ denoted by $\partial(G_w)$ to be the set of simple roots $\beta$ not contained in $\text{Supp}(w)$ and non commuting with $w$.

2 Minuscule Schubert varieties

In this paragraph we recall the notion of minuscule weight and study the related homogeneous and Schubert varieties. Our basic reference will be [LMS].

**Definition 2.1.** — Let $\varpi$ be a fundamental weight, 

(i) we say that $\varpi$ is minuscule if we have $\langle \alpha^\vee, \varpi \rangle \leq 1$ for all positive root $\alpha \in \Delta^+$;

(ii) we say that $\varpi$ is cominuscule if $\langle \alpha_0^\vee, \varpi \rangle = 1$ where $\alpha_0$ is the longest root.

With the notation of N. Bourbaki [Bon], the minuscule and cominuscule weights are:

<table>
<thead>
<tr>
<th>Type</th>
<th>minuscule</th>
<th>cominuscule</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>$\varpi_1 \cdots \varpi_n$</td>
<td>same weights</td>
</tr>
<tr>
<td>$B_n$</td>
<td>$\varpi_n$</td>
<td>$\varpi_1$</td>
</tr>
<tr>
<td>$C_n$</td>
<td>$\varpi_1$</td>
<td>$\varpi_n$</td>
</tr>
<tr>
<td>$D_n$</td>
<td>$\varpi_1$, $\varpi_{n-1}$ and $\varpi_n$</td>
<td>same weights</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$\varpi_1$ and $\varpi_6$</td>
<td>same weights</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$\varpi_7$</td>
<td>same weight</td>
</tr>
<tr>
<td>$E_8$</td>
<td>none</td>
<td>none</td>
</tr>
<tr>
<td>$F_4$</td>
<td>none</td>
<td>none</td>
</tr>
<tr>
<td>$G_2$</td>
<td>none</td>
<td>none</td>
</tr>
</tbody>
</table>
**Definition 2.2.** — Let \( \varpi \) be a minuscule weight and let \( P_{\varpi} \) be the associated parabolic subgroup. The homogeneous variety \( G/P_{\varpi} \) is then said to be minuscule. The Schubert varieties of a minuscule homogeneous variety are called minuscule Schubert varieties.

An element \( w \in W \) is said to be minuscule if \( G/P_{\varpi}w \) is a minuscule homogeneous variety.

**Remark 2.3.** — To study minuscule homogeneous varieties and their Schubert varieties, it is sufficient to restrict ourselves to simply-laced groups.

In fact the variety \( G/P_{\varpi}n \) with \( G = \text{Spin}_{2n+1} \) is isomorphic to the variety \( G'/P'_{\varpi_{n+1}} \) with \( G' = \text{Spin}_{2n+2} \) and there is a one to one correspondence between Schubert varieties thanks to this isomorphism. The same situation occurs with \( G/P_{\varpi}1 \), \( G = \text{Sp}_{2n} \) and \( G'/P'_{\varpi_1} \), \( G' = \text{SL}_{2n} \).

### 3 Quivers and configuration varieties

#### 3.1 Quiver associated to a reduced writing

Let \( P_{\varpi} \) be a parabolic subgroup of \( G \) associated to a minuscule weight \( \varpi \), let us consider an element \( \bar{w} \in W/W_{P_{\varpi}} \) and let \( \bar{w} \) be the shortest element in the class \( \bar{w} \). To any reduced writing \( w = s_{\beta_1} \cdots s_{\beta_r} \) of \( \bar{w} \) in terms of simple reflections (for all \( i \in [1, r] \), we have \( \beta_i \in S \)) we associate a quiver with colored vertices. Let us first give the following

**Definition 3.1.** — For a fixed reduced writing \( (\Box) \) of \( w \), we define the successor \( s(i) \) (resp. the predecessor \( p(i) \)) of an element \( i \in [1, r] \) by \( s(i) = \min\{j \in [1, r] \mid j > i \text{ and } \beta_j = \beta_i\} \) resp. by \( p(i) = \max\{j \in [1, r] \mid j < i \text{ and } \beta_j = \beta_i\} \).

Now we can define the quiver \( Q_{\bar{w}} \) associated to the reduced writing \( (\Box) \) of \( w \) (we will denote by \( \tilde{w} \) the data of \( w \) with such a reduced writing):

**Definition 3.2.** — Let us denote by \( Q_{\bar{w}} \) the quiver whose set of vertices is in bijection with the set \( [1, r] \) and whose arrows are given in the following way: there is an arrow from \( i \) to \( j \) if \( \langle \beta_j, \beta_i \rangle \neq 0 \) and \( i < j < s(i) \).

This quiver comes with a coloration of its edges by simple roots thanks to the application \( \beta : [1, r] \rightarrow S \) such that \( \beta(i) = \beta_i \).

**Remark 3.3.** — It is equivalent to give the reduced writing \( \bar{w} \) or the quiver \( Q_{\bar{w}} \). This quiver seems to be the same as the one defined by S. Zelikson [Zel] for ADE types.

#### 3.2 Configuration varieties and Bott-Samelson resolution

In this paragraph, we associate to any quiver \( Q_{\bar{w}} \), a configuration variety \( \tilde{X}(\bar{w}) \) (see also [Ma]). Let \( x \) be an element in \( G/B \) and \( \beta_i \) a simple root. Let us denote by \( B_{\beta_i} \) the parabolic subgroup generated by \( B \) and \( U_{-\beta_i} \). We have a projection morphism \( \pi_{\beta_i} : G/B \rightarrow G/B_{\beta_i} \) whose fibers are isomorphic to \( \mathbb{P}^1 \) and we denote by \( \mathbb{P}(x, \beta_i) \) the projective line \( \pi_{\beta_i}^{-1}(\pi_{\beta_i}(x)) \).
### Definition 3.4

Let $Q_{\tilde{w}}$ a quiver associated to a writing $\tilde{w}$ of $w$. We define the configuration variety $\bar{X}(\tilde{w})$ by

$$
\bar{X}(\tilde{w}) = \left\{ (x_1, \ldots, x_r) \in \prod_{i=1}^{r} G/B / x_0 = 1 \text{ and } x_i \in \mathbb{P}(x_{i-1}, \beta_i) \text{ for all } i \in [1, r] \right\}.
$$

### Remark 3.5

(i) If we denote by $P_{\beta_i}$ the maximal parabolic not containing $U_{-\beta_i}$, the restriction of the morphism $G/B \to G/P_{\beta_i}$ to $\mathbb{P}(x, \beta_i)$ is an isomorphism so that $\bar{X}(\tilde{w})$ is isomorphic to

$$
\left\{ (x_1, \ldots, x_r) \in \prod_{i=1}^{r} G/P_{\beta_i} / x_0 = 1 \text{ and } x_i \in \mathbb{P}(x_{i-1}, \beta_i) \text{ for all } i \in [1, r] \right\}.
$$

(ii) In his article [Ma], P. Magyar shows that the variety $\tilde{X}(\tilde{w})$ is isomorphic to the classical Bott-Samelson variety described for example in [De].

Because the element $w$ is the shortest in the class $\tilde{w}$ and because the writing $\tilde{w}$ is reduced, the last root $\beta_r$ has to be the only simple root $\beta$ such that $\langle w^\vee, \beta \rangle = 1$ and $P_r = P_{w}$. The image of the projection morphism $\pi : \bar{X}(\tilde{w}) \to G/P_r$ is the minuscule Schubert variety $X(\tilde{w})$. The morphism $\pi : \tilde{X}(\tilde{w}) \to X(\tilde{w})$ is birational.

### 3.3 Cycles on the configuration variety

#### 3.3.1 A basis of the Chow ring

In this paragraph, we describe some particular elements in the Chow ring $A_*(\bar{X}(\tilde{w}))$. We describe a basis of this ring as the cones of ample divisors and effective curves. We calculate the canonical divisor in terms of these basis.

For all $k \in [1, r]$, let us denote by $X_k$ the image of $\bar{X}(\tilde{w})$ in the product $\prod_{i=1}^{k} G/P_{\beta_i}$ and $X_0 = \{x_0\}$. We have natural projection morphism $f_k : X_k \to X_{k-1}$ for all $k \in [1, r]$ which are $\mathbb{P}^1$-fibration. The morphism $\sigma_k : X_{k-1} \to X_k$ defined by $\sigma_{k-1}(x_1, \ldots, x_{k-1}) = (x_1, \ldots, x_{k-1}, x_{p(k)})$ with $x_{p(k)} = 1$ if $p(k)$ does not exist is a section of $f_k$. We recover in this way the structure of $\tilde{X}(\tilde{w})$ as a tower of $\mathbb{P}^1$-fibrations with sections described in [De].

Let us define the divisors $Z_i = f_r^{-1} \cdots f_{i+1}^{-1} \sigma_i(X_{i-1})$. The divisors $(Z_i)_{i \in [1, r]}$ have normal crossing (cf. for example [De]). We have

$$
Z_i = \left\{ (x_1, \ldots, x_r) \in \bar{X}(\tilde{w}) / x_i = x_{p(i)} \right\}
$$

in the configuration variety with $x_{p(i)} = 1$ if $p(i)$ does not exist. Then for any subset $K$ of $[1, r]$, one defines $Z_K = \bigcap_{i \in K} Z_i$. Let us recall the following (see for example [De]):

**Fact 3.6.** — The image by $\pi$ of $Z_K$ is the Schubert subvariety $X(y)$ where $y$ is the longest element that can be written as a subword of $\tilde{w}$ without the terms $s_i$ for $i \in K$.  

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Denote by $\xi_i$ the class of $Z_i$ in $A^*(\tilde{X}(\bar{w}))$. These classes form a basis of the Chow ring.

**Theorem 3.7.** — (Demazure [De] Par. 4. prop. 1) The Chow ring $A^*(\tilde{X}(\bar{w}))$ of $\tilde{X}(\bar{w})$ is isomorphic over $\mathbb{Z}$ to

$$\mathbb{Z}[\xi_1, \cdots, \xi_r] / \left( \xi_i \cdot \sum_{j=1}^{i} (\alpha_j^\vee, \alpha_i) \xi_j \text{ for all } i \in [1, r] \right).$$

Let us recall the following result (which is no longer true if the writing is not reduced):

**Proposition 3.8.** — [LT] The divisors $(\xi_i)_{1 \leq i \leq r}$ form a basis of the cone of effective divisors.

Let us denote by $T_i$ the pull-back on $\tilde{X}(\bar{w})$ of the relative tangent sheaf of the fibration $f_i$. Define the classical sequence of roots $(\alpha_i)_{i \in [1, r]}$ associated to $\bar{w}$ by $\alpha_1 = \beta_1$, $\alpha_2 = s_{\beta_1}(\beta_2)$, $\cdots$, $\alpha_r = s_{\beta_1} \cdots s_{\beta_{r-1}}(\beta_r)$. We denote by $C_i$ the curve $Z_K$ with $K = [1, r] \setminus \{i\}$ and recall some formulae in the ring $A^*(\tilde{X}(\bar{w}))$ given in [Pe1] corollary 3.8 and propositions 3.3 and 3.11:

**Proposition 3.9.** — We have the following formulae:

$$[C_i] \cdot \xi_j = \begin{cases} 0 & \text{for } i > j \\ 1 & \text{for } i = j \\ \langle \beta_i^\vee, \beta_j \rangle & \text{for } i < j \end{cases}, \quad [C_i] \cdot T_j = \begin{cases} 0 & \text{for } i > j \\ \langle \beta_i^\vee, \beta_j \rangle & \text{for } i \leq j \end{cases} \text{ and } T_i = \sum_{k=1}^{r} \langle \alpha_k^\vee, \alpha_i \rangle \cdot \xi_k.

This proposition and the formula $c_1(\tilde{X}(\bar{w})) = \sum_{i=1}^{r} T_i$ gives:

$$c_1(\tilde{X}(w)) = -K_{\tilde{X}(\bar{w})} = \sum_{k=1}^{r} \left( \sum_{i=k}^{r} \langle \alpha_k^\vee, \alpha_i \rangle \right) \xi_k.$$

### 3.3.2 Ample divisors

In this paragraph, we describe the ample divisors on $\tilde{X}(\bar{w})$. This has already been done in [LT] but we rephrase it in terms of configuration varieties. We also get a description of the Mori cone (see [Mat] for references on this cone).

We have natural morphisms $p_i : \tilde{X}(\bar{w}) \to G/P_{\beta_i}$, and as $P_{\beta_i}$ is maximal, the Picard group of $G/P_{\beta_i}$ is generated by a very ample divisor $O_{G/P_{\beta_i}}(1)$ and we define on $\tilde{X}(\bar{w})$ the invertible sheaf $L_i = p_i^*(O_{G/P_{\beta_i}}(1))$. These sheaves will form a basis of the ample cone.

We also define particular curves $Y_i$ for $i \in [1, r]$ on $\tilde{X}(\bar{w})$ by:

$$Y_i = \left\{ (x_1, \cdots, x_r) \in \tilde{X}(\bar{w}) / x_j = 1 \text{ for } j \neq i \right\}.$$

The following lemma shows that $Y_i$ is a curve isomorphic to $\mathbb{P}(1, \beta_i)$.

**Lemma 3.10.** — For any $x_i \in \mathbb{P}(1, \beta_i)$, the element $(x_j)_{j \in [1, r]}$ of $\prod_{j=1}^{r} G/P_{\beta_j}$ such that $x_j = 1$ for all $j \neq i$ is in the configuration variety $\tilde{X}(\bar{w})$.  

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Proof — We only have to prove that for any \( x_i \) in \( \mathbb{P}(1, \beta_i) = B_{\beta_i}/B \), we have \( 1 \in \mathbb{P}(x_i, \beta_{i+1}) \).

The element \( x_i \) can be lifted in \( b_i \in B_{\beta_i} \). The elements of \( \mathbb{P}(x_i, \beta_{i+1}) \) are the classes of elements of the form \( b_i b_{i+1} \in B_{\beta_i} B_{\beta_{i+1}} \). If \( \beta_{i+1} \neq \beta_i \) (this is always the case if the writing is reduced), then \( B_{\beta_i} \subset P_{\beta_{i+1}} \). In this case we set \( b_{i+1} = 1 \) so that the class of \( b_i b_{i+1} \in \mathbb{P}(x_i, \beta_{i+1}) \) is \( 1 \) in \( G/P_{\beta_{i+1}} \). If \( \beta_{i+1} = \beta_i \), then we set \( b_{i+1} = b_i^{-1} \) to get the result. □

The definitions of the curves \( Y_i \) and the line bundles \( L_i \) yield to following:

**Proposition 3.11.** — We have the formula \( L_i \cdot [Y_j] = \delta_{i,j} \). In other words the families \( (L_i)_{i \in [1,r]} \) and \(( [Y_i] )_{i \in [1,r]} \) are dual to each other.

Let us prove that the family \(( [Y_i] )_{i \in [1,r]} \) forms a basis of \( A_1(\bar{X}(\bar{w})) \).

**Proposition 3.12.** — For all \( i \in [1,r] \), we have \( [Y_i] = [C_i] - [C_{s(i)}] \) (where \([C_{s(i)}] = 0 \) if \( s(i) \) doesn’t exist).

In consequence, the classes \(( [Y_i] )_{i \in [1,r]} \) form a basis of \( A_1(X_Q) \) over \( \mathbb{Z} \) and the classes \(( L_i )_{i \in [1,r]} \) form a basis of \( A^1(X_Q) \) over \( \mathbb{Z} \).

Proof — On the first hand, the curve \( C_i \) is given by the equations \( x_j = x_{p(j)} \) for \( j \neq i \). This means that for \( j < i \), we have \( x_j = 1 \) and for all \( j \) with \( \beta_j \neq \beta_i \), we also have \( x_j = 1 \). The only indices \( k \) for which \( x_k \) may be different from 1 are such that \( k = s^n(i) \) for some \( n \in \mathbb{N} \). For such a \( k \), we have the equality \( x_k = x_i \). Denote by \( n(i) \) the biggest integer \( n \) such that \( s^n(i) \) exists. The curve \( C_i \) (resp. \( C_{s(i)} \)) is the diagonal in the product

\[
\prod_{k=0}^{n(i)} \mathbb{P}(1, \beta_{s_k(i)}) \quad \text{resp.} \quad \prod_{k=1}^{n(i)} \mathbb{P}(1, \beta_{s_k(i)}).
\]

On the other hand, the curve \( Y_i \) corresponds to the first factor of the first product. In this product we thus have the required equality. □

We can now describe the ample cone (see also [LT]) and the cone of effective curves.

**Corollary 3.13.** — The cone of ample divisors is generated by the classes \( L_i \) and the cone of effective curves is generated by the classes \([Y_i]\). All ample divisors are very ample.

Proof — Let \( D \) be ample on \( \bar{X}(\bar{w}) \), then \( a_i = D \cdot [Y_i] \) is a positive integer. Because of proposition 3.11, we have \( D = \sum_{i=1}^{r} a_i L_i \) and \( D \) lies in the cone generated by the \( L_i \).

Conversely, any divisor \( \sum_{i=1}^{r} a_i L_i \) with \( a_i > 0 \) gives the embedding of \( \bar{X}(\bar{w}) \) obtained by composing the inclusion in the product \( \prod_{i=1}^{r} G/P_{\beta_i} \) with the Veronese morphism given by the very ample sheaf \( \bigotimes_{i=1}^{r} \mathcal{O}_{G/P_{\beta_i}}(a_i) \).

In the same way we get the result on effective curve. □

Finally we calculate the divisors \( L_i \) in terms of the basis \(( \xi_k )_{k \in [1,r]} \):
**Proposition 3.14.** — The $k$th coordinate of $L_i$ in the basis $(\xi_i)_{i \in [1,r]}$ is 0 if $k > i$, 1 if $k = i$ and is given by the following formulae if $k < i$ and $\beta_k = \beta_i$ (resp. $\beta_k \neq \beta_i$):

$$1 + \sum_{j=k+1, \beta_j = \beta_i}^{i} \langle \alpha_k^\vee, \alpha_j \rangle \quad \text{(resp. } \sum_{j=k+1, \beta_j = \beta_i}^{i} \langle \alpha_k^\vee, \alpha_j \rangle \text{)}.$$  

In particular we have the following simple formula

$$L_r = \sum_{k=1}^{r} \xi_k.$$  

**Proof** — Let us recall from [Pe1] lemma 4.5 that the following classes of curves $[\hat{C}_i] = [C_i] + \sum_{k=i+1}^{r} \langle \alpha_i^\vee, \alpha_k \rangle [C_k]$ form a dual basis to $(\xi_i)_{i \in [1,r]}$. The $k$th coordinate is thus given by the intersection $L_i \cdot [\hat{C}_k]$.

For this we will need the formula coming directly from propositions 3.11 and 3.12

$$L_i \cdot [C_j] = \begin{cases} 1 & \text{for } i > j \text{ and } \beta_i = \beta_j \\ 0 & \text{otherwise.} \end{cases}$$

Applying this gives the first formula. For the case of $L_r$, the formula is a consequence of the following formula from [Pe1] corollary 2.18:

$$\sum_{j=k+1, \beta_j = \beta_r}^{r} \langle \alpha_k^\vee, \alpha_j \rangle = \begin{cases} 1 & \text{if } \beta_k \neq \beta_r \\ 0 & \text{if } \beta_k = \beta_r. \end{cases}$$

\[\square\]

### 4 Geometry of the quiver

In this paragraph, we give an explicit description of the quiver $Q_{\bar{w}}$ given by the reduced writing $w = s_{\beta_1} \cdots s_{\beta_r}$ of the shortest element in the class $\bar{w} \in W/W_{P_{\bar{w}}}$. We also define some invariants of the quiver and deduce some consequences on the geometry of the Schubert variety.

#### 4.1 Minuscule conditions on the quivers

Set $w_i = s_{\beta_i} \cdots s_{\beta_r}$ for $i \in [1,r]$ and $w_{r+1} = 1$ and let us first recall the following result from [LMS] (proof of theorem 3.1):

**Fact 4.1.** — We have $\langle \beta_i^\vee, w_i(-\varpi) \rangle = -1$ for all $i \in [2, r+1]$. In consequence we have for all $i \in [2, r]$:

$$w_i(-\varpi) = -\varpi + \beta_r + \cdots + \beta_i.$$  

The following proposition describes all possible quivers for minuscule Schubert varieties.

**Proposition 4.2.** — Geometry of the quiver.
(i) There is no arrow from the vertex $r$ and $\beta_r$ is the unique simple root with $\langle \beta_r^\vee, w \rangle = 1$.

