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FLAGS, SCHUBERT POLYNOMIALS, DEGENERACY LOCI, AND DETERMINANTAL FORMULAS

WILLIAM FULTON

1. Introduction. The principal goal of this paper is a formula for degeneracy loci of a map of flagged vector bundles. If $h: E \to F$ is a map of vector bundles on a variety X,

$$E_1 \subset E_2 \subset \cdots \subset E_s = E, \qquad F = F_t \twoheadrightarrow F_{t-1} \twoheadrightarrow \cdots \twoheadrightarrow F_1 \tag{1.1}$$

are flags of subbundles and quotient bundles, and integers $r(q, p)$ are specified for each $1 \leqslant p \leqslant s$ and $1 \leqslant q \leqslant t$, then there is a degeneracy locus

$$\Omega_{\mathbf{r}}(h) = \{x \in X: \operatorname{rank}(E_p(x) \to F_q(x)) \leqslant r(q, p)\ \forall p, q\}. \tag{1.2}$$

Under appropriate conditions on the rank function $\mathbf{r}$, which guarantee that, for generic h, $\Omega_{\mathbf{r}}(h)$ is irreducible, we prove a formula for the class $[\Omega_{\mathbf{r}}(h)]$ of this locus in the Chow or cohomology ring of X, as a polynomial in the Chern classes of the vector bundles. When expressed in terms of Chern roots, these polynomials are the "double Schubert polynomials" introduced and studied by Lascoux and Schützenberger.

The simplest such rank conditions are when $s = t$ (but with repeats allowed in the chains of sub and quotient bundles), and one restricts the ranks of maps $E_p \to F_{s+1-p}$ for $1 \leqslant p \leqslant s$. In this case the polynomials have simple determinantal

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expressions, as "multi-Schur functions". When all F_i coincide, special choices of these rank functions recover the Kempf-Laksov determinantal formula and the Giambelli-Thom-Porteous formula.

Before describing our results in more detail, it may help to give a brief sketch of previous work in this area. In the nineteenth century many geometers and algebraists considered an $m \times \ell$ matrix $(a_{i,j}(x))$ with $a_{i,j}$ a general homogeneous polynomial of positive degree $s_i + t_j$ for some nonnegative integers s_i and t_j. (In modern language this is the situation where all bundles are direct sums of line bundles on projective space.) The problem is to determine the degree of the locus where various upper left $q \times p$ submatrices had ranks bounded by some numbers $r(q, p)$. One motivation was from enumerative geometry, where formulas for such loci were used to describe possible singularities of curves and surfaces in space. Another was algebraic, carrying on the program of elimination theory which went back to Bézout: to describe the set of solutions of a system of equations. Indeed, the case of a matrix with one row is precisely Bézout's theorem; the case of larger matrices was challenging because the loci are often described by many more equations than their codimensions.

S. Roberts in 1867 found a formula for the degree of the locus where the matrix fails to have maximal rank, in terms of what we would now recognize as a Schur polynomial, and many other algebraists and geometers (e.g., Cremona, Schubert, C. Segre, Stuyvaert, Vahlen, and Veronese) worked on other cases. This culminated in the results of Giambelli in the early twentieth century. (See [G1] and [G3], which also contain references to the preceding work.) Giambelli simplified previous formulas and generalized them to allow several ranks to be specified for submatrices which had either the full number m of rows or the full number ℓ of columns, although in this generality he had to assume the forms $a_{i,j}$ all have the same degree. Our formula gives a general solution to this classical problem, giving the degree of this locus, in every case where it is irreducible, and for all possible degrees, as a value of a certain double Schubert polynomial. (See Corollary 8.3.)

In the modern era, Thom and Porteous began the generalization to maps of arbitrary vector bundles, giving the formula for the locus where the rank is bounded by some integer. More general determinantal formulas were given by Kempf and Laksov [KL], [F], and Pragacz [P], where one can find a modern treatment of such formulas. These too are very special cases of our determinantal formula. As the matrices get larger, the percentage of cases which are determinantal, (and so, in particular, the number covered by previously known formulas) goes exponentially to zero. At the end of this paper we describe the relation with previous classical and modern results in more detail.

Two questions arise immediately when looking at such loci: to determine which rank conditions determine irreducible varieties (locally, for generic matrices), and when they do, how to specify them efficiently, prescribing only a smaller set of rank conditions from which the others follow. The answer to the first question is easy: the possible rank conditions are precisely those for which there is a matrix with exactly these ranks, and these correspond to permutations, with a permutation matrix being a typical member of its locus. (For example, there is no 2×2 matrix with upper left entry zero, and the ranks of the first row, first column, and the whole matrix being 1, and the corresponding locus of 2×2 matrices has two irreducible components; in general, without conditions on the ranks, there can be many components of many dimensions.)

The second question is more interesting since, in most examples that arise in practise, only a few of the $m \cdot \ell$ possible rank conditions are prescribed, and the others are consequences of these. To study this we introduce the "essential set" of a permutation, which is in fact the set of southeast corners of the diagram (introduced by Rothe in 1800) of a permutation. In the special case of the Grassmannian, where the diagram is very simple, it has been known for a long time that the corners control the situation, and they play a key role for example in Zelevinsky's resolution of singularities of Schubert varieties in Grassmannians; see [BFL]. The Grassmannian case is exactly the case where the essential set lies entirely in one row. We show that in general a rank function is determined by its restriction to the essential set. It remains a challenge in general to describe which such sets with rank functions arise, but the rank conditions which are easiest to describe—and which correspond to the previous formulas in the literature—are those such that the essential set of points (q, p) for which the ranks of the upper $q \times p$ submatrix are prescribed is spread out in a string from southwest to northeast; i.e., there are no two ranks specified for (q, p) and (q', p') with $q < q'$ and $p < p'$. We prove that these correspond exactly to the permutations that Lascoux and Schützenberger call "vexillary" permutations—a fact which we hope may shed some light on that notion. Moreover, these are exactly the permutations for which they prove a simple, Schur-type determinantal expression for the Schubert polynomials. Thus, our general degeneracy formula becomes a determinantal formula in the "vexillary" case, and in particular the known formulas are recovered.

We now describe our result in more details. The crucial case is that of complete flags, when $s = t = n$ and E_i and F_i have rank i; so we discuss that for simplicity. In this case the degeneracy loci are parametrized by permutations w in the symmetric group S_n. Let $\ell(w)$ be the length of w, i.e., the number of inversions, and let $\mathbf{r}_w(q, p)$ be the cardinality of the set $\{i \leqslant q \colon w(i) \leqslant p\}$. Let

$$(1.3) \qquad x_i = c_1(\mathrm{Ker}(F_i \twoheadrightarrow F_{i-1})) \qquad \text{and} \qquad y_i = c_1(E_i/E_{i-1}), \quad 1 \leqslant i \leqslant n.$$

Let $\Omega_w = \Omega_{\mathbf{r}_w}(h)$ be the locus where the rank of $E_p \to F_q$ is at most $\mathbf{r}_w(q, p)$ for all p and q. The expected (and maximum, if nonempty) codimension of Ω_w is $\ell(w)$. When X is smooth and Ω_w has the expected codimension, our formula is

$$(1.4) \qquad [\Omega_w] = \mathfrak{S}_w(x, y)$$

where $\mathfrak{S}_w(x, y) = \mathfrak{S}_w(x_1, \ldots, x_n, y_1, \ldots, y_n)$ is the double Schubert polynomial for w, a homogeneous polynomial in the $2n$ variables of degree $\ell(w)$. It is defined as follows. When $w = w_o$ is the permutation of longest length $N = n(n-1)/2$, i.e., $w_o(i) = n + 1 - i$ for $1 \leqslant i \leqslant n$, this double Schubert polynomial is

$$(1.5) \qquad \mathfrak{S}_{w_o}(x, y) = \prod_{i+j \leqslant n} (x_i - y_j).$$

The general Schubert polynomial is determined by the property that, if w is a permutation with $w(i) > w(i+1)$ and w' is the permutation of length one less obtained from w by interchanging the values of $w(i)$ and $w(i+1)$, i.e., $w' = w \cdot s_i$, where s_i is the simple transposition interchanging i and $i+1$, then

$$(1.6) \qquad \mathfrak{S}_{w'}(x, y) = (\partial_i \mathfrak{S}_w)(x, y)$$

where, for any polynomial P in variables $x_1, \ldots, x_n$ and any $1 \leqslant i \leqslant n-1$,

$$(1.7) \qquad \partial_i P = \frac{P(x_1, \ldots, x_n) - P(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n)}{x_i - x_{i+1}}.$$

This operator ∂_i is applied with $\mathfrak{S}_w(x, y)$, regarded as a polynomial in the x variables alone. In other words, if w is obtained from w_o by the sequence of simple transpositions using the sequence of integers $i_1, \ldots, i_r$, with $r = N - \ell(w)$, then

$$(1.8) \qquad \mathfrak{S}_w(x, y) = \partial_{i_r} \circ \cdots \circ \partial_{i_1}(\mathfrak{S}_{w_o})(x, y).$$

The ordinary Schubert polynomial $\mathfrak{S}_w(x)$ is obtained from the double Schubert polynomial by setting all $y_i = 0$.

For a simple example, take $w = 2\ 4\ 3\ 1$; i.e., $w(1) = 2$, $w(2) = 4$, $w(3) = 3$ and $w(4) = 1$. Then Ω_{2431} is the locus where

$$\operatorname{rank}(E_1 \to F_3) = 0 \qquad \text{and} \qquad \operatorname{rank}(E_3 \to F_2) \leqslant 1$$

(the other conditions in (1.2) follow from these) whose expected codimension is $\ell(w) = 2$. Since w is obtained from $w_o = 4321$ by the sequence $4321 \mapsto 4231 \mapsto 2431$, first interchanging the second and third, then the first and second, we have

$$\begin{aligned} \mathfrak{S}_{2431}(x, y) &= \partial_1 \circ \partial_2((x_1 - y_1)(x_1 - y_2)(x_1 - y_3)(x_2 - y_1)(x_2 - y_2)(x_3 - y_1)) \\ &= \partial_1((x_1 - y_1)(x_1 - y_2)(x_1 - y_3)(x_2 - y_1)(x_3 - y_1)) \\ &= (x_1 - y_1)(x_2 - y_1)(x_3 - y_1)(x_1 + x_2 - y_2 - y_3). \end{aligned}$$

This is equal to $c_1(F_3 - E_1) \cdot c_1(F_2 - E_3 + E_1)$. Similarly, for the first nonvexillary permutation 2143, the locus where $E_1 \to F_1$ vanishes and $E_3 \to F_3$ has rank at most

2 is described by the formula

$$\begin{aligned}\mathfrak{S}_{2143}(x, y) &= (x_1 - y_1)(x_1 + x_2 + x_3 - y_1 - y_2 - y_3)\\ &= c_1(F_1 - E_1)\cdot c_1(F_3 - E_3).\end{aligned}$$

In this example, as in many examples for small n, it is an easy exercise to verify the formula directly, using standard ad hoc geometric constructions. The formula for a general degeneracy locus $\Omega_{\mathbf{r}}(h)$ is easily deduced from the case of complete flags, the expected codimension $d(\mathbf{r})$ being the length $\ell(w)$ of the corresponding permutation, and the polynomial $P_{\mathbf{r}}$ in the Chern classes which describes the locus is equal to the corresponding Schubert polynomial in the Chern roots of the bundles involved.

Without the genericity or smoothness assumptions, we construct classes $\mathbf{\Omega}_{\mathbf{r}}$ of codimension $d(\mathbf{r})$, supported on the locus $\Omega_{\mathbf{r}}$, whose image in the Chow group of X is $P_{\mathbf{r}} \cap [X]$, with standard functorial properties. When the locus has the expected dimension, $\mathbf{\Omega}_{\mathbf{r}}$ is a positive cycle whose support is $\Omega_{\mathbf{r}}$. If in addition X is Cohen-Macaulay, then, using the natural subscheme structure on $\Omega_{\mathbf{r}}$, we have $\mathbf{\Omega}_{\mathbf{r}} = [\Omega_{\mathbf{r}}]$. In the complex case the classes can be constructed in the relative (local) cohomology group $\mathrm{H}^{2d(\mathbf{r})}(X, X - \Omega_{\mathbf{r}})$.

When specialized to the flag manifold of flags in an n-dimensional vector space with the E_i a fixed flag of subspaces, formula (1.4) implies that of Bernstein-Gel'fand-Gel'fand [BGG] and Demazure [D] relating classes of the Schubert varieties to polynomials in Chern classes of the universal line bundles. Our formula generalizes theirs in much the same way that the determinantal formula generalizes Giambelli's formula for Schubert varieties in the Grassmannians. Doing the general case with arbitrary flags, however, not only gives new formulas, but, as will be no surprise to group theorists, makes the proof easier. Unlike most previous proofs of special cases, for example, we require no Gysin formulas or construction of resolutions of singularities of the loci involved.

Since the cohomology of the flag manifold is the quotient of a ring of polynomials by an ideal generated by symmetric polynomials, a formula for Schubert varieties, as in [BGG] or [D], is only determined up to this ideal. Lascoux and Schützenberger introduced Schubert polynomials as a set of representatives for these classes with particularly nice algebraic properties. The present work can be seen as a complete geometric vindication of their insight: the Schubert polynomials are the only polynomials that satisfy the general degeneracy formula. In fact, this characterization can be used to prove some known and new identities involving Schubert polynomials.

The proof of the formula (1.4) is straightforward when $w = w_o$ is the permutation of longest length. The problem in general is how to start with this formula for this smallest degeneracy locus and relate it to larger loci. One needs an operation on cycles in a flag bundle which increases their dimension. In general, while maps from a variety to itself will not do this, correspondences will. Here, we construct the operators ∂_i from correspondences which are $\mathbb{P}^1$-bundles. The reduction to $\mathbb{P}^1$-bundles is to be expected since most results about Schubert varieties and flag manifolds since that of Bott and Samelson [BS] (e.g., [BGG], [D], [K], [R], [S]) come down to verifications for $\mathbb{P}^1$-bundles.

The paper is organized as follows. The next section reviews basic facts about Schubert and double Schubert polynomials. The third section describes the relation between rank conditions and permutations, and the corresponding loci in the space of matrices. This is globalized to bundles in the next two sections. In §6 we recall basic facts about Schubert varieties in flag manifolds and prove the local version of the basic $\mathbb{P}^1$-bundle correspondence. (With the possible exception of the "essential set" in §3, most of the results of these sections are minor variations of known facts about homogeneous varieties.) The "Giambelli formula" for degeneracy loci in flag bundles, which is the universal case of the theorem, is proved in §7, and the deduction of the general theorem is carried out in §8. Here, we use standard intersection theory techniques to discuss cases where the degeneracy loci are bigger than expected. In §9 we show that simple rank conditions correspond to vexillary permutations. This yields the general determinantal formula in §10, where we give some known and new applications.

Acknowledgements. The simplicity of the proof comes from the realization of the operators of [BGG] and [D] as correspondences, a fact which has been noticed by others, but which R. MacPherson showed us several years ago. This appears clearly in Springer's Bourbaki talk [S]. (It is also possible to prove (1.4) by globalizing the constructions of Demazure [D], constructing suitable resolutions of singularities of these degeneracy loci, but the algebra is more complicated than in [D] in the general case when the subbundles E_i are not trivial.) The $\mathbb{P}^1$-correspondence makes these constructions unnecessary and, in particular, gives simpler proofs of the results of [BGG] and [D]. We hope that this treatment will help geometers and algebraists appreciate these fundamental papers, without overly offending Lie group experts by the omission of roots, weights, tori, parabolic subgroups, etc.

