1. Introduction. An $n$-by-$n$ permutation matrix can be represented by an $n$-by-$n$ array of squares with one dot in each row and column and all other squares empty. The diagram of a permutation matrix (defined in 1800 by Rothe) is obtained by shading every row from the dot and eastwards and shading every column from the dot and southwards. The essential set is the set of southeast corners of the connected components of the diagram. The essential set, together with a rank function, was introduced by Fulton [7] in a pioneering paper from 1992, to study the irreducible loci in spaces of matrices. Although we are not going to pursue this geometric issue much, but rather concentrate on the essential set per se, we need a short recollection of the background.

Fulton studies varieties given by ideals of minors, subject to certain rank conditions, in a generic matrix. One concern is, given a prescribed rank function $r(i, j)$, to determine if there exist matrices such that the rank of the upper-left $i$-by-$j$ submatrices is $r(i, j)$ for every position $(i, j)$. He observes that if such matrices exist, then there is in particular some permutation matrix with this property, so it is enough to consider permutation matrices. More generally, he is interested in prescribing the ranks only for a few of the submatrices, such that all the other ranks follow from these: in other words, to find a subset of the upper-left submatrices of a permutation matrix such that the permutation is uniquely determined by their ranks. Fulton shows that the essential set of a permutation matrix is such a set; that is, the permutation is determined by its essential set and the ranks of the corresponding upper-left submatrices. In Section 2 we present more detailed definitions and the relevant results of Fulton.

Our first result, in Section 3, is an elementary algorithm that retrieves a permutation from its ranked essential set. The algorithm is flexible, in that the input set may be bigger and include any other squares, as well as be smaller as long as it contains a certain "core" of the essential set. By this flexibility, the algorithm ought to be useful as a practical tool. In general, the core is much smaller than the essential set, so this is a significant strengthening.

Fulton used the essential sets mainly for vexillary (2143-avoiding) permutations. He gave a characterization of the ranked essential set for vexillary permutations, which may be expressed by three simple conditions, and he pointed out that it might be useful to have an analogous characterization of the ranked sets of squares that arise as ranked essential sets of all permutations. As an application of the algorithm, we are able to state such a characterization, which keeps
two of Fulton's three conditions for the vexillary case and replaces the third one with a new and, alas, more complicated condition. We also remark on how the algorithm can be used directly to determine if a given ranked set contains the essential set of a permutation. This is done in Section 4.

In Section 5, we use a combinatorial technique to describe several classes of permutations in terms of their essential set. For example, the Baxter permutations are precisely those whose essential set has at most one square in each row and column. Another example is 321-avoiding vexillary permutations, which can be described as having the essential set entirely contained in one row or in one column. As a corollary we obtain a direct interpretation of the formula for the number of 321-avoiding vexillary permutations of Billey, Jockusch, and Stanley [2].

In Section 6 we discuss how some of Fulton's results on minor ideals and the essential set can be generalized to higher dimensions, that is, if we work not with matrices but with higher-dimensional arrays of variable entries. It turns out that with natural definitions of permutations, determinants, and essential sets, it is true also in this general situation that a permutation is determined by its essential set, and that the minor ideal defined by the permutation is generated by the minors stemming from the essential set.

There are several interesting problems left open, and we collect some of them at the end of the paper. In another paper [5], the same authors have studied various enumerative aspects of the essential set.

We thank Dan Laksov for drawing our attention to this problem and Bruce Sagan for pointing out the connection between our algorithm and Viennot's shadowing procedure. We are also grateful to the referee for several helpful suggestions.

2. Preliminaries. The combinatorial object that we are studying is the essential set of a permutation, together with its rank function. They are defined as follows. First, let every permutation \( w \in S_n \) be represented by its dotted permutation matrix, regarded as an \( n \times n \) array of squares in the plane, where square \((i, w(i))\) has a dot for all \( i \in [1,n] \) and all other squares are white, so there is exactly one dot in each row and column. We will number the rows from north to south and the columns from west to east.