(ii) If a vertex $i < r$ of the quiver is such that $s(i)$ does not exist, then there is a unique arrow from $i$. If $k$ is the end of the arrow we have $\langle \beta_i^\vee, \beta_k \rangle = -1$.

(iii) If a vertex $i$ of the quiver is such that $s(i)$ exists, then there are exactly two arrows from $i$. If $k_1$ and $k_2$ are the end of these arrows we have $\langle \beta_i^\vee, \beta_{k_1} \rangle = \langle \beta_i^\vee, \beta_{k_2} \rangle = -1$.

Proof — (i) The previous fact shows that we have $\langle \beta_r^\vee, w \rangle = 1$.

(ii) Let $i$ be such a vertex. In particular we have $\beta_i \neq \beta_r$ and $\langle \beta_i^\vee, w \rangle = 0$. The previous fact gives $\sum_{k=i+1}^r \langle \beta_i^\vee, \beta_k \rangle = -1$.

We conclude because every term of this sum have to be either 0 or $-1$.

(iii) Let $i$ be such a vertex. The same calculation as above shows that $\sum_{k=s(i)+1}^r \langle \beta_i^\vee, \beta_k \rangle = -1$ if $\beta_i \neq \beta_r$ and $\sum_{k=i+1}^r \langle \beta_i^\vee, \beta_k \rangle = 0$ if $\beta_i = \beta_r$. In particular we always have $\sum_{k=i+1}^{s(i)} \langle \beta_i^\vee, \beta_k \rangle = 0$.

We conclude because the only positive term is $\langle \beta_i^\vee, \beta_{s(i)} \rangle = 2$ and every other term of this sum have to be either 0 or $-1$. $\square$

Remark 4.3. — (i) One can deduce from this result (see [Pe2]) the fact already proved by J. R. Stembridge [St] that the reduced writing $\tilde{w}$ is unique modulo commutation relations (there is no braid relation). We can thus write $Q_w$ instead of $Q_{\tilde{w}}$ and call it the quiver associated to the minuscule Schubert variety $X(w)$.

Furthermore, because of this unicity and the fact that between $i$ and $s(i)$ there are always two vertices, the writing deduced from a quiver satisfying the conditions of the preceding proposition is always reduced and the quivers satisfying the conditions are always quivers associated to a minuscule Schubert variety.

(ii) A minuscule quiver is always connected: there is a path from any vertex $i$ to the last vertex $r$.

4.2 Combinatoric description of the minuscule quivers

It is easy from proposition 4.2 to describe the quivers $Q_w$ of a minuscule homogeneous variety $G/P_w$ (see appendix for a list of these quivers). We now describe the quivers of minuscule Schubert varieties in $G/P_w$ as subquivers of $Q_w$. Define a natural partial order on the quiver:

Definition 4.4. — (i) We denote by $\preceq$ the partial order on the vertices of the quiver generated by the relations $i \preceq j$ if there exists an arrow from $i$ to $j$. 

...
Let $A$ be a totally unordered set of vertices of the quiver $Q_\varpi$ for the partial order $\preceq$. We denote by $Q_A$ the full subquiver of $Q_\varpi$ with vertices $i \in Q_\varpi$ such that there exists $a \in A$ with $i \preceq a$ and by $Q_A$ the full subquiver of $Q_\varpi$ whose vertices are not vertices of $Q_A$.

**Proposition 4.5.** — The quiver of Schubert varieties in $G/P_\varpi$ are in one to one correspondence with the subquivers $Q_A$ of $Q_\varpi$ for $A$ any totally unordered set of vertices of $Q_\varpi$.

With this correspondence, the bruhat order is given by the inclusion of quivers.

**Proof** — Let $X(w) \subset G/P_\varpi$ a Schubert variety and denote by $w_0 \in W$ the only element such that $X(w_0) = G/P_\varpi$. There exists a sequence $(\beta_1, \cdots, \beta_i)$ of simple roots such that $w_0 = s_{\beta_1} \cdots s_{\beta_i} w$. Taking a reduced writing $w = s_{\beta_{i+1}} \cdots s_{\beta_r} w$ of $w$ we get a reduced writing of $w_0$ which is unique modulo commutation relations.

The vertices of the quiver $Q_\varpi$ are indexed by $[1, r]$. Denote by $A = \{i_1, \cdots, i_k\}$ the set of maximal elements for the partial order $\preceq$ of the set $[1, i]$. The set $A$ is totally unordered and the quiver associated to $X(w)$ is $Q_A$.

The fact that this is a one to one correspondence comes from the unicity of the reduced writing. □

**Remark 4.6.** — As a corollary one can prove (see [Pe2]) the following classical result on the Bruhat order (see for example [LMS]): the Bruhat order in $W/W_P$ is generated by simple reflexions. In other words, all Schubert divisors are mobile.

Finally let us define some particular vertices of these quivers. In the following definition $Q_w$ is the quiver of a minuscule Schubert variety $X(w)$.

**Definition 4.7.** — (i) We call pic any vertex of $Q_w$ minimal for the partial order $\preceq$. We denote by $p(Q_w)$ the set of pics of $Q_w$.

(ii) We call hole of the quiver $Q_w$ any vertex $i$ of $Q_w$ such that $p(i)$ does not exist and there are exactly two vertices $j_1 \preceq i$ and $j_2 \preceq i$ in $Q_w$ with $\langle \beta^*_i, \beta_j \rangle \neq 0$ for $k = 1, 2$.

We will also call a hole of $Q_w$ any $i \in Q_\varpi \setminus Q_w$ such that $s(i)$ does not exist in $Q_\varpi$ and $\beta_i \in \partial(G_w)$. Such a hole will be called a virtual hole. We denote by $t(Q_w)$ the set of holes of $Q_w$.

(iii) The height $h(i)$ of a vertex $i$ is the largest $n$ such that there exists a sequence $(i_k)_{k \in [1, n]}$ of vertices with $i_1 = 1$, $i_n = r$ and such that there is an arrow from $i_k$ to $i_{k+1}$ for all $k \in [1, n-1]$.

**Remark 4.8.** — (i) If $Q_w = Q_A$ as in definition 4.4 then $t(Q_w) = A$.

(ii) The height is well defined because there is at least one path from any vertex $i$ to the last vertex $r$.

The following proposition gives a recursive way to calculate the height of a vertex.

**Proposition 4.9.** — Let $Q$ be a quiver associated to a minuscule Schubert variety and $i$ a vertex of this quiver. We have the following cases:
• if \( s(i) \) does not exist, then there exists a unique \( k \succ i \) with \( \langle \beta_k^\vee, \beta_i \rangle = -1 \) and we have

\[
h(i) = h(k) + 1.
\]

• If \( s(i) \) exists, then there exists a non negative integer \( n \) and a sequence \( (j_k, j'_k)_{k \in [0, n+1]} \) of vertices with \( j_0 = i, j'_k = s(j_k) \) for \( k \in [0, n] \), \( \beta_{j_{n+1}} \neq \beta_{j'_{n+1}} \), \( \langle \beta_{j_k}, \beta_{j_{k+1}} \rangle = -1 \) and \( \langle \beta_{j_k'}, \beta_{j'_{k+1}} \rangle = -1 \) for \( k \in [0, n] \) and \( j_0 \preceq \cdots \preceq j_n \preceq j_{n+1}, j'_n \preceq \cdots \preceq j'_0 \). In this case we have

\[
h(j_k) = 2n + 2 - k + h(s(i)) \quad \text{and} \quad h(j'_k) = k + h(s(i)) \quad \forall k \in [0, n+1].
\]

**Proof** — We prove these formulae by descending induction on \( i \). If \( i = r \) then \( h(i) = 1 \). Assume the proposition is true for all \( j > i \). In the first case, any sequence of arrows from \( i \) to \( r \) has to pass through the vertex \( k \) and we have \( h(i) = h(k) + 1 \).

The first to prove in the second case is the existence of the sequence \( (j_k, j'_k)_{k \in [0, n+1]} \). Let us denote by \( j_1 \) and \( j'_1 \) the two vertices \( j \) such that there is an arrow from \( i \) to \( j \). If \( \beta_{j_1} \neq \beta_{j'_1} \) then set \( n = 0 \) and we are done. Otherwise, assume (for example) that \( j_1 < j'_1 \) then \( j'_1 = s(j_1) \). Indeed, if it was not the case there would exist \( k \in [j_1, j'_1] \) and in particular \( k < s(i) \) with \( \beta_k = \beta_{j_1} \) thus \( \langle \beta_{j'_1}^\vee, \beta_k \rangle = -1 \). By construction of the quiver, there must be an arrow from \( i \) to \( k \) and thus at least three arrows from \( i \). This is impossible by proposition 4.12. We can construct with \( (j_1, j'_1) \) a couple \( (j_2, j'_2) \) in the same way and by induction a sequence \( (j_k, j'_k)_{k \in [0, n+1]} \). As long as \( \beta_{j_k} = \beta_{j'_k} \) we can go on. This has to stop because the quiver is finite.

The formula is now clear because any sequence from \( i \) to \( r \) as to go through \( j_1 \) or \( j'_1 \). As the height of \( j_1 \) is bigger than the one of \( j'_1 = s(j_1) \) by induction we must have \( h(i) = h(j_1) + 1 \) and we conclude one more time by induction. \( \square \)

**Remark 4.10.** — By changing the order of commuting factors, we may assume in the preceding proposition that \( k = i + 1 \) in the first case and that \( j_k = i + k \) and \( j'_k = i + 2n + 3 - k \) for all \( k \in [0, n+1] \) in the second one.

We can now describe the stabiliser of a Schubert variety \( X(w) \) thanks to its quiver \( Q_w \).

**Proposition 4.11.** — Let \( J \) be the set of simple roots not in \( \beta(t(Q_w)) \). The stabiliser of \( X(w) \) is the parabolique subgroup \( P_J \) generated by \( B \) and the groups \( U_{-\beta} \) with \( \beta \in J \).

**Proof** — A simple root \( \beta \) is such that \( U_{-\beta} \subset P \) if and only if \( s_\beta w < w \) (for the Bruhat order). But from unicity of reduced writing and our characterisation (proposition 4.12) of quivers associated to a reduced writing, we see that this implies that \( \beta \) to be in \( \beta(t(Q_w)) \). \( \square \)

**Corollary 4.12.** — Let \( Q_{w'} \) be the quiver of a Schubert subvariety \( X(w') \) of \( X(w) \) stable under \( P_J = \text{Stab}(X(w)) \). Then \( \beta(t(Q_{w'})) \subset \beta(t(Q_w)) \).
4.3 Weil and Cartier divisors

In this paragraph, we describe thanks to the quiver the Weil and Cartier divisors of a minuscule Schubert variety \( X(w) \). We also compute the canonical sheaf of \( X(w) \).

**Proposition 4.13.** — The group \( \text{Weil}(X(w)) \) of Weil-divisors is the free \( \mathbb{Z} \)-module generated by the classes \( D_i := \pi_* \xi_i \) for \( i \in p(Q_w) \).

The Picard group \( \text{Pic}(X(w)) \subset \text{Weil}(X(w)) \) is isomorphic to \( \mathbb{Z} \) and is generated by the element \( L(w) := \pi_* L_r = \mathcal{O}_{G/P_{\beta_r}}(1)|_{X(w)} \). We have the formula

\[
L(w) = \sum_{i \in p(Q_w)} D_i.
\]

**Proof** — It is well known (see for example [Br]) that the Picard group is isomorphic to \( \mathbb{Z} \) and generated by \( L(w) \) and that the group of Weil divisors is free generated by the divisorial Schubert varieties. These varieties are the images by \( \pi : \tilde{X}(\tilde{w}) \to X(w) \) of the non contracted divisors \( Z_i \). Now the divisor \( Z_i \) is the configuration variety obtained from the quiver \( Q \) with the vertex \( i \) removed (the arrows are reorganised as in definition 3.1). According to the fact 3.6, the image of \( Z_i \) is not contracted if and only if the quiver is reduced (ie correspond du a reduced writing). It is clear that this can only be the case for \( i \in p(Q_w) \).

The last formula is an application of corollary 3.14. We recover the well known fact that all Schubert divisors are of multiplicity one in the minuscule case. \( \square \)

**Corollary 4.14.** — A minuscule Schubert variety is locally factorial if and only if its quiver has a unique pic.

4.4 Canonical divisor

The Schubert varieties are singular and in general not Gorenstein (see [WY] for a characterisation of Gorenstein Schubert varieties for \( GL_n \)). We can therefore not define the canonical divisor as a Cartier divisor.

The canonical divisor \( K_{X(w)} \) of a Schubert variety \( X(w) \) is well defined as a Weil divisor thanks to the divisor of a 1-form on \( X(w) \). The properties of Schubert varieties (they are normal, Cohen-Macaulay with rational singularities) and the Bott-Samelson resolution \( \pi : \tilde{X}(\tilde{w}) \to X(w) \) enables however to calculate \( K_{X(w)} \) by \( K_{X(w)} = \pi_*(K_{\tilde{X}(\tilde{w})}) \) (see for example [BK] paragraph 3.4).

Let us denote by \( h(w) \) the lowest height of a pic in \( Q_w \) (the quiver associated to \( X(w) \)), we have the following:

**Proposition 4.15.** — We have the formula

\[
-K_{X(w)} = \sum_{i \in p(Q_w)} (h(i) + 1)D_i = (h(w) + 1)L(w) + \sum_{i \in p(Q_w)} (h(i) - h(w))D_i.
\]
PROOF — The second part of the formula comes from the first one and proposition 4.13.

To prove the first part, we use the fact that $K_X(w) = \pi_*(K_{\tilde{X}(\tilde{w})})$ and the formula of paragraph 4.3.1. We are left to prove the following:

**Lemma 4.16.** — We have the formula:

$$\sum_{k=i}^{r} \langle \alpha_i^\vee, \alpha_k \rangle = \sum_{k \neq i} \langle \alpha_i^\vee, \alpha_k \rangle = h(i) + 1$$

**Proof** — We proceed by descending induction and use the proposition 4.9 and the remark 4.10. We have the following two cases:

- if $s(i)$ does not exist, then with $\langle \beta_{i+1}, \beta_i \rangle = -1$.
- If $s(i)$ exists, then there exists a non negative integer $n$ and a sequence $(j_k, j'_k)_{k \in [0, n+1]}$ of vertices with $j_0 = i$, $j'_k = s(j_k)$ for $k \in [0, n]$, $\beta_{j_{n+1}} \neq \beta_{j'_{n+1}}$, $\langle \beta_k, \beta_{j_{k+1}} \rangle = -1$ and $\langle \beta'_{j_k}, \beta'_{j'_{k+1}} \rangle = -1$ for $k \in [0, n]$ and $j_0 \leq \cdots \leq j_n \leq j'_{n+1} \leq j'_n \leq \cdots \leq j'_0$. Furthermore, we may assume that $j_k = i + k$ and $j'_k = i + 2n + 3 - k$ for all $k \in [0, n+1]$.

We proceed by descending induction on $i$. If $i = r$, there is only one term and the sum is $\langle \alpha_r^\vee, \alpha_r \rangle = 2 = h(r) + 1$. We assume that the formula is true for all $j \geq i + 1$. Let us use the following sequence $\tilde{\alpha}_j = s_{\beta_r} \cdots s_{\beta_{r-j+2}}(\beta_{r-j+1})$ already introduced in [Pell] and satisfying the equality $\langle \tilde{\alpha}_{r+1-k}, \tilde{\alpha}_{r+1-i} \rangle = \langle \alpha_k, \alpha_i \rangle$.

Calculating $\tilde{\alpha}_{r+1-i} = s_{\beta_r} \cdots s_{\beta_{i+1}}(\beta_i)$ we find

$$\tilde{\alpha}_{r+1-i} = \begin{cases} 
\tilde{\alpha}_{r-i} + s_{\beta_r} \cdots s_{\beta_{i+2}}(\beta_i) & \text{in the first case;} \\
\tilde{\alpha}_{r-i-n} + \tilde{\alpha}_{r-i-n-1} - \tilde{\alpha}_{r-i-2n-2} & \text{in the second case;}
\end{cases}$$

Let us now calculate the sum

$$\sum_{k \neq i} \langle \alpha_k^\vee, \alpha_i \rangle = \sum_{k \neq i} \langle \tilde{\alpha}_{r+1-k}^\vee, \tilde{\alpha}_{r+1-i} \rangle.$$

In the first case, denote $\alpha = s_{\beta_r} \cdots s_{\beta_{i+2}}(\beta_i)$. If $j \geq i + 2$, then we have $\langle \beta_j^\vee, \beta_j \rangle = 0$ so that $\alpha = \beta_i$. Furthermore, the root $\tilde{\alpha}_{r+1-j}$ is a sum of simple roots contained in the set $\{\beta_j, \cdots, \beta_r\}$ so for $j \geq i + 2$, we have $\langle \tilde{\alpha}_{r+1-j}^\vee, \alpha \rangle = 0$. In this case the sum equals:

$$\sum_{k \neq i} \langle \tilde{\alpha}_{r+1-k}^\vee, \tilde{\alpha}_{r+1-i} \rangle = \sum_{k \neq i} \langle \tilde{\alpha}_{r+1-k}^\vee, \tilde{\alpha}_{r-i} \rangle + \sum_{k \neq i} \langle \tilde{\alpha}_{r+1-k}^\vee, \alpha \rangle = \sum_{k \neq i+1} \langle \tilde{\alpha}_{r+1-k}^\vee, \tilde{\alpha}_{r-i} \rangle + \langle \tilde{\alpha}_{r-i}^\vee, \alpha \rangle + \langle \tilde{\alpha}_{r-i}^\vee, \alpha \rangle = h(i+1) + 1 - \langle \beta_i^\vee, \beta_i+1 \rangle + \langle (\beta_i + \beta_{i+1})^\vee, \beta_i \rangle + \langle \beta_i+1^\vee, \beta_i \rangle = h(i+1) + 1 + 1 + 1 - 1 = h(i+1) + 2.$$
In the second case, it is an easy exercise to see that the roots \( \{ \beta_k \}_{k \in [i, i+n+1]} \) form a diagram of type \( D_{n+2} \) (with the notations of \([Bou]\) the root \( \beta_{i+k} \) is the \((k+1)\)-th root of the diagram). We can then calculate

\[
\tilde{\alpha}_{r+1-i-k} = \begin{cases} 
\sum_{j=0}^{2n+2-k} \beta_{i+j} & \text{for all } k \in [0, n+1] \\
\sum_{j=0}^{2n+3-k} \beta_{i+j} & \text{for all } k \in [n+2, 2n+3]
\end{cases}
\]

The sum is in this case:

\[
\sum_{k \geq i} \langle \tilde{\alpha}_{r+1-k}, \tilde{\alpha}_{r-i} \rangle = \sum_{k \geq i} \langle \tilde{\alpha}_{r+1-k}, \tilde{\alpha}_{r-i} \rangle + \sum_{k \geq i} \langle \tilde{\alpha}_{r+1-k}, \tilde{\alpha}_{r-i} \rangle - \sum_{k \geq i} \langle \tilde{\alpha}_{r+1-k}, \tilde{\alpha}_{r-i} \rangle
\]

\[
= \sum_{k \geq i+n+1} \langle \tilde{\alpha}_{r+1-k}, \tilde{\alpha}_{r-i} \rangle + \sum_{k \geq i+n+2} \langle \tilde{\alpha}_{r+1-k}, \tilde{\alpha}_{r-i} \rangle - \sum_{k \geq i+n+3} \langle \tilde{\alpha}_{r+1-k}, \tilde{\alpha}_{r-i} \rangle
\]

The description of \( \tilde{\alpha}_{r+1-i-k} \) shows that all roots \( \tilde{\alpha}_{r+1-i-k} \) for \( k \in [0, n] \) (resp. \( k \in [1, 2n+2] \)) have intersection 1 with \( \tilde{\alpha}_{r-i} \) and \( \tilde{\alpha}_{r-i-1} \) (resp. \( \tilde{\alpha}_{r-i-2} \)) and we have the intersections \( \langle \tilde{\alpha}_{r-i}, \tilde{\alpha}_{r-i} \rangle = 0 \) and \( \langle \tilde{\alpha}_{r+1-i}, \tilde{\alpha}_{r-i} \rangle = 0 \). This gives us the following formulae:

\[
\sum_{k=0}^{n} \langle \tilde{\alpha}_{r+1-k}, \tilde{\alpha}_{r-i} \rangle = n+1; \quad \sum_{k=0}^{n+1} \langle \tilde{\alpha}_{r+1-k}, \tilde{\alpha}_{r-i} \rangle = n+1; \quad \sum_{k=0}^{2n+2} \langle \tilde{\alpha}_{r+1-k}, \tilde{\alpha}_{r-i} \rangle = 2n+2.
\]

Using the induction hypothesis we get:

\[
\sum_{k \geq i} \langle \tilde{\alpha}_{r+1-k}, \tilde{\alpha}_{r+1-i} \rangle = h(i+n+1) + h(i+n+2) + 1 - h(i+2n+3) - 1 + n+1 + n+1 - 2n+2
\]

\[
= h(i+n+1) + h(i+n+2) + 1 - h(i+2n+3).
\]

We conclude in both cases thanks to proposition \([Bou]\) \( \square \).