We also benefitted from discussions with P. Pragacz, who proved a special case of our formula five years ago and recently told us of the work of Giambelli and its relation to our formula. (See §10.) Thanks are also due to R. Stanley for providing the useful notes of Macdonald [M], which make the ideas of Lascoux and Schützenberger accessible. I am grateful to Pragacz, A. Lascoux, D. Grayson and a referee for several suggestions and corrections, particularly of a historical nature, in response to a preliminary version of this paper.

The geometric aspects of this paper generalize readily to other semisimple groups. We plan a sequel on the orthogonal and symplectic analogues of the formulas presented here.

2. Schubert polynomials. We start by recalling the definitions of Schubert polynomials, which were introduced and studied in the series of papers [L1–3], [LS1–3],

[W], and elucidated and developed further in [M]. We will follow [M], summarizing those notions and properties we need. Let $A[x]$ denote the ring of polynomials in variables $x_1, \ldots, x_n$ with coefficients in a commutative ring A, which may be taken to be the integers. Formula (1.7) defines "divided difference" operators ∂_i on $A[x]$, $1 \leqslant i \leqslant n-1$. These operators lower the degrees of polynomials by one and satisfy the equations

$$\partial_i \circ \partial_i = 0; \tag{2.1}$$

$$\partial_i \circ \partial_j = \partial_j \circ \partial_i \quad \text{if } |i-j| \geqslant 2; \tag{2.2}$$

$$\partial_i \circ \partial_j \circ \partial_i = \partial_j \circ \partial_i \circ \partial_j \quad \text{if } |i-j| = 1; \qquad \text{and} \tag{2.3}$$

$$\partial_i(P \cdot Q) = P \cdot \partial_i(Q) \quad \text{if } P \text{ is symmetric in } x_i \text{ and } x_{i+1}. \tag{2.4}$$

In particular, $\partial_i(P) = 0$ if P is symmetric in x_i and x_{i+1}.

From (2.1)–(2.3) and the fact that the symmetric group S_n is generated by the simple transpositions s_i which interchange i and $i+1$, for $1 \leqslant i \leqslant n-1$, subject to relations corresponding to (2.1)–(2.3) (or see [M, (2.5)]), it follows that one can define operators ∂_w on $A[x]$ for any $w \in S_n$ by writing w as a product of $\ell(w)$ such transpositions: $w = s_{i_1} \cdot \ldots \cdot s_{i_{\ell(w)}}$ and setting

$$\partial_w = \partial_{i_1} \circ \ldots \circ \partial_{i_{\ell(w)}}. \tag{2.5}$$

Moreover, for any sequence of integers $i_1, \ldots, i_\ell$ between 1 and $n-1$,

$$\partial_{i_1} \circ \ldots \circ \partial_{i_\ell} = \begin{cases} \partial_w & \text{if } w = s_{i_1} \cdot \ldots \cdot s_{i_1} \text{ has length } \ell \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

Define the *Schubert polynomial* $\mathfrak{S}_w(x)$, homogeneous of degree $\ell(w)$, in $\mathbb{Z}[x]$ by writing

$$w = w_o \cdot s_{i_1} \cdot \ldots \cdot s_{i_r}$$

where $r = N - \ell(w)$ and $w_o(i) = n + 1 - i$, and setting

$$\mathfrak{S}_w(x) = \partial_{i_r} \circ \ldots \circ \partial_{i_1}(x_1^{n-1} \cdot x_2^{n-2} \cdot \ldots \cdot x_{n-1}). \tag{2.7}$$

Equivalently, $\mathfrak{S}_w(x) = \partial_{w^{-1} \cdot w_o}(x_1^{n-1} \cdot x_2^{n-2} \cdot \ldots \cdot x_{n-1})$. Note that, since the module generated by all $x_1^{i_1} \cdot x_2^{i_2} \cdot \ldots \cdot x_n^{i_n}$ with $i_j \leqslant n-j$ is preserved by the divided difference operators, the variable x_n does not appear in $\mathfrak{S}_w(x)$.

More generally, define the *double Schubert polynomial* $\mathfrak{S}_w(x, y)$, which is homogeneous of degree $\ell(w)$ in two sets of variables $x_1, \ldots, x_n$, and $y_1, \ldots, y_n$ by defining

$$\mathfrak{S}_w(x, y) = \partial_{i_r} \circ \ldots \circ \partial_{i_1}\left(\prod_{i+j \leqslant n} (x_i - y_j)\right) \tag{2.8}$$

$$= \partial_{w^{-1} \cdot w_o}\left(\prod_{i+j \leqslant n} (x_i - y_j)\right)$$

with the same choices of $i_1, \ldots, i_r$ as before. Here, the y_i's are regarded as constants; i.e., the operators are defined on the ring $A[x]$, where A is the ring $\mathbb{Z}[y]$. Note that $\mathfrak{S}_w(x, 0) = \mathfrak{S}_w(x)$.

The double Schubert polynomials can be expressed in terms of ordinary Schubert polynomials [M, (6.3)] as follows.

LEMMA 2.9. *For any permutation w in S_n,*

$$\mathfrak{S}_w(x, y) = \sum (-1)^{\ell(v)} \mathfrak{S}_u(x) \mathfrak{S}_v(y),$$

the sum over those pairs (u, v) of permutations in S_n such that $v^{-1} \cdot u = w$ and $\ell(u) + \ell(v) = \ell(w)$.

COROLLARY 2.10. $\mathfrak{S}_{w^{-1}}(x, y) = (-1)^{\ell(w)} \mathfrak{S}_w(y, x)$.

COROLLARY 2.11. *$\mathfrak{S}_w(x, y)$ is symmetrical in x_i, x_{i+1} if and only if $w(i) < w(i+1)$. Also, $\mathfrak{S}_w(x, y)$ is symmetrical in y_i, y_{i+1} if and only if $w^{-1}(i) < w^{-1}(i+1)$.*

Proof. As in [M, (4.3)(iii)], $\mathfrak{S}_w(x, y)$ is symmetrical in x_i, x_{i+1} exactly when $\partial_i \mathfrak{S}_w(x, y) = 0$, i.e., $\ell(w \cdot s_i) = \ell(w) + 1$ or $w(i) < w(i+1)$. The assertion for the y variables follows from this and the preceding corollary. □

It follows that, if $w(i) < w(i+1)$ for $i > d$, then $\mathfrak{S}_w(x, y)$ does not involve the variables x_i for $i > d$; similarly, if $w^{-1}(i) < w^{-1}(i+1)$ for $i > e$, then $\mathfrak{S}_w(x, y)$ does not involve the variables y_i for $i > e$. In addition, if $S_n \subset S_{n+1}$ as usual (with $w(n+1) = n+1$), then $\mathfrak{S}_w(x, y)$ is unchanged.

In §9 we will discuss some of the Schubert polynomials which can be expressed more simply than by the iterated procedure of the definition. For now we note only that

$$\mathfrak{S}_{id}(x, y) = 1. \tag{2.12}$$

$$\mathfrak{S}_{s_k}(x, y) = x_1 + \cdots + x_k - y_1 - \cdots - y_k, \quad 1 \leqslant k < n. \tag{2.13}$$

Another useful formula relating Schubert polynomials [M (4.15″)], although one we will not need for our theorem, is Monk's formula

$$\mathfrak{S}_{s_k}(x) \cdot \mathfrak{S}_w(x) = \sum \mathfrak{S}_{w \cdot t}(x) \tag{2.14}$$

where the sum is over all transpositions $t = t_{ij}$ of integers $i < j$ such that $i \leqslant k < j$ and $w(i) < w(j)$, but $w(s)$ does not lie between $w(i)$ and $w(j)$ for any $i < s < j$; equivalently, $\ell(w \cdot t) = \ell(w) + 1$.

We will need a slight generalization of these operators. Let $c_1, \ldots, c_n$ be any elements in A, let I be the ideal in $A[x]$ generated by the elements $e_i(x) - c_i$, where $e_i(x)$ is the ith elementary symmetric polynomial in $x_1, \ldots, x_n$. From (2.4) it follows that the operators ∂_w map I to itself; so they define operators, for which we use the same notation:

$$\partial_w \colon A[x]/I \to A[x]/I. \tag{2.15}$$

3. Rank conditions and permutations. We consider m by ℓ matrices, for fixed ℓ and m, with entries in an arbitrary field. For $1 \leqslant p \leqslant \ell$ and $1 \leqslant q \leqslant m$, let $A_{[q,p]}$ denote the upper left q by p submatrix of an m by ℓ matrix A. Our first aim is to describe the possible rank functions of upper left corners of matrices, i.e., to characterize those functions r on the set $[1, m] \times [1, \ell]$ such that there is a matrix A with

$$r(q, p) = \operatorname{rank}(A_{[q,p]}) \qquad \text{for all } q \text{ and } p.$$

For such r we will then describe the variety V_r of matrices A such that $\operatorname{rank}(A_{[q,p]}) \leqslant r(q, p)$ for all p and q.

One can describe the possible rank functions r in elementary terms as follows. The first row $r(1, 1), \ldots, r(1, m)$ of ranks consists of some 0's followed by some 1's. Let w_1 be the smallest p such that $r(1, p) = 1$, setting $w_1 = \infty$ if there is no such p. Let w_2 be the smallest p such that $r(2, p) \neq r(1, p)$ and set $w_2 = \infty$ if the second row of ranks is the same as the first now. Continuing in this way, let w_q be the smallest p such that $r(q, p) \neq r(q-1, p)$, or $w_q = \infty$ if there is no such p. The qth row of the rank function then agrees with the $(q-1)$th row until the w_qth column, and from there on it is one larger. This gives a sequence $w_. = (w_1, \ldots, w_m)$ with $w_i \in [1, \ell] \cup \{\infty\}$ and the finite numbers in this sequence all distinct. For an example, for

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 1 \\ 0 & 3 & 3 & 3 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad r = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 2 & 3 \end{bmatrix},$$

which has the sequence $w_. = (2, 5, \infty, 1)$.

As we have seen, the sequence $w_.$ determines the rank function. More precisely, $r(q, p)$ is the number of $i \leqslant q$ such that $w_i \leqslant p$. Conversely, any such sequence comes from a unique rank function. In fact, given $w_.$, define a matrix A with a 1 as the (q, p) entry if $w_q = p$, and a 0 otherwise; $r(q, p)$ is the number of 1's in $A_{[q,p]}$. Summarizing, we have the following lemma.

LEMMA 3.1. *Let $r: [1, m] \times [1, \ell] \to \mathbb{N}$. The following are equivalent.*

(i) *There is an m by ℓ matrix A with* $\operatorname{rank}(A_{[q,p]}) = r(q, p)$ *for all p and q.*

(ii) *There is an m by ℓ matrix A whose entries are* 0*'s and* 1*'s, with at most one* 1 *in any row or column, with* $\operatorname{rank}(A_{[q,p]}) = r(q, p)$ *for all q and p.*

(iii) *There is a sequence $w_{\bullet} = (w_1, \ldots, w_m)$, with each w_i in $[1, \ell] \cup \{\infty\}$ and the finite numbers in this sequence all distinct, with*

$$r(q, p) = \operatorname{Card}\{i \leqslant q: w_i \leqslant p\}.$$

We call r a *rank function* if the conditions of the lemma hold. The rank function r, the matrix A in (ii), and the sequence $w_{\bullet}$ in (iii) uniquely determine each other. One may also note that, if $M_{m,\ell}$ is the set of all m by ℓ matrices, B_m^- is the group of invertible lower triangular m by m matrices, and B_ℓ^+ the group of invertible upper triangular ℓ by ℓ matrices, then for A and A' in $M_{m,\ell}$

$$\operatorname{rank}(A'_{[q,p]}) = \operatorname{rank}(A_{[q,p]})\ \forall p, q \quad \Leftrightarrow \quad A' = P \cdot A \cdot Q \text{ for some } P \in B_m^-,\ Q \in B_\ell^+.$$

For any r let

$$V_r = \{A \in M_{m,\ell}: \operatorname{rank}(A_{[q,p]}) \leqslant r(q, p) \quad \text{for } 1 \leqslant p \leqslant \ell,\ 1 \leqslant q \leqslant m\}. \tag{3.2}$$

This is a closed subscheme of $M_{m,\ell} \cong \mathbb{A}^{\ell m}$ defined by the vanishing of all $r(q, p) + 1$ minors of $X_{[q,p]}$ for all p, q, where $X = (x_{i,j})$ is the generic m by ℓ matrix. Let V_r^o be the open subset of V_r consisting of those matrices with $\operatorname{rank}(A_{[q,p]}) = r(q, p)$ for all p and q.

PROPOSITION 3.3. *If r is a rank function, then*

(a) *V_r is the Zariski closure of V_r^o;*

(b) *V_r is an irreducible variety;*

(c) $\operatorname{codim}(V_r, M_{m,\ell}) = d(w_{\bullet})$, *where $w_{\bullet}$ is the sequence corresponding to r, and*

$$d(w_{\bullet}) = \operatorname{Card}\{(q, p) \in [1, m] \times [1, \ell]: p < w_q \text{ and } p \neq w_i \text{ for } i < q\};$$

(d) *V_r is reduced and Cohen-Macaulay.*

The proposition will be proved by reducing the assertions to corresponding general results about Schubert varieties in a flag manifold. To do this we next relate the above rank conditions to permutations; this relation will be used throughout the paper.

Given ℓ and m, then $n = \ell + m$. To prove the proposition, the idea is to consider the projection from the general linear group $\pi: \mathrm{GL}_n \to M_{m,\ell}$ which takes A to $A_{[m,\ell]}$ and prove the analogous results for $\pi^{-1}(V_r)$. Given a rank function r as above, let $w_{\bullet}$ be the sequence defined in Lemma 3.1 and define a permutation w in S_n by induction on i, setting

$$(3.4)\quad w(i) = \begin{cases} w_i & \text{if } i \leqslant m \text{ and } w_i \leqslant \ell \\ \min\{[\ell+1, n] - \{w(1), \dots, w(i-1)\}\} & \text{if } i \leqslant m \text{ and } w_i = \infty \\ \min\{[1, n] - \{w(1), \dots, w(i-1)\}\} & \text{if } i > m. \end{cases}$$

Then w is a permutation with

$$(3.5)\quad w(i) < w(i+1) \quad \text{if } i > m \qquad \text{and} \qquad w^{-1}(j) < w^{-1}(j+1) \quad \text{if } j > \ell.$$

Conversely, any $w \in S_n$ satisfying (3.5) comes from a unique sequence $w_\bullet$ and hence from a unique rank function r. For example, if $m = 5$, $\ell = 6$, and $w_\bullet$ is the sequence $(3, \infty, 1, 6, \infty)$, then w is the permutation 3 7 1 6 8 2 4 5 9 10 11. Note also that with $d(w_\bullet)$ as in (c) of the proposition, then

$$(3.6)\qquad d(w_\bullet) = \ell(w) = \operatorname{Card}\{i < j \colon w(i) > w(j)\}.$$

We next consider the general case of rank functions determined by permutations $w \in S_n$. For any permutation w in S_n and any $1 \leqslant p, q \leqslant n$, let

$$(3.7)\qquad r_w(q, p) = \operatorname{Card}\{i \leqslant q \colon w(i) \leqslant p\}.$$

We call r_w the *rank function* of w. As above, it can be used to put conditions on an n by n matrix A by requiring that for all q and p the rank of its upper left q by p submatrix is at most $r_w(q, p)$. For example, if A_w is the permutation matrix with a 1 in the ith row and $w(i)$th column, the rank of its upper left q by p submatrix is exactly $r_w(q, p)$. The rank functions determined by permutations are exactly those which are the ranks of *nonsingular* n by n matrices.