Shading the squares in each row from the dot and eastwards, and shading the squares in each column from the dot and southwards, we get the diagram of the permutation as the white (nonshaded) squares. We call a white square a white corner if it has no white neighbor either to the east or to the south. In other words, the white corners are the southeast corners of the components of the diagram. The essential set \( \mathcal{E}(w) \) of a permutation \( w \) is defined to be the set of white corners of the diagram of \( w \). Equivalently,

\[
\mathcal{E}(w) = \{(i, j) \in [1, n-1] \times [1, n-1]: w(i) > j, w^{-1}(j) > i, w(i+1) \leq j, w^{-1}(j+1) \leq i\}.
\]
For every square \((i, j)\) in the dotted matrix, its rank is defined by

\[ r_w(i, j) = |\{q < i : w(q) < j\}| = |\{\text{dots northwest of } (i, j)\}|. \]

The name "rank" stems from the fact that \(r_w(i, j)\) is equal to the matrix rank of the \(i\)-by-\(j\) upper-left submatrix of the ordinary permutation matrix of \(w\), where the dots are replaced by ones and the blank squares by zeros.

We will discuss two results of Fulton. First, the property that makes the essential set such a useful subset of the \(n^2\) squares.

**Lemma 2.1** (Fulton [7], Lemma 3.10). (a) For any \(w \in S_n\) and a generic \(n\)-by-\(n\) matrix \(A = (x_{ij})\), the ideal in the polynomial ring of the variables \(x_{11}, \ldots, x_{nn}\) generated by all minors of size \(r_w(i, j) + 1\) taken from the upper-left \(i\)-by-\(j\) submatrix of \(A\), for all \(i, j \in [1, n]\), is generated by these same minors using only those \((i, j)\) that are in \(\mathcal{E}(w)\).

(b) A permutation \(w \in S_n\) is determined by the restriction of its rank function \(r_w\) to \(\mathcal{E}(w)\). \(\square\)

Part (a) has a surprisingly simple proof, just by checking in four short steps that nothing outside the essential set adds anything new to the ideal. Part (b) follows from (a), since knowing the ideal implies knowing \(r_w(i, j)\) for every position \((i, j)\), and hence knowing the permutation \(w\). Observe the following subtle point: there may be other permutations than \(w\) that have coinciding rank functions in \(\mathcal{E}(w)\), but no such permutation will have the same essential set as \(w\); for example, \(321 \in S_3\) coincides with the rank of the white corner of \(312 \in S_3\). In the next section, we will provide an alternative proof of (b) by an algorithm for retrieving the permutation from its ranked essential set. In the final section, we...
will give higher-dimensional generalizations of both parts—(a) in Proposition 6.1 and (b) in Proposition 6.2.

Second, we present the characterization of the ranked essential sets arising from vexillary permutations. The vexillary permutations are an important class of permutations in the theory of Schubert polynomials and related areas (see Macdonald's book [11]). Fulton gave a set of sufficient conditions [7, Proposition 9.6] for an essential set to correspond to a vexillary permutation. In the same proposition, he also stated that all but one of the conditions (see (2) below) were necessary. We have strengthened that condition, (C1)(b), to obtain a set of necessary and sufficient conditions. We should also mention that we have reformulated the other conditions of Fulton to suit our purposes.

**Proposition 2.2 (Fulton).** Let $E \subseteq [1, n] \times [1, n]$ be a set of squares with an integer valued function $r(i, j)$ defined on $E$. Then $E$ is the essential set with rank function $r$ of some vexillary $n$-by-$n$ permutation matrix if and only if:

(C1) for each $(i, j) \in E$ we have
(a) $\min\{i, j\} > r(i, j) > 0$, and
(b) $r(i, j) + n > i + j$;

(C2) for every distinct pair $(i, j), (i', j') \in E$ such that $i > i', j < j'$ and $E \cap [i', i] \times [j, j'] = \{(i, j), (i', j')\}$, we have

$$i - i' > r(i, j) - r(i', j') > j - j';$$

(V) $E$ does not contain any pair $(i_1, j_1), (i_2, j_2)$ where $i_1 < i_2$ and $j_1 < j_2$.