This lemma completes the proof of the proposition. \( \square \)

**Corollary 4.17.** — The Schubert variety \( X(w) \) is Gorenstein if and only if all the pics of its quiver have the same height. In this case we have \( K_{X(w)} = (h(w)+1)L(w) \).

**Proof** — The variety \( X(w) \) is Gorenstein if and only if its canonical divisor is Cartier. The preceding formula shows that this is equivalent to the fact that all the pics of its quiver have the same height. \( \square \)

**Remark 4.18.** — For \( GL(n) \), we recover a particular case of the result of A. Woo and A. Yong \([WY]\) on Gorenstein Schubert varieties.
5 Generalisation of Bott-Samelson’s construction

In this paragraph, we are going to construct some varieties $\hat{X}(\hat{w})$ together with a birational morphism $\hat{\pi} : \hat{X}(\hat{w}) \to X(w)$. These constructions generalise the $IH$-small resolutions constructed by A. Zelevinsky [Zé] and P. Sankaran and P. Vanchinathan [SV1].

Recall that the Bott-Samelson varieties can be seen as a tower of $\mathbb{P}^1$-fibrations coming from a reduced writing $\hat{w}$ of an element $w$. Many generalisations of this construction (for example in [Zé] and [SV1] but also in [Pe3] and even in the general construction of C. Contou-Carrère [CC]) are constructed as towers of locally trivial fibrations with fibers isomorphic to a fixed homogeneous variety thanks to a more general decomposition $\hat{w}$ of $w$ as a product of elements in the Weyl group. For the varieties $\hat{X}(\hat{w})$ we make the same construction but we allow locally trivial fibrations with fiber a locally factorial or gorenstein Schubert variety.

5.1 Elementary construction

Let us explain the following elementary construction. As in [De], the construction of $\hat{X}(\hat{w})$ will simply be a successive application of this elementary construction. Let $u \in W$ and $Y$ a variety with action of $P_Y$ a parabolic subgroup of $G$ containing $B$ and assume that $P^u \cap G_u \subset P_Y$. We define

$$\hat{Y}(u) = \left( \frac{P_u \cap G_u u (P^u \cap G_u)}{P^u \cap G_u} \right) \times (P^u \cap G_u) Y.$$

**Lemma 5.1.** — (i) The variety $\hat{Y}(u)$ is a locally trivial fibration over $X(u)$ with fibers isomorphic to $Y$.

(ii) Define the parabolic $P_{\hat{Y}(u)}$ by $\Sigma(P_{\hat{Y}(u)}) = (\Sigma(P_Y) \cap \text{Supp}(u)^c) \cup \partial(G_u) \cup (\Sigma(P_u) \cap \text{Supp}(u))$. There is an action of $P_{\hat{Y}(u)}$ on $\hat{Y}(u)$.

**Proof** — (i) The first part of the proposition comes from the isomorphism between $X(u)$ and $\left( \frac{P_u \cap G_u u (P^u \cap G_u)}{P^u \cap G_u} \right)$. (ii) For the second part, let us remark that one can replace the groups $P_\cap G_u$ and $P^u \cap G_u$ by bigger groups $A$ and $B$ such that the natural map $\left( \frac{P_u \cap G_u u (P^u \cap G_u)}{P^u \cap G_u} \right) \to A \cup B / B$ is an isomorphism and $B \subset P_Y$. For example, we take $A$ and $B$ such that $\Sigma(A) = (\Sigma(P_Y) \cap \text{Supp}(u)^c) \cup \partial(G_u) \cup (\Sigma(P_u) \cap \text{Supp}(u))$ and $\Sigma(B) = (\Sigma(P_Y) \cap \text{Supp}(u)^c) \cup (\Sigma(P^u) \cap \text{Supp}(u))$. We have the required isomorphism and $B \subset P_Y$ (simply because $\Sigma(P_Y) \subset \Sigma(B)$). Then $\hat{Y}(u)$ is isomorphic to $A \cup B \times \hat{Y}(u)$ and $A$ acts on $\hat{Y}(u)$.

5.2 Construction of the resolution

Let us give the following:

**Definition 5.2.** — (i) Let $w \in W$, a writing $w = w_1 \cdots w_n$ where for all $i \in [1, n]$, we have $w_i \in W$ is called a generalised decomposition and denoted by $\hat{w}$. If moreover we have the equality $l(w) = \sum_{i=1}^{n} l(w_i)$ then we will say that the generalised decomposition is reduced.
(iii) Let us associate to any generalised decomposition a sequence of parabolic subgroups \((P_i)_{i \in [1,n]}\) defined by \(P_n = P_{w_n}\) and
\[
\Sigma(P_i) = (\Sigma(P_{i+1}) \cap \text{Supp}(w_i))^c \cup \partial(G_{w_i}) \cup (\Sigma(P_{w_i}) \cap \text{Supp}(w_{i+1})).
\]

(iv) We will say that a generalised reduced decomposition is admissible if for all \(i \in [1, n-1]\) we have \(P_{w_i} \cap G_{w_i} \subset P_{i+1}\).

We proceed by induction. The variety \(\tilde{X}_n(\tilde{w})\) is well defined and we have \(P_{\tilde{X}_n(\tilde{w})} = P_{w_n}\). Assume that \(\tilde{X}_{i+1}(\tilde{w})\) is well defined and that \(P_{\tilde{X}_{i+1}(\tilde{w})} = P_{i+1}\). To prove that \(\tilde{X}_i(\tilde{w})\) exists, we have to prove that \(P_{w_i} \cap G_{w_i} \subset P_{\tilde{X}_{i+1}(\tilde{w})}\) but it is the case by hypothesis. The fact that \(P_{\tilde{X}_i(\tilde{w})} = P_i\) comes from lemma 5.1.

Now in the case of a good generalised reduced decomposition, we have to prove that \(P_i = P_{\tilde{X}_i(\tilde{w})} = P_{w_i} \cdots w_n\). We know from lemma 5.1 that \(\Sigma(P_{\tilde{X}_i(\tilde{w})}) = (\Sigma(\tilde{X}_{i+1}(\tilde{w})) \cap \text{Supp}(w_i))^c \cup \partial(G_{w_i}) \cup (\Sigma(P_{w_i}) \cap \text{Supp}(w_{i+1})).\) Let \(\beta \in \Sigma(P_{w_i} \cdots w_n)\). If \(\beta \in \text{Supp}(w_i)\) then \(\beta\) has to be a hole of the quiver of \(w_i\) so that \(\beta \in \Sigma(P_{w_i})\) and \(\beta \in \Sigma(P_{\tilde{X}_i(\tilde{w})})\). If \(\beta \not\in \text{Supp}(w_i)\) and \(\beta \not\in \partial(G_{w_i})\) then \(\beta\) commutes with \(w_i\) and we have \(\beta \in \Sigma(P_{w_i+1} \cdots w_n)\) and \(\beta \in \Sigma(P_{\tilde{X}_i(\tilde{w})})\). Finally if \(\beta \in \partial(G_{w_i})\) we also have \(\beta \in \Sigma(P_{\tilde{X}_i(\tilde{w})})\).

Conversely, let \(\beta \in \Sigma(P_{\tilde{X}_i(\tilde{w})})\). If \(\beta \in \text{Supp}(w_i)\) then \(\beta \in \Sigma(P_{w_i})\) so \(\beta\) corresponds to a hole of the quiver of \(w_i\) and thus have to be a hole of the quiver of \(w_i \cdots w_n\). If \(\beta \in \partial(G_{w_i})\) we are done by hypothesis and finally if \(\beta\) is neither in \(\text{Supp}(w_i)\) nor in \(\partial(G_{w_i})\) then \(\beta \in \Sigma(P_{w_i+1} \cdots w_n)\) by induction hypothesis. Thus \(\beta\) corresponds to a hole of the quiver of \(w_i+1 \cdots w_n\) and does not appear in the quiver of \(w_i\). It is thus still a hole of the quiver of \(w_i \cdots w_n\). \(\square\)

**Definition 5.4.** — We denote by \(\tilde{X}(\tilde{w})\) the variety \(\tilde{X}_1(\tilde{w})\).

**Corollary 5.5.** — The variety \(\tilde{X}(\tilde{w})\) is a tower of local trivial fibrations \(f_i\) with fibers isomorphic to \(X(w_i)\).

**Lemma 5.6.** — Let \(\tilde{w}\) an admissible reduced generalised decomposition of a minuscule element \(w\) of \(W\). Let \(i \in [1, n-1]\) and assume that for any couple \((\beta, \beta') \in \text{Supp}(w_i) \times \text{Supp}(w_{i+1})\) we have \(\langle \beta'', \beta' \rangle = 0\).

Then \(w_iw_{i+1} = w_{i+1}w_i\), the generalised writing \(\tilde{w}'\) given by \(w = w'_i \cdots w'_n\) where \(w'_k = w_k\) for \(k \not\in \{i; i+1\}\), \(w'_i = w_{i+1}\) and \(w'_{i+1} = w_i\) is admissible and reduced and the morphisms \(\tilde{\pi} : \tilde{X}(\tilde{w}) \to X(w)\) and \(\tilde{\pi}' : \tilde{X}(\tilde{w}') \to X(w)\) are the same.
Proof — We simply have to look at the following situation. Let $A$ and $B$ be parabolic subgroups of a group $G$ and $C$ and $D$ be parabolic subgroups of a group $G'$. Assume that $B$ and $D$ act on a variety $X$ and consider the variety $\overline{AB} \times B \overline{CD} \times D X$ ($B$ acts on $\overline{CD} \times D X$ thanks to its action on $X$). It is isomorphic to $(\overline{AB} \times \overline{CD}) \times B \times D X$ and the construction is completely symmetric.

Let us remark that the variety $\hat{X}(\hat{w})$ is also isomorphic to the variety $\hat{X}(\hat{w}')$ where $\hat{w}'$ is such that $w_k' = w_k$ for $k < i$, $w_i' = w_i w_{i+1}$ and $w_k'' = w_{k+1}$ for $k > i + 1$.

Thanks to this lemma, we may assume that the support of any element $w_i$ is connected (otherwise replace it by a product of elements having a connected support).

5.3 Link with the Bott-Samelson resolution

In this paragraph, we show that the Bott-Samelson resolution $\hat{X}(\hat{w})$ of a minuscule element factorises through any pseudo-resolution $\hat{X}(\hat{w})$ constructed above. And we view this variety as a projection from the Bott-Samelson resolution $\hat{X}(\hat{w})$.

Let $\hat{w}$ be an admissible generalised reduced writing of $w$ and let us fix for any $i \in [1, n]$ a (unique) reduced writing $w_i = s_{1,i} \cdots s_{r_i,i}$ denoted $\hat{w}_i$.

Lemma 5.7. — For any $i \in [1, n]$, the writing

$$w = \left( \prod_{k=1}^{i} \prod_{j=1}^{r_k} s_{j,k} \right) \cdot \prod_{k=i+1}^{n} w_k$$

denoted $\hat{w}_i'$ is an admissible generalised reduced writing of $w$.

Proof — Let us denote by $w_k'$ for $k \in [1, N]$ the terms of the generalised writing. It is clear that it is reduced. Let us prove that it is admissible. Because the writing $\hat{w}$ is admissible, it is clear that the inclusion $P^{w_{N-k}} w_{N-k} \subset P_{N-k}$ holds for $k \leq i + 2$. But for $k \geq i + 1$ then $w_{N-k}'$ is a simple reflexion $s_\beta$ and we have $G_{w_{N-k}'} = SL_2(\beta)$ and $P^{w_{N-k}} w_{N-k}'$ is contained in the Borel $B$ so that the inclusion in $P_{N-k+1}$ is trivial.

Remark 5.8. — Let us remark that the classical Bott-Samelson $\hat{X}(\hat{w})$ resolution is given by $\hat{X}(\hat{w}_1')$.

Proposition 5.9. — There is a morphism $\hat{X}(\hat{w}_i') \to \hat{X}(\hat{w}_{i+1}')$ for all $i \in [1, n]$ (for $i = n$, let us fix $\hat{X}(\hat{w}_{n+1}') = \hat{X}(\hat{w})$) and the morphism $\pi : \hat{X}(\hat{w}) \to X(w)$ from the Bott-Samelson resolution to the Schubert variety factors through these morphisms. In particular we will denote by $\overline{\pi}$ the morphism from $\hat{X}(\hat{w})$ to $\hat{X}(\hat{w})$.

Proof — The variety $\hat{X}(\hat{w}_i')$ is the quotient of the product

$$\left( \prod_{k=1}^{i} \prod_{j=1}^{r_k} (P_{s_{j,k}} \cap G_{s_{j,k}}) s_{j,k} (P^{s_{j,k}} w_{N-k} \cap G_{s_{j,k}}) \right) \times \prod_{k=i+1}^{n} (P_{w_k} \cap G_{w_k}) w_k (P_{w_k} \cap G_{w_k})$$

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by the product

\[ \left( \prod_{k=1}^{i} \prod_{j=1}^{r_k} P_{s_{j,k}}^{\ast} \cap G_{s_{j,k}} \right) \times \prod_{k=i+1}^{n} P_{w_k}^{\ast} \cap G_{w_k}. \]

The action respects multiplication and in particular the multiplication map on the \( i \)th factor

\[ \prod_{j=1}^{r_i} (P_{s_{j,i}} \cap G_{s_{j,i}})^{s_{j,i}} (P_{s_{j,i}} \cap G_{s_{j,i}}) \to (P_{w_i} \cap G_{w_i}) w_i (P_{w_i} \cap G_{w_i}) \]

and the identity map on all the other factors is still defined modulo the action giving a map \( \tilde{X}(\tilde{w}_i) \to \tilde{X}(\tilde{w}_{i+1}) \). This map is simply the identity on all but one fibration (the one of fiber \( X(w_i) \)) and on this fibration is it given by the map \( \tilde{X}(\tilde{w}_i) \to X(w_i) \) from the Bott-Samelson resolution to the Schubert variety.

The last affirmation is a simple consequence of the associativity of the product: the morphism from \( \tilde{X}(\tilde{w}) = \tilde{X}(\tilde{w}_1') \) is given by the product of all the terms and the factorisations are given by making the product in a certain order. \( \square \)

**Remark 5.10.** — (i) The morphism \( \tilde{\pi} : \tilde{X}(\tilde{w}) \to X(w) \) is \( P_{\tilde{X}(\tilde{w})} \)-equivariant and in particular if the generalised reduced decomposition \( \tilde{w} \) is good, it is \( P_w \)-equivariant.

(ii) Let us explain this construction with the quiver \( Q_w \) of \( w \) and the configuration variety. Denote by \( m(Q_w) \) the set of maximal elements of \( Q_w \) for the partial order \( \preceq \).

The classical morphism \( \pi : \tilde{X}(\tilde{w}) \to X(w) \) is given by the projection from the configuration variety \( \tilde{X}(\tilde{w}) \in \prod_{i \in m(Q_w)} G/P_{\beta_i} \) on the product \( \prod_{i \in m(Q_w)} G/P_{\beta_i} \).

To give a generalised decomposition \( \tilde{w} \) of \( w \) is equivalent to give a partition of the vertices of the quiver by subquivers \( (Q_w)_{i \in [1,n]} \). Then the morphism from \( \tilde{X}(\tilde{w}) \) to \( \tilde{X}(\tilde{w}_1') \) is given by the projection on the product \( \prod_{k=1}^{i} \prod_{j \in Q_{w_k}} G/P_{\beta_j} \times \prod_{k=i+1}^{n} \prod_{j \in Q_{w_k}} G/P_{\beta_j} \).

We can give a generalisation of fact 3.6 for the morphism \( \tilde{\pi} \). Recall that we have an admissible generalised reduced decomposition \( \tilde{w} \) given by \( w = w_1 \cdots w_n \) and a reduced decomposition of each \( w_i \) giving a reduced writing \( \tilde{w} \) of \( w \) given by \( s_{\beta_i} \cdots s_{\beta_k} \). In terms of quiver, the quiver \( Q_w \) of the reduced writing \( \tilde{w} \) has a partition by subquivers \( Q_{w_i} \) isomorphic to quivers of the elements \( w_i \). We denote by \( p_{\tilde{w}}(Q_w) \) the set of vertices of \( Q_w \) which are pics for the quiver \( Q_{w_i} \) to which they belong.

**Corollary 5.11.** — The variety \( Z_K \) is not contracted by \( \tilde{\pi} : \tilde{X}(\tilde{w}) \to \tilde{X}(\tilde{w}) \) if and only if for each \( j \in [1,n] \) the part of the subword \( \prod_{i \in [1,p] \setminus K} s_i \) corresponding to a subword of \( w_j \) is reduced.

In particular, assume that all the \( w_i \) are minuscule elements, the group of divisors of \( \tilde{X}(\tilde{w}) \) has a basis given by \( \tilde{\pi}_w[Z_i] \) for \( i \in p_{\tilde{w}}(Q_w) \) and the group of 1-cycles of \( \tilde{X}(\tilde{w}) \) has a basis indexed by \( [1,n] \) given by \( \tilde{\pi}_w[C_i] \) for \( i \) the maximal vertex of the quiver \( Q_{w_i} \) for \( i \in [1,n] \).

**Proof** — This comes from the description fiberwise of the morphism from \( \tilde{X}(\tilde{w}) \) to \( \tilde{X}(\tilde{w}) \) and the lemma 3.6. To verify that the described elements form a basis, let us recall that \( \tilde{X}(\tilde{w}) \) is a tower of locally trivial fibration with fibers \( X(w_i) \) for \( i \in [1,n] \) so we know from the case of minuscule Schubert varieties that these elements form a basis. \( \square \)
5.4 Constructing generalised reduced decomposition

In this paragraph, we are going to give a way of constructing good generalised reduced decomposition of an element \( w \) in a product of minuscule elements \( (w_i)_{i \in [1,n]} \).

**Definition 5.12.** — Let \( A \subset p(Q_w) \) be a subset of the set of pic of \( Q_w \), we denote by \( Q(A_w) \) the full subquiver of \( Q_w \) containing the vertices \( i \) of \( Q_w \) such that \( i \neq j \) for all \( j \in p(Q_w) \setminus A \).

It is different from \( Q_w \) as soon as \( A \) is different from \( p(Q_w) \).

**Proposition 5.13.** — (i) Each connected component \( C \) of the quiver \( Q_w(A) \) is isomorphic to the quiver of a minuscule Schubert variety and in particular has a unique maximal element \( m(C) \) for the partial order \( \preceq \).

(ii) When \( A \) has a unique element then \( Q_w(A) \) is connected.

(iii) The quiver \( \hat{Q}_w(A) \) obtained from \( Q_w \) by removing the vertices of \( Q_w(A) \) is also the quiver of a minuscule Schubert variety.

(iv) The set \( p(Q_w(A)) \) is \( A \) and the set \( p(\hat{Q}_w(A)) \) is \( p(Q_w) \setminus A \).