A permutation is clearly determined by its rank function. We need to know that it is determined by its restriction to an appropriate smaller set. Define the *essential set* $\mathscr{E}ss(w)$ to be the subset of $[1, n-1] \times [1, n-1]$

$$(3.8)\quad \mathscr{E}ss(w) = \{(q, p) \colon w(q) > p,\ w(q+1) \leqslant p,\ w^{-1}(p) > q, \text{ and } w^{-1}(p+1) \leqslant q\}.$$

Plotting pairs in $[1, n] \times [1, n]$ as in matrices, a point (q, p) is in $\mathscr{E}ss(w)$ when there is no point on the graph $\{(i, w(i))\}$ of w which is due west or due north of or at (q, p), and there is no point on the graph which is due east of or due south of or at $(q+1, p+1)$. This can be expressed in terms of the diagram of the permutation. (See [M, (1.20)].) The *diagram* of w is the subset

$$D(w) = \{(i, j) \in [1, n] \times [1, n] \colon w(i) > j \text{ and } w^{-1}(j) > i\},$$

which is a set of cardinality $\ell(w)$. Then $\mathcal{E}ss(w)$ is the set of southeast corners of $D(w)$; i.e.,

$$\mathcal{E}ss(w) = \{(q, p) \in D(w)\colon (q + 1, p), (q, p + 1), \text{ and } (q + 1, p + 1) \notin D(w)\}.$$

For an example, take $w = 4\ 8\ 6\ 2\ 7\ 3\ 1\ 5$; Figure 3.9 outlines the boxes labelling essential points with the corresponding values of the ranks inside; the dots indicate the points on the graph of w, the shaded squares are due south or east of a point on the graph, and the unshaded squares make up the diagram.

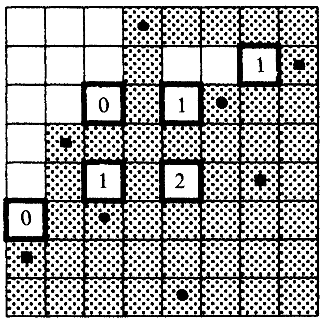

FIGURE 3.9

The rank number $r_w(q, p)$ for a square in the essential set is the number of shaded squares directly north of the square, which is the same as the number directly west of the square. For some other examples of special types, see the end of §9.

LEMMA 3.10. (a) *For any w in S_n and any n by n matrix A, the ideal generated by all minors of size $r_w(q, p) + 1$ taken from the upper left q by p corner of A, for all $1 \leqslant p, q \leqslant n$, is generated by these same minors using only those (q, p) which are in $\mathcal{E}ss(w)$.*

(b) *A permutation w in S_n is determined by the restriction of its rank function r_w to $\mathcal{E}ss(w)$.*

(c) *If $w(m) \neq m$, then there is some $(q, p) \in \mathcal{E}ss(w)$ with $p + q \geqslant m$.*

Proof. (a) We start with the ideal generated by the indicated minors for (q, p) in $\mathcal{E}ss(w)$ and show that the minors for other (q, p) are in this ideal by successively eliminating one of the conditions defining $\mathcal{E}ss(w)$.

Case 1: $w(q) > p$, $w^{-1}(p) > q$, *and* $w(q + 1) \leqslant p$. If $w^{-1}(p + 1) > q$, take the largest k such that $w^{-1}(p + i) > q$ for $1 \leqslant i \leqslant k$. Since $w(q) > p$, we must have $p + k < n$. Then $(q, p + k)$ is in $\mathcal{E}ss(w)$, and $r_w(q, p) = r_w(q, p + k)$; so we have all $r_w(q, p) + 1$ minors of $A_{[q, p+k]}$, and hence of $A_{[q, p]}$, in the ideal.

Case 2: $w(q) > p$ *and* $w^{-1}(p) > q$. If $w(q + 1) > p$, take the largest k such that $w(q + i) > p$ for $1 \leqslant i \leqslant k$. Then Case 1 applies to the pair $(q + k, p)$, and $r_w(q, p) = r_w(q + k, p)$; so we have all $r_w(q, p) + 1$ minors of $A_{[q+k, p]}$, and so of $A_{[q, p]}$, in the ideal.

Case 3: $w(q) > p$. If $w^{-1}(p) \leqslant q$, take the largest k such that $w^{-1}(p-i) \leqslant q$ for $0 \leqslant i < k$. This time, $r_w(q, p) = r_w(q, p-k) + k$. If $k = p$, there are no minors of size $r_w(q, p) + 1$. If $k < p$, Case 2 applies to $(q, p-k)$, and to conclude this case it suffices to note that increasing a side of a rectangle by one and adding one to the size of the minors gives no new generators for the ideal.

Case 4: no conditions. If $w(q) \leqslant p$, take the largest k such that $w(q-i) \leqslant p$ for $0 \leqslant i < k$. Then $r_w(q, p) = r_w(q-k, p) + k$, and, as in the preceding case, there are no minors if $k = q$, and if $k < q$ adding k rows and increasing the size of the minors by k gives no new generators for the ideal.

(b) Apply (a) to the permutation matrix A_w of w. Knowing all $r_w(q, p)$ means that one knows the number of 1's in all upper left q by p rectangles, which determines w.

(c) Interchanging w and w^{-1} if necessary, we may assume that $w(m) = p < m$. At least one of the p distinct numbers $w(q)$, as q varies from $m-p$ to $m-1$, must be greater than p. For this q and p, we have $w(q) > p$ and $w^{-1}(p) > q$. By the proof of Cases 1 and 2 of (a), there is a $(q', p') \in \mathcal{E}ss(w)$ with $q' \geqslant q$ and $p' \geqslant p$. And $p' + q' \geqslant p + q \geqslant m$. □

LEMMA 3.11. *Let $w \in S_n$ and let*

$$V_w = \{A \in \mathrm{GL}_n \colon \operatorname{rank}(A_{[q,p]}) \leqslant r_w(q, p) \text{ for all } 1 \leqslant p, q \leqslant n\}.$$

Then V_w is the closure of the set where all inequalities are replaced by equalities, and V_w is an irreducible, reduced, Cohen-Macaulay subvariety of GL_n of codimension $\ell(w)$.

We will deduce this from the corresponding result on Schubert varieties in the flag manifold $F\ell(n)$ in §6. Now we use it to prove Proposition 3.3. Let $n = \ell + m$ and consider the projection $\pi\colon \mathrm{GL}_n \to M_{m,\ell}$ taking A to $A_{[m,\ell]}$. If w is the permutation constructed from the rank function r in (3.4), it follows from the preceding lemma that $\pi^{-1}(V_r) = V_w$, as sets and as subschemes. The morphism π is smooth, with fibres open subsets of $\mathbb{A}^{n^2-\ell m}$. And π is surjective; in fact, it has a section which takes a matrix $B = (b_{i,j})$ to the matrix $A = (a_{i,j})$ with

$$a_{i,j} = \begin{cases} b_{i,j} & \text{if } i \leqslant m \text{ and } j \leqslant \ell \\ 1 & \text{if } i + j = n + 1 \\ 0 & \text{otherwise.} \end{cases}$$

The assertions about $V_r \subset M_{m,\ell}$ then follow from the corresponding assertions about $V_w \subset \mathrm{GL}_n$. □

We will need a generalization of these ideas. Suppose ℓ and m are given together with two sequences

$$1 \leqslant a_1 < a_2 < \cdots < a_s \leqslant \ell \qquad \text{and} \qquad 1 \leqslant b_1 < b_2 < \cdots < b_t \leqslant m.$$

Suppose we are given a collection $\mathbf{r} = (r_{j,i})$, $1 \leqslant j \leqslant t$, $1 \leqslant i \leqslant s$, of integers. We say that $\mathbf{r}$ is a *permissible* collection of rank numbers if there is a permutation w in some S_n with $n \geqslant \max(a_s, b_t)$, so that

$$\mathscr{E}ss(w) \subset \{b_1, \ldots, b_t\} \times \{a_1, \ldots, a_s\} \tag{3.12}$$

$$\text{and} \qquad r_{j,i} = r_w(b_j, a_i) \quad \text{for all } 1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant t.$$

By Lemma 3.1 such w, if it exists, is unique up to the inclusions of S_n in S_{n+1}; and one may always take $n \leqslant a_s + b_t$. Let $d(\mathbf{r}) = \ell(w)$. From Proposition 3.3 and Lemma 3.10 we have the following corollary.

COROLLARY 3.13. *If* $\mathbf{r}$ *is permissible, then the locus of* m *by* ℓ *matrices* A *such that* $\operatorname{rank}(A_{[b_j, a_i]}) \leqslant r_{j,i}$ *for all* i *and* j *forms an irreducible, reduced, Cohen-Macaulay variety of codimension* $d(\mathbf{r})$.

These loci for permissible $\mathbf{r}$ are exactly the irreducible loci that can be defined by rank conditions using only upper corners chosen from the prescribed rows and columns.

We conclude this section with a simple lemma which justifies our name of essential sets: none of the rank conditions can be omitted.

LEMMA 3.14. *Let* $w \in S_n$ *and* $(q_0, p_0) \in \mathscr{E}ss(w)$. *Then for any* ℓ *and* m *such that* $\mathscr{E}ss(w) \subset [1, m] \times [1, \ell]$, *there is an* m *by* ℓ *matrix* A *such that*

$$\operatorname{rank}(A_{[q,p]}) \leqslant r_w(q, p) \quad \text{for all } (q, p) \in \mathscr{E}ss(w) - \{(q_0, p_0)\};$$

$$\operatorname{rank}(A_{[q_0,p_0]}) = r_w(q_0, p_0) + 1.$$

Proof. Define $A = (a_{i,j})$ by the rule

$$a_{i,j} = \begin{cases} 1 & \text{if } i \leqslant q_0,\ j \leqslant p_0, \text{ and } w(i) = j \\ 1 & \text{if } i = q_0 \text{ and } j = p_0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that A has at most one 1 in each row and column since $w(q_0) > p_0$ and $w^{-1}(p_0) > q_0$. Hence, the rank of $A_{[q,p]}$ is the number of 1's in the upper left q by p corner. It is then evident from the definition of A that $\operatorname{rank}(A_{[q,p]}) \leqslant r_w(q, p)$ whenever $q < q_0$ or $p < p_0$. On the other hand, if $q \geqslant q_0$ and $p \geqslant p_0$, then $\operatorname{rank}(A_{[q,p]}) = r_w(q_0, p_0) + 1$. This proves the assertions for, if (q, p) and (q_0, p_0) are any two pairs in the essential set of a permutation w, with $q_0 < q$ and $p_0 < p$, then

$$r_w(q_0, p_0) < r_w(q, p). \tag{3.15}$$

□

Remark 3.16. It is not necessarily the case, however, that one can find a permutation w' whose essential set is $\mathscr{E}ss(w)\setminus\{(q_0, p_0)\}$ and whose rank function is the restriction of r_w to this set. For example, if $w = 4\ 8\ 6\ 2\ 7\ 3\ 1\ 5$ is the permutation from Figure 3.9, one can verify that there is no such w' for $(q_0, p_0) = (3, 5)$.

4. Degeneracy loci. Given, on an arbitrary variety or scheme X, a morphism of filtered vector bundles

$$E_1 \subset E_2 \subset \cdots \subset E_n \xrightarrow{h} F_n \twoheadrightarrow F_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow F_1$$

with $\operatorname{rank}(E_i) = \operatorname{rank}(F_i) = i$, and a permutation $w \in S_n$, we have the degeneracy subscheme $\Omega_w = \Omega_w(h) = \Omega_w(h, E_\bullet, F_\bullet)$ defined by the conditions

$$\operatorname{rank}(E_p \to F_q) \leqslant r_w(q, p) \qquad \text{for all } 1 \leqslant p, q \leqslant n \tag{4.1}$$

with $r_w(q, p)$ as in (3.7). This locus has a natural scheme structure given by the vanishing of the induced maps from $\bigwedge^{r_w(q,p)+1}(E_p)$ to $\bigwedge^{r_w(q,p)+1}(F_q)$.

There is a subset of these n^2 conditions which defines the same locus and scheme more efficiently, given by the essential set of w, as follows.

Proposition 4.2. *The scheme Ω_w is defined by the conditions*

$$\operatorname{rank}(E_p \to F_q) \leqslant r_w(q, p)$$

for all p and q in $\{1, \ldots, n\}$ such that $(q, p) \in \mathscr{E}ss(w)$, i.e.,

$$w(q) > p, \qquad w^{-1}(p) > q, \qquad w(q+1) \leqslant p, \qquad \textit{and} \qquad w^{-1}(p+1) \leqslant q.$$

Proof. The assertion is a local on X; so one is reduced to the case where the bundles are trivial, which amounts to the situation considered in Lemma 3.10. □

It follows that any set of (q, p)'s containing the essential set can be used. For example, using all those with $w^{-1}(p) > q$ amounts to the following description. (See Remark 6.2.)

Corollary 4.3. *For $w \in S_n$ define the nest of sets*

$$\{1, \ldots, n\} \supset \mathscr{S}_1 \supset \mathscr{S}_2 \supset \cdots \supset \mathscr{S}_{n-1}$$

where $\mathscr{S}_q = \{1, \ldots, n\} - \{w(i)\colon 1 \leqslant i \leqslant q\}$. Arrange the integers in $\mathscr{S}_q$ in order:

$$\mathscr{S}_q = \{s_q(1) < s_q(2) < \cdots < s_q(n-q)\}.$$

Then Ω_w is the locus where

$$\dim(\operatorname{Ker}(E_{s_q(i)} \to F_q)) \geqslant i \qquad \text{for all } 1 \leqslant q \leqslant n-1,\ 1 \leqslant i \leqslant n-q.$$

For example, for $w = 2\ 4\ 3\ 1$, Proposition 4.2 gives Ω_w as the locus where $E_1 \to F_3$ is zero and $E_3 \to F_2$ has rank at most 1. The sequence of subsets is $\{1, 3, 4\} \supset \{1, 3\} \supset \{1\}$, which leads, somewhat less efficiently, to the same conditions.

From the first description (4.1) or that of Proposition 4.2, with w^{-1} the inverse permutation to w, we have

$$\Omega_{w^{-1}}(E_\bullet, F_\bullet, h) = \Omega_w(E_\bullet^\vee, F_\bullet^\vee, h^\vee) \tag{4.4}$$

where the dual flags and map are those of the sequence

$$F_1^\vee \subset \cdots \subset F_n^\vee \xrightarrow{h^\vee} E_n^\vee \twoheadrightarrow \cdots \twoheadrightarrow E_1^\vee .$$

More generally, suppose we are given a map $h: E \to F$ of vector bundles of ranks ℓ and m on a variety X with partial flags

$$A_1 \subset A_2 \subset \cdots \subset A_s \subset E \xrightarrow{h} F \twoheadrightarrow B_t \twoheadrightarrow B_{t-1} \twoheadrightarrow \cdots \twoheadrightarrow B_1 \tag{4.5}$$

of ranks $1 \leqslant a_1 < a_2 < \cdots < a_s \leqslant \ell$ and $m \geqslant b_t > b_{t-1} > \cdots > b_1 \geqslant 1$. Let $\mathbf{r} = (r_{j,i})$ be a permissible collection of rank numbers for these numbers, as in (3.12). One has a degeneracy locus

$$\Omega_{\mathbf{r}}(h) = \{x \in X: \operatorname{rank}(A_i(x) \to B_j(x)) \leqslant r_{j,i}\ \forall i, j\}. \tag{4.6}$$

By the preceding section the permissible ranks are those for which these loci are, at least locally and for h generic, irreducible.