To be precise, Fulton gave the condition

$$n > \max\{i : (i, j) \in \mathcal{E}(w)\} + \max\{j : (i, j) \in \mathcal{E}(w)\}$$

instead of (C1)(b) for sufficiency. As stated here, Proposition 2.2 follows from the chess theorem presented in Section 4. Conditions (C1) and (C2) are necessary for the essential set of any permutation and condition (V) is the special condition for vexillary permutations.

**Example.** The permutation $5736241 \in S_7$ is a vexillary permutation (see Figure 2). Note that the essential set of $5736241$ does not satisfy (2) above, but it does satisfy (C1)(b).

In Section 4 we will give a characterization of ranked essential sets arising from arbitrary permutations by replacing condition (V) above with a new condition. In Section 5 we will describe several classes of permutations in terms of their essential set (without considering ranks) in the spirit of condition (V).

3. The retrieval algorithm. We shall present and analyze an elementary algorithm for explicitly determining a permutation from its ranked essential set.

It will be convenient to work with concepts from partially ordered sets, so let $\mathbb{P} = \{1, 2, 3, \ldots\}$ be the set of positive integers, and let $\mathbb{P}^2$ be partially ordered by
The permutation $5736241 \in S_7$ has its essential set in a string from southwest to northeast and is therefore vexillary by Proposition 2.2.

$(m_1, m_2) \leq (m'_1, m'_2)$ if and only if $m_1 \leq m'_1$ and $m_2 \leq m'_2$. To stress the analogy with permutation matrices, we shall refer to the elements of $\mathbb{P}^2$ as squares.

A finite set $B$ of squares in $\mathbb{P}^2$ is called a shape. The slice $S_{i,m}$ of $B$ is the set of all squares in $B$ such that the $i$th coordinate is $m$. For example, if $B$ is a matrix, then $S_{1,m}$ is the $m$th row and $S_{2,m}$ is the $m$th column.

A subset $w$ of a shape $B$ is a proper dotting if it contains at most one square of every slice of $B$. The rank of a square $c$ in $B$ with a proper dotting $w$ is

$$r_w(c) \text{ def } |\{c' \in w : c' \leq c\}|.$$  


Let $B_0$ be a copy of $B$ with labels. We shall obtain a proper dotting $w$ of $B$ while constructing a finite sequence of labeled shapes $B_0, B_1, B_2, \ldots$, such that all labels of every $B_k$ will agree with the ranks given by the restriction of the final dotting $w$ to the subshape $B_k$.

Odd step $2i-1$. The labeled shape $B_{2i-1}$ is obtained from $B_{2i-2}$ by removing every square $c$ such that $c < c'$ where $c' B_{2i-2}$ is a square labeled zero. In these squares, there cannot be any dot in $w$.

Even step $2i$. After the previous step, $B_{2i-1}$ has no square labeled zero. Put a dot in $w$ in every minimal square (in the partial ordering) of $B_{2i-1}$. To obtain $B_{2i}$, we now, for every such minimal square $d$, first decrease the label by one for every labeled square $c \geq d$ and then remove from $B_{2i-1}$ both slices containing $d$. This makes sure that the dotting will be proper, and the labels will take into account that some dots have been removed. If any label becomes negative during the relabeling, then the algorithm halts with no output.
FIGURE 3

A run of the algorithm on the essential set of the permutation in Figure 1: $B_0$, $B_1$, $B_2$, and $B_3$ are depicted, with the revealed dots indicated. $B_4$ will then be a single square that must have a dot.

We will show that this algorithm is very flexible and is connected with another construction in algebraic combinatorics, namely, Viennot's shadowing procedure [13] (which has an interesting “matrix ball” generalization by Fulton in [8]). Viennot's procedure gives a geometric description of the Robinson-Schensted (R-S) correspondence and works as follows. Let $w$ be the proper set of dots. Do the shading as we have done before, shading every row and column from the dot and onwards. Then, the first Viennot layer $V_1$ of dots consists of the dots that are not separated from $(0,0)$ by any shaded slices. If this first layer of dots and all their slices are removed, then we see a second Viennot layer $V_2$, etc.