**Proof** — (i) (a) Let us prove that in any connected component \( C \) there is only one maximal element for the partial order \( \preceq \). Let \( j_1 \) and \( j_2 \) be two such maximal elements. By connectivity, there exists a sequence of vertices \( i_0 = j_1, i_1, \ldots, i_n = j_2 \) such that there is an arrow linking \( i_k \) and \( i_{k+1} \) (in one sense or another). Let us take a minimal such sequence (that is to say \( n \) is minimal) an let \( x \) be the smallest integer in \([0,n]\) such that \( i_x \preceq i_{x-1} \) and \( i_x \preceq i_{x+1} \). Such an element exists because \( j_1 \) and \( j_2 \) are maximal. By minimality of \( n \) we have \( i_{x-1} \neq i_{x+1} \) are different and thanks to proposition 4.2 the vertex \( s(i_x) \) exists. The arrows arriving to \( s(i_x) \) come from \( i_{x-1} \in C, i_{x+1} \in C \) and maybe from a third vertex \( k \preceq i_x \) (and thus \( k \in C \)). The vertex \( s(i_x) \) has to be in \( C \). If we replace \( i_x \) by \( s(i_x) \), we get a new sequence of length \( n \) but with \( i_{x-1} \) the first term such that \( i_{x-1} \preceq i_{x-2} \) and \( i_{x-1} \preceq i_x \). By induction we get a sequence of length \( n \) such that the smallest \( x \in [0,n] \) with \( i_x \preceq i_{x-1} \) and \( i_x \preceq i_{x+1} \) is \( x = 1 \). This tells us that \( s(i_1) \in C \) and \( j_1 = i_0 \preceq s(i_1) \) in \( C \) which is a contradiction to the maximality of \( j_1 \).

(b) Let us now prove that any connected component \( C \) of \( Q(A) \) satisfies the conditions of 4.2. Let \( k \) be a vertex of \( C \) such that \( s(k) \) does not exist or is not in \( C \).

In the first case, this means that there is at most one arrow from \( k \) and denote by \( j \) the end vertex of this arrow. If \( j \) is not in \( C \) (or does not exist) then \( k \) is the maximal element of \( C \). Otherwise \( j \) is in \( C \) and there is exactly one arrow from \( k \) in \( C \).

In the second case, we have two vertices \( k_1 \) and \( k_2 \) such that the arrows arriving to \( s(k) \) come from \( k_1, k_2 \) and eventually a third one \( k_3 \preceq k \) which has to be in \( C \). As \( s(k) \not\in C \), at least one of the two vertices \( k_1 \) and \( k_2 \) has to be out of \( C \). If both are out of \( C \) then \( k \) is the unique maximal element of \( C \). Otherwise exactly one vertex from \( \{k_1, k_2\} \) is in \( C \).

If \( k \) is a vertex of \( C \) such that \( s(k) \not\in C \), then we have two vertices \( k_1 \) and \( k_2 \) such that the arrows arriving to \( s(k) \) come from \( k_1, k_2 \) and eventually a third one \( k_3 \preceq k \) which has to be in \( C \). These two elements have to be in \( C \) otherwise \( s(k) \) would not be in \( C \).
(c) We are left to prove that if \( m(C) \) is the maximal element of \( C \) then \( \beta(m(C)) \) is a simple minuscule root for some group.

If the Dynkin diagram is of type \( A_n \) this is always true because any simple root is minuscule. Even if the set \( \beta(C) \) of simple roots is of type \( A_n \) we are done. Let us assume that \( \beta(C) \) contains a trivalent root \( \gamma \) and a root on each branch of the Dynkin diagram (remark that because \( C \) is connected, so is \( \beta(C) \)).

Denote by \( m(C) \) the maximal element of \( C \) and by \( \beta \) the simple root \( \beta(m(C)) \). If this root was not a minuscule root of the Dynkin diagram \( \beta(C) \) (a sub-Dynkin diagram of the one of \( G \)) then we would have the following situation:

\[
\begin{array}{c}
\beta_0 \quad \beta_1 \quad \beta_{n-1} \quad \beta_n \quad \beta_{n+1} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\beta_{n+2}
\end{array}
\]

such that \( \beta_1 = \beta, \beta_n = \gamma \) and all the simple roots \( \beta_i \) are in \( \beta(C) \). There are two distinct simple roots \( \beta_{n+1} \) and \( \beta_{n+2} \) in \( \beta(C) \) and not in \([\beta, \gamma]\) such that \( \langle \gamma', \beta_{n+i} \rangle \neq 0 \) for \( i = 1, 2 \). There is a simple root \( \beta_0 \) in \( \beta(C) \) different from all the \( \beta_i \) for \( i \in [1, n+2] \) such that \( \langle \beta', \beta_0 \rangle \neq 0 \).

Denote by \( i_k \) the biggest (for \( \succeq \) ) vertex in \( C \) such that \( \beta(i_k) = \beta_k \) for all \( k \in [0, n+2] \). Then because \( C \) satisfies the properties of proposition \([1,2] \), we see that in \( C \), for all \( k \in [2, n+1] \) there exists a unique arrow from \( i_k \) and it goes to \( i_{k-1} \). In the same way there exists a unique arrow from \( i_{n+2} \) and it goes to \( i_n \) and from \( i_0 \) to \( i_1 = m(C) \). This means that in \( C \) we have the following subquiver:

\[
\begin{array}{c}
i_0 \quad i_1 \quad i_n \quad i_{n+1} \quad i_{n+2}
\end{array}
\]

Now let us consider the subquiver \( Q' \) of \( Q \) corresponding to the vertices \( i \) such that \( i \succeq i_0 \) or \( i \succeq i_0 \) or \( i \succeq i_0 \). This is a quiver corresponding to a minuscule Schubert variety. Each time there is a hole \( i \) in the quiver, we can add a new vertex \( j \) such that \( \beta(j) = \beta(i) \) to obtain a quiver which still corresponds to a minuscule Schubert variety. We can thus add a vertex \( i_{n+3} \) with \( \beta(i_{n+3}) = \beta \) and by induction vertices \( i_{n+2+k} \) with \( \beta(i_{n+2+k}) = \beta(i_k) \) for all \( k \in [1, n-1] \). In this new quiver we have the following subquiver:
But we also could have choosen \( \beta(\ell_{2n+2}) = \beta(\ell_n) \) proving that in the quiver \( Q_\infty \) of the minuscule homogeneous variety there exists a vertex \( j \) with \( s(j) = i \). But then between \( j \) and \( i_n \) there would be three vertices (namely \( i_{n+1}, i_{n+2} \) and \( i_{2n+2} \)) having an arrow to \( i_n \). This contradicts proposition \( \text{4.2} \) for \( Q_\infty \).

(ii) If \( A \) has a unique element then all vertices of \( Q_w(A) \) are bigger than the unique element of \( A \) and \( Q_w(A) \) is connected.

(iii) The quiver \( \tilde{Q}_w(A) \) is obtained from \( Q_w \) by removing all the vertices smaller than \( m(C) \) for any connected component \( C \) of \( Q_w(A) \). It is thus (see proposition \( \text{4.5} \)) the quiver of a minuscule Schubert variety.

(iv) Clear from the definitions. \( \square \)

To construct a partition of the quiver \( Q_w \) of a minuscule element \( w \) into quivers \( (Q_{w_i})_{i \in [1,n]} \) with \( w_i \) a minuscule elements, it suffices to give a partition \( (A_i)_{i \in [1,n]} \) of the set \( p(Q_w) \) of the pics of the quiver. Indeed, given such a partition \( (A_i)_{i \in [1,n]} \), we define by induction a sequence \( (Q_i)_{i \in [0,n]} \) of quivers with \( Q_0 = Q \) and \( Q_{i+1} = \tilde{Q}_i(A_{i+1}) \) \( Q_{w_1} = Q_w(A_1) \). We then define by \( Q_{w_i} \) the quiver \( Q_{i-1}(A_i) \). The quivers \( (Q_{w_i})_{i \in [1,n]} \) form a partition of \( Q_w \) by quivers associated to minuscule elements \( w_i \).

**Remark 5.14.** — (i) For such partitions (giving a reduced generalised decomposition \( \hat{w} \)), we have \( p_{\hat{w}}(Q_w) = p(Q_w) \).

(ii) For such constructions, the vertices of \( Q_{w_i} \) are the vertices \( x \) such that there exists a pic \( p \in A_i \) with \( p \prec x \) and \( p' \neq x \) for any pic \( p' \) in \( A_j \) with \( j > i \).

For such partitions \( (Q_{w_i})_{i \in [1,n]} \) of the quiver \( Q_w \) coming from partitions \( (A_i)_{i \in [1,n]} \) of \( p(Q_w) \) we have a reduced generalised decomposition \( w = w_1 \cdots w_n \) denoted \( \hat{w} \).

**Proposition 5.15.** — The reduced generalised writing \( \hat{w} \) is good.

**Proof** — We have to prove that the inclusions \( \Sigma(P_{w_{i+1} \cdots w_n}) \subset \Sigma(P_{w_{i} \cdots w_n} \cap G_{w_i}) \) and \( \partial(G_{w_i}) \subset \Sigma(P_{w_{i} \cdots w_n}) \). But the set \( \Sigma(P_{w_{i} \cdots w_n}) \) is the set \( \beta(\{i \in Q_w / i \text{ is a hole of } Q_{w_{i} \cdots w_n}\}) \) where \( Q_{w_i} \) is the subquiver of \( Q_w \) whose vertices are in \( \cup_{k \geq i} Q_{w_k} \) (see proposition \( 4.11 \)).

The set \( \Sigma(P_{w_{i} \cdots w_n} \cap G_{w_i}) \) is the set \( \beta(m_i) \cap \text{Supp}(w_i) \) where \( m_i \) is the maximal vertex of \( Q_{w_i} \). So for the first inclusion we only have to prove that for any simple root \( \beta \in \text{Supp}(w_i) \cap \Sigma(P_{w_{i+1} \cdots w_n}) \) we have \( \beta = \beta(m_i) \). But as \( \beta \in \text{Supp}(w_i) \), there exists \( j \in Q_{w_i} \) such that \( \beta(j) = \beta \). Let \( j \) be the biggest such vertex. If \( j \) was not the biggest element \( m_i \) in \( Q_{w_i} \) then there would exist in \( Q_{w_i} \) an element \( k \) with an arrow from \( j \) to \( k \). But then there are two cases, if \( s(j) \) exists, it is a hole of \( Q_{w_{i+1} \cdots w_n} \) and there are two vertices \( k_1 \) and \( k_2 \) having an arrow to \( s(j) \) so between \( j \)
and \( s(j) \) there are three vertices \( k, k_1 \) and \( k_2 \) and this is impossible thanks to proposition 4.12.

If \( s(j) \) does not exist, then \( j \) is a virtual hole of \( Q_{w_1 \cdots w_n} \) and there is a vertex \( k' \) such that \( \langle \beta(k'), \beta(j) \rangle \neq 0 \) so that \( s(j) \) does not exist but there are two vertices \( k \) and \( k' \) having an arrow coming from \( j \) and this is impossible thanks to proposition 4.12.

For the second inclusion, let \( \beta \) be a simple root in \( \partial(G_{w_i}) \), then there exists a vertex \( j \in Q_{w_i} \) with \( \langle \beta(j), \beta \rangle \neq 0 \). If \( \beta \) is not in the support of \( w_1 \cdots w_n \) then \( \beta \) is the simple root of a virtual hole and \( \beta \in \Sigma(P_{w_1 \cdots w_n}) \). If \( \beta \) is in this support then there exists a vertex \( k \) such that \( \beta(k) = \beta \).

Let \( k \) be the smallest such vertex, we have an arrow from \( j \) to \( k \) thus \( k \) is not a pic of \( Q_{w_1 \cdots w_n} \) and thus not a pic of \( Q_{w_1+1 \cdots w_n} \) (see the previous proposition). In particular there exists a vertex \( x \in Q_{w_1 \cdots w_n} \) with an arrow from \( x \) to \( k \). But then \( k \) is the smallest vertex with \( \beta(k) = \beta \) in \( Q_{w_1 \cdots w_n} \) and there are two arrows arriving to \( k \) thus it is a hole of \( Q_{w_1 \cdots w_n} \) and we are done. \( \Box \)

We now give here three types of partitions of \( Q_w \) constructed in this way.

**Construction 1.** — Choose any order \( \{i_1, \ldots, i_n\} \) on the set \( p(Q_w) \) of the pics of \( Q_w \) and set \( A_k = \{i_k\} \).

**Construction 2.** — Define a partition \( (A_i)_{i \in [1,n]} \) by induction: \( A_1 \) is the set of pics with minimal height and \( A_{i+1} \) is the set of pics in \( p(Q_w) \setminus \bigcup_{k=1}^{i} A_k \) with minimal height.

Before giving the last construction let us fix some notations and prove the following proposition. Recall that \( p_w(Q_w) = p(Q_w) \) (with these constructions) is the set of all vertices \( j \) of \( Q_w \) such that there exists an integer \( i \in [1,n] \) with \( j \in p(Q_{w_i}) \). Let us denote by \( m_w(Q_w) \) the set of vertices \( j \) of \( Q_w \) such that \( j \) is a maximal element of \( Q_{w_i} \) for some \( i \in [1,n] \).

The partial order \( \preceq \) induces a partial order on \( m_w(Q_w) \). Let us finally prove the following:

**Proposition 5.16.** — Let \( i \in m_w(Q_w) \), there exists a unique minimal element \( f(i) \) in \( m_w(Q_w) \) for \( \preceq \) such that \( i \preceq f(i) \).

**Proof.** — Let us prove the following

**Lemma 5.17.** — Let \( j \) and \( k \) in \( m_w(Q_w) \) such that there exists \( x \in Q_w \) with \( x \preceq j \) and \( x \preceq k \). Then we have either \( j \preceq k \) or \( k \preceq j \)

**Proof.** — We proceed by induction on \( a + b \) where \( a \) and \( b \) are the indexes in \([1,n] \) such that \( j \in Q_{w_n} \) and \( k \in Q_{w_n} \). Let \( x \) be a maximal element (for \( \preceq \)) such that \( x \preceq j \) and \( x \preceq k \) and suppose that \( x \) is different from \( j \) and \( k \).

If there is only one arrow from \( x \) say going to a vertex \( y \), then we must have \( y \preceq j \) and \( y \preceq k \) contradicting the maximality. Let \( y_1 \) and \( y_2 \) be the two arriving elements of the two arrows from \( x \). If \( \beta(y_1) = \beta(y_2) \) then \( y_1 \preceq y_2 \) (or the converse) and we have \( y_1 \preceq j \) and \( y_1 \preceq k \) contradicting the maximality. We thus have \( \beta(y_1) \neq \beta(y_2) \) and \( y_1 \preceq j \) but \( y_1 \neq k \) and \( y_2 \neq j \) but \( y_2 \preceq k \). This also implies that \( s(x) \) exists because \( y_1 \) and \( y_2 \) are connected to the biggest element \( r \) of the quiver and so the segments \([\beta(y_1), \beta(r)]\) and \([\beta(y_1), \beta(r)]\) are contained
in \( \beta(\{z \in Q_w \mid z \succ x, z \neq x\} \) and thus \( \beta(x) \) is in this set. So \( s(x) \) exists and we have \( s(x) \neq j \) and \( s(x) \neq k \).

Now let \( c, d \) and \( e \) in \([1, n]\) such that \( s(x) \in Q_{w_e}, j_1 \in Q_{w_d} \) and \( j_2 \in Q_{w_c} \). We must have \( c \geq d \) and \( c \geq e \) because \( s(x) \succ j_1, j_2 \). We must also have \( a \geq d \) and \( b \geq e \). But if \( p \) is a pic in \( A_c \) such that \( s(x) \succ p \), we must have \( p \preceq j_1 \) or \( p \preceq j_2 \) which implies (see remark 5.14) that \( d \geq c \) or \( e \geq c \). We thus have \( c = d \) or \( c = e \). Assume for example that \( c = d \) and denote by \( m \) the maximal element of \( Q_{w_c} \). If \( c = d = a \) then \( j_2 \preceq s(x) \preceq m = j \) and \( j_2 \preceq k \) a contradiction to the maximality of \( x \). So \( c = d < a \), but we have \( x \preceq m \) and \( x \preceq j \) and by induction, we must have \( m \preceq j \). Then we have \( j_2 \preceq s(x) \preceq m \preceq j \) and \( j_2 \preceq k \) one more time a contradiction to the maximality of \( x \). \( \square \)

The preceding lemma proves that for \( i \in m_\hat{w}(Q_w) \) (and even for any \( i \in Q_w \)) the set \( \{ j \in m_\hat{w}(Q_w) \mid j \succ i \} \) is totally ordered and thus there exists a minimal element \( f(i) \). \( \square \)

We can now give the last construction which is a particular case of construction 1. Because in construction 1 there is a bijection between \( p(Q_w) \) and \( m_\hat{w}(Q_w) \) we define thanks to the preceding proposition the function \( f \) on the set \( p(Q_w) \) simply by the following: if \( p \in p(Q_w) \) is in \( Q_{w_1} \) and if \( m \in m_\hat{w}(Q_w) \) is the maximal element of \( Q_{w_1} \) then \( f(m) \) is the maximal element of some \( Q_{w_j} \).

There is only one pic \( q \) in \( Q_{w_j} \) and we define \( f(p) = q \).

**Construction 3.** — Choose an order \( \{i_1, \cdots, i_n\} \) on the set \( p(Q_w) \) of the pics of \( Q_w \) such that \( h(i_k) \leq h(f(i_k)) \) for all \( k \in [1, n - 1] \) and set \( A_k = \{i_k\} \).

This choice on the order is equivalent to choose an order \( \{i_1, \cdots, i_n\} \) on the set \( p(Q_w) \) of the pics of \( Q_w \) such that if \( i_k \) and \( i_{k+1} \) are adjacent in the quiver then \( h(i_k) \leq h(i_{k+1}) \).

These constructions may produce non connected subquivers \( Q_{w_i} \), but thanks to lemma 5.6 we may assume (replacing these quivers by their connected components) that all the quivers \( Q_{w_i} \) are quivers of minuscule Schubert varieties.

Construction 3 will give all relative minimal models of \( X(w) \) and construction 2 will give the relative canonical model of \( X(w) \).

### 6 Relative Mori theory of minuscule Schubert varieties

In this paragraph we prove the above assertion on the relative canonical and minimal models of a minuscule Schubert variety \( X(w) \). We only consider generalised reduced writing \( \hat{w} \) of \( w \) obtained thanks one of the three previous constructions.

#### 6.1 Ample divisors and effective curves

Recall that we described a basis of divisors and 1-cycles on \( \hat{X}(\hat{w}) \) in the following way: the group of divisors of \( \hat{X}(\hat{w}) \) has a basis given by \( D_i = \pi_*[Z_i] \) for \( i \in p_\hat{w}(Q_w) = p(Q_w) \) and the group of 1-cycles of \( \hat{X}(\hat{w}) \) has a basis given by \( \pi_*[C_i] \) for \( i \in m_\hat{w}(Q_w) \). Recall also that (cf. [FMPSS])
the Chow groups of $\hat{X}(\hat{w})$ are generated by $B$-orbits, free over $\mathbb{Z}$ and the Picard group is dual to the group of 1-cycles.

We have seen in remark 5.10 that the morphism $\pi : \hat{X}(\hat{w}) \to \hat{X}(\hat{w})$ is the projection from $\hat{X}(\hat{w})$ to the product $\prod_{i \in m_{Qw}(Q_w)} G/P_i$. We have on $\hat{X}(\hat{w})$ a projection $p_i : \hat{X}(\hat{w}) \to G/P_i$ for all $i \in m_{\hat{w}}(Q_w)$. Let us define the Cartier divisor $M_i = p_i^*(O_{G/P_i}(1))$ for all $i \in m_{\hat{w}}(Q_w)$. We have $L_i = \pi_* M_i$ for all $i \in m_{\hat{w}}(Q_w)$. Because of the description of $\hat{X}(\hat{w})$ as a a tower of locally trivial fibrations with fibers isomorphic to minuscule Schubert varieties $X(w_i)$, we have the following:

**FACT 6.1.** — The family $(M_i)_{i \in m_{\hat{w}}(Q_w)}$ is a basis of $\text{Pic}(\hat{X}(\hat{w}))$.

Recall that we gave a basis $(Y_i)_{i \in [1,r]}$ of the cone of effective 1-cycles $\hat{X}(\hat{w})$ in paragraph 6.2. Because $[Y_i] = [C_i] - [C_{s(i)}]$ we see that we have the following:

**FACT 6.2.** — The family $(\pi_* [Y_i])_{i \in m_{\hat{w}}(Q_w)}$ is a basis of the group of 1-cycles on $\hat{X}(\hat{w})$ and is dual to the basis $(M_i)_{i \in m_{\hat{w}}(Q_w)}$.

**PROOF** — The pull-back of $M_i$ by $\pi$ is $L_i$ and we have $L_i \cdot [Y_j] = \delta_{i,j}$ so by projection formula we get the result.

**PROPOSITION 6.3.** — The family $([\pi_* Y_i])_{i \in m_{\hat{w}}(Q_w)}$ is a basis of the cone of effective classes and the family $(M_i)_{i \in m_{\hat{w}}(Q_w)}$ is a basis of the closure of the ample cone.