5. Flag bundles. If E is a vector bundle of rank n on a variety or scheme X, the flag bundle $F\ell(E)$ comes equipped with a morphism $\rho: F\ell(E) \to X$ and a universal flag $U_\bullet$ of subbundles of ρ^*E:

$$U_1 \subset U_2 \subset \cdots \subset U_{n-1} \subset \rho^*E$$

with U_i of rank i. It has the universal property that, if $f: Y \to X$ is any morphism and $V_\bullet$ is a (complete) flag of subbundles of f^*E, there is a unique morphism $\tilde{f}: Y \to F\ell(E)$ such that $\tilde{f}^*U_i = V_i$ as subbundles of f^*E for all i. The flag bundle may be constructed as a sequence of projective bundles of ranks $n-1$, $n-2, \ldots, 1$, starting with $\rho_1: \mathbb{P}(E) \to X$ with universal line subbundle M_1, then forming $\rho_2: \mathbb{P}(\rho_1^*E/L_1) \to \mathbb{P}(E)$ with universal line subbundle $M_2/\rho_2^*M_1$, then $\rho_3: \mathbb{P}(\rho_2^*\rho_1^*E/M_2) \to \mathbb{P}(\rho_1^*E/L_1)$, and so on. In particular, this shows that $F\ell(E)$ is smooth of rank $N = n(n-1)/2$ over X.

The flag bundle $F\ell(E)$ also has a universal sequence of quotient bundles

$$\rho^*E \twoheadrightarrow Q_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow Q_1$$

where $Q_i = \rho^*E/U_{n-i}$ is a vector bundle of rank i.

Suppose we are given a complete flag $E_\bullet$ of subbundles of the bundle E. We then have on $F\ell(E)$ the situation

$$\rho^*E_1 \subset \cdots \subset \rho^*E_{n-1} \subset \rho^*E = \rho^*E \twoheadrightarrow Q_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow Q_1 . \tag{5.1}$$

For w in the symmetric group S_n, we therefore have a degeneracy locus Ω_w in $F\ell(E)$, which we denote by $\Omega_w(E_\bullet)$. By the local description in the next section it follows that, if X is irreducible, $\Omega_w(E_\bullet)$ is an irreducible subvariety of $F\ell(E)$ of codimension $\ell(w)$.

Assume now that X is nonsingular and let $A = A^\bullet X$ be the Chow ring of X. There is a canonical homomorphism

$$\psi\colon A[x_1, \dots, x_n] \to A^\bullet(F\ell(E)), \tag{5.2}$$

$$x_i \mapsto c_1(\mathrm{Ker}(Q_i \to Q_{i-1}) = c_1(U_{n+1-i}/U_{n-i})$$

(with $Q_0 = U_0 = 0$ and $U_n = Q_n = E$). We need the following well-known lemma ([C2, Exp. 4], [F, §14]).

LEMMA 5.3. *The homomorphism ψ is surjective, with kernel the ideal I generated by the polynomials $e_i(x_1, \dots, x_n) - c_i(E)$, $1 \leqslant i \leqslant n$, where e_i is the ith elementary symmetric function in n variables.*

Proof. From the construction of $F\ell(E)$ as a succession of projective bundles, it follows that the images of the $n!$ monomials $x_1^{i_1} \cdot x_2^{i_2} \cdot \ldots \cdot x_n^{i_n}$, with $i_j \leqslant n - j$, form a basis for $A^\bullet(F\ell(E))$ over A. Since these monomials generate $A[x]/I$, the map is an isomorphism. □

6. Schubert varieties. When X is a point, the flag bundle becomes the classical flag manifold. We make the connection with a classical description of (generalized) Schubert varieties in the flag manifold. In particular, we make the translation between our notation, which emphasizes the quotient bundles, with the simpler geometric description emphasizing subspaces. (The necessity for this translation is the same as for projective space: the first Chern class of the quotient line bundle is represented by a hyperplane, while the line subbundle has negative Chern class.)

Let $F\ell(V) = F\ell(n)$ be the flag manifold of (complete) flags $W_\bullet$ in a vector space V of dimension n. Fix a flag $V_\bullet$ of vector spaces in V. For each permutation w in S_n, define $X_w^o = X_w^o(V_\bullet)$ to be

$$\{W_\bullet \in F\ell(V)\colon \dim(W_q \cap V_p) = r_w(q, p) \text{ for } 1 \leqslant p, q < n\}$$

where $r_w(q, p) = \mathrm{Card}\{j \leqslant q\colon w(j) \leqslant i\}$. Let $X_w = X_w(V_\bullet)$ be the closed subscheme of $F\ell(V)$ defined by the corresponding inequalities $\dim(W_q \cap V_p) \geqslant r_w(q, p)$.

The flag manifold has a universal flag $V \twoheadrightarrow Q_{n-1} \twoheadrightarrow \ldots \twoheadrightarrow Q_1$ of quotient bundles of the trivial bundle $V = V_{F\ell(V)}$ on $F\ell(V)$. The flag of subspaces $V_\bullet$ of V determines a flag of subbundles of the trivial bundle (denoted by the same letters); so by the preceding section we have degeneracy loci $\Omega_w(V_\bullet)$. The following lemma records the identification of these loci and some basic facts about them.

LEMMA 6.1. (a) *For any* $w \in S_n$, $\Omega_w(V_\bullet) = X_{w \cdot w_0}$, *where* w_0 *is the permutation which takes* i *to* $n + 1 - i$.

(b) *Each* X_w^o *is a locally closed irreducible subvariety of* $F\ell(V)$, *isomorphic to affine space of dimension* $\ell(w)$.

(c) *Each* X_w *is an irreducible variety of dimension* $\ell(w)$, *and* X_w *is the closure of* X_w^o.

(d) *Each* X_w *is a reduced, Cohen-Macaulay variety.*

(e) *The classes* $[X_w]$ *of the Schubert varieties form an additive basis for the Chow ring* $A^\bullet X$.

Proof. For (a), $\Omega_w(V_\bullet)$ is defined by the conditions that the rank of $V_p \to Q_q$ is at most $r_w(q, p)$, which is the same as saying that the dimension of the kernel $V_p \cap W_{n-q}$ is at least $p - r_w(q, p)$, i.e., that for all p and q

$$\begin{aligned}\dim(W_q \cap V_p) &\geqslant p - r_w(n-q, p)\\ &= \operatorname{Card}\{i \leqslant p \colon w^{-1}(i) > n - q\}\\ &= \operatorname{Card}\{i \leqslant p \colon w_0 w^{-1}(i) \leqslant q\}\\ &= r_{w\cdot w_0}(q, p),\end{aligned}$$

which is the condition to be in $X_{w \cdot w_0}$.

The claims in (b) and (c) are part of the general Bruhat decomposition; see Remark 6.2. We write out (b) for use in the following lemma. Take a basis $e_1, \dots, e_n$ for V so that V_p is spanned by $e_1, \dots, e_p$. Every flag $W_\bullet$ in X_w^o can be uniquely described by specifying that W_q is the subspace spanned by the first q rows of an n by n matrix A which has a 1 in the ith row and $w(i)$th column, for each i between 1 and n, and has zero entries to the right of and below each of these 1's. That is, $A = (a_{i,j})$ with $a_{i,w(i)} = 1$ and $a_{i,j} = 0$ if $j > w(i)$ or $i > w^{-1}(j)$. Note that the number of free entries in such a matrix is

$$\operatorname{Card}\{(i, j) \colon j < w(i) \text{ and } i < w^{-1}(j)\},$$

which is the number $\ell(w)$ of inversions. This gives an isomorphism of X_w^o with $\mathbb{A}^{\ell(w)}$. For (c), since X_w is the disjoint union of all X_v^o with $v = w \cdot u$ and $\ell(v) = \ell(w) - \ell(u)$, it suffices to verify that X_v^o is contained in X_w in case $u = s_i$, which is a simple verification using the preceding description.

Part (e) follows formally from (b) and (c) and the cellular decomposition of each X_w; see [F, §14]. Part (d) was proved by Musili and Seshadri [MS], and Ramanathan [R]. □

We can use this to complete the proof of Lemma 3.11, and hence Proposition 3.3. Fix a basis $e_1, \ldots, e_n$ for a vector space V and let $\rho\colon \mathrm{GL}_n \to F\ell(n)$ be the map which takes $A \in \mathrm{GL}_n$ to the flag

$$\langle A \cdot e_1 \rangle \subset \langle A \cdot e_1, A \cdot e_2 \rangle \subset \cdots \subset \langle A \cdot e_1, \ldots, A \cdot e_{n-1} \rangle .$$

Let $V_i = \langle e_{n+1-i}, \ldots, e_n \rangle$. We claim that $V_w = \rho^{-1}(X_v)$ with $v = w_0 \cdot w^{-1}$. In fact, A is in $\rho^{-1}(X_v)$ when, for all p and q,

$$\dim(\langle A \cdot e_1, \ldots, A \cdot e_q \rangle \cap \langle e_{p+1}, \ldots, e_n \rangle) \geqslant \operatorname{Card}\{i \leqslant q\colon v(i) \leqslant n - p\},$$

which is equivalent to

$$\begin{aligned} \operatorname{rank}(A_{[q,p]}) &\leqslant q - \operatorname{Card}\{i \leqslant q\colon v(i) \leqslant n - p\} \\ &= \operatorname{Card}\{i \leqslant q\colon w_0 \cdot v(i) \leqslant p\} = r_{w_0 \cdot v}(q, p), \end{aligned}$$

which is the condition to be in V_w. Since ρ is smooth and surjective, the results of the preceding lemma for $X_v \subset F\ell(n)$ imply the corresponding results for $V_w \subset \mathrm{GL}_n$, noting that $\ell(w) = N - \ell(v)$.

Remark 6.2. If we identify the permutation w with the permutation matrix with a 1 in the $(i, w(i))$ place and zeros elsewhere, and set B to be the subgroup of all upper triangular matrices in the general linear group G, then the choice of reference flag $V_.$ identifies $F\ell(V)$ with G/B, with a coset $g \cdot B$ mapping to the flag $W_. = g \cdot V_.$. Then the locus X_w° is identified with the Bruhat cell BwB/B. For details in this language, see [C1, Exp. 13].

There are many other notations that have been used for the generalized Schubert varieties in flag manifolds. Ehresmann [E] and Monk [Mo] use a notation like that in Corollary 4.3, labelling Schubert varieties by nests of subsets, but making the transformation $w \mapsto w \cdot w_0$ as above. In addition, since they consider the associated flags in projective spaces, they correspondingly subtract 1 from all the numbers.

Finally, we need an elementary lemma which will be used for the local description of the basic correspondences in the next section. For an integer m between 1 and $n - 1$, define a subvariety $Z(m)$ of $F\ell(V) \times F\ell(V)$ by

$$Z(m) = \{(W_., W'_.)\colon W_i = W'_i \text{ for all } i \neq m\}.$$

Let p_1 and p_2 be the two projections from $Z(m)$ to $F\ell(V)$. In terms of the universal subbundles, each of these projections identifies $Z(m)$ with the projective bundle $\mathbb{P}(U_{m+1}/U_{m-1})$ over $F\ell(V)$.

LEMMA 6.3. *Let $w \in S_n$ and let $1 \leqslant m < n$.*

(a) *If $w(m) < w(m+1)$, then p_1 maps $p_2^{-1}(X_w)$ birationally onto $X_{w'}$, where $w' = w \cdot s_m$.*

(b) *If $w(m) > w(m+1)$, then p_1 maps $p_2^{-1}(X_w)$ into X_w.*

Proof. In the situation of (a), we show in fact that p_1 maps $p_2^{-1}(X_w^o)\backslash\Delta$ isomorphically onto $X_{w'}^o$, where Δ is the diagonal in $F\ell(V) \times F\ell(V)$. In fact, if $W_\bullet$ is a flag in X_w^o described by a matrix A as in the preceding proof, then the flags $(W'_\bullet, W_\bullet)$ in $p_2^{-1}(W_\bullet)\backslash\{W_\bullet\}$ are those where $W'_\bullet$ is given by matrices of the form A', where, if v_i and v'_i are the ith rows of A and A', then $v'_i = v_i$ for $i \neq m, m+1$, $v'_{m+1} = v_m$, and $v'_m = t \cdot v_m + v_{m+1}$ for some scalar t. (Note that the fibres of p_2 are all projective lines.) One verifies easily that, when $w(m) < w(m+1)$, these $W'_\bullet$ are precisely the flags in $X_{w'}^o$, and each such flag has such a description for a unique t. On the other hand, if $w(m) > w(m+1)$, all such flags are in X_w^o, which implies (b). □

7. A Giambelli formula for flag bundles. Let $F\ell(E)$ be the bundle of complete flags in a vector bundle E of rank n on a nonsingular variety X, with projection ρ from $F\ell(E)$ to X, with its universal flag $U_1 \subset \cdots \subset U_{n-1} \subset \rho^*E$ of subbundles, and with universal sequence $\rho^*E \twoheadrightarrow Q_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow Q_1$ of quotient bundles, as in §5. For each integer k between 1 and $n-1$, let $Z_k \subset F\ell(E) \times_X F\ell(E)$ be the subvariety of pairs $(W_\bullet, W'_\bullet)$ for which $W_i = W'_i$ for all $i \neq n-k$. Let p_1 and p_1 be the two projections from Z_k to $F\ell(E)$:

(7.1)
$$\begin{tikzcd} & Z_k \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ F\ell(E) \arrow[dr, "\rho"'] & & F\ell(E) \arrow[dl, "\rho"] \\ & X. & \end{tikzcd}$$

Via either projection, Z_k can be identified with the $\mathbb{P}^1$-bundle

$$Z_k = \mathbb{P}(\mathrm{Ker}(Q_{k+1} \to Q_{k-1})) = \mathbb{P}(U_{n-k+1}/U_{n-k-1}).$$

LEMMA 7.2. *With the identification of $A^\bullet(F\ell(E))$ with $A[x_1, \ldots, x_n]/I$ of Lemma 5.3, the endomorphism $p_{1*} \circ p_2^*$ of $A^\bullet(F\ell(E))$ is the operator ∂_k described in* §2.

Proof. We consider first the situation when $n = 2$, $k = 1$, in which case $F\ell(E)$ is the $\mathbb{P}^1$-bundle $\mathbb{P}(E)$, with universal quotient line bundle Q_1, and $Z_1 = \mathbb{P}(E) \times_X \mathbb{P}(E)$ is just the fibre product. In this case $A^\bullet(F\ell(E))$ is spanned over A by 1 and $x_1 = c_1(Q_1)$, and it suffices to verify that $p_{1*} \circ p_2^*(1) = 0$ and that $p_{1*} \circ p_2^*(x_1) = 1$. The first of these assertions is obvious, and for dimension reasons we must have $p_{1*} \circ p_2^*(x_1) = d \in A^0(\mathbb{P}(E))$ for some integer d. To verify that $d = 1$, we may restrict to a fibre; i.e., we may assume X is a point. In this case x_1 is represented by a closed point $P \in \mathbb{P}(E) = \mathbb{P}^1$; so p_1 maps $p_2^{-1}(P)$ isomorphically onto $\mathbb{P}(E)$, which proves the assertion.