A formal, recursive, definition of the Viennot layers is

$$
V_i = \left\{ d \in w \setminus \bigcup_{j < i} V_j \mid d > d' \in w \Rightarrow d' \in \bigcup_{j < i} V_j \right\}, \quad \text{for } i = 1, 2, 3, \ldots.
$$

In Viennot's context, the shape is a permutation matrix, and the connection with the Robinson-Schensted correspondence is that the row number (resp., column number) for the westmost (northmost) dot in Viennot layer number $i$ gives the value in position $i$ in the first row of the first (second) tableaux in the R-S correspondence. However, for our purposes we will allow any proper dotting of any arbitrary shape, and we are only interested in which dots are in each Viennot layer. We shall see that the algorithm reveals the dots one Viennot layer at a time.

Next, we define the core of a given properly dotted shape $B$. The $i$th white filling, $W_i$, is the union of connected components of white squares between the two Viennot layers $V_{i-1}$ and $V_i$ (we let $V_0$ denote $\{(0,0)\}$):

$$
W_i = \{ \text{white squares } c \mid \exists d \in V_{i-1} : d < c, \nexists d' \in V_i : d' < c \}, \quad \text{for } i = 1, 2, 3, \ldots.
$$

The core of the dotting of $B$ is

$$
\bigcup_i \{ c \in W_i \mid c' > c \Rightarrow c' \notin W_i \},
$$

the set of maximal squares of the white fillings.
Example. The permutation $4271635 \in S_7$ shown in Figure 1 has core $(3,1)$, $(1,3)$, $(3,6)$, $(5,5)$. We see also in Figure 3 that the core is enough to run the algorithm. For the permutation $5736241 \in S_7$ in Figure 2, the core is equal to the essential set. One can prove that this is true for all vexillary permutations.

**Lemma 3.1.** For a permutation matrix, the core is contained in the essential set.

**Proof.** A maximal element of a white filling is necessarily a maximal element of its connected component of white squares, and hence a member of the essential set. ■

**Theorem 3.2.** Suppose the algorithm produces a dotting $w$ for a given input shape $B$ with labeled subset $E$, such that the ranks of squares in $E$ agree with the labels. Then $w$ is the unique such proper dotting with its core contained in $E$. If the algorithm fails to produce a dotting or produces a dotting whose ranks do not agree with the labels in $E$, then there is no such dotting.

**Proof.** Suppose there is some proper dotting $w$ such that the ranks of squares in $E$ agree with the labels, and such that the core of $w$ is contained in $E$. It is then clear by induction that the algorithm will find this $w$: The odd step $2i - 1$ will remove $W_i$, the $i$th white filling of $w$, its maximal elements being labeled zero at this point. The even step $2i$ will reveal $V_i$, the $i$th Viennot layer of $w$, and remove it and its shaded slices and adjust the labels so that they agree with the current ranks as determined by the remaining dots. ■

Thanks to this theorem and the lemma above, we have immediately the following strengthening of Fulton’s result that a permutation is determined by its ranked essential set.

**Corollary 3.3.** A permutation matrix is determined by the restriction of its rank function to its core (and hence in particular to its essential set). □

This is a significant strengthening, in the sense that the core in general is much smaller than the essential set. In another article [5], the authors have shown that the average size of the essential set is $n^2/36$. The core, though, is of size less than $n$. Say that a square $(i,j)$ is between two other squares $(i_1, j_1)$ and $(i_2, j_2)$ if $i_1 < i < i_2$ and $j < j_1 < j_2$, and say that two dots in a Viennot layer are neighboring if no other dot in the same layer is between them.

**Lemma 3.4.** A permutation matrix has at most one member of the core for every pair of neighboring dots in a Viennot layer. Therefore, the size of the core is at most $n$ minus the number of (nonempty) Viennot layers.