**PROOF** — The embedding of $\hat{X}(\hat{w})$ in $\prod_{i \in m_{\hat{w}}(Q_w)} G/P_i$ is given by $\bigotimes_{i \in m_{\hat{w}}(Q_w)} M_i$. The cone generated by the $M_i$ is thus contained in the ample cone.

Conversely, let $A$ be an ample sheaf and let $a_i = A \cdot [\pi_* Y_i]$ for $i \in m_{\hat{w}}(Q_w)$. We must have $a_i > 0$ and the divisor $A - \sum_i a_i M_i$ is numerically trivial and we get the result.

By duality we have the result on curves.

Proposition 6.3 has the following relative version.

**PROPOSITION 6.4.** — We have the formula

$$M_i = \sum_{k \in P(Q_w), \ k \neq i} D_k.$$

**PROOF** — Because the variety $\hat{X}(\hat{w})$ is normal with rational singularities, we have $M_i = \pi_* L_i$ and we obtain the formula in the same way as in proposition 6.3 thanks to the fact that all quivers $Q_{w_i}$ are associated to minuscule Schubert varieties.

Let us generalise corollary 6.4 and give a criterion for a variety $\hat{X}(\hat{w})$ to be locally factorial.

**COROLLARY 6.5.** — The variety $\hat{X}(\hat{w})$ is locally factorial if and only if for all $i \in [1,n]$ the quiver $Q_{w_i}$ has a unique pic.
6.2 Canonical divisor of \( \hat{X}(\hat{\omega}) \)

As in proposition 4.15, we have \( K_{\hat{X}(\hat{\omega})} = \tilde{\pi}^* K_{\hat{X}(\hat{\omega})} \). The same calculus gives the:

**FACT 6.6.** — We have

\[
-K_{\hat{X}(\hat{\omega})} = \sum_{k \in p(Q_w)} (h(k) + 1)D_k.
\]

For the three constructions, the pics of a fixed quiver \( Q_{w_i} \) have all the same height so we can define \( h(w_i) \) to be the height of any pic of \( Q_{w_i} \). Set \( h(w_{n+1}) = -1 \). Proposition 6.4 gives us the following (by induction on the order of \( m_{\hat{\omega}}(Q_w) \)):

**COROLLARY 6.7.** — We have the formula

\[
-K_{\hat{X}(\hat{\omega})} = \sum_{i \in m_{\hat{\omega}}(Q_w)} (h(w_i) - h(w_{f(i)}))M_i
\]

and in particular \( \hat{X}(\hat{\omega}) \) is Gorenstein.

6.3 Types of singularities

In this paragraph we are going to prove that the variety \( \hat{X}(\hat{\omega}) \) has terminal singularities in case of constructions 1, 2 and 3.

For this we use the resolution \( \tilde{X}(\tilde{\omega}) \) of \( \hat{X}(\hat{\omega}) \) and compare the canonical divisor \( K_{\tilde{X}(\tilde{\omega})} \) to the pull-back of the canonical divisor \( K_{\hat{X}(\hat{\omega})} \) by \( \tilde{\pi} \). We need the following fact coming directly from the formula of paragraph 3.3.1 and lemma 4.16:

**FACT 6.8.** — We have the formula

\[
-K_{\tilde{X}(\tilde{\omega})} = \sum_{i = 1}^r (h(i) + 1)\xi_i.
\]

We can now prove the following:

**PROPOSITION 6.9.** — The variety \( \hat{X}(\hat{\omega}) \) has terminal (and hence canonical) singularities.

**PROOF** — Let us calculate the pull-back of \( K_{\hat{X}(\hat{\omega})} \) by \( \tilde{\pi} \):

\[
-\tilde{\pi}^* K_{\hat{X}(\hat{\omega})} = \sum_{i \in m_{\hat{\omega}}(Q_w)} (h(w_i) - h(w_{f(i)}))L_i
\]

but thanks to proposition 3.14 and the fact that the quivers \( Q_{w_i} \) are the quivers of minuscule Schubert varieties we have

\[
L_i = \sum_{k \neq i} \xi_k
\]

giving

\[
-\tilde{\pi}^* K_{\hat{X}(\hat{\omega})} = \sum_{i = 1}^r \left( (h(w_i) + 1) \sum_{k \in Q_{w_i}} \xi_k \right)
\]
We get for the difference:

\[
K_{\tilde{X}(\bar{w})} - \bar{\pi}^* K_{\tilde{X}(\bar{w})} = \sum_{i=1}^{n} \sum_{k \in \mathbb{Q}_{w_i}} (h(w_i) - h(k))\xi_k.
\]

But \(h(w_i)\) is the highest height of an element in \(\mathbb{Q}_{w_i}\) so \(h(w_i) - h(k) \geq 0\) with equality for \(k\) a pic of the quiver that is to say if and only if \(\xi_i\) is not contracted by \(\bar{\pi}\). \(\square\)

### 6.4 Description of the relative minimal and canonical models

We can now prove our results on relative minimal and canonical models of minuscule Schubert varieties.

**Theorem 6.10.** — (i) The varieties \(\tilde{X}(\bar{w})\) obtained from construction \(\mathbb{A}\) are relative minimal models of \(X(w)\).

(ii) The variety \(\tilde{X}(\bar{w})\) obtained from construction \(\mathbb{B}\) is the relative canonical model of \(X(w)\).

**Proof** — (i) We have to prove that any curve \(C\) contracted by \(\tilde{\pi} : \tilde{X}(\bar{w}) \to X(w)\) satisfies \([C] \cdot K_{\tilde{X}(\bar{w})} \geq 0\). The class \([C]\) can be writen as \([C] = \sum_i a_i[\tilde{\pi}_*Y_i]\) with \(a_i \geq 0\) and \(a_n = 0\) (because the curve is contracted). We just have to prove the non negativity of the intersections \(K_{\tilde{X}(\bar{w})} \cdot [\tilde{\pi}_*Y_j]\) for \(j \in [1, n-1]\). We have \(K_{\tilde{X}(\bar{w})} \cdot [\tilde{\pi}_*Y_j] = h(w_{f(j)}) - h(w_j)\) and by construction \(\mathbb{B}\) this intersection is non negative.

(ii) It suffices to prove that the contracted curves have a positive intersection with the canonical divisor and this comes from the previous calculus and construction \(\mathbb{B}\). \(\square\)

Let \(\tilde{X}(\bar{w})\) a variety obtained from one of the construction \(\mathbb{A} \mathbb{B}\) or \(\mathbb{C}\). We can completely describe the extremal rays of the relative cone of effective 1-cycles on \(\tilde{X}(\bar{w})\) (i.e. the effective 1-cycles contracted by \(\tilde{\pi}\)).

**Fact 6.11.** — (i) The extremal rays of \(\tilde{X}(\bar{w})\) are given by the classes \([\tilde{\pi}_*Y_j]\) such that \(h(w_{f(j)}) < h(w_j)\).

(ii) If \(\tilde{X}(\bar{w})\) is obtained from construction \(\mathbb{A}\) then there is no extremal ray. However, if \(D\) is an effective divisor, then the \((K_{\tilde{X}(\bar{w})} + D)\)-extremal rays are given by the classes \([\tilde{\pi}_*Y_j]\) such that \((K_{\tilde{X}(\bar{w})} + D) \cdot [\tilde{\pi}_*Y_j] < 0\).

**Proof** — (i) Let \([C]\) the class of an effective curve. Then there exists non negative integers \(a_i\) such that \([C] = \sum_i a_i[\tilde{\pi}_*Y_i]\). Denote by \(\mu_j\) (resp. \(\nu_k\) and \(\omega_l\)) the classes \([\tilde{\pi}_*Y_j]\) such that \(K_{\tilde{X}(\bar{w})} \cdot [\tilde{\pi}_*Y_i] < 0\) (resp. \(> 0\) and \(= 0\)). For each \(j\) and \(k\) there is a linear combination with positive coefficient \(x_{j,k}\mu_j + y_{j,k}\nu_k\) such that \(K_{\tilde{X}(\bar{w})} \cdot (x_{j,k}\mu_j + y_{j,k}\nu_k) = 0\). It is easy to check that if \(K_{\tilde{X}(\bar{w})} \cdot [C] < 0\) then \([C]\) has to be a linear combination with non negative coefficient of classes \((\mu_j), (x_{j,k}\mu_j + y_{j,k}\nu_k)\) and \((\omega_l)\) proving the result.

(ii) The same proof with \(K_{\tilde{X}(\bar{w})} + D\) instead of \(K_{\tilde{X}(\bar{w})}\) works. It works for all varieties \(\tilde{X}(\bar{w})\) obtained from construction \(\mathbb{A} \mathbb{B}\) or \(\mathbb{C}\). \(\square\)
Let us consider a fixed order \((p_1, \ldots, p_n)\) on the pics of the quiver \(Q_w\) and \(\tilde{w}\) the good reduced generalised writing it induces and let us denote by \(\tilde{w}'\) the good reduced generalised writing induced by the order \((q_1, \ldots, q_n)\) on the pics where \(q_k = p_k\) for \(k \notin \{i, i + 1\}\), \(q_i = p_{i+1}\) and \(q_{i+1} = q_i\). Denote by \(\tilde{w}''\) the good reduced generalised writing given by \(w''_k = w_k\) for \(k < i\), \(w''_i = w_i w_{i+1}\) and \(w_k = w_{k+1}\) for \(k > i\) that is to say obtained by the partition \((A_k)_{k \in [1, n-1]}\) of \(p(Q_w)\) given by \(A_k = \{p_k\}\) for \(k < i\), \(A_i = \{p_i, p_{i+1}\}\) and \(A_k = \{p_{k+1}\}\) for \(k > i\). Denote by \(k_i\) (resp. \(k_{i+1}\)) the maximal vertex of \(Q_{w_i}\) (resp. \(Q_{w_{i+1}}\)).

We have two morphisms (for example because of remark 5.10)

\[
f : \tilde{X}(\tilde{w}) \to \tilde{X}(\tilde{w}'') \quad \text{and} \quad f' : \tilde{X}(\tilde{w}') \to \tilde{X}(\tilde{w}'').
\]

**Proposition 6.12.** — Let \(\tilde{X}(\hat{w})\) obtained from construction 7 8 or 9

(i) If \(k_i \not\in k_{i+1}\) (i.e. if \(i + 1 \neq f(i)\)) then the morphisms \(f\) and \(f'\) are isomorphisms.

(ii) If \(f(i) = i + 1\) and \(\hat{\pi}_s[Y_k] : K_X > 0\), then \(f\) (resp. \(f'\)) is the small contraction corresponding to the extremal ray \(\mathbb{R}\pi_*[Y_k]\) (resp. \(\mathbb{R}\pi_*[Y_{k'}]\)) and \(f\) is the flip of \(f'\).

(iii) If \(f(i) = i + 1\) and \(\hat{\pi}_s[Y_k] : K_X = 0\), denote \(D = D_{i+1}\) (resp. \(D' = D'_{i}\)) then for \(\varepsilon > 0\), the class \(\hat{\pi}_s[Y_{k'}]\) (resp. \(\pi_*[Y_k]\)) is extremal for \(K_{\tilde{X}(\hat{w})} + \varepsilon D\) (resp. \(K_{\tilde{X}(\hat{w})} + \varepsilon D'\)) and \(f\) (resp. \(f'\)) is the small contraction corresponding to the extremal ray \(\mathbb{R}\hat{\pi}_s[Y_{k'}]\) (resp. \(\mathbb{R}\pi_*[Y_k]\)). The morphism \((f', D')\) is the flop of \((f, D)\).

**Proof.** — (i) Because \(k_i\) and \(k_{i+1}\) are non comparable for \(\xi_i\), we have thanks to lemma 5.17 that \(w_i\) and \(w_{i+1}\) satisfy the hypothesis of lemma 5.6 and we have the result.

In the cases where \(f(i) = i + 1\), we already know that \(\tilde{X}(\hat{w})\) and \(\tilde{X}(\tilde{w}')\) are locally factorial with terminal singularities. We also know that \(\tilde{X}(\hat{w}'')\) is normal and the morphisms are birational and Mori-small (the group of Weil divisors has a basis given by the pics). Because of our description of Picard groups we also have \(\rho(X/X'') = \rho(X'/X'') = 1\).

(ii) We are left to study the divisors \(K_{\tilde{X}(\hat{w})}\) and \(K_{\tilde{X}(\hat{w}')}\) on the fibers of \(f\) and \(f'\). We will not describe these fibers in details in this proposition but more details will be given in the next paragraph. Because all the sheaves \(\mathcal{M}_{k_j}\) for \(j \neq i\) are already defined on \(\tilde{X}(\hat{w}'')\) they are trivial on the fibers of \(f\) and the sheaf \(\mathcal{M}_{k_i}\) on \(\tilde{X}(\hat{w})\) is relatively ample with respect to \(f\). The restriction of \(K_{\tilde{X}(\hat{w})}\) to the fibers of \(f\) is given by \((K_{\tilde{X}(\hat{w})} \cdot \hat{\pi}_s[Y_k])\mathcal{M}_{k_i}\) so that \(K_{\tilde{X}(\hat{w})}\) is ample, anti-ample or trivial according to the positivity of the intersection \(K_{\tilde{X}(\hat{w})} \cdot \hat{\pi}_s[Y_k]\) and thus according to the height of the pics. This proves in case (ii) that \(K_{\tilde{X}(\hat{w})}\) is \(f\)-ample.

Furthermore, the fiber of \(f\) is contained in \(G/P_{\beta(k_i)}\) thus the classes of contracted curves are proportional to \(\hat{\pi}_s[Y_{k_i}]\).

In the same we get that \(-K_{\tilde{X}(\hat{w}')}\) is \(f'\)-ample and the result.

(iii) Proposition 6.14 tells us that \(D = D_{i+1}\) satisfies \(\hat{\pi}_s[Y_k] \cdot D < 0\) so the class \(\hat{\pi}_s[Y_k]\) is extremal for \(K_{\tilde{X}(\hat{w})} + \varepsilon D\).

The Bott-Samelson variety \(\tilde{X}(\hat{w})\) is a resolution of the birational morphism between \(\tilde{X}(\hat{w})\) and \(\tilde{X}(\hat{w}')\) and \(D\) is the image of \(\xi_{p_i+1}\) whose image in \(\tilde{X}(\hat{w}')\) is \(D'\) so that \(D'\) is the strict transform of \(D\).
We are left to study the divisors $K_{\hat{X}(\hat{w})}$, $K_{\hat{X}(\hat{w}')}$, $D$ and $D'$ on the fibers of the morphisms $f$ and $f'$. The calculus in (i) proves the triviality of $K_{\hat{X}(\hat{w})}$ and $K_{\hat{X}(\hat{w}')}$. For $D$ and $D'$, the same argument as for the canonical sheaves proves that their restriction to the fibers of $f$ (resp. $f'$) is a positive multiple of $-M_{k_i}$ (resp. $M_{k'_i}$) concluding the proof.

**Remark 6.13.** — If $f(i) = i + 1$ and $[Y_{k_i}] \cdot K_{\hat{X}(\hat{w})} < 0$ then by symmetry $f'$ is the flip of $f$.

**Corollary 6.14.** — (i) The varieties $\hat{X}(\hat{w})$ obtained from construction 1 are linked by flips and flops and any variety obtained from $\hat{X}(\hat{w})$ by flips and flops comes from this construction.

(ii) The varieties $\hat{X}(\hat{w})$ obtained from construction 2 are linked by flips and any variety obtained from $\hat{X}(\hat{w})$ by flips comes from this construction.

**Proof** — (i) We know that the extremal rays (or more generally the $(K_{\hat{X}(\hat{w})} + D)$-extremal rays) of $\hat{X}(\hat{w})$ are generated by the classes $\pi_*[Y_{k_i}]$ with $K_{\hat{X}(\hat{w})} \cdot \pi_*[Y_{k_i}] < 0$ (resp. $(K_{\hat{X}(\hat{w})} + D) \cdot \pi_*[Y_{k_i}] < 0$). But the associated flip or flop gives a variety obtained by construction 1.

(ii) The same argument works in this case because $h(w_f(i)) = h(w_i)$ and we stay in the class of varieties obtained from construction 2.

**Remark 6.15.** — For any variety obtained from construction 1 we proved existence and termination of flips and flops (in the sense of K. Matsuki [Mat]).

**Corollary 6.16.** — The relative minimal models of $X(w)$ are exactly the varieties obtained from construction 3.

**Proof** — We use theorem 12-1-8 of [Mat], the fact that varieties obtained from construction 3 are relative minimal models and existence and termination of flops for these varieties.

### 7 Small IH-resolutions of minuscule Schubert varieties

In this section we prove that the morphism $\hat{X}(\hat{w}) \to X(w)$ obtained from construction 3 is IH-small. We then discuss the smoothness of $\hat{X}(\hat{w})$ and describe all IH-small resolutions of minuscule Schubert varieties. Let us first recall the definition of an IH-small morphism:

**Definition 7.1.** — A morphism $\pi : Y \to X$ is said to be IH-small if for all $k > 0$, we have

$$\text{codim}_X \{x \in X \mid \dim(\pi^{-1}(x)) = k\} > 2k.$$  

A small morphism $\pi : Y \to X$ is a small resolution of $X$ if $Y$ is smooth.

In this section we will use a case by case analysis instead of a global proof for all minuscule Schubert varieties in the same time. A combinatorial direct proof on the quiver is possible but it would lead to a too complicated combinatorical discussion and we prefer avoiding it.
7.1 Necessary condition

Let us first prove the following proposition showing that among the morphisms \( \hat{\pi} : \hat{X}(\hat{w}) \to X(w) \) obtained from construction 1 only the one coming from construction 3 can be small:

**Proposition 7.2.** — Let \( \hat{\pi} : \hat{X}(\hat{w}) \to X(w) \) obtained from construction 1 but not from construction 3. Then the morphism \( \hat{\pi} : \hat{X}(\hat{w}) \to X(w) \) is not IH-small.

**Proof** — If \( \hat{\pi} : \hat{X}(\hat{w}) \to X(w) \) was small then, because \( \hat{X}(\hat{w}) \) has terminal singularities, it would be a relative minimal model (this is a consequence of the proof by B. Totaro \[To\] of theorem 7.9). This is not the case by corollary 6.16.

One can give an explicit subvariety in \( X(w) \) not satisfying the IH-small condition for \( \hat{\pi} \).

Let \( i \) be a pic of \( Q_w \) such that \( h(f(i)) < h(i) \) (such a pic exists because the resolution is not obtained from construction 3). Let us consider the smallest (for \( \succeq \) ) vertex \( j \in Q_w \) such that \( j \succeq i \) and \( j \succeq f(i) \). Then one can prove that the image \( \hat{\pi}(Z_k) \) of the divisor \( Z_j \subset \hat{X}(\hat{w}) \) in \( \hat{X}(\hat{w}) \) is of codimension \( h(f(i)) - h(j) + 1 \) and that its image \( \pi(Z_j) \) in \( X(w) \) is of codimension \( h(i) - h(j) + h(f(i)) - h(j) + 1 \). The fiber above \( \pi(Z_j) \) contains \( \hat{\pi}(Z_j) \) and is of dimension at least \( h(i) - h(j) \). But we have

\[
\text{Codim}_{X(w)}(\pi(Z_j)) = h(i) - h(j) + h(f(i)) - h(j) + 1 \leq 2(h(i) - h(j)).
\]

\[\square\]

On the contrary when we choose a good order on the pics then we obtain the following

**Theorem 7.3.** — The morphisms \( \hat{\pi} : \hat{X}(\hat{w}) \to X(w) \) obtained from construction 3 are IH-small.

In subsections 7.2, 7.3 and 7.4 we are going to give a proof of this theorem. Let us first give an easy corollary:

**Corollary 7.4.** — The morphism \( \hat{\pi} : \hat{X}(\hat{w}) \to X(w) \) obtained from construction 3 is IH-small.

**Proof** — Indeed, any morphism \( \hat{\pi}' \) obtained from construction 3 factors through the morphism \( \hat{\pi} \) and it is easy to verify that this implies that, as \( \hat{\pi}' \) is IH-small, \( \hat{\pi} \) is IH-small. \[\square\]

7.2 Fibers

We will adapt the technics of \[SV1\] in our setting. The idea that choosing a good order in the pics will produce IH-small morphisms comes from A. Zelevinsky’s paper \[Ze\].