The general case can be deduced from this special case as follows. Let $m = n - k$

and let Y be the bundle of incomplete flags in E of ranks $1, 2, \ldots, m-1, m+1, \ldots, n-1$, with universal subbundles W_i of rank i. Then $F\ell(E)$ can be identified with the bundle $\mathbb{P}(F)$ over Y, where $F = W_{m+1}/W_{m-1}$, and we have identifications

$$\begin{tikzcd} F\ell(E) \arrow[d, equal] & Z_k \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, equal] & F\ell(E) \arrow[d, equal] \\ \mathbb{P}(F) & \mathbb{P}(F) \times_Y \mathbb{P}(F) \arrow[l, "p_1"'] \arrow[r, "p_2"] & \mathbb{P}(F). \end{tikzcd}$$

The universal quotient line bundle on $\mathbb{P}(F)$ is identified with the bundle $U_{m+1}/U_m = \operatorname{Ker}(Q_k \to Q_{k-1})$; so its first Chern class is x_k. By the case just considered, $p_{1*} \circ p_2^*(\alpha) = 0$ and $p_{1*} \circ p_2^*(\alpha \cdot x_k) = \alpha$ for all $\alpha \in A^\cdot Y$. Since, as in the proof of Lemma 5.3, any class in $A^\cdot X$ is a sum of two such classes, and ∂_k has the same values on such classes (noting that any α in $A^\cdot Y$ is symmetric in x_k and x_{k+1}), the lemma follows. □

Now suppose we are given a complete flag $E_\bullet$ of subbundles of the bundle E. We have seen that $\Omega_w(E_\bullet)$ is an irreducible subvariety of $F\ell(E)$ of codimension $\ell(w)$; so it determines a class $[\Omega_w(E_\bullet)]$ in $A^{\ell(w)}(F\ell(E))$.

LEMMA 7.3. (a) *If $w(k) > w(k+1)$, then the endomorphism $p_{1*} \circ p_2^*$ of $A^\cdot(F\ell(E))$ takes $[\Omega_w(E_\bullet)]$ to $[\Omega_{w'}(E_\bullet)]$, where $w' = w \cdot s_k$.*
(b) *If $w(k) < w(k+1)$, then $p_{1*} \circ p_2^*([\Omega_w(E_\bullet)]) = 0$.*

Proof. Note that, with $w' = w \cdot s_k$, $\ell(w') = \ell(w) - 1$ when $w(k) > w(k+1)$, and $\ell(w') = \ell(w) + 1$ otherwise. To prove (a) it suffices to verify that p_1 maps $p_2^{-1}(\Omega_w(E_\bullet))$ birationally onto $\Omega_{w'}(E_\bullet)$ when $w(k) < w(k+1)$. To prove (b), it suffices to show that p_1 maps $p_2^{-1}(\Omega_w(E_\bullet))$ into $\Omega_w(E_\bullet)$ if $w(k) > w(k+1)$, for then p_1 maps $p_2^{-1}(\Omega_w(E_\bullet))$ to a smaller dimensional variety. It suffices to verify these assertions in each fibre of ρ; i.e., we may assume X is a point. In this case, the assertions translate to those of Lemma 6.3 in the preceding section. Indeed, the present Z_k becomes the $Z(m)$ of that lemma, with $m = n - k$, and, by Lemma 6.1(a), $\Omega_w(E_\bullet)$ becomes X_v, where $v = w \cdot w_0$. It suffices to note that $w(k) > w(k+1)$ precisely when $v(m) < v(m+1)$, and that, if $w' = w \cdot s_k$, then $w' \cdot w_0 = v'$, where $v' = v \cdot s_m$. □

From these two lemmas we have the essential relation between degeneracy loci, as follows.

LEMMA 7.4. *Let $w \in S_n$, $1 \leqslant k < n$. Set $w' = w \cdot s_k$. Then*

$$\partial_k([\Omega_w(E_\bullet)]) = \begin{cases} [\Omega_{w'}(E_\bullet)] & \text{if } w(k) > w(k+1) \\ 0 & \text{if } w(k) < w(k+1). \end{cases}$$

We can now prove the following formula for the degeneracy locus $\Omega_w(E_\bullet)$ in the flag bundle $F\ell(E)$. It is a special case of the theorem stated in the introduction and will be used for the general proof.

PROPOSITION 7.5. *For any complete flag $E_\bullet$ of subbundles in a vector bundle E of rank n on a nonsingular variety X and any $w \in S_n$, the class of $\Omega_w(E_\bullet)$ in $A^{\ell(w)}(F\ell(E))$ is given by the formula*

$$[\Omega_w(E_\bullet)] = \mathfrak{S}_w(x_1, \dots, x_n, y_1, \dots, y_n)$$

where $\mathfrak{S}_w$ is the double Schubert polynomial corresponding to w, x_i is the first Chern class of $\mathrm{Ker}(Q_i \to Q_{i-1})$, *and y_i is the first Chern class of E_i/E_{i-1}.*

Proof. Suppose first that $w = w_o$ is the permutation which takes i to $n + 1 - i$; so $\Omega_{w_o}(E_\bullet)$ is the subvariety of $F\ell(E)$ given by the vanishing of the maps from ρ^*E_p to Q_{n-p} for $1 \leqslant p < n$. Thus, $\Omega_{w_o}(E_\bullet)$ is the image of the section from X to $F\ell(E)$ given by the flag $E_\bullet$; in particular, it is smooth of codimension $N = \ell(w_o) = n(n-1)/2$. In fact, $\Omega_{w_o}(E_\bullet)$ is the zero of a section of the vector bundle K, where

$$K = \mathrm{Ker}\left(g\colon \bigoplus_{p=1}^{n-1} \mathrm{Hom}(E_p, Q_{n-p}) \to \bigoplus_{p=1}^{n-2} \mathrm{Hom}(E_p, Q_{n-p-1})\right) \tag{7.6}$$

and g takes $\sum \alpha_p$, with $\alpha_p\colon E_p \to Q_{n-p}$, to $\sum \beta_p$, where $\beta_p\colon E_p \to Q_{n-p-1}$ is defined to be $\alpha_{p+1} \circ i_p - j_{n-p} \circ \alpha_p$, with i_p the given inclusion of E_p in E_{p+1} and j_{n-p} the projection from Q_{n-p} to Q_{n-p-1}. This map g is easily checked to be surjective; so K is a bundle of rank $N = n(n-1)/2$. A map h determines a section of K (by setting α_p the map from E_p to F_{n-p} induced by h), whose zero locus is exactly Ω_{w_0}. Hence.

$$[\Omega_{w_o}] = c_N(K) = \prod_{i+j \leqslant n} (x_i - y_j) = \mathfrak{S}_{w_o}(x, y), \tag{7.7}$$

and the formula is proved in this case.

For the general case, write $w = w_o \cdot s_{k_1} \cdot \dots \cdot s_{k_r}$, where $r = N - \ell(w)$. Applying Lemma 7.4 r times, together with the case just proved, we have

$$\begin{aligned}
[\Omega_w(E_\bullet)] &= \partial_{k_r} \circ \dots \circ \partial_{k_1}([\Omega_{w_0}(E_\bullet)]) \\
&= \partial_{k_r} \circ \dots \circ \partial_{k_1}(\mathfrak{S}_{w_0}(x, y)) \\
&= \mathfrak{S}_w(x, y),
\end{aligned}$$

the last by the definition of the double Schubert polynomials. □

For the flag manifold $F\ell(V)$, when $V_\bullet$ is a fixed flag of subspaces of a vector space V, all the y_i vanish, and the proposition is a "Giambelli formula", by which we mean a formula which writes the class of a Schubert variety as a polynomial in the standard line bundles. Here, the class of $\Omega_w = \Omega_w(V_\bullet)$ is written as the Schubert polynomial in the Chern classes x_i of the quotient line bundles $\mathrm{Ker}(Q_i \to Q_{i-1})$. In the notation of Lemma 6.1,

$$[X_{w \cdot w_0}] = [\Omega_w] = \mathfrak{S}_w(x_1, \ldots, x_n). \tag{7.8}$$

Note that $x_1 + \cdots + x_k$ is the Schubert polynomial of the permutation s_k, which represents the locus where the map from V_k to Q_k is not an isomorphism, i.e., the locus of those flags $W_\bullet$ for which W_{n-k} meets V_k nontrivially. Thus, (7.8) writes an arbitrary Schubert class in terms of these basic Schubert classes. Formula (7.8), together with (2.14), gives a corresponding "Pieri formula", due to Monk [Mo]:

$$[\Omega_{s_k}] \cdot [\Omega_w] = \sum [\Omega_{w \cdot t}] \tag{7.9}$$

where the sum is over all transpositions $t = t_{ij}$ of integers such that $i \leqslant k < j$ and $w(i) < w(j)$, but $w(s)$ does not lie between $w(i)$ and $w(j)$ for any $i < s < j$. Translating via (6.1) and letting X_m be the locus of $W_\bullet$ such that W_m meets V_{n-m} nontrivially, this can be written

$$[X_m] \cdot [X_v] = (x_1 + \cdots + x_{n-m}) \cdot [X_v] = \sum [X_{v \cdot t}], \tag{7.10}$$

the sum over transpositions $t = t_{ij}$ with $i \leqslant m < j$, $v(i) > v(j)$, but $v(s)$ does not lie between $v(i)$ and $v(j)$ for $i < s < j$. More general "Pieri" formulas have been proved by Giambelli [G2], Chevalley (see [D, §4]), and Lascoux and Schützenberger [LS1].

Both papers [D] and [BGG] give formulas that are in a sense dual to formula (7.8). These compute the expansion of an arbitrary polynomial in the variables x_i in terms of the basis of $A^\bullet(F\ell(V))$ given by the Schubert varieties. (For some groups G, some Schubert varieties cannot be written as polynomials in Chern classes of the corresponding line bundles.) This takes the following form.

COROLLARY 7.11. *For any homogeneous polynomial P of degree d in $\mathbb{Z}[x_1, \ldots, x_n]$,*

$$P(x_1, \ldots, x_n) = \sum_{\ell(w)=d} (\partial_w P)[\Omega_w].$$

Proof. Note that each $\partial_w P$ is an integer when $\ell(w) = d$. By the proposition, the corollary is equivalent to the assertion that any P in $\mathbb{Z}[x]$ is congruent to $\sum (\partial_w P) \cdot \mathfrak{S}_w(x)$ modulo the ideal I. Either by direct calculation ([M, (5.6)]) or by the proposition and Lemma 6.1(e), the Schubert polynomials form a basis for $\mathbb{Z}[x]/I$; so it suffices to prove the corollary when $P = \mathfrak{S}_v(x)$ for some permutation v of length d. So we are reduced to proving that $\partial_w \mathfrak{S}_v(x) = \delta_{w,v}$ when w and v have length d. This is a simple consequence of the definition of Schubert polynomials; see [M, (4.2)]. □

There is also a dual formula. Verifying that the classes $[X_w] = [\Omega_{w \cdot w_0}]$ form a basis which is dual to the $[\Omega_w]$ (see [Mo]) this can be written

$$\int_X P(x_1, \ldots, x_n) \cdot [X_w] = \partial_w P. \tag{7.12}$$

Remark 7.13. If $C_w = \Omega_{w \cdot w_o}(p_1^*(S.), p_2^*(Q.))$ is the degeneracy locus in $F\ell(E) \times_X F\ell(E)$ using the map from the subbundles in the first factor to the quotient bundles in the second, and the permutation $w \cdot w_o$, one can show using Lemma 7.4 and the Bott-Samelson construction as in [S, §2], that the correspondence C_w determines the operator ∂_w, i.e.,

$$\partial_w(\alpha) = C_w^*(\alpha) = p_{1*}(C_w \cdot p_2^* \alpha)$$

for all α in $A^\cdot(F\ell(E))$. Equivalently, $C_u^* \cdot C_v^* = C_{u \cdot v}^*$ or 0 according as $\ell(u) + \ell(v) = \ell(u \cdot v)$ or not. We do not need this generalization.

8. The degeneracy locus formula. We next generalize the preceding formula to the case of partial flags and then apply this to prove the general degeneracy formula.

If E is a bundle of rank n on X and $0 < b_1 < \cdots < b_t < n$ is a sequence of integers, there is a partial flag bundle, which we denote by $F\ell(E; b.)$ of quotient sequences of E of ranks given by the b_i. On $F\ell(E; b.)$ there is a universal quotient sequence

$$E \twoheadrightarrow F_t \twoheadrightarrow F_{t-1} \twoheadrightarrow \cdots \twoheadrightarrow F_1$$

where F_i has rank b_i. Suppose $A_1 \subset A_2 \subset \cdots \subset A_s \subset E$ is a partial flag of subbundles of E of ranks $0 < a_1 < \cdots < a_s < n$. For any collection $\mathbf{r} = (r_{j,i})$ of nonnegative integers, $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant t$, we have the degeneracy locus $\Omega_{\mathbf{r}}(A.) \subset F\ell(E; b.)$ defined by the conditions

$$\operatorname{rank}(A_i \to F_j) \leqslant r_{j,i} \text{ for all } i \text{ and } j.$$

We suppose that $\mathbf{r}$ is permissible in the sense of (3.12) and let $w \in S_n$ be the corresponding permutation. Let $d(\mathbf{r}) = \ell(w)$. By Corollary 2.11, $\mathfrak{S}_w(x, y)$ is symmetric in the variables in each of the groups

$$x_1, \ldots, x_{b_1}; x_{b_1+1}, \ldots, x_{b_2}; \ldots; x_{b_{t-1}+1}, \ldots, x_{b_t};$$

$$y_1, \ldots, y_{a_1}; y_{a_1+1}, \ldots, y_{a_2}; \ldots; y_{a_{s-1}+1}, \ldots, y_{a_s}.$$

If each of these groups of variables is regarded as the Chern roots of the bundles F_1, $\operatorname{Ker}(F_2 \to F_1)$, $\ldots$, $\operatorname{Ker}(F_t \to F_{t-1})$, and A_1, A_2/A_1, $\ldots$, A_s/A_{s-1} respectively, it follows that we may write the double Schubert polynomial $\mathfrak{S}_w(x, y)$ as a polynomial in the Chern classes of these vector bundles, where we have set $x_j = 0$ for $j > b_t$ and $y_i = 0$ for $i > a_s$. We denote this polynomial by $P_{\mathbf{r}}(b., a.)$.