**Proof.** Let $(q_1, p_1), (q_2, p_2), \ldots, (q_k, p_k)$ be the dots of the $i$th Viennot layer $V_i$ in order of increasing $q$’s and decreasing $p$’s. If you draw a picture, it is obvious that every square in the $i$th white filling $W_i$ must be, in the poset sense, less than or equal to at least one of the (not necessarily white) squares $(q_2 - 1, p_1 - 1)$,
(q_3 - 1, p_2 - 1), \ldots, (q_k - 1, p_{k-1} - 1). By definition of the white filling, there cannot be any square in W_i in a row or column where there is a dot of an earlierViennot layer V_l, l < j, since any white square before the dot must belong to some earlier white filling, and all squares after the dot are shaded. For every j = 1, \ldots, k - 1, let q'_j and p'_j be the largest possible in the intervals q_j \leq q'_j < q_{j+1} and p_{j+1} \leq p'_j < p_j such that no dot of any earlier Viennot layer lies in row q'_j or column p'_j. The square (q'_j, p'_j) must belong to some white filling W_l for l \leq i, and they have been chosen maximal. The core members of the W_i are its maximal elements, hence, precisely those among the k - 1 squares (q'_1, p'_1), \ldots, (q'_{k-1}, p'_{k-1}) that belong to W_i. Summation over all Viennot layers proves the lemma.

4. The chess theorem. The origin of this article is a question posed twice by Fulton in [7] with a hope that the answer would help to extend his determinantal formula for degeneracy loci from the vexillary case to general permutations: Is there a characterization of which ranked essential sets arise from arbitrary permutations, in analogy with the characterization of ranked essential sets in the case of vexillary permutations?

By using the retrieval algorithm, one can prove such a result.

4.1. Fulton's question and the chess theorem. Fulton characterized the ranked essential sets of vexillary permutations (Proposition 2.2) by the conditions (C1), (C2), and (V). By replacing (V) with a new condition (C3) below, we obtain a characterization of ranked essential sets of arbitrary permutations. We have nick-named our result the chess theorem, for CHaracterization of ranked ESSential sets, inspired by the chessboard-like setting with white squares and dark squares.

**Theorem 4.1 (Chess theorem).** Let E \subseteq \{1, n\} x \{1, n\} be a set of squares with integer labels r(i, j). E is the essential set with rank function r of an n-by-n permutation matrix if and only if:

(C1) For each (i, j) \in E, we have
   (a) \min\{i, j\} > r(i, j) > 0, and
   (b) r(i, j) + n \geq i + j.

(C2) For every distinct pair (i, j), (i', j') \in E such that i \geq i', j \leq j' and E \cap [i', i] x [j, j'] = \{(i, j), (i', j')\}, we have
   \[i - i' > r(i, j) - r(i', j') > j - j'.\]

(C3) For every pair (i, j), (i', j') \in E such that i < i', j < j' and E \cap [i + 1, i'] x [j + 1, j'] = \{(i', j')\}, and r(i'', j''), r(i''', j''') as defined below, we have
   \[r(i', j') \geq i - i'' + j - j''' + r(i'', j'') + r(i''', j''') - r(i, j).\]

In condition (C3), let (i'', j'') be the square of E with the largest i'' satisfying i'' \leq i, j'' \geq j' and E \cap [i'', i'] x [j'', j'] = \{(i'', j'')\} (if no such square exists, set
The conditions in the "chess theorem" are equivalent to lower bounds on how many dots there must be in certain areas. The numbers in the figure are these lower bounds.

\[ r(i'', j'') = 0; \] symmetrically, let \((i'''', j'''')\) be the square of \(E\) with the largest \(j'''\) satisfying \(j''' \leq j\), \(i''' \geq i'\) and \(E \cap [i', i'''] \times [j'''', j] = \{(i'''', j'''')\}\) (if no such square exists, set \(r(i'''', j'''') = 0)\).

**Proof (Sketch).** The proof can be found in its entirety in Linusson’s Ph.D. thesis [10]. It is several pages long, and the technicalities are not very enlightening, so let us here restrict ourselves to a brief outline.

To prove necessity, the conditions in the "chess theorem" can be described as lower bounds on how many dots there must be in certain regions of the permutation matrix where the strict inequalities mean at least one dot in the region and the nonstrict inequalities mean that there is a nonnegative number in the region. See Figure 4. For (C1) the interpretation is as follows: \(r(i, j) > 0\) means that there are at least zero dots in the northwest area; \(i > r(i, j)\) means that at least one of the northern rows has its dot in the northeast area; the same is true for \(j > r(i, j)\) and the southwest area; and finally, \(r(i, j) + n - (i + j)\) counts the number of dots in the southeast area. Conditions (C2) and (C3) are interpreted in a resembling way.