Let us recall that the varieties \( \hat{X}(\hat{w}) \) where constructed by induction. The first step being given by the morphism \( p : \hat{\mathcal{P}uQ} \times Q X(v) \to X(w) \) where \( P \) is the stabiliser of \( X(w) \), \( v \) is obtained from \( w \) by removing the first pic and \( u \) corresponds to the removed vertices. The group \( Q \) is the intersection of \( P \) with the stabiliser of \( X(v) \). By induction there exists a resolution
\( \pi' : \tilde{X}(\tilde{v}) \rightarrow X(v) \) equivariant under the stabiliser of \( X(v) \). The resolution is given by the fiber product \( \hat{\pi} : \tilde{X}(\tilde{w}) = \mathcal{P}uQ \times Q \tilde{X}(\tilde{v}) \rightarrow X(w) \).

Let us prove a lemma giving a description of the fibers of \( \pi \) and a formula on their dimension. This lemma is directly inspired by lemma 2.1 of [SV1]. If \( w' \leq w \) in the Bruhat order, let us denote by \( U(w') \) the \( P \)-orbit (\( P \) is the stabiliser of \( X(w) \)) of \( e_{w'} \) (the fixed point of the torus corresponding to the Schubert cell of \( w' \)) in \( X(w) \). Because \( \pi \) is \( P \)-equivariant, all the fibers of points in \( U(e_{w'}) \) are isomorphic and to calculate \( f_{\pi,w'} \) the dimension of the fiber \( \pi^{-1}(e_{w'}) \), it is enough to calculate \( \dim(\pi^{-1}(U(e_{w'}))) - \dim(U(e_{w'})) \).

**Lemma 7.5.** — Define the set

\[
S(w',w) = \left\{ (u',v') \in W \left| \begin{array}{c}
u' \leq u \text{ and } v' \leq v, \text{ in the Bruhat order and } \\
P_X(u'v') = X(w'), P_X(u') = X(u') \text{ and } Q_X(v') = X(v')
\end{array} \right. \right\}.
\]

(i) We have \( p^{-1}(U(w')) = \bigcup_{(u',v') \in S(w',w)} PuQ \times Qe_{v'} \).

(ii) We have \( \pi^{-1}(U(w')) = \bigcup_{(u',v') \in S(w',w)} PuQ \times Q \pi^{-1}(Qe_{v'}) \).

(iii) This gives the formula

\[
f_{\pi,w'} = \text{Card}(Q_{u'}) + f_{\pi',v'} + \text{Card}(Q_{v'}) - \text{Card}(Q_{w'}) = \text{Card}(Q_{u'}) + f_{\pi',v'} - \text{Codim}_{X(w')}(X(v'))
\]

for some \( (u',v') \in S(w',w) \).

**Proof** — (i) Let \( (u',v') \in S(w',w) \), we have the inclusions:

\[
p(PuQ \times Qe_{v'}) \subset Pe_{w'}Qe_{v'} \subset Pe_{w'}X(v') \subset PX(u'v') = X(w').
\]

Furthermore \( p(PuQ \times Qe_{v'}) \) is a \( P \)-orbit an contains \( Pe_{w'}e_{v'} = Pe_{w'} \) so \( PuQ \times Qe_{v'} \) is contained in \( p^{-1}(U(w')) \).

Conversely, if \( (x,y) \in PuQ \times Q X(v) \) is such that \( p(x,y) = xy \in U(w') \), then there are elements \( u' \) and \( v' \) in the Weyl group such that we have \( PX(u') = X(u') \), \( QX(v') = X(v') \) and \( PxQ \times QpyP_{v'}/P_{v'Q} = PuQ \times Q X(v') \). But then there exists \( (p,q) \in P \times Q \) such that \( pe_{w'} = x \) and \( qe_{v'} = y \) so that we have \( (x,y) \in PuQ \times Q e_{v'} \). Furthermore, there exists \( p' \in P \) such that \( p'xy = e_{w'} \) thus \( p'pe_{w'}qe_{v'} = e_{w'} \). This implies, because \( QX(v') = X(v') \), that \( PX(u'v') = X(w') \).

(ii) Comes directly from (i).

(iii) Because \( \dim U(w') = \text{Card}(Q_{w'}) \), we only need to prove that

\[
\dim(\pi^{-1}(U(w'))) = \text{Card}(Q_{u'}) + f_{\pi',v'} + \text{Card}(Q_{v'})
\]

for some \( (u',v') \in S(w',w) \), it is true thanks to (ii).

**Remark 7.6.** — (i) In the case where \( PuQ/Q \) is an homogeneous variety, we recover lemma 2.1 of [SV1]. In this case we must have \( u' = u \) because \( X(u) \) is the only \( P \)-stable Schubert subvariety of \( X(u) \). We then have \( \text{Card}(Q_{u'}) = \text{Codim}_{X(w)}(X(v)) = \text{Card}(Q_{w}) - \text{Card}(Q_{v}) \).
(n) More generally, the Schubert variety $X(u')$ is a Schubert subvariety of $X(u)$ with the same stabiliser so that if $i$ is a hole of its quiver then $\beta(i) \in \beta(t(Q_u))$. The Schubert variety $X(v')$ is a Schubert subvariety of $X(v)$ with stabiliser $\text{Stab}(w) \cap \text{Stab}(v)$. If $i$ is a hole of $Q_{v'}$, then $\beta(i)$ has to be in the union $\beta(t(Q_u)) \cup \beta(t(Q_w))$.

Let us now describe the condition $PX(u'v') = X(w')$. The quiver of the Schubert variety $X(u'v')$ is obtained by gluing the quiver of $X(u')$ above the quiver of $X(v')$. Furthermore, the quiver of the Schubert variety $PX(a)$ if $X(a)$ is a Schubert subvariety of $X(w)$, is the smallest subquiver $Q$ of $Q_w$ containing the quiver $Q_a$ and such that $\beta(t(Q)) \subset \beta(t(Q_w))$. In particular, if we denote by $A$ the set of vertices $i \in Q_a$ such that $i$ is not the successor of an element of $Q_a$ and $\beta(i) \in \beta(t(Q_w))$ then the set non virtual holes of $PX(a)$ is $\hat{A}$ and the virtual holes are associated to simple roots not in the support of $\beta(t(Q_w)) \setminus \beta(A)$.

If $i$ is a hole of $Q_{w'}$ such that $\beta(i)$ is not in the support of $u$. Let $j$ be the smallest vertex in $Q_{v'}$ such that $\beta(j) = \beta(i)$. Then $j$ will be a vertex of $Q_{v',w'}$ with no predecessor and has to be a hole of $PX(u'v') = X(w')$. We must thus have $j = i$. In particular all the holes of $Q_{w'}$ associated to simple roots not in the support of $u$ have to be holes of $v'$.

The proof of the theorem 7.3 will go as follows. Because the morphism $\pi$ is $P$-equivariant where $P$ is the stabiliser of $X(w)$, we need to prove that for any $w' \in W$ such that $X(w')$ is stable under $P$ in $X(w)$, we have $\text{Codim}_{X(w)}(X(w')) > 2f_{\pi,w'}$.

For the classical cases ($A_n$ and $D_n$) we will introduce two functions $\Gamma$ and $q$ such that

$$\text{Codim}_{X(w)}(X(w')) = \Gamma(w', w) + q(w', w)$$

The function $q$ will take only non negative values and will be positive if $w' \neq w$.

We then proceed by induction on the number of pics of $w$ (or on the number of fibrations in $\widehat{X}(\widehat{w})$) and prove the more stronger result:

$$\Gamma(w', w) \geq 2f_{\pi,w'}.$$ 

Because of the previous lemma, it is enough to prove that for all $(u', v') \in S(w', w)$ we have

$$\Gamma(w', w) \geq 2(\text{Card}(Q_{u'}) + f_{\pi',v'} - \text{Codim}_{X(w')}(X(v')))$$

Let $\theta \in W$ such that $X(\theta)$ is the closure of the orbit of $X(v')$ in $X(v)$ under $\text{Stab}(X(v))$. We have $f_{\pi',v'} = f_{\pi',\theta}$ and by induction hypothesis we have $2f_{\pi',\theta} \leq \Gamma(\theta, v')$. We are thus reduced to prove:

$$2(\text{Card}(Q_{u'}) - \text{Codim}_{X(w')}(X(v'))) \leq \Gamma(w', w) - \Gamma(\theta, v').$$

We prove this formula in the following paragraph in the $A_n$ and $D_n$ case.

### 7.3 The case of $A_n$ and $D_n$

To prove the result on smallness, we will need to describe the elements of $S(w', w)$ and calculated the dimension in the formula of lemma 7.3. The only two difficult cases of minuscule
Schubert varieties will be the cases of grassmannians (the varieties constructed are the varieties of Zelevinsky [Ze]) and of maximal isotropic subspaces in an even dimensional vector space endowed with a non degenerate quadratic form (some of these cases have been treated in [SV] and we complete their study). Indeed the other minuscule Schubert varieties are those of $E_6$ and $E_7$ for which we will make a case by case analysis (which is tedious by not hard with lemma [6]) and the case of Schubert varieties in a quadric which are very simple.

### 7.3.1 The $A_n$ case

Let us consider a minuscule quiver $Q_w$ of a Schubert variety in the grassmannian $G(p,q)$ of $p$-dimensional subvector spaces of a $q$-dimensional vector space. We may assume that all the simple roots are in the support of $w$ (otherwise we simply restrict the group) so that there will be no virtual hole in $Q_w$. The set of simple roots $\beta(t(Q_w))$ can be written as $\{\alpha_{k_1}, \ldots, \alpha_{k_s}\}$ in the notation of [Bou]. Let us denote by $t_1, \ldots, t_s$ the holes such that $\beta(t_i) = \alpha_{k_i}$. Because of proposition [4.2] for all $i \in [2, s]$ there exists exactly one pic $p_i$ between the holes $t_{i-1}$ and $t_i$. Furthermore there must be a pic $p_1$ (resp. $p_{s+1}$) with $\beta(p_1) = \alpha_k$ (resp $\beta(p_{s+1}) = \alpha_k$) with $k < k_1$ (resp. $k > k_s$). In particular we see that the number $n$ of pics equals $s + 1$.

Let us now define the following sequences $(a_i(w))_{i \in [1,s+1]}$ and $(b_i(w))_{i \in [0,s]}$ of integers (we will sometimes simply denote them by $a_i$ and $b_i$ omitting $w$):

\[
\begin{align*}
    a_i(w) &= h(p_i) - h(t_i) \text{ and } b_i(w) = h(p_{i+1}) - h(t_i) \text{ for } i \in [1, s], \\
    a_{s+1}(w) &= p - \sum_{i=1}^{s} a_i(w), \\
    b_0(w) &= q - \sum_{i=1}^{s} b_i(w).
\end{align*}
\]

It is an easy game on the quiver and the description of configuration varieties to verify that if the sequences of integers associated to the quiver $Q_w$ are $(a_i(w))_{i \in [1,s+1]}$ and $(b_i(w))_{i \in [0,s]}$ then we have:

\[
X(w) = \{ V \in G(p,q) / \dim(V \cap K^n_i) \geq m_i \text{ for all } i \in [1,s] \}
\]

where $n_i = \sum_{k=1}^{i} (a_k + b_{k-1})$ and $m_i = \sum_{k=1}^{i} a_k$.

Let $X(w')$ be a Schubert subvariety of $X(w)$ with the same stabiliser. Then we must have $\beta(t(Q_{w'})) \subset \beta(t(Q_w))$. For any hole $t_i$ of $Q_w$ let us define the depth of $w'$ in $t_i$ to be the integer

\[
c_i = \text{Card}\{ j \in Q_w \setminus Q_{w'} / \beta(j) = \beta(t_i) \}.
\]

The same game on the quiver and the description of configuration varieties shows that the associated sequences are given by

\[
\begin{align*}
    a_i(w') &= a_i(w) + c_i - c_{i-1} \text{ for all } i \in [1, s+1] \\
    b_i(w') &= b_i(w) + c_i - c_{i+1} \text{ for all } i \in [0, s]
\end{align*}
\]

with $c_0 = c_{s+1} = 0$. We have

\[
X(w) = \{ V \in G(p,q) / \dim(V \cap K^n_i) \geq m_i + l_i \text{ for all } i \in [1, s] \}
\]
with \( l_i = \sum_{k=1}^{i} c_k \). These description enables us to give the following fact to calculate the codimension of \( X(w') \) in \( X(w) \) (we set \( c_0 = c_{s+1} = 0 \)):

**Fact 7.7.** — We have the formula:

\[
\text{Codim}_{X(w)}(X(w')) = \text{Card}(Q_w) - \text{Card}(Q_{w'}) = \Gamma(w', w) + q(w', w)
\]

where

\[
\Gamma(w', w) = \sum_{i=1}^{s} c_i (a_i + b_i) \quad \text{and} \quad q(w', w) = \frac{1}{2} \sum_{i=1}^{s+1} (c_i - c_{i-1})^2.
\]

**Proof** — This can be seen with a simple calculation. We will describe it geometrically on the quiver. We have the following quiver (see the appendix for a description of the quivers):

\[
\begin{align*}
&\text{...} \quad p_1 \quad p_2 \quad p_s \quad p_{s+1} \\
&b_0 \quad a_1 \quad t_1 \quad b_1 \quad \text{...} \quad a_s \quad t_s \quad b_s \quad a_{s+1} \\
&c_1 \quad \text{...} \quad c_s
\end{align*}
\]

The codimension being given by the difference of the number of vertices, we find

\[
\text{Codim}_{X(w)}(X(w')) = \sum_{i=1}^{s} c_i (a_i + b_i) - \sum_{i=1}^{s-1} c_i c_{i+1} + \sum_{i=1}^{s} c_i^2
\]

and simple calculation gives the formula. Remark that we have \( q(w', w) > 0 \) for \( w' \neq w \). □

Let us assume that \( v \) is obtained from \( w \) by removing the \( k^{th} \) pic of \( Q_w \). We obtain on the quivers the following situation:
This simply means that the sequences of integers \((a_i(v))_{i \in [1,s]}\) and \((b_i(v))_{i \in [0,s-1]}\) are given by

\[
\begin{align*}
  a_i(v) &= \begin{cases} 
    a_i(w) & \text{for } i < k-1 \\
    a_k(w) + a_{k-1}(w) & \text{for } i = k-1 \\
    a_{i+1}(w) & \text{for } i \geq k
  \end{cases} \\
  b_i(v) &= \begin{cases} 
    b_i(w) & \text{for } i < k-1 \\
    b_k(w) + b_{k-1}(w) & \text{for } i = k-1 \\
    b_{i+1}(w) & \text{for } i \geq k
  \end{cases}
\end{align*}
\]

where \(a_0(w) = b_0(w)\). Furthermore, the quiver \(Q_u\) has no hole meaning that the variety \(PuQ/Q\) is smooth and \(u'\) has to be equal to \(u\). We only need to determine \(v'\).

Let us now consider the quiver \(Q\) obtained by intersecting in \(Q_w\) the quivers \(Q_v\) and \(Q_{w'}\). The quivers of \(v'\) has to be a subquiver of this quiver such that (see remark 7.6) all the holes of \(Q_{w'}\) are holes of \(Q_{v'}\) and \(Q_{v'}\) may have one more hole corresponding to the hole of \(v\) which is not a hole of \(w\).

The quiver \(Q_{v'}\) has \(s+1\) holes and the sequences \((a_i(v'))_{i \in [1,s+2]}\) and \((b_i(v'))_{i \in [0,s+1]}\) are given
holes of $v$.

It is now an easy calculation (and straightforward on the quiver) that the depth $c'_i$ of $\theta$ in the holes of $v$ is $c_{k-1} - y = c_k - x$ for the $(k-1)^{th}$ hole, and $c_i$ for all the holes before the $(k-1)^{th}$ hole and $c_{i+1}$ for all the holes after the $(k-1)^{th}$ hole. We can now calculate:

$$\text{Codim}_{X(w)}(X(v)) - \text{Codim}_{X(w')} (X(v')) = a_k(w)b_{k-1}(w) - (a_k(w') - x)(b_{k-1}(w') - y)$$

where $x \in [0, c_k]$, $y \in [0, c_{k-1}]$ and $c_{k-1} - y = c_k - x$. Indeed, the last formula is given by the fact that the only hole different from those of $Q_{w'}$ has to be associated to the same root as the hole of $v$ which is not a hole of $w$. This gives the equality

$$\sum_{i=1}^{k-1} (a_i(v) + b_{i-1}(v)) = \sum_{i=1}^{k} (a_i(v') + b_{i-1}(v'))$$

and the equality $c_{k-1} - y = c_k - x$. The fact that $x \in [0, c_k]$ and $y \in [0, c_{k-1}]$ are equivalent to the fact that $X(v')$ is a Schubert subvariety of $X(v)$.

The Schubert subvariety $X(\theta)$ is contained in $X(v)$, contains $X(v')$ and is stable by the stabiliser of $X(v)$. It must have the same hole as $v'$ except for those not corresponding to holes of $v$. In our case we have to fill the holes $k - 1$ and $k + 1$ of $v'$ to obtain $\theta$:

The quiver $Q_{\theta}$ of $\theta$ has $s - 1$ holes and the integers $(a_i(\theta))_{i \in [1, s]}$ and $(b_i(\theta))_{i \in [0, s-1]}$ are given by:

$$a_i(v') = \begin{cases} a_i(v') & \text{for } i \leq k - 2 \\ a_{k-1}(v') + a_k(v') & \text{for } i = k - 1 \\ a_{k+1}(v') + a_{k+2}(v') & \text{for } i = k \\ a_{i+2}(v') & \text{for } i \geq k + 1 \end{cases}$$

and

$$b_i(v') = \begin{cases} b_i(v') & \text{for } i < k - 1 \\ b_{k-2}(v') + b_{k-1}(v') & \text{for } i = k - 1 \\ b_k(v') + b_{k+1}(v') & \text{for } i = k \\ b_{i+2}(w) & \text{for } i \geq k \end{cases}$$

It is now an easy calculation (and straightforward on the quiver) that the depth $c'_i$ of $\theta$ in the holes of $v$ is $c_{k-1} - y = c_k - x$ for the $(k-1)^{th}$ hole, and $c_i$ for all the holes before the $(k-1)^{th}$ hole and $c_{i+1}$ for all the holes after the $(k-1)^{th}$ hole. We can now calculate:

$$\text{Codim}_{X(w)}(X(v)) - \text{Codim}_{X(w')} (X(v')) = a_k(w)b_{k-1}(w) - (a_k(w') - x)(b_{k-1}(w') - y)$$
and finally:

$$\text{Codim}_{X(w)}(X(v)) - \text{Codim}_{X(w')}(X(v')) = xa_k(w) + yb_{k-1}(w) - xy.$$ 

On the other hand we calculate

$$\Gamma(w', w) - \Gamma(\theta, v) = \sum_{i=1}^{s} c_i(a_i(w) + b_i(w)) - \sum_{i=1}^{s-1} c'_i(a_i(v) + b_i(v))$$

$$= \sum_{i=1}^{s} c_i(a_i(w) + b_i(w)) - \sum_{i<k-1} c_i(a_i(w) + b_i(w)) - \sum_{i>k-1} c_{i+1}(a_{i+1}(w) + b_{i+1}(w)) - c'_{k-1}(a_{k-1}(w) + b_{k-1}(w))$$

$$= c_{k-1}(a_{k-1}(w) + b_{k-1}(w)) + c_k(a_k(w) + b_k(w)) - (c_k - x)(a_k(w) + b_k(w)) - (c_{k-1} - y)(a_{k-1}(w) + b_{k-1}(w))$$

and finally

$$\Gamma(w', w) - \Gamma(\theta, v) = x(a_k(w) + b_k(w)) + y(a_{k-1}(w) + b_{k-1}(w)).$$

Now the fact that the $k^{th}$ pic of $w$ was smaller than the adjacent pics means that we have $a_{k-1}(w) \geq b_{k-1}(w)$ and $b_k(w) \geq a_k(w)$ so that we get the inequality (because $x$ and $y$ are non negative):

$$2(\text{Codim}_{X(w)}(X(v)) - \text{Codim}_{X(w')}(X(v'))) \leq \Gamma(w', w) - \Gamma(\theta, v).$$

And the theorem is proved in this case.