PROPOSITION 8.1. *The codimension of* $\Omega_{\mathbf{r}}(A.)$ *in* $F\ell(E, b.)$ *is* $d(\mathbf{r})$, *and*

$$[\Omega_{\mathbf{r}}(A.)] = P_{\mathbf{r}}(b., a.) \text{ in } A^{d(\mathbf{r})}(X).$$

Proof. There is a composite $X' \to X$ of projective bundle maps, such that on X' the sequence $A.$ fills in to a complete flag $E.$ of subbundles of E with $A_i = E_{a_i}$ for each i. Since the degeneracy locus is preserved by such a pullback and the pullback on the Chow ring is injective, it suffices to prove the result when $A.$ is part of a complete flag $E.$. Similarly, the bundle $F\ell(E)$ of complete flags maps to $F\ell(E, b.)$, and the above sequence of quotients pulls back to corresponding elements of the universal quotient sequence: $F_i = Q_{b_i}$; again, it suffices to prove the formula after pulling back to $F\ell(E)$. But now on $F\ell(E)$, by Proposition 4.2, the locus $\Omega_{\mathbf{r}}(A.)$ is equal to $\Omega_w(E.)$; so the assertions of the proposition become those of Proposition 7.5. □

Now we can state the general result. For simplicity we consider only schemes of finite type over a field, remarking that, following [F, §20], only notational changes are necessary to extend to schemes of finite type over any regular base. Consider

$$A_1 \subset A_2 \subset \cdots \subset A_s \xrightarrow{h} B_t \twoheadrightarrow B_{t-1} \twoheadrightarrow \cdots \twoheadrightarrow B_1$$

where the A_i and B_j are bundles of ranks a_i and b_j on a scheme X. Let $\mathbf{r} = (r_{j,i})$ be a permissible collection of rank numbers; so we have the degeneracy locus $\Omega_{\mathbf{r}}(h)$ defined by the conditions

$$\operatorname{rank}(A_i \to B_j) \leqslant r_{j,i} \text{ for all } i \text{ and } j.$$

Let w be the permutation giving rise to $\mathbf{r}$. As above, $\mathfrak{S}_w(x, y)$ is symmetric in the variables in each of the groups displayed above; so can be written at a polynomial $P_{\mathbf{r}}(b., a.)$ in the Chern classes of the bundles B_1, $\operatorname{Ker}(B_2 \to B_1), \ldots, \operatorname{Ker}(B_t \to B_{t-1})$, and $A_1, A_2/A_1, \ldots, A_s/A_{s-1}$.

We claim that, if X is smooth and h is suitably generic, then $\Omega_{\mathbf{r}}(h)$ is a subvariety of codimension $d(\mathbf{r})$ in X, and $[\Omega_{\mathbf{r}}(h)] = P_{\mathbf{r}}(b., a.)$ in $A^{d(\mathbf{r})}(X)$. The following theorem is a version of this assertion to allow singular varieties and arbitrary maps h, with no assumption of genericity.

THEOREM 8.2. *If X is a purely d-dimensional scheme, there is a class $\boldsymbol{\Omega}_{\mathbf{r}}(h)$ in $A_{d-d(\mathbf{r})}(\Omega_{\mathbf{r}}(h))$, satisfying the following.*

(a) *The image of $\boldsymbol{\Omega}_{\mathbf{r}}(h)$ in $A_{d-d(\mathbf{r})}(X)$ is $P_{\mathbf{r}}(b., a.) \cap [X]$.*

(b) *Each component of $\Omega_{\mathbf{r}}(h)$ has codimension at most $d(\mathbf{r})$ in X. If* $\operatorname{codim}(\Omega_{\mathbf{r}}(h), X) = d(\mathbf{r})$, *then $\boldsymbol{\Omega}_{\mathbf{r}}(h)$ is a positive cycle whose support is $\Omega_{\mathbf{r}}(h)$.*

(c) *The formation of $\boldsymbol{\Omega}_{\mathbf{r}}(h)$ commutes with pullback by flat or local complete intersection morphisms, and with push-forward by proper morphisms.*

(d) *If* $\operatorname{codim}(\Omega_{\mathbf{r}}(h), X) = d(\mathbf{r})$ *and X is Cohen-Macaulay (for example smooth), then $\Omega_{\mathbf{r}}(h)$ is Cohen-Macaulay and*

$$\boldsymbol{\Omega}_{\mathbf{r}}(h) = [\Omega_{\mathbf{r}}(h)].$$

In addition, the classes are uniquely determined by properties (c) and (d). For complex varieties, one can construct $\mathbf{\Omega}_{\mathbf{r}}(h)$ in the relative cohomology group $H^{2d(\mathbf{r})}(X, X - \Omega_{\mathbf{r}}(h))$, and similarly for étale cohomology for arbitrary varieties. In fact, for arbitrary X there is a class $\mathbf{\Omega}_{\mathbf{r}}(h)$ in the bivariant Chow group $A^{d(\mathbf{r})}(\Omega_{\mathbf{r}}(h) \to X)$, whose image in the Chow cohomology group $A^{d(\mathbf{r})}X$ is $P_{\mathbf{r}}(b_{\bullet}, a_{\bullet})$ and which gives the class in the theorem by operating on the fundamental class $[X]$. This strengthens the assertions of (a) and (c). Since the arguments are essentially the same as those in [F, §14 and §17], we will concentrate on those aspects that are special to the present situation.

Construction and proof. Let $E = A_s \oplus B_t$. The graph of h gives an embedding of A_s in E, and there is a canonical projection of E onto the second factor B_t. The sequence $E \twoheadrightarrow B_t \twoheadrightarrow \cdots \twoheadrightarrow B_1$ determines a section s_h: $X \to F\ell(E; b_{\bullet})$ such that this sequence is the pullback by s_h of the universal quotient sequence. It follows that $\Omega_{\mathbf{r}}(h)$ is the inverse image by s_h of the subscheme $\Omega_{\mathbf{r}}(A_{\bullet})$ of $F\ell(E; b_{\bullet})$. We define $\mathbf{\Omega}_{\mathbf{r}}(h)$ to be the refined pullback of the cycle $[\Omega_{\mathbf{r}}(A_{\bullet})]$ by the section s_h:

$$\mathbf{\Omega}_{\mathbf{r}}(h) = s_h^{!}([\Omega_{\mathbf{r}}(A_{\bullet})]).$$

In this formula $s_h^{!}$: $A_*(\Omega_{\mathbf{r}}(A_{\bullet})) \to A_*(\Omega_{\mathbf{r}}(h))$ is the refined Gysin homomorphism defined by the regular embedding s_h [F, §6.2]. The proof of property (c) in the theorem is then standard intersection theory, as in [F, §14].

To prove (a) one can argue as follows. First, since the formula pulls back, it suffices to prove the corresponding formula for $[\Omega_{\mathbf{r}}(A_{\bullet})]$. In the case when X is nonsingular, this is Proposition 8.1. To deduce it for quasiprojective X of pure dimension, one can realize the bundles and maps as pullbacks from some nonsingular variety (a Hom bundle over products of flag manifolds and projective spaces); again, the formula pulls back. Finally, for arbitrary X one can use Chow's lemma to find a birational, proper morphism $X' \to X$ with X' quasiprojective; since by (c) the formula is compatible with proper pushforward, the fact that it holds on X' implies it on X.

The assertions in (b) also follow from the fact that we know them for the locus $\Omega_{\mathbf{r}}(A_{\bullet})$ and the fact that s_h is a regular embedding. Part (d) also follows from this, together with the fact that, if X is Cohen-Macaulay, Lemma 6.1(d) implies that $\Omega_{\mathbf{r}}(A_{\bullet})$ is also Cohen-Macaulay. It follows from this that the pullback by a regular section, when the codimension is preserved, commutes with taking the cycle of a subscheme; the point is that a regular sequence locally defining $s_h(X)$ in $F\ell(E; b_{\bullet})$ must remain a regular sequence on the Cohen-Macaulay variety $\Omega_{\mathbf{r}}(A_{\bullet})$ if the codimension is preserved. □

We will conclude this section with the application to the case most studied in the classical literature. Suppose $(a_{i,j}(x))$ is an $m \times \ell$ matrix of homogeneous polynomials in variables $x_0, \ldots, x_d$ (for simplicity over an algebraically closed field) with

$$\deg(a_{i,j}(x)) = s_i + t_j > 0$$

where $s_1, \ldots, s_m$ and $t_1, \ldots, t_\ell$ are given nonnegative integers. Let $\mathbf{r} = (r_{q,p})$ be a rank function; i.e., there is a permutation w in S_n for some $n \leqslant m + \ell$ so that $r_{q,p} = r_w(q, p)$. The locus of interest is the locus of points in projective space $\mathbb{P}^d$ such that the rank of the upper left $q \times p$ minor of this matrix is at most $r_{q,p}$ for all q and p. Set

$$\Omega_{\mathbf{r}}((a_{ij})) = \{[x] \in \mathbb{P}^d : \operatorname{rank}((a_{i,j}(x))_{[q,p]}) \leqslant r_{q,p} \text{ for all } q \text{ and } p\}.$$

We will show that for all ranks for which this locus is irreducible (for generic matrices), the degree is given by the corresponding double Schubert polynomial. Let $d(\mathbf{r}) = \ell(w)$ and assume that $d(\mathbf{r}) \leqslant d$. Bertini's theorem and Proposition 3.3 imply that, for generic forms, the locus is reduced, and irreducible if $d(\mathbf{r}) < d$. The matrix $(a_{i,j}(x))$ gives a map of vector bundles from the direct sum of the line bundles $\mathcal{O}(-s_i)$ on $\mathbb{P}^d$ to the direct sum of the line bundles $\mathcal{O}(t_j)$. Applying Theorem 8.2 with $A_q = \bigoplus_{i=1}^{q} \mathcal{O}(-s_i)$ and $B_p = \bigoplus_{i=1}^{p} \mathcal{O}(t_i)$, we deduce the following corollary.

COROLLARY 8.3. *For a generic forms $a_{i,j}$ of degree $s_i + t_j$, the locus $\Omega_{\mathbf{r}}((a_{ij}))$ is a reduced subvariety (irreducible if $d(\mathbf{r}) < d$) of pure dimension $d - d(\mathbf{r})$ and degree*

$$\deg \Omega_{\mathbf{r}}((a_{ij})) = \mathfrak{S}_w(t_1, \ldots, t_\ell, 0, \ldots, -s_1, \ldots, -s_m, \ldots, 0, \ldots,).$$

We will discuss some special cases of this formula at the end of §10.

9. Vexillary permutations and multi-Schur polynomials. For any permutation w, one has sets

$$I_i(w) = \{j > i : w(j) < w(i)\},$$

$$J_i(w) = \{j < i : w(j) > w(i)\}.$$

The *code* of the permutation w is the sequence $(c_1, c_2, \ldots)$ of cardinalities of the sets $I_1(w), I_2(w), \ldots$; c_i is the number of points in the ith row of the diagram of w. A permutation is determined by its code, by the recipe: $w(1) = c_1 + 1$, and $w(i)$ is the $(c_i + 1)$st element in the complement of $\{w(1), \ldots, w(i-1)\}$ in $\{1, \ldots, n\}$. The cardinalities of the sets $I_i(w)$, when arranged in decreasing order, form a partition $\lambda = \lambda(w)$, called the *shape* of w. The cardinalities of the sets $J_i(w)$ similarly rearrange to a partition $\mu(w)$. The permutation w is called *vexillary*, or *single-shaped*, if $\lambda(w)$ and $\mu(w)$ are conjugate partitions. There are several conditions which are equivalent to this, one being that there be no $a < b < c < d$ with $w(b) < w(a) < w(d) < w(c)$. Thus, for example, all permutations of S_n for $n \leqslant 4$ are vexillary except for $w =$ 2 1 4 3. For larger n, however, only a small proportion of permutations are vexillary. In fact, the probability of a permutation being vexillary decreases exponentially to zero as n goes to infinity; see [M, §1].

There are useful determinantal formulas for the Schubert polynomials of vexillary permutations. We will see that vexillary permutations are also those corresponding to a simple class of rank conditions on a matrix. For this we will need the following variation on the results of [LS], [W], and [M]. To this end we introduce some notation, which is a little different from that in [M]. Let $x_1, x_2, \ldots, y_1, y_2, \ldots$ be commuting variables. For nonnegative integers u and v and any integer ℓ, let

$$h_\ell(u, v) = \text{coefficient of } t^\ell \text{ in } \prod_{i=1}^{u} (1 - y_i t) \Big/ \prod_{i=1}^{v} (1 - x_i t); \tag{9.1}$$

$$e_\ell(u, v) = \text{coefficient of } t^\ell \text{ in } \prod_{i=1}^{v} (1 + x_i t) \Big/ \prod_{i=1}^{u} (1 + y_i t). \tag{9.2}$$

Let $\lambda = (p_1^{m_1}, p_2^{m_2}, \ldots, p_k^{m_k})$ be a partition with $m_i \geqslant 1$ and $p_1 > p_2 > \cdots > p_k > 0$. Set $m = m_1 + \cdots + m_k$. Define, for $1 \leqslant i \leqslant m$,

$$\rho_\lambda(i) = \min\{s \colon i \leqslant m_1 + \cdots + m_s\}. \tag{9.3}$$

For any nonnegative integers $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$, define *multi-Schur polynomials* by the formulas

$$s_\lambda((u_1, v_1)^{m_1}, \ldots, (u_k, v_k)^{m_k}) = \det(h_{\lambda_i - i + j}(u_{\rho_\lambda(i)}, v_{\rho_\lambda(i)})), \tag{9.4}$$

$$s_\lambda^*((u_1, v_1)^{m_1}, \ldots, (u_k, v_k)^{m_k}) = \det(e_{\lambda_i - i + j}(u_{\rho_\lambda(i)}, v_{\rho_\lambda(i)})) \tag{9.5}$$

where the determinants on the right are m by m, indexed by i and j varying from 1 to m. Note that the function ρ_λ simply assures that the first m_1 rows use the variables (u_1, v_1), the next m_2 rows the variables (u_2, v_2), etc.

PROPOSITION 9.6. (a) *Let a_i, b_i and r_i be nonnegative integers, for $1 \leqslant i \leqslant k$, satisfying the conditions*

$$a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k, \qquad b_1 \geqslant b_2 \geqslant \cdots \geqslant b_k,$$

$$0 < a_1 - r_1 < a_2 - r_2 < \cdots < a_k - r_k, \quad b_1 - r_1 > b_2 - r_2 > \cdots > b_k - r_k > 0.$$

For any $n \geqslant a_k + b_1$ there is a unique permutation w in S_n with

$$\mathscr{E}ss(w) = \{(b_1, a_1), (b_2, a_2), \ldots, (b_k, a_k)\}$$

and

$$r_w(b_i, a_i) = r_i \qquad \text{for } 1 \leqslant i \leqslant k.$$

(b) *The permutation w of* (a) *is vexillary, and every vexillary permutation arises from a unique collection of such integers.*

(c) *The shape* $\lambda(w)$ *of* w *is* $(p_1^{m_1}, p_2^{m_2}, \dots, p_k^{m_k})$, *where*

$$p_1 = a_k - r_k, p_2 = a_{k-1} - r_{k-1}, \dots, p_k = a_1 - r_1;$$

$$m_1 = b_k - r_k, m_2 = (b_{k-1} - r_{k-1}) - (b_k - r_k), \dots, m_k = (b_1 - r_1) - (b_2 - r_2).$$

(d) *The Schubert polynomial of* w *is a multi-Schur polynomial*:

$$\mathfrak{S}_w(x, y) = s_\lambda((a_k, b_k)^{m_1}, \dots, (a_1, b_1)^{m_k}).$$

(e) *The conjugate partition* $\mu(w) = \lambda(w)'$ *is* $(q_1^{n_1}, \dots, q_k^{n_k})$ *with*

$$q_1 = b_1 - r_1, q_2 = b_2 - r_2, \dots, q_k = b_k - r_k;$$

$$n_1 = a_1 - r_1, n_2 = (a_2 - r_2) - (a_1 - r_1), \dots, n_k = (a_k - r_k) - (a_{k-1} - r_{k-1}).$$

(f) *The Schubert polynomial of* w *is also given by the formula*

$$\mathfrak{S}_w(x, y) = s_\mu^*((a_1, b_1)^{n_1}, \dots, (a_k, b_k)^{n_k}).$$

Proof. The uniqueness of w is implied by Lemma 3.10. For the existence, define the integers p_i, m_i, and the partition λ as in (c). Let $f_i = b_{k+1-i}$ for $1 \leqslant i \leqslant k$. We claim that

$$f_i \geqslant m_1 + \cdots + m_i \qquad \text{for } 1 \leqslant i \leqslant k; \tag{9.7}$$

$$0 \leqslant f_i - f_{i-1} \leqslant m_i + p_{i-1} - p_i \qquad \text{for } 2 \leqslant i \leqslant k. \tag{9.8}$$

In fact,

$$m_1 + \cdots + m_i = b_{k+1-i} - r_{k+1-i} = f_i - r_{k+1-i},$$

and (9.7) follows since each r_{k+1-i} is nonnegative. For (9.8),

$$\begin{aligned} m_i + p_{i-1} - p_i &= (b_{k+1-i} - r_{k+1-i}) - (b_{k+2-i} - r_{k+2-i}) + (a_{k+2-i} - r_{k+2-i}) \\ &\quad - (a_{k+1-i} - r_{k+1-i}) \\ &= (b_{k+1-i} - b_{k+2-i}) + (a_{k+2-i} - a_{k+1-i}) \\ &\geqslant b_{k+1-i} - b_{k+2-i} = f_i - f_{i-1} \geqslant 0. \end{aligned}$$

Now for any permutation w with code $(c_1, c_2, \dots)$, define, for each nonzero c_i, a number e_i by the formula

$$e_i = \max\{j \geqslant i : c_j \geqslant c_i\}.$$

The numbers e_i, arranged in increasing order, form the *flag* of w. In [M, (1.38)] it is proved that for $\lambda = (p_1^{m_1}, \ldots, p_k^{m_l})$ and $f_1 \leqslant \cdots \leqslant f_k$ satisfying (9.7) and (9.8), there is a unique vexillary permutation w whose shape is λ and whose flag is $(f_1^{m_1}, \ldots, f_k^{m_k})$. The code of w is determined as follows: first, m_1 p_1's are put in the rightmost spaces in the interval $[1, f_1]$; then, m_2 p_2's are put in the rightmost spaces still available in $[1, f_2]$, and so on until the last m_k p_k's are put in the rightmost spaces left in $[1, f_k]$. The remaining spaces are filled with zeros.