Verifying that the conditions are necessary is the easy direction. The proof of sufficiency is done by showing that the conditions are enough to make it possible to run the algorithm and obtain a number of dots in the matrix, which is then shown to constitute a permutation matrix with \(E\) as essential set.

Fulton also considers the more general situation when the dotting does not have full rank and the matrix might be a rectangle. The full proof of Theorem 4.1 also shows that this case is covered if we replace (C1)(b) by condition (R) below.
COROLLARY 4.2. Given \( k \leq m, n \), let \( E \subseteq [1, m] \times [1, n] \) be a set of squares with integer labels \( r(i, j) \). \( E \) is the essential set with rank function \( r \) of a properly dotted \( m \times n \) rectangle with at least \( k \) dots if and only if conditions (C1)(a), (C2), and (C3) of Theorem 4.1, as well as (R) below, all are fulfilled.

(R) For every square \( (i, j) \in E \) we have \( n + m - k + r(i, j) \geq i + j \).

Note that the inequalities in conditions (C1)(a), (C2), and (C3) are not dependent of \( k \). Hence the number of dots in the \( m \times n \) rectangle is the largest number \( k \leq m, n \) satisfying (R).

4.2. A related question. The following related question was suggested by the referee: Given a function \( r \) on an arbitrary subset \( E \) of \([1, m] \times [1, n]\), when does this \( r \) determine an irreducible locus in the corresponding space of matrices? Equivalently, when does there exist a subset \( E' \subseteq E \) such that restriction of \( r \) to \( E' \) is the ranked essential set of a permutation and the other rank conditions on \( E \) are implied by those on \( E' \)?

It seems to be a difficult task to find a characterization of such sets, but the retrieval algorithm supplies an easy way to answer the question for any given specific example. Take the ranked set \( E \) as input to the retrieval algorithm. Thanks to Theorem 3.2 and Lemma 3.1, we know that if \( E \) indeed contains the ranked essential set of some permutation \( w \), then the algorithm will produce \( w \) as output. So, testing a ranked set \( E \) is done by simply running the algorithm and then checking if the essential set of the dotting obtained is a subset of \( E \) and if the rank function is correct for all squares in \( E \).

5. Essential sets of certain classes of permutations. Certain classes of permutations can be characterized by the shape of the essential set. An easy example is provided by the alternating permutations, that is, \( w \) such that \( w(1) > w(2) < w(3) > w(4) \), etc. A permutation is alternating if and only if there are descents exactly at positions \( 1, 3, 5, \) etc. As observed by Fulton, there is a descent at position \( i \) if and only if there is a white corner in row \( i \); hence, a permutation \( w \in S_n \) is alternating if and only if there are white corners in all odd rows (except for the last one if \( n \) is odd), and nowhere else. See Figure 1.

5.1. Vexillary permutations. An important example of characterization by essential set is Fulton's description of the vexillary permutations as having no white corners \((i, j)\) and \((i', j')\) such that \( i < i' \) and \( j < j' \). His proof is algebraic in nature, but we would like to point out here that the result can be obtained elementarily from the alternative characterization of vexillary permutations as 2143-avoiding (see Macdonald [11]). The argument goes as follows.

Suppose \( w \) contains a 2143-pattern: \( i_1 < i_2 < i_3 < i_4 \) with \( w(i_2) < w(i_1) < w(i_4) < w(i_3) \). Then \((i_1, w(i_2))\), the unique square that is at the same time to the left of the first dot and above the second one, must be a white square. Thanks to the shading of these dots, there must be a white corner \((i, j)\) with \( i < i_2, j < w(i_1) \). Similarly, we must have a white square at \((i_3, w(i_4))\), and hence a white corner \((i', j')\) with \( i_3 < i', w(i_4) \leq j' \). In particular, we have \( i < i' \) and \( j < j' \). For the other
direction, just observe that a white corner \((i, j)\) implies the existence of dots
\[
(i_1 \leq i, w(i_1) = j + 1) \text{ and } (i_2 = i + 1, w(i_2) \leq j),
\]
while a white corner \((i', j')\) gives dots
\[
(i_3 = i', w(i_3) > j') \text{ and } (i_4 > i', w(i_4) = j'),
\]
forming a 2143-pattern.