### 7.3.2 The $D_n$ case

Let us consider a minuscule quiver $Q_w$ of a Schubert variety in the grassmannian $G_{iso}(p, 2p)$ of $p$-dimensional isotropic subvector spaces of a $2p$-dimensional vector space endowed with a non degenerate quadratic form. We may assume that all the simple roots are in the support of $w$ (otherwise we simply restrict the group) so that there will be no virtual hole in $Q_w$. The set of simple roots $\beta(t(Q_w))$ can be written as $\{\alpha_{k_1}, \cdots, \alpha_{k_s}\}$ in the notation of [Bou]. Let us denote by $t_1, \cdots, t_s$ the holes such that $\beta(t_i) = \alpha_{k_i}$. Because of proposition 4.2 for all $i \in [2, s]$ there exists exactly one pic $p_i$ between the holes $t_{i-1}$ and $t_i$. Furthermore there must be a pic $p_1$ with $\beta(p_1) = \alpha_k$ with $k < k_1$. If $\alpha_{k_1} \not\in \{\alpha_{p-1}, \alpha_p\}$ then there must be pic $p_{s+1}$ with $\beta(p_{s+1}) = \alpha_{p-1}$ or $\alpha_p$ and in this case there are $s+1$ pics, otherwise there are $s$ pics. In the last case, we have $\alpha_{k_s} = \alpha_{p-1+i}$ with $i = 0$ or $i = 1$. We define $p_{s+1}$ to be the smallest vertex (for $\preceq$) of $Q_w$ with $\beta(p_{s+1}) = \alpha_{p-i}$.

Let us now define the following sequences $(a_i(w))_{i \in [1, s+1]}$ and $(b_i(w))_{i \in [0, s]}$ of integers (we will sometimes simply denotethem by $a_i$ and $b_i$ omitting $w$):

In the first case:

$$\begin{align*}
\begin{cases}
  a_i(w) = h(p_i) - h(t_i) \text{ for } i \in [1, s] \\
  b_i(w) = h(p_{i+1}) - h(t_i) \text{ for } i \in [1, s-1], \\
  b_s(w) = h(p_{s+1}) - h(t_s) + \frac{1}{2} \\
  a_{s+1}(w) = p - \sum_{i=1}^{s} a_i(w), \\
  b_0(w) = \frac{1}{2}h(p_{s+1}) - \sum_{i=1}^{s} b_i(w).
\end{cases}
\end{align*}$$

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In the second case:

\[
\begin{align*}
    a_i(w) &= h(p_i) - h(t_i) \text{ and } b_i(w) = h(p_{i+1}) - h(t_i) \text{ for } i \in [1, s-1], \\
    a_s(w) &= h(p_s) - h(p_{s+1}) \text{ and } b_s(w) = h(p_{s+1}) - h(p_{s+1}) - \frac{1}{2} \\
    a_{s+1}(w) &= p - \sum_{i=1}^s a_i(w), \\
    b_0(w) &= \frac{1}{2}h(p_{s+1}) - \sum_{i=1}^s b_i(w).
\end{align*}
\]

It is an easy game on the quiver and the description of configuration varieties to verify that if the associated sequences of integers to the quiver $Q_w$ are $(a_i(w))_{i \in [1, s+1]}$ and $(b_i(w))_{i \in [0, s]}$ then we have:

\[X(w) = \{ V \in \mathbb{G}_{iso}(p, 2p) / \dim(V \cap \mathbb{K}^{n_i}) \geq m_i \text{ for all } i \in [1, s] \}\]

where the $\mathbb{K}^k$ form a complete flag of fixed isotropic subspaces, $n_i = \sum_{k=1}^i (a_k + b_{k-1})$ and $m_i = \sum_{k=1}^i a_k$ for $i \in [1, s]$ in the first case, and $n_i = \sum_{k=1}^i (a_k + b_{k-1})$ and $m_i = \sum_{k=1}^i a_k$ for $i \in [1, s-1]$, $n_s = p$, $m_s = 1 + \sum_{k=1}^s a_k$ and $\mathbb{K}^{n_s}$ is associated to the root $\alpha_k$ in the second case.

Let $X(w')$ be a Schubert subvariety of $X(w)$ with the same stabiliser. Then we must have $\beta(t(Q_w')) \subset \beta(t(Q_w))$. For any hole $t_i$ of $Q_w$ let us define the depth of $w'$ in $t_i$ to be the integer

\[c_i = \begin{cases} 
    \text{Card}\{j \in Q_w \setminus Q_{w'}/ \beta(j) = \beta(t_i)\} \text{ for } \beta(t_i) \notin \{\alpha_{p-1}, \alpha_p\} \\
    2\text{Card}\{j \in Q_w \setminus Q_{w'}/ \beta(j) = \beta(t_i)\} \text{ for } \beta(t_i) \in \{\alpha_{p-1}, \alpha_p\}.
\end{cases}\]

The same game on the quiver and the description of configuration varieties shows that the associated sequences are given by

\[
\begin{align*}
    a_i(w') &= a_i(w) + c_i - c_{i-1} \text{ for all } i \in [1, s+1] \\
    b_i(w') &= b_i(w) + c_i - c_{i+1} \text{ for all } i \in [0, s]
\end{align*}
\]

with $c_0 = 0$ and $c_{s+1} = c_s$. We have

\[X(w) = \{ V \in \mathbb{G}(p, q) / \dim(V \cap \mathbb{K}^{n_i}) \geq m_i + l_i \text{ for all } i \in [1, s] \}\]

with $l_i = \sum_{k=1}^i c_k$. These description enables us to give the following fact to calculate the codimension of $X(w')$ in $X(w)$ (we set $c_0 = c_{s+1} = 0$):

**Fact 7.8.** We have the formula:

\[\text{Codim}_{X(w)}(X(w')) = \text{Card}(Q_w) - \text{Card}(Q_{w'}) = \Gamma(w', w) + q(w', w)\]

where

\[\Gamma(w', w) = \sum_{i=1}^s c_i(a_i + b_i) \quad \text{and} \quad q(w', w) = \frac{1}{2} \sum_{i=1}^s (c_i - c_{i-1})^2.\]

**Proof** — This can be seen with a simple calculation. We will describe it geometrically on the quiver. In the first case, we have the following quiver:
The codimension being given by the difference of the number of vertices, we find

$$\text{Codim}_{X(w)}(X(w')) = \sum_{i=1}^{s-1} c_i(a_i + b_i) + c_s(a_s + b_s - \frac{1}{2}) - \frac{c_s(c_s - 1)}{2} - \sum_{i=1}^{s-1} c_i c_{i+1} + \sum_{i=1}^{s} c_i^2$$

and simple calculation gives the formula. In the second case, we have the following quiver:
The codimension being given by the difference of the number of vertices, we find

\[
\text{Codim}_{X(w)}(X(w')) = \sum_{i=1}^{s-1} c_i(a_i + b_i) + a_s c_s - \sum_{i=1}^{s-1} c_i c_{i+1} + \sum_{i=1}^{s-1} c_i^2 + \frac{c_s(c_s + 1)}{2}
\]

and simple calculation gives the formula. Remark that we have \(q(w', w) > 0\) for \(w' \neq w\). \(\square\)

Let us assume that \(v\) is obtained from \(w\) by removing the \(k^{th}\) pic of \(Q_w\). We obtain on the quivers the four following situation:

Case 1

\[
\begin{array}{c}
\cdots \quad a_{k+1} \quad b_{k-1} \quad a_k \quad b_k \quad u \\
\cdots \\
\end{array}
\]

Case 1 bis

\[
\begin{array}{c}
\cdots \quad a_s \quad b_s \\
\cdots \\
\end{array}
\]

Case 2

\[
\begin{array}{c}
\cdots \quad a_s \quad b_s \\
\cdots \\
\end{array}
\]

Case 3

\[
\begin{array}{c}
\cdots \quad a_{s+1} \quad b_{s+1} \\
\cdots \\
\end{array}
\]

Case 1 is strictly equivalent to the \(A_n\) case and we get the result with the same calculation in this case. Case 1 bis has been done with these techniques in \([SV1]\), we will not make the calculation one more time. For case 2, the sequences of integers \((a_i(v))_{i \in [1, s+1]}\) and \((b_i(v))_{i \in [0, s]}\) are given by:

\[
a_i(v) = \begin{cases} 
    a_i(w) & \text{for } i < s \\
    a_s(w) + b_s(w) - \frac{1}{2} & \text{for } i = s \\
    a_{s+1}(w) - (b_s(w) - \frac{1}{2}) & \text{for } i = s + 1
\end{cases}
\]

\[
b_i(v) = \begin{cases} 
    b_i(w) + b_s(w) - \frac{1}{2} & \text{for } i = 0 \\
    b_i(w) & \text{for } 0 < i < s \\
    \frac{1}{2} & \text{for } i = s.
\end{cases}
\]
For case 3, the sequences of integers \((a_i(v))_{i \in [1,s]}\) and \((b_i(v))_{i \in [0,s-1]}\) are given by:

\[
a_i(v) = \begin{cases} 
    a_i(w) & \text{for } i < s - 1 \\
    a_s(w) + a_{s-1}(w) + b_{s-1}(w) & \text{for } i = s - 1 \\
    a_{s+1}(w) - b_{s-1}(w) & \text{for } i = s 
\end{cases}
\]

and

\[
b_i(v) = \begin{cases} 
    b_i(w) & \text{for } i < s - 1 \\
    b_s(w) & \text{for } i = s - 1 
\end{cases}
\]

Furthermore, in the case 2, the quiver \(Q_u\) has no hole meaning that the variety \(\mathcal{P} u Q / Q\) is smooth and \(u'\) has to be equal to \(u\). In this case we only need to determine \(v'\).

Let us now consider the quiver \(Q\) obtained by intersecting in \(Q_w\) the quivers \(Q_v\) and \(Q_{u'}\). The quiver of \(v'\) has to be a subquiver of this quiver such that (see remark 7.6) all the holes of \(Q_{u'}\) are holes of \(Q_{v'}\) and \(Q_{v'}\) may have one more hole corresponding to the hole of \(v\) which is not a hole of \(w\).

In case 2, the quiver \(Q_{v'}\) has \(s+1\) holes and the sequences \((a_i(v'))_{i \in [1,s+2]}\) and \((b_i(v'))_{i \in [0,s+1]}\) are given by

\[
a_i(v') = \begin{cases} 
    a_i(w') & \text{for } i \leq s \\
    b_s(w') - \frac{1}{2} - x & \text{for } i = s + 1 
\end{cases}
\]

and

\[
b_i(v') = \begin{cases} 
    b_i(w') & \text{for } i < s \\
    x & \text{for } i = s \\
    \frac{1}{2} & \text{for } i = s + 1 
\end{cases}
\]

where \(x \in [0,c_s]\). In case 3, the quiver \(Q_{v'}\) has \(s\) holes and the sequences \((a_i(v'))_{i \in [1,s+1]}\) and \((b_i(v'))_{i \in [0,s]}\) are given by

\[
a_i(v') = \begin{cases} 
    a_i(w') & \text{for } i \leq s - 1 \\
    a_s(w) + b_{s-1}(w) - c_{s-1} + y & \text{for } i = s 
\end{cases}
\]

and

\[
b_i(v') = \begin{cases} 
    b_i(w') & \text{for } i < s - 1 \\
    c_{s-1} - y & \text{for } i = s - 1 \\
    b_s(w') & \text{for } i = s 
\end{cases}
\]

We also get that \(u'\) has a unique hole,

\[
\begin{align*}
    a_0(u') = x \text{ and } a_1(u') = a_s(w) + b_{s-1}(w) - x = a_1(u) + b_0(u) - x \\
    b_0(u') = x \text{ and } b_1(u') = b_1(u) = b_s(w)
\end{align*}
\]

In case 3, we have \(y \in [c_{s-1} - c_s, c_{s-1}], \ x \in [b_{s-1}(w') - c_k, b_{s-1}(w')]\) and \(b_{s-1} - x = c_s - y\). Indeed, the last formula is given by the fact that last hole of \(Q_{u'}\) has to be the same hole as
the last hole of $Q_{w'}$. This implies that $b_{s-1}(v') + b_0(u') = b_{s-1}(w')$ and the equality. The fact that $y \in [c_{s-1} - c_s, c_{s-1}]$ comes from the fact that $X(v')$ is a Schubert subvariety of $X(v)$ and that $X(u'v') = X(w')$.

The Schubert subvariety $X(\theta)$ is contained in $X(v)$, contains $X(v')$ and is stable by the stabiliser of $X(v)$. It must have the same hole as $v'$ except for those not corresponding to holes of $v$. In our case we have to fill the $s^{th}$ hole (in case 2) or the $(s-1)^{th}$ hole (in case 3) of $v'$ to obtain $\theta$:

In case 2, the integers $(a_i(\theta))_{i\in[1,s+1]}$ and $(b_i(\theta))_{i\in[0,s]}$ associated to $\theta$ are given by

$$a_i(\theta) = \begin{cases} a_i(v') & \text{for } i \neq s \\ a_{s+1}(v') + a_s(v') & \text{for } i = s \end{cases} \quad \text{and} \quad b_i(\theta) = \begin{cases} b_i(v') & \text{for } i \neq s - 1 \\ b_{s-1}(v') + b_s(v') & \text{for } i = s - 1 \end{cases}$$

In case 3, the integers associated to $\theta$ are given by $(a_i(\theta))_{i\in[1,s]}$ and $(b_i(\theta))_{i\in[0,s-1]}$ are given by

$$a_i(\theta) = \begin{cases} a_i(v') & \text{for } i \neq s - 1 \\ a_{s-1}(v') + a_s(v') & \text{for } i = s - 1 \end{cases} \quad \text{and} \quad b_i(\theta) = \begin{cases} b_i(v') & \text{for } i \neq s - 2 \\ b_{s-2}(v') + b_{s-1}(v') & \text{for } i = s - 2 \end{cases}$$

It is now an easy calculation (and straightforward on the quiver) that in case 2, the depth $c'_i$ of $\theta$ in the holes of $v$ is $c_s - x$ for the $s^{th}$ hole and $c_i$ for the other holes. In case 3, the depth $c'_i$ of $\theta$ in the holes of $v$ is $y = c_s + x - b_{s-1}$ for the $(s-1)^{th}$ hole and $c_i$ for the other holes.

We can now calculate in case 2:

$$\dim X(v') - \text{Codim}_{X(w')}X(v')) = \frac{(b_s(w) + \frac{1}{2})(b_s(w) - \frac{1}{2})}{2} - \frac{(b_s(w') - x + \frac{1}{2})(b_s(w') - x - \frac{1}{2})}{2}$$

and because $b_s(w') = b_s(w)$ we get:

$$\dim X(v') - \text{Codim}_{X(w')}X(v')) = xb_s(w) - \frac{x^2}{2}.$$
a simple calculation gives

\[ \Gamma(w', w) - \Gamma(\theta, v) = x(a_s(w) + b_s(w)). \]

Now the fact that the \( s \)th pic of \( w \) was smaller than the \( (s - 1) \)th pic means that \( a_s(w) \geq b_s(w) \) so that we get the inequality :

\[ 2(\dim X(w) - \text{Codim}_{X(w)}(X(v'))) \leq \Gamma(w', w) - \Gamma(\theta, v). \]

The theorem is proved in case 2. In case 3, we have:

\[
\dim X(w') - \text{Codim}_{X(w')}(X(v')) = \frac{x(x+1)}{2} + x(a_s(w) + b_{s-1}(w) - x) - \frac{x(x+1)}{2} - x a_s(w') \\
= x(a_s(w) + b_{s-1}(w) - x - a_s(w) - c_s + c_{s-1})
\]

and finally:

\[ \dim X(w') - \text{Codim}_{X(w')}(X(v')) = x(c_{s-1} - y). \]

On the other hand we have,

\[ \Gamma(w', w) - \Gamma(\theta, v) = \sum_{i=1}^{s} c_i(a_i(w) + b_i(w)) - \sum_{i=1}^{s-1} c_i'(a_i(v) + b_i(v)) \]

a simple calculation gives

\[ \Gamma(w', w) - \Gamma(\theta, v) = (c_{s-1} - y)(a_{s-1}(w) + b_{s-1}(w)) + (b_{s-1} - x)(a_s(w) + b_s(w)). \]

Now the fact that the \( (s - 1) \)th pic of \( w \) is smaller than the \( (s - 2) \)th pic means that we have \( a_{s-1}(w) \geq b_{s-1}(w) \). Furthermore, we have \( x \leq b_{s-1}(w') = b_{s-1}(w) \) and \( a_s(w) + b_s(w) \geq 0 \) so that we get the inequality :

\[ 2(\dim X(w') - \text{Codim}_{X(w')}(X(v'))) \leq \Gamma(w', w) - \Gamma(\theta, v). \]

The theorem is proved in case 3.

### 7.4 Exceptional cases

We are left to deal with three cases: quadrics and minuscule varieties for \( E_6 \) and \( E_7 \).

#### 7.4.1 Quadrics

For quadrics, let us remark that all Schubert varieties except one are locally factorial (the quivers have only one pic) so that in all cases except one we have \( \widehat{X}(\widehat{w}) = X(w) \) and there is nothing to prove. The only non locally factorial Schubert variety (we are in \( \mathbb{K}^{2p} \) with a non degenerate quadratic form) is given by:

\[ X(w) = \{ x \in \mathbb{P}(\mathbb{K}^{2p}) / x \text{ is isotropic and } x \in W_{p-2}^{\perp} \} \]

for a fixed isotropic subspace \( W_{p-2} \) of dimension \( p - 2 \). The associated quiver has \( p \) vertices and is given by
In particular, the resolution $\tilde{\pi} : \tilde{X}(\tilde{w}) \to X(w)$ is given by $p : \overline{PuQ} \times Q X(v) \to X(w)$ where $\overline{PuQ}/Q$ is of dimension 1. The fiber of the morphism $\tilde{\pi}$ is at most 1. On the other hand, as it is $\text{Stab}(X(w))$-equivariant, the fiber is strictly positive on Schubert subvarieties stable under $\text{Stab}(X(w))$. These subvarieties are of codimension at least 3 and the result follows.

More generally, if the morphism $\tilde{\pi}$ is of the form $p : \overline{PuQ} \times Q X(v) \to X(w)$ and its fiber is of dimension at most one then the morphism $\tilde{\pi}$ has to be $IH$-small.

7.4.2 The $E_6$ case

We have seen that if the Schubert variety is locally factorial (i.e. its quiver has a unique pic) or if the morphism $\tilde{\pi}$ is of the form $p : \overline{PuQ} \times Q X(v) \to X(w)$ and its fiber is of dimension at most one then the morphism $\tilde{\pi}$ has to be $IH$-small. We are now going to list the morphisms $\tilde{\pi}$ obtained from construction 3 not satisfying these properties and verify that $\tilde{\pi}$ is small.

For $E_6$, all the morphisms $\tilde{\pi}$ not verifying the preceding properties are of the form $p : \overline{PuQ} \times Q X(v) \to X(w)$. We list here the quiver of $X(w)$ indicating on each quiver the quivers of $u$ and $v$.

Case 1
Case 2
Case 3
Case 4
Case 5

The dimension of the fiber in these morphisms is at most $f = 2$ except in the second case where it is at most $f = 3$. The Schubert subvarieties $X(w')$ stable under $\text{Stab}(X(w))$ of codimension not bigger than $2f$ are the following:

Case 1
Case 2
Case 3
Case 4
Case 5
These quivers $Q$ are obtained from the quiver $Q_w$ by removing all the vertices smaller than a hole $i$ of $Q_w$. It is now easy to see that any subvariety $Z_K$ of the Bott-Samelson resolution $\tilde{\pi} : \tilde{X}(\tilde{w}) \to X(w)$ such that $\tilde{\pi}(Z_K) = X(w')$ is contained in the divisor $Z_i$. We thus have $\tilde{\pi}^{-1}(X(w')) = Z_i$ and $\tilde{\pi}^{-1}(X(w'))$ is contained in the image of $Z_i$ in $\tilde{X}(\tilde{w})$. Seeing $\tilde{X}(\tilde{w})$ as a configuration variety, the image of $Z_i$ in $\tilde{X}(\tilde{w})$ is the configuration variety $\overline{P_uQ} \times Q X(v')$ where $Q_{v'} = Q \cap Q_v$. In particular, the dimension of the fiber of $\tilde{\pi}$ above $X(w')$ is 1, 1, 2, 1, 1, 1 in the different cases and the morphism $\tilde{\pi}$ is always $IH$-small.

7.4.3 The $E_7$ case

We proceed in the same way in this case and list the quivers having at least two pics and for which the fiber is at least two.