It is also shown in [M, §1] that every vexillary permutation arises in this way (the trivial permutation $w = \mathrm{id}$ corresponding to the trivial case when $k = 0$). We next verify that, for this permutation w, $r_w(b_i, a_i) = r_i$ for all i. Equivalently, we show that $r_w(b_{k+1-i}, a_{k+1-i}) = r_{k+1-i}$. Now since $b_{k+1-i} = f_i$,

$$r_w(b_{k+1-i}, a_{k+1-i}) = \operatorname{Card}\{s \leqslant f_i\colon w(s) \leqslant a_{k+1-i}\}.$$

We first claim that

$$r_{k+1-i} = \operatorname{Card}\{s \leqslant f_i\colon c_s < p_i\}. \tag{9.9}$$

This follows from the fact that $r_{k+1-i} = f_i - (m_1 + \cdots + m_i)$ and from the prescription for constructing the code of w. To conclude the proof it therefore suffices to verify that, for any s with $s \leqslant f_i$, the condition $c_s < p_i$ is equivalent to the condition $w(s) \leqslant a_{k+1-i}$. We show that in fact, for $s \leqslant f_i$,

$$c_s = p_i \quad \Rightarrow \quad w(s) > a_{k+1-i}; \tag{9.10}$$

$$c_s = p_j, \quad j > i \quad \Rightarrow \quad a_{k+1-j} < w(s) \leqslant a_{k+2-j}; \tag{9.11}$$

$$c_s = 0 \quad \Rightarrow \quad 1 \leqslant w(s) \leqslant a_1. \tag{9.12}$$

We verify this by induction on s. Suppose $s = 1$, so that $w(1) = c_1 + 1$. If $c_1 = p_j$, then, by (9.9), $r_{k+1-j} = 0$; so $p_j = a_{k+1-j}$ and $w(1) = a_{k+1-j} + 1$, which yields (9.10) and (9.11). If $c_1 = 0$, then $w(1) = 1 \leqslant a_1$, which finishes the proof for $s = 1$. Now let $s > 1$ and assume the assertions for smaller s. If $c_s = p_j$, by (9.9) and induction,

$$\begin{aligned} c_s = p_j = a_{k+1-j} - r_{k+1-j} &= a_{k+1-j} - \operatorname{Card}\{t < s\colon c_t < p_j\} \\ &= a_{k+1-j} - \operatorname{Card}\{t < s\colon w(t) < p_j\}. \end{aligned}$$

By the construction of w from its code, it follows that $w(s) > a_{k+1-j}$, which proves (9.10) and half of (9.11). If $j > i$, to show that $w(s) < a_{k+1-i}$ we need to know that

$$c_s + \operatorname{Card}\{t < s\colon w(t) \leqslant a_{k+2-j}\} < a_{k+2-i}. \tag{9.13}$$

The left side of (9.13) is

$$p_j + \operatorname{Card}\{t < s\colon c_t < p_{j-1}\} = (a_{k+1-j} - r_{k+1-j}) + r_{k+2-j},$$

which is strictly less than a_{k+2-j} by hypothesis. Finally, if $c_s = 0$, to show that $w(s) \leqslant a_1$ it suffices to verify that

$$\text{Card}\{t < s\colon w(t) \leqslant a_1\} < a_1\,. \tag{9.14}$$

By induction, the left side of (9.14) is

$$\text{Card}\{t < s\colon c_t = 0\} \leqslant \text{Card}\{t < f_k\colon c_t = 0\} = r_1 < a_1\,.$$

The conjugate to $(p_1^{m_1}, \dots, p_k^{m_k})$ is $(q_1^{n_1}, \dots, q_k^{n_k})$, where $q_i = m_1 + \cdots + m_{k+1-i}$ and $p_i = n_1 + \cdots + n_{k+1-i}$, from which (c) follows.

To complete the proof of (a)–(c) and (e), we must verify that the essential set of this permutation w is the set $\{(b_1, a_1), \dots, (b_k, a_k)\}$. We show first that (b_j, a_j) is in $\mathscr{E}ss(w)$. Since interchanging each a_i with each b_i leads to the inverse permutation w^{-1}, it suffices to show that $w(b_j) > a_j$ and $w(b_j + 1) \leqslant a_j$. Let $i = k + 1 - j$, so that $f_i = b_j$. The above description of w shows that, for $s = f_i$, c_s is p_j for some $j \leqslant i$; so by (9.10) and (9.11), $w(s) > a_{k+i-1} = a_j$. Similarly, for $s = f_i + 1$, $c_s = p_j$ for $j > i$; so $w(s) \leqslant a_{k+2-j} \leqslant a_{k+1-i}$. To show that $\mathscr{E}ss(w)$ is contained in the set $\{(b_j, a_j)\}$, it suffices again by symmetry to show that, if $s \neq b_j$ for all j, then $w(s) < w(s + 1)$. But if $s \neq f_i$ for all i, then $c_s \leqslant c_{s+1}$ by construction of the code, and then $w(s) < w(s + 1)$ by construction of w from its code.

In [M, (6.16)] it is shown, although in different notation, that the double Schubert polynomial of w is $s_\lambda((a_k, b_k)^{m_1}, \dots, (a_1, b_1)^{m_k})$, which proves (d). The dual form (f) follows from the duality theorem for multi-Schur polynomials [M, (3.8″)]. □

The Young (or Ferrars) diagram of the partition λ of the vexillary permutation arising from the numbers a_i, b_i, and r_i is determined as in Figure 9.15.

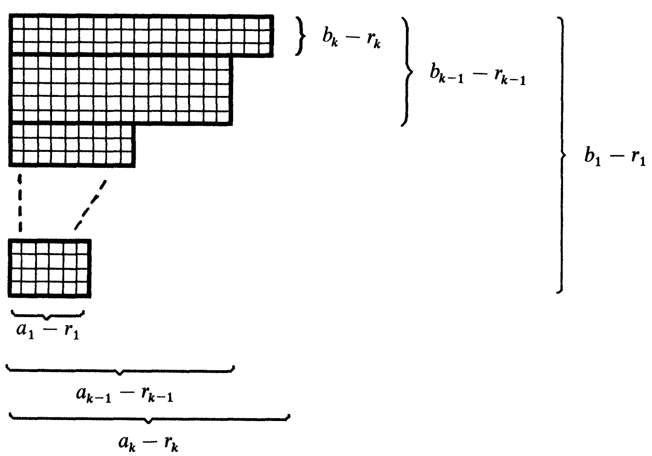

FIGURE 9.15

Remark 9.16. It is sometimes useful (and this occurs in the literature) to allow rank conditions as in the proposition but to allow inequalities throughout:

$$0 \leqslant a_1 - r_1 \leqslant \cdots \leqslant a_k - r_k \qquad \text{and} \qquad b_1 - r_1 \geqslant \cdots \geqslant b_k - r_k \geqslant 0$$

(as well as $a_1 \leqslant \cdots \leqslant a_k$ and $b_1 \geqslant \cdots \geqslant b_k$). With these more general assumptions there is still a unique (vexillary) permutation w with

$$\mathscr{E}ss(w) \subset \{(b_1, a_1), \ldots, (b_k, a_k)\} \qquad \text{and} \qquad r_w(b_i, a_i) = r_i \quad \forall i.$$

The partition of w and its conjugate can be defined by the same formulas (c) and (e), but now the p_i (and q_i) need not be distinct, and some of the multiplicities m_i (and n_i) can be zero. The point is that the rank condition at a point (b_i, a_i) can be omitted, as it follows from the others, if either $a_i - r_i = a_{i-1} - r_{i-1}$ or $b_i - r_i = b_{i+1} - r_{i+1}$. The formulas (d) and (f) for the Schubert polynomials, however, do not always hold in this greater generality. For an example, let $k = 2$, $a_1 = 1$, $a_2 = 2$, $b_1 = 3$, $b_2 = 2$, $r_1 = 0$, and $r_2 = 1$; using these numbers in the formula gives $\lambda = (1^1, 1^2) = (1^3)$, but $s_\lambda((2, 2)^1, (1, 3)^2) \neq s_\lambda((1, 3)^3)$, the latter being the correct answer obtained after throwing away the extra condition.

Remark 9.17. It follows from (a) of the proposition that a permutation is vexillary exactly when its essential set is strung along a southwest to northeast path; i.e., it has no two pairs (q, p) and (q', p') with $q < q'$ and $p < p'$. For a vexillary permutation w, it follows from (a) that the essential set $\mathscr{E}ss(w)$ is minimal in the strongest possible sense: if any (q_0, p_0) is omitted, there is another permutation w', which is necessarily also vexillary , with essential set $\mathscr{E}ss(w')$ equal to the complement of $\{(q_0, p_0)\}$ in $\mathscr{E}ss(w)$, and rank function $r_{w'}$ equal to r_w on $\mathscr{E}ss(w')$. (See Remark 3.16.) It would be useful to have criteria as in (a) for permutations which are not vexillary, characterizing those sets which can be essential sets for a permutation w, and what rank functions are possible.

Among the vexillary permutations are those called *Grassmannian*. In addition to the identity permutation, they are permutations with just one descent; i.e., there is one b such that $w(i) < w(i + 1)$ unless $i = b$. They are characterized by the fact that their essential sets lie in one row.

PROPOSITION 9.18. *Let a_i, r_i, $1 \leqslant i \leqslant k$, be nonnegative integers satisfying the conditions $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k$ and*

$$0 < a_1 - r_1 < a_2 - r_2 < \cdots < a_k - r_k, \qquad r_1 < r_2 < \cdots < r_k < b.$$

Then there is a unique permutation w in S_n for any $n \geqslant a_k + b$ with

$$\mathscr{E}ss(w) = \{(b, a_1), (b, a_2), \ldots, (b, a_k)\}$$

and

$$r_w(b, a_i) = r_i \qquad \text{for } 1 \leqslant i \leqslant k.$$

The permutation w is Grassmannian, and any Grassmannian permutation arises uniquely in this way. Its Schubert polynomial is

$$\mathfrak{S}_w(x, y) = s_\lambda((a_k, b)^{m_1}, \ldots, (a_1, b)^{m_k})$$

where $\lambda = (p_1^{m_1}, \ldots, p_k^{m_k})$ *with*

$$p_1 = a_k - r_k, p_2 = a_{k-1} - r_{k-1}, \ldots, p_k = a_1 - r_1;$$

$$m_1 = b - r_k, m_2 = r_k - r_{k-1}, \ldots, m_k = r_2 - r_1.$$

Dually,

$$\mathfrak{S}_w(x, y) = s_\mu^*((a_1, b)^{n_1}, \ldots, (a_k, b)^{n_k})$$

where $\mu = (q_1^{n_1}, \ldots, q_k^{n_k})$ *with*

$$q_1 = b - r_1, q_2 = b - r_2, \ldots, q_k = b - r_k;$$

$$n_1 = a_1 - r_1, n_2 = (a_2 - r_2) - (a_1 - r_1), \ldots, n_k = (a_k - r_k) - (a_{k-1} - r_{k-1}).$$

Proof. Using Proposition 9.6, we need only show that the vexillary permutations that arise this way are precisely the Grassmannian permutations. This follows from the fact that $w(q) > w(q+1)$ if and only if there is some p with (q, p) in $\mathscr{E}ss(w)$. $\square$

There is another class of vexillary permutations, called *dominant*: they are the permutations w such that the cardinalities of the sets $I_1(w), I_2(w), \ldots,$ form a weakly decreasing sequence. These too have a simple description by the rank conditions: they are precisely those permutations whose rank functions are identically zero on their essential sets. In fact we have the following proposition.

PROPOSITION 9.19. *Let a_i, b_i, $1 \leqslant i \leqslant k$, be nonnegative integers satisfying the conditions*

$$0 < a_1 < a_2 < \cdots < a_k, \qquad b_1 > b_2 > \cdots > b_k > 0.$$

Then there is a unique permutation w in S_n for any $n \geqslant a_k + b_1$, with

$$\mathscr{E}ss(w) = \{(b_1, a_1), (b_2, a_2), \ldots, (b_k, a_k)\}$$

and

$$r_w(b_i, a_i) = 0 \qquad \text{for } 1 \leqslant i \leqslant k.$$

The permutation w is dominant, and any dominant permutation arises in this way. Its

Schubert polynomial is

$$\mathfrak{S}_w(x, y) = s_\lambda((a_k, b_k)^{m_1}, \ldots, (a_1, b_1)^{m_k})$$

where $\lambda = (a_k^{m_1}, \ldots, a_1^{m_k})$ *with*

$$m_1 = b_k\,, m_2 = b_{k-1} - b_k, \ldots, m_k = b_1 - b_2\,.$$

Dually,

$$\mathfrak{S}_w(x, y) = s_\mu^*((a_1, b_1)^{n_1}, \ldots, (a_k, b_k)^{n_k})$$

where $\mu = (b_1^{n_1}, \ldots, b_k^{n_k})$ *with*

$$n_1 = a_1\,, n_2 = a_2 - a_1, \ldots, n_k = a_k - a_{k-1}\,.$$

Proof. We use the description of the code discussed in the proof of Proposition 9.6. For the code to be weakly increasing, the insertion of the m_i p_i's in the rightmost available spaces in $[1, f_i]$ must exactly fill the spaces of $[f_{i-1} + 1, f_i]$. This means that $m_i = f_i - f_{i-1}$, or that $m_1 + \cdots + m_i = f_i$ for all i, which happens precisely when $r_{k+1-i} = 0$. The rest of the proposition follows from Proposition 9.6. The fact that any permutation whose rank function vanishes on its essential set must be vexillary, which was stated before the proposition, follows from (3.15). □

We conclude this section with some examples of each of these types of permutations, with notation as in Figure 3.9.