In a similar way, one can prove some other connections between certain shapes of essential sets and certain familiar classes of permutations. All the proofs are rather straightforward verifications, so we have chosen to omit them.

By antianalogy with the vexillary case, let us define a permutation to be \textit{antivexillary} if it has no white corners \((i, j)\) and \((i', j')\) such that \(i' > i\) and \(j > j'\). Thus, an antivexillary permutation has its white corners spread in the northwest to southeast direction, while the vexillary permutations have their white corners spread in the southwest to northeast direction. The antivexillary permutations admit a surprising characterization in terms of forbidden patterns.

**Proposition 5.1.** A permutation \(w\) is antivexillary if and only if it is both 321-avoiding and 351624-avoiding.

A permutation \(w \in S_n\) is said to be a \textit{Baxter permutation} if for every \(1 \leq a < b < c < d \leq n\) we have

(\text{B1}) \ w(d) + 1 = w(a) \text{ and } w(c) > w(a),

(\text{B2}) \ w(a) + 1 = w(d) \text{ and } w(b) > w(d).

This class of permutations was first defined by Baxter [1] in his work on fix points in compositions of commuting functions. Baxter permutations have also been thoroughly studied from an enumerative point of view [3]. Here we give a new, very simple, characterization of Baxter permutations in terms of their essential set.

**Proposition 5.2.** A permutation is a Baxter permutation if and only if its essential set has at most one square in each row and column.

Alternating Baxter permutations are also frequently considered in the literature; see, for example, [4]. Combining the conditions for being alternating and being Baxter gives the following proposition.

**Proposition 5.3.** A permutation is an alternating Baxter permutation if and only if its essential set has at most one square in each odd row, and no square in the even rows.

Billey, Jockusch, and Stanley [2] obtained a "curious" enumerative consequence of their analysis of the Schubert polynomials of 321-avoiding permutations: the number of \(w \in S_n\) that are \textit{both vexillary and 321-avoiding} is
1 + 2(2^n - (n + 1)) - \binom{n+1}{3}. By characterizing these permutations in terms of the essential set, we can give an immediate interpretation of each of these terms.

**Proposition 5.4.** A permutation \( w \) is both 321-avoiding and 2143-avoiding (vexillary) if and only if it has all its white corners either in one single row or in one single column. 

The identity permutation is both 321-avoiding and vexillary, so it takes care of the first term, 1, in the expression \( 1 + 2(2^n - (n + 1)) - \binom{n+1}{3} \). All other permutations have at least one white corner. Having all white corners in one single row is equivalent to having exactly one descent. The number of permutations with one descent is easily seen to be \( 2^n - (n + 1) \): choose any subset of \([1, n]\) except for intervals \([1, k]\), \( k = 0, 1, \ldots, n \), and order it in increasing order, then continue with the complement in increasing order. By transposition, there are equally many permutations with all white corners in one single column, so this takes care of the second term, \( 2(2^n - (n + 1)) \). We must now subtract the number of permutations that have been added twice; they are those with only one white square all together. As is most easily seen from the picture (Figure 5), these are the permutations of the word-form

\[
1 \cdot \cdot \cdot i(j + 1) \cdot \cdot \cdot k(i + 1) \cdot \cdot \cdot j(k + 1) \cdot \cdot \cdot n.
\]

We can choose \( i < j < k \) arbitrarily in the interval \([0, n]\), so this takes care of the last term, \( \binom{n+1}{3} \), of the expression.

6. **The essential set in higher dimensions.** Recall Fulton's lemma from Section 2, which stated that (a) given a permutation \( w \), the ideal generated by certain minors defined by the rank \( r_w \) is generated by the subset of these minors stemming from \( \mathcal{E}(w) \); and (b) the permutation is determined by its ranked essential set. In Section 3, we strengthened the (b) part to read that the permutation is determined by its ranked core. We shall now see that both parts (a) and (b) hold also in higher dimensions in a natural way.