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
<th>Case 4</th>
<th>Case 5</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="case1.png" alt="Diagram" /></td>
<td><img src="case2.png" alt="Diagram" /></td>
<td><img src="case3.png" alt="Diagram" /></td>
<td><img src="case4.png" alt="Diagram" /></td>
<td><img src="case5.png" alt="Diagram" /></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case 6</th>
<th>Case 7</th>
<th>Case 7 bis</th>
<th>Case 8</th>
<th>Case 9</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="case6.png" alt="Diagram" /></td>
<td><img src="case7.png" alt="Diagram" /></td>
<td><img src="case7bis.png" alt="Diagram" /></td>
<td><img src="case8.png" alt="Diagram" /></td>
<td><img src="case9.png" alt="Diagram" /></td>
</tr>
</tbody>
</table>
All the resolutions are of type $\overline{PuQ} \times^Q X(v')$ except case 8. The maximal dimension $f$ of the fiber in all these cases is given by

<table>
<thead>
<tr>
<th>case</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>7 bis</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>11 bis</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>4</td>
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<td>2</td>
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<td>3</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

We have circled the vertices $i$ such that the quivers $Q_{w'}$ obtained from the quiver $Q_w$ by removing all the vertices smaller (for $\preceq$) than the hole $i$ of $Q_w$ are the quivers of the Schubert subvarieties $X(w')$ stable under $\text{Stab}(X(w))$ of codimension not superior to $2f$. The codimension of $X(w')$ in $X(w)$ is given by:

<table>
<thead>
<tr>
<th>case</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>7 bis</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>11 bis</th>
<th>12</th>
<th>13</th>
<th>15</th>
<th>16</th>
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</thead>
<tbody>
<tr>
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<td>3</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3 or 8</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>3 or 6</td>
<td>4</td>
<td>3</td>
<td>3</td>
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</tr>
</tbody>
</table>

Remark that in case 5 and 14 there is no such Schubert subvariety so that the morphism is already small. In all the other cases and as for the $E_6$ case, it is easy to see that any subvariety $Z_K$ of the Bott-Samelson resolution $\overline{\pi} : \overline{X(\tilde{w})} \to X(w)$ such that $\overline{\pi}(Z_K) = X(w')$ is contained in the divisor $Z_i$. We thus have $\overline{\pi}^{-1}(X(w')) = Z_i$ and $\overline{\pi}^{-1}(X(w'))$ is contained in the image of $Z_i$ in $\overline{X(\tilde{w})}$. Seeing $\overline{X(\tilde{w})}$ as a configuration variety, the image of $Z_i$ in $\overline{X(\tilde{w})}$ is the configuration variety $\overline{PuQ} \times^Q X(v')$ where $Q_{v'} = Q_w \cap Q_v$. In particular, the dimension of the fiber of $\overline{\pi}$ above $X(w')$ is given by:

<table>
<thead>
<tr>
<th>case</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>7 bis</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>11 bis</th>
<th>12</th>
<th>13</th>
<th>15</th>
<th>16</th>
<th>17</th>
</tr>
</thead>
<tbody>
<tr>
<td>dim</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1 or 3</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1 or 2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

and the morphism $\overline{\pi}$ is always $IH$-small in these cases. We are left with case 8 for which the resolution is of the form $\overline{PtQ} \times^Q RuS \times^S X(v)$ and the partitions of the quivers are given by:
The maximal dimension $f$ of the fiber in all these cases is given by

<table>
<thead>
<tr>
<th>case</th>
<th>8.1</th>
<th>8.2</th>
<th>8.3</th>
<th>8.4</th>
<th>8.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

We have cerced and numeroted the vertices $i$ such that the quivers $Q_w'$ obtained from the quiver $Q_w$ by removing all the vertices smaller (for $\preceq$) than a fixed subset of the holes of $Q_w$ are the quivers of the Schubert subvarieties $X(w')$ stable under $\text{Stab}(X(w))$ of codimension not superior to $2f$. Let $A$ be a non empty subset of $\{1, 2, 3\}$ and let $Q_w'$ be the quiver obtained by removing the vertices smaller than the vertices in $A$. The codimension of $X(w')$ in $X(w)$ is given by (here $A$ is $\{1\}$, $\{2\}$, $\{1, 2\}$ or in the last two cases $\{3\}$):

<table>
<thead>
<tr>
<th>case</th>
<th>8.1</th>
<th>8.2</th>
<th>8.3</th>
<th>8.4</th>
<th>8.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{codim}$</td>
<td>3, 3 or 5</td>
<td>3, 3 or 5</td>
<td>3, 3 or 5</td>
<td>3, 3, 5 or 8</td>
<td>3, 3, 5 or 8</td>
</tr>
</tbody>
</table>

Suppose that $A$ is a subset of $\{1, 2\}$, it is easy to see that any subvariety $Z_K$ of the Bott-Samelson resolution $\pi : \widetilde{X}(\bar{w}) \to X(w)$ such that $\pi(Z_K) = X(w')$ is contained in the variety $Z_A$. The fiber $\pi^{-1}(X(w'))$ is thus contained in the image of $Z_A$ in $\widetilde{X}(\bar{w})$. Seeing $\widetilde{X}(\bar{w})$ as a configuration variety, this image in $\widetilde{X}(\bar{w})$ is the configuration variety $\text{PitQ} \times Q \text{RuS} \times S X(v')$ where $Q_w'$ and $Q_v'$ are obtained respectively from $Q_u$ and $Q_v$ by removing the vertices smaller than one vertex in $A \cap Q_u$ respectively $A \cap Q_v$. In particular, the dimension of the fiber of $\pi$ above $X(w')$ is given by

<table>
<thead>
<tr>
<th>case</th>
<th>8.1</th>
<th>8.2</th>
<th>8.3</th>
<th>8.4</th>
<th>8.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{dim}$</td>
<td>1, 1 or 1</td>
<td>1, 1 or 2</td>
<td>1, 1 or 1</td>
<td>1, 1 or 1</td>
<td>1, 1 or 1</td>
</tr>
</tbody>
</table>

and the morphism $\pi$ is always $IH$-small in these cases. We are left with the case where $A = \{3\}$. In this case it is not hard to see that any subvariety $Z_K$ of the Bott-Samelson resolution $\pi : \widetilde{X}(\bar{w}) \to X(w)$ such that $\pi(Z_K) = X(w')$ is contained in the variety $Z_{\{2, 3\}}$ or in $Z_4$. The fiber $\pi^{-1}(X(w'))$ is thus contained in the image of $Z_{\{2, 3\}}$ or of $Z_4$ in $\widetilde{X}(\bar{w})$. Seeing $\widetilde{X}(\bar{w})$ as a configuration variety, the image of $Z_{\{2, 3\}}$ in $\widetilde{X}(\bar{w})$ is the configuration variety $\text{PitQ} \times Q \text{RuS} \times S X(v')$ where $Q_w'$ and $Q_v'$ are obtained respectively from $Q_u$ and $Q_v$ by removing the vertices smaller than one vertex in $\{2, 3\} \cap Q_u$ respectively $\{2, 3\} \cap Q_v$. The image of $Z_4$ is the configuration variety $\text{PitQ} \times Q \text{RuS} \times S X(v')$ where $Q_v'$ is obtained from $Q_v$ by removing the
vertices smaller than the vertex 4. In particular, the fiber of \( \tilde{\pi} \) above \( X(w') \) has two components whose dimensions are given by

<table>
<thead>
<tr>
<th>case</th>
<th>8.4 ; ( Z_{(2,3)} )</th>
<th>8.4 ; ( Z_4 )</th>
<th>8.5 ; ( Z_{(2,3)} )</th>
<th>8.5 ; ( Z_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>dim</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

and the morphism \( \tilde{\pi} \) is always \( IH \)-small.

### 7.5 Small resolution

Let us now describe all \( IH \)-small resolutions of minuscule Schubert varieties whenever they exist. Having adopted (and described) the relative minimal models point of view, we use the following result of B. Totaro [To] using a key result of J. Wisniewski [Wi]:

**Theorem 7.9.** — Any \( IH \)-small resolution of \( X \) is a small relative minimal model for \( X \).

Looking for \( IH \)-small resolution we only have to check in our list of minimal models. Furthermore, because of theorem 7.3 the morphism \( \tilde{\pi} : \tilde{X}(\tilde{w}) \to X(w) \) from any minimal model to \( X(w) \) is \( IH \)-small so that we get the following

**Corollary 7.10.** — The \( IH \)-small resolutions of \( X(w) \) are given by the morphisms \( \tilde{\pi} : \tilde{X}(\tilde{w}) \to X(w) \) obtained from construction 3 with \( \tilde{X}(\tilde{w}) \) smooth.

We now give a combinatorial description of these varieties. Let \( Q_v \) be a quiver associated to a minuscule Schubert variety \( X(v) \) and \( i \) a vertex of \( Q_v \).

**Definition 7.11.** — The vertex \( i \) of \( Q_v \) is called minuscule if \( \beta(i) \) is a minuscule simple root of the sub-Dynkin diagram of \( G \) defined by \( \text{Supp}(v) \).

**Theorem 7.12.** — The variety \( \tilde{X}(\tilde{w}) \) obtained from construction 4 is smooth if and only if for all \( i \), the unique pic \( p_i \) of \( Q_{w_i} \) is minuscule in \( Q_{w_i} \).

**Proof** — We have seen that the variety \( \tilde{X}(\tilde{w}) \) is a sequence of locally trivial fibrations with fiber Schubert varieties \( X(w_i) \). The theorem will follow from: \( \square \)

**Proposition 7.13.** — A minuscule Schubert variety \( X(w) \) is smooth if and only if \( Q_w \) has a unique pic \( p \) and \( p \) is minuscule in \( Q_w \).

**Proof** — We know from [BP] that a minuscule Schubert variety \( X(w) \) is smooth if and only if it is homogeneous under its stabiliser. It is easy to verify that the quiver of any minuscule homogeneous variety has a unique pic which is minuscule.
Conversely, according to proposition 4.11, the variety is homogeneous under its stabiliser if and only if the quiver $Q_w$ has no non virtual hole. Now we have seen that for $A_n$, the quiver of any Schubert variety is of the form (we have circled the non virtual holes of the quiver):

and the only case where there is a unique pic is when there is no hole. In this case we have the quiver of a grassmannian and it is smooth see appendix). For the case of maximal isotropic subspaces (say associated to the simple root $\alpha_n$ with the notations of [Bou]), the quiver is of the form (we have circled the non virtual holes of the quiver):

and there are three cases when there is only one pic namely

In the second case, one of the two vertices $i_{n-1}$ and $i_n$ such that $i_k$ is the smallest element (for $\preceq$) with $\beta(i_k) = \alpha_k$ with the notations of [Bou] is a hole of the quiver. In the first case the quiver is the quiver of the isotropic grassmannian and in the third one it is the quiver of a projective space. For the quadric case, the quiver has one of the four following forms:
and in the first and last cases we get respectively the quiver of a quadric or the quiver of a projective space. In the two intermediate cases, there is one hole in the quiver.

Finally, it is an easy verification on the quivers of $E_6$ and $E_7$ to check that the proposition is true (cf. appendix).

\[ \square \]

7.6 Stringy polynomials

Another way of proving the non existence of $IH$-small resolutions is the following: because of theorem 7.9 of any $IH$-small resolution $\tilde{X}$ of a variety $X$ if it exists will factor through the relative canonical model $\hat{X}$ (which will always exists if $\tilde{X}$ does). Furthermore, the resolution $\tilde{X} \to \hat{X}$ will be $IH$-small and in particular crepant. We can thus use the stringy polynomial $E(\hat{X}, u, v)$ defined by V. Batyrev in \[Ba\]. If $\hat{X}$ admits a crepant resolution then this polynomial (which in general is a formal power serie) is a true polynomial. To prove the non existence of $IH$-resolution, it would be enough to prove that $E(\hat{X}, u, v)$ is not a polynomial.

Let us give an example where we make the full calculation. Let us first recall the following definitions (for more details and more general definitions, see \[Ba\]).

Let $X$ be a normal irreducible variety, we define the following notations:

$$E(X, u, v) = \sum_{u,v} e^{p,q}(X) u^p v^q \quad \text{with} \quad e^{p,q}(X) = \sum_i (-1)^i h^{p,q}(H^i_c(X, \mathbb{C}))$$

where $H^i_c(X, \mathbb{C})$ is the $i^{th}$ cohomology group with compact support and $h^{p,q}(H^i_c(X, \mathbb{C}))$ is the dimension of its $(p, q)$-type component. The polynomial $E(X, u, v)$ is what V. Batyrev call the Euler polynomial (or $E$-polynomial).

Assume now that $X$ is a gorenstein normal irreducible variety with at worst terminal singularities. Let $\pi : Y \to X$ a resolution of singularities such that the exceptional locus is a divisor $D$ whose irreducible components $(D_i)_{i \in I}$ are smooth divisors with only normal crossing. We then have

$$K_Y = \pi^* K_X + \sum_{i \in I} a_i D_i \quad \text{with} \quad a_i > 0.$$
For any subset $J \subset I$ we define

$$D_J = \begin{cases} \bigcap_{j \in J} D_j & \text{if } J \neq \emptyset \\ Y & \text{if } J = \emptyset \end{cases} \quad \text{and} \quad D_J^o = D_J \setminus \bigcup_{i \in I \setminus J} (D_J \cap D_i).$$

**Definition 7.14.** — The stringy function associated with the resolution $\pi : Y \to X$ is the following:

$$E_{st}(X, u, v) = \sum_{J \subset I} E(D_J^o, u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{a_j+1} - 1}.$$

Then V. Batyrev proves the following

**Theorem 7.15.** — (i) The function $E_{st}(X, u, v)$ is independent of the resolution $\pi : Y \to X$ with exceptional locus of pure codimension 1 given by smooth irreducible divisors with normal crossing.

(ii) If $X$ admits a crepant resolution $\pi : Y \to X$ (that is to say $\pi^* K_X = K_Y$) then $E_{st}(X, u, v) = E(Y, u, v)$ and it is a polynomial.

(iii) In particular, if $X$ admits a crepant resolution, the stingy Euler number

$$e_{st}(X) = \lim_{u,v \to 0} E_{st}(X, u, v) = \sum_{J \subset I} e(D_J^o) \prod_{j \in J} \frac{1}{1 + a_j}$$

is an integer.

We now give an example of a minuscule Schubert variety which is singular non locally factorial and does not admit an $IH$-small resolution.

**Example 7.16.** — Let $G$ be $SO(12)$ and $w$ given by the following reduced writing $\bar{w}$ (the symmetry $s_i$ is the simple reflection associated to the $i^{th}$ simple root with the notation of [Bou]):

$$w = s_2 s_4 s_1 s_3 s_6 s_2 s_8 s_3 s_4 s_4 s_6.$$ 

The associated Schubert variety is the following ($G_{iso}(k, 12)$ is the isotropic grassmannian, we denote by $G_{iso}^1(6, 12)$ and $G_{iso}^2(6, 12)$ the homogenous varieties associated to the simple roots $\alpha_5$ and $\alpha_6$):

$$X(w) = \{ V \in G_{iso}^2(6, 12) / \dim(V \cap W_3) \geq 1 \text{ and } \dim(V \cap W_6) \geq 3 \}$$

where $W_3 \in G_{iso}(3, 12)$ and $W_6 \in G_{iso}^1(6, 12)$. The quiver $Q_w$ is

```
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The variety $X(w)$ has for resolution the Bott-Samelson resolution $\tilde{X}(\tilde{w})$. Moreover, because the morphism $\pi : \tilde{X}(\tilde{w}) \to X(w)$ is $B$-equivariant, the exceptional locus has to be $B$-invariant and thus an union of $Z_K$. The only non contracted divisors $Z_i$ of $\tilde{X}(\tilde{w})$ in $X(w)$ are $Z_1$ and $Z_2$. Furthermore, the variety $Z_{\{1,2\}}$ is not contracted so that the exceptional locus $D$ is the union

$$D = \bigcup_{i=3}^{11} Z_i.$$

All $Z_i$ are smooth and intersect transversally. Denote by $D_1$ and $D_2$ the images of $Z_1$ and $Z_2$ in $X(w)$. The ample generator of the Picard group of $X(w)$ is given by $L = D_1 + D_2$. We have

$$\pi^* L = \sum_{i=1}^{11} Z_i.$$

Formulae of paragraph 3.3.1 and lemma 4.16 give us:

$$-K_{\tilde{X}(\tilde{w})} = \sum_{i=1}^{11} (h(i) + 1)Z_i$$

and proposition 4.15 gives us:

$$-K_X = 7D_1 + 7D_2 = 7L.$$

In particular, we have

$$K_{\tilde{X}(\tilde{w})} - \pi^* K_X = \sum_{i=1}^{11} (6 - h(i))Z_i = (Z_3 + Z_4 + Z_5) + 2(Z_6 + Z_7) + 3(Z_8 + Z_9) + 4Z_{10} + 5Z_{11}.$$

Remark that for $J \subset [3,9]$, the variety $Z_J^o$ is a sequence of 9 locally trivial fibrations in $\mathbb{A}^1$ of in points (there are exactly $|J|$ points) over $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, we have $e(Z_J^o) = 4$ for all $J \subset [3,9]$.

Now we have the easy formula

$$\sum_{J\subset I} \prod_{j\in J} x_j = \prod_{i\in I} (1 + x_i).$$

We can thus calculate in our situation:

$$e_{st}(X(w)) = 4(1 + \frac{1}{2})^3(1 + \frac{1}{3})^2(1 + \frac{1}{4})^2(1 + \frac{1}{5})(1 + \frac{1}{6}) = \frac{105}{2}.$$

We conclude that $X(w)$ has no $IH$-small resolution as given by theorem 7.12.

This kind of calculation can be generalised, this with be done in a subsequent paper. For example, the same calculation in the general case where $X(w)$ is gorenstein (or equivalently all the pics $p \in p(Q_w)$ have the same height $h(w)$) gives the following result: let us define for $i \in Q_w$ its coheight $\text{coh}(i) = h(w) - h(i)$. Then we have

$$e_{st}(X(w)) = \prod_{i\in Q_w} \left(1 + \frac{1}{1 + \text{coh}(i)}\right).$$

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8 Appendix

In this appendix we give the quivers of minuscule homogeneous varieties and describe the quivers of minuscule Schubert varieties.

8.1 Quivers of minuscule homogeneous varieties

The following quiver is the quiver of the grassmannian of \( p \)-dimensional subvector spaces of an \( n \)-dimensional vector space. The morphism \( \beta \) associating to any vertex a simple root is simply the vertical projection on the Dynkin diagram.

\[
\begin{array}{c}
\vdots \\
\vdots \\
\beta \\
\end{array}
\]

It is easy to verify that this diagramm satisfies the geometric conditions of proposition 4.2 so that it corresponds to a Schubert variety of dimension \( p(n - p) \) of the grassmannian. It must be the quiver of the grassmannian.

In the same way, the quiver of the grassmannian of maximal isotropic subspaces in a \( 2n \)-dimensional vector space endowed with a non degenerate quadratic form is given by (one more time, the morphism \( \beta \) is given by the vertical projection on the Dynkin diagram) one of the following form depending on the parity of \( n \):
In the text we use for the quiver of the Grassmannian and for the quiver of the Grassmannian of maximal isotropic subspaces the following schematic version of the quivers:

For the even dimensional quadrics, we get the following quiver (we often draw it like the diagram on the right even if $\beta$ is not exactly given by the projection and if we forget some arrows):
Finally for $E_6$ and $E_7$ we only draw the simplified versions where some arrows (easily detected) have been omitted and where the map $\beta$ is not exactly the projection:
8.2 Quivers of minuscule Schubert varieties

Thanks to the description of the quivers of minuscule homogeneous varieties and the proposition 4.5, we know that the quivers of a minuscule Schubert variety is of the following form (we have circled the successors of elements in the set $A$ described in proposition 4.5):

or for the quadrics:

and finally for $E_6$ and $E_7$ let us give a complete list of the quivers (except the empty quiver):
It is easy to verify (thanks to our results) that the only Schubert varieties admitting a $IH$-small resolution are the following (we only list there number in the previous list): 1, 6, 7, 9, 11, 13, 17, 19, 20, 21, 22, 23, 24, 25, 26 and the 0-dimensional one.

Let us now list the Schubert varieties for the $E_7$ case:
It is easy to verify (thanks to our results) that the only Schubert varieties admitting a IH-small resolution are the following (we only list there number in the previous list): 1, 24, 27, 28, 31, 34, 37, 40, 44, 46, 48, 49, 50, 51, 52, 53, 54, 55 and the 0-dimensional one.

References


References


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