Vexillary

$w = 9\ 6\ 8\ 3\ 7\ 4\ 10\ 2\ 1\ 5$
$\ell(w) = 31$
$k = 8$
a_i: 1 2 5 5 5 7 8
b_i: 8 7 7 5 3 3 1
r_i: 0 0 2 1 0 1 0
$\lambda(w) = (8, 6, 5, 4, 3, 2^2, 1)$

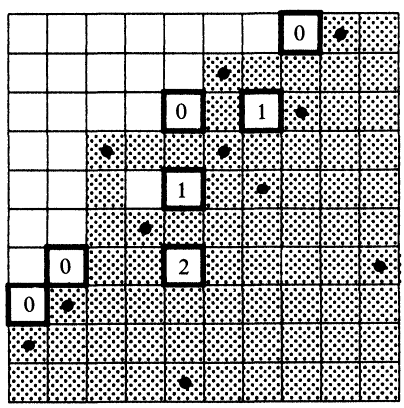

FIGURE 9.20

Grassmannian

$w = 2\ 4\ 5\ 7\ 9\ 10\ 1\ 3\ 6\ 8$
$\ell(w) = 16$
$k = 4$
a_i: 1 3 6 8
b_i: 0 1 3 4
$\lambda(w) = (4^2, 3, 2^2, 1)$

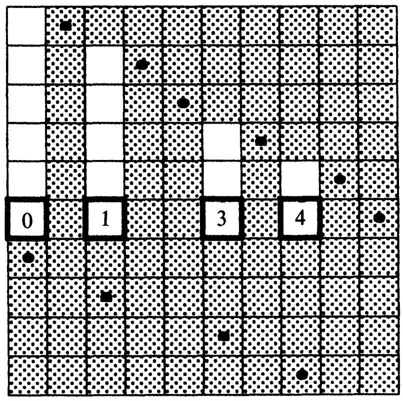

FIGURE 9.21

Dominant

$w = 10\ 7\ 5\ 4\ 2\ 1\ 3\ 6\ 8\ 9$
$\ell(w) = 23$
$k = 5$
a_i: 1 3 4 6 9
b_i: 5 4 3 2 1
$\lambda(w) = (9, 6, 4, 3, 1)$

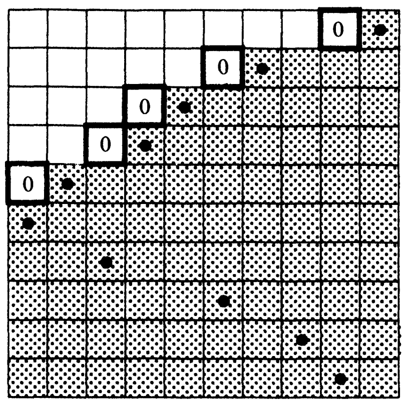

FIGURE 9.22

10. Determinantal formulas and applications. Combined with the algebra of the preceding section, the theorem of §8 gives a general determinantal formula. Suppose we are given vector bundles

$$A_1 \subset A_2 \subset \cdots \subset A_k \qquad \text{and} \qquad B_1 \twoheadrightarrow B_2 \twoheadrightarrow \cdots \twoheadrightarrow B_k$$

on a scheme X, of ranks $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k$ and $b_1 \geqslant b_1 \geqslant \cdots \geqslant b_k$, and a morphism $h: A_k \to B_1$ of bundles. (Note that equalities are allowed in these bundles, and we have changed to number the sequence of quotient bundles from the top down.) Let $r_1, \ldots, r_k$ be nonnegative integers satisfying

$$0 < a_1 - r_1 < a_2 - r_2 < \cdots < a_k - r_k,$$

$$b_1 - r_1 > b_2 - r_2 > \cdots > b_k - r_k > 0.$$

Define $\Omega_{\mathbf{r}} = \Omega_{\mathbf{r}}(h)$ to be the subscheme defined by the conditions that the rank of the map from A_i to B_i is at most r_i for $1 \leqslant i \leqslant k$. Let λ be the partition $(p_1^{m_1}, p_2^{m_2}, \ldots, p_k^{m_k})$, where

$$p_1 = a_k - r_k, p_2 = a_{k-1} - r_{k-1}, \ldots, p_k = a_1 - r_1;$$

$$m_1 = b_k - r_k, m_2 = (b_{k-1} - r_{k-1}) - (b_k - r_k), \ldots, m_k = (b_1 - r_1) - (b_2 - r_2).$$

Let $\mu = \lambda' = (q_1^{n_1}, \ldots, q_k^{n_k})$ be the conjugate partition, i.e.,

$$q_1 = b_1 - r_1, q_2 = b_2 - r_2, \ldots, q_k = b_k - r_k;$$

$$n_1 = a_1 - r_1, n_2 = (a_2 - r_2) - (a_1 - r_1), \ldots, n_k = (a_k - r_k) - (a_{k-1} - r_{k-1}).$$

Let $d(\mathbf{r}) = |\lambda| = \sum m_i p_i = |\mu| = \sum n_i q_i$ be the number partitioned. Let $m = m_1 + \cdots + m_k = b_1 - r_1$, and, for $1 \leqslant i \leqslant m$, let

$$\rho(i) = \max\{s \in [1, k]\colon i \leqslant b_s - r_s = m_1 + \cdots + m_{k+1-s}\}.$$

For $1 \leqslant i \leqslant n = n_1 + \cdots + n_k = a_k - r_k$, let

$$\rho'(i) = \min\{s \in [1, k]\colon i \leqslant a_s - r_s = n_1 + \cdots + n_s\}.$$

THEOREM 10.1. *If X is purely d-dimensional, there is a class $\mathbf{\Omega}_{\mathbf{r}}$ in $A_{d-d(\mathbf{r})}(\Omega_{\mathbf{r}})$, such that the image of $\mathbf{\Omega}_{\mathbf{r}}$ in $A_{d-d(\mathbf{r})}(X)$ is $P_{\mathbf{r}} \cap [X]$, where*

$$P_{\mathbf{r}} = \det(c_{\lambda_i - i + j}(A_{\rho(i)}^{\vee} - B_{\rho(i)}^{\vee}))_{1 \leqslant i, j \leqslant m}$$

$$= \det(c_{\mu_i - i + j}(B_{\rho'(i)} - A_{\rho'(i)}))_{1 \leqslant i, j \leqslant n}.$$

The properties (b)–(d) *of Theorem* 8.2 *are also valid for these classes.*

The rest of discussion in §8 about the general classes $\mathbf{\Omega}_{\mathbf{r}}$ applies to these classes as well. The proof follows from the general Theorem 8.2, together with Proposition 9.6. In interpreting the formulas from §9, the variables x_i and y_i become Chern roots of the bundles; in particular, if u and v are the ranks of the bundles A_s and B_s, then $h_\ell(u, v) = c_\ell(A_s^{\vee} - B_s^{\vee})$ and $e_\ell(u, v) = c_\ell(B_s - A_s)$. $\square$

In the special case where $r_i = a_i - i$ for all i, $\Omega_{\mathbf{r}}$ is the locus where the dimension of the kernel of $A_i \to B_i$ is at least i for all i. In this case a formula like that in Theorem 10.1 was announced by Pragacz [P, (8.3)]. Theorem 10.1 gives general conditions under which it holds:

$$b_1 - a_1 + 1 > b_2 - a_2 + 2 > \cdots > b_k - a_k + k > 0.$$

The polynomial giving this locus is then

$$\det(c_{b_i-a_i+j}(B_i - A_i))_{1\leqslant i,j\leqslant k}.$$

When all the bundles B_i are equal, our general determinantal formula specializes as follows. Suppose we are given vector bundles and a map

$$A_1 \subset A_2 \subset \cdots \subset A_k \xrightarrow{h} B,$$

the bundles of ranks $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k$ and b, and nonnegative integers $r_1, \ldots, r_k$ satisfying

$$0 < a_1 - r_1 < a_2 - r_2 < \cdots < a_k - r_k, \qquad r_1 < r_2 < \cdots < r_k < b.$$

Define $\Omega_{\mathbf{r}}$ to be the subscheme defined by the conditions that the rank of the map from A_i to B is at most r_i for $1 \leqslant i \leqslant k$. Let λ be the partition $(p_1^{m_1}, \ldots, p_k^{m_k})$ with

$$p_1 = a_k - r_k, \qquad p_2 = a_{k-1} - r_{k-1}, \ldots, p_k = a_1 - r_1;$$

$$m_1 = b - r_k, \qquad m_2 = r_k - r_{k-1}, \ldots, m_k = r_2 - r_1.$$

Let $\mu = \lambda' = (q_1^{n_1}, \ldots, q_k^{n_k})$ be the conjugate partition, i.e.,

$$q_1 = b - r_1, \ldots, q_k = b - r_k;$$

$$n_1 = a_1 - r_1, \ldots, n_k = (a_k - r_k) - (a_{k-1} - r_{k-1}).$$

Let $d(\mathbf{r}) = |\lambda| = |\mu|$, $m = b - r_1$, $n = a_k - r_k$. Defining $\rho(i)$ and $\rho'(i)$ as before, we have by Proposition 9.18, the following proposition.

Proposition 10.2. *If X is purely d-dimensional, there is a class $\mathbf{\Omega}_{\mathbf{r}}$ in $A_{d-d(\mathbf{r})}(\Omega_{\mathbf{r}})$ such that the image of $\mathbf{\Omega}_{\mathbf{r}}$ in $A_{d-d\mathbf{r})}(X)$ is $P_{\mathbf{r}} \cap [X]$, where*

$$P_{\mathbf{r}} = \det(c_{\lambda_i-i+j}(A_{\rho(i)}^{\vee} - B^{\vee}))_{1\leqslant i,j\leqslant m}$$

$$= \det(c_{\mu_i-i+j}(B - A_{\rho'(i)}))_{1\leqslant i,j\leqslant n}.$$

Properties (b)–(d) *of Theorem* 8.2 *are also valid for these classes.*

The determinantal formula of Kempf and Laksov ([KL], [F]) is a special case of the preceding proposition (and of the formula of Pragacz mentioned above), applied to the case when $a_i - r_i = i$ for all i between 1 and k. In this case $\mu_i = b - a_i + i$, and since $\rho'(i) = i$ and $n = k$, the formula reads

$$\mathbf{\Omega}_{\mathbf{r}} = \det(c_{b-a_i+j}(B - A_i))_{1\leqslant i,j\leqslant k} \cap [X].$$

When $k = 1$, the theorem specializes to the Giambelli-Thom-Porteous formula as follows. We are given a map $h: A \to B$ of vector bundles of ranks a and b, and a nonnegative integer $r < \min(a, b)$. Now Ω_r is the locus where the rank of h is at most r. In this case $n = a - r$, $\mu_i = b - r$ for all i, and the formula is

$$\boldsymbol{\Omega}_{\mathbf{r}} = \det(c_{b-r-i+j}(B - A))_{1 \leqslant i,j \leqslant a-r} \cap [X].$$

One can also specialize Theorem 10.1 to the case where all the prescribed ranks r_i are zero, using Proposition 9.19 for dominant permutations. With $\mu = (b_1^{n_1}, \ldots, b_k^{n_k})$ and

$$n_1 = a_1, n_2 = a_2 - a_1, \ldots, n_k = a_k - a_{k-1},$$

$n = a_k$, and $\rho'(i) = \min\{s \in [1, k]: i \leqslant a_s\}$, the formula for the locus where each $A_i \to B_i$ vanishes is

$$\boldsymbol{\Omega}_{\mathbf{0}} = \det(c_{\mu_i - i + j}(B_{\rho'(i)} - A_{\rho'(i)}))_{1 \leqslant i,j \leqslant n}. \tag{10.3}$$

The special cases of Theorem 8.3 which were considered by Giambelli ([G1], [G3]) also fall under the special "vexillary" class considered in this section. Giambelli considers rank conditions which one can place on an $m \times \ell$ matrix $(a_{i,j}(x))$ of forms by putting rank conditions along the bottom row and right column. Since these occur in a southwest-to-northeast pattern, they always correspond to a collection of rank conditions to which our general determinantal formula of Proposition 10.2 applies. In [G1] he considers a special case that puts a bound on the rank of the whole $m \times \ell$ matrix and, in addition, requires that some left $m \times \nu$ and some top $\mu \times \ell$ matrix be singular.

In [G3] Giambelli consider general rank conditions along the bottom row and right column. For this a positive integer $c \leqslant \min(m, \ell)$ is specified with integers μ_i and ν_i satisfying

$$0 \leqslant \mu_1 < \cdots < \mu_c < m \qquad \text{and} \qquad 0 \leqslant \nu_1 < \cdots < \nu_c < \ell.$$

Giambelli considers the condition that the rank of the entire matrix is at most c that the rank of the upper $\mu_i \times \ell$ minor is at most $i - 1$, and that the rank of the left $m \times \nu_i$ minor is at most $i - 1$ for all $1 \leqslant i \leqslant c$. The codimension of the locus is

$$t = m\ell - c \cdot (m + \ell - 1) + \sum_{i=1}^{c} \mu_i + \sum_{i=1}^{c} \nu_i.$$

In this more general situation, however, Giambelli had to assume that all the entries $a_{i,j}$ in the matrix are forms of the same degree p, but in this case he gives the following formula for the degree of the locus:

$$\deg \boldsymbol{\Omega}_{\mathbf{r}}((a_{i,j})) = p^t \cdot \det\left[\binom{m + \ell - 2 - \mu_i - \nu_j}{m - 1 - \mu_i}\right]_{1 \leqslant i,j \leqslant c}.$$

Pragacz points out that this is quite different from our formulas and therefore gives an interesting formula involving the corresponding Schubert function. Note that Giambelli's formula is the determinant of a $c \times c$ matrix, while ours give determinants of a larger $m \times m$ or $\ell \times \ell$ matrix. If w is the permutation corresponding to the rank condition $\mathbf{r}$ (see Remark 9.16) and we set the variables x_i equal to x and the variables y_i equal to $-y$, then the combination of our formula with Giambelli's gives

$$\mathfrak{S}_w(x, \dots, x, -y, \dots, -y) = (x+y)^t \cdot \det\left[\binom{m+\ell-2-\mu_i-\nu_i}{m-1-\mu_i}\right]_{1 \leqslant i,j \leqslant c}.$$

For example, for the ordinary Schubert polynomial this gives the formula

$$\mathfrak{S}_w(1, \dots, 1) = \det\left[\binom{m+\ell-2-\mu_i-\nu_j}{m-1-\mu_i}\right]_{1 \leqslant i,j \leqslant c}.$$

Giambelli's formulas in [G1] also combine with ours to give identities among some simpler multi-Schur polynomials.

As we mentioned, the general degeneracy formula can be used to prove some of the known identities among Schubert polynomials. For this one takes again the bundles to be sums of line bundles, so that their Chern classes are independent variables through some large degree, and so that there are maps between the bundles with degeneracy loci irreducible of the expected codimension. For example, the variety X can be taken to be a product of $2n$ large projective spaces and the line bundles to be the pullbacks of the universal sub or quotient line bundles on the factors. In this context, for example, Corollary 2.10 follows from the fact that $\Omega_{w^{-1}}(h) = \Omega_w(h^\vee)$. The duality formulas in [M, (3.8)] can also be proved this way.

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Department of Mathematics, University of Chicago, 5734 University Avenue, Chicago, Illinois 60637