6.1. **Minors and the essential set in higher dimensions.** The basic objects will be the \( d \)-dimensional permutation matrices, by which we mean a dotting of \([1, n]^d\)
with \( n \) dots such that when any coordinate is fixed, there is exactly one way of giving values to the other coordinates to find a dot. Equivalently, the \( d \)-dimensional permutation matrices can be seen as the geometric description of a sequence of \( d - 1 \) permutations in \( S_n \). Each one of the \( d - 1 \) projections of the \( d \)-dimensional permutation matrix onto a two-dimensional coordinate plane that contains the first coordinate axis will be an ordinary permutation matrix, and conversely, every such ordered set of \( d - 1 \) ordinary permutation matrices from \( S_n \) determine a unique \( d \)-dimensional permutation matrix. Hence we can present a \( d \)-dimensional permutation \( w \) as \( w = (w_1, w_2, \ldots, w_{d-1}) \in S_{n}^{d-1} \).

Accordingly, we define the \( d \)-determinant of a hypercubic \( d \)-dimensional matrix \( M = (m_{i_1, \ldots, i_d}) \) of size \( n \) to be

\[
\sum_{w \in S_{n}^{d-1}} \prod_{j=1}^{n} m_{j, w(j), \ldots, w_{d-1}(j)} \cdot \text{sgn}(w),
\]

where

\[
\text{sgn}(w) = \prod_{i=1}^{d-1} \text{sgn}(w_i).
\]

So, now we have determinants, and hence minors, in higher dimensions. Observe that the above notion of a higher-dimensional determinant is not the same as the one proposed in the recent book by Gelfand, Kapranov, and Zelevinsky [9]. However, in their book they mention our definition as having been studied during the second half of the nineteenth century, for which they refer to Ernst Pascal's *Die Determinanten* [12] from 1900, and further references therein.

Let the rank of a \( d \)-dimensional matrix be the size of its biggest nonvanishing minor. For a \( d \)-dimensional permutation \( w \), let \( V_w \) be the algebraic variety of hypercubic \( d \)-dimensional matrices \( A \) such that the rank of each submatrix \( A([1, i_1] \times [1, i_2] \times \cdots \times [1, i_d]) \) is at most equal to the rank of the corresponding submatrix of \( w \). Thus, \( V_w \) can be described as the zeros of a polynomial ideal generated by minors in a generic matrix of sizes given by the rank function of \( w \). Our goal, in the spirit of Fulton's lemma, is a result stating that this minor ideal is indeed generated by the minors coming from the essential set of \( w \), so we now need a definition of essential set in higher dimensions.

Let \( \mathbb{P}^d \), for \( d \geq 2 \), be partially ordered by \((m_1, \ldots, m_d) \leq (m'_1, \ldots, m'_d)\) if and only if \( m_i \leq m'_i \) for all \( i \in [1, d] \). For simplicity, we still refer to the elements of \( \mathbb{P}^d \) as "squares." As before, a shape is a finite set \( B \) of squares in \( \mathbb{P}^d \), and the slice \( S_{i, m} \) of \( B \) is the set of all squares in \( B \) such that the \( i \)th coordinate is \( m \). A dotted subset \( w \) of a shape \( B \) is a proper dotting if it contains at most one square of every slice of \( B \). The rank of a square \( c \) in \( B \) with a proper dotting \( w \) is

\[
r_w(c) \overset{\text{def}}{=} |\{c' \in w : c' \leq c \}|.
\]
Clearly, this agrees with the rank function for \(d\)-dimensional matrices, if dots are taken as ones and empty squares as zeros.

It should be obvious that the situation of \(d\)-dimensional permutation matrices is the special case where the shape is \([1, n]^d\) and the number of dots in \(w\) is \(n\), which is maximal for a proper dotting. To complete the analogy, we define the essential set of a \(d\)-dimensional permutation matrix \(w\) by shading, for every dotted square \(c\), every square \(c' \geq c\) such that \(c\) and \(c'\) belong to a common slice; now the essential set \(\mathcal{E}(w)\) is the set of “white corners,” that is, the maximal elements of the connected components of white squares. Clearly, the rank function is constant on connected components of white squares.

**Proposition 6.1.** For any \(w\) in \(S_{n}^{d-1}\) and a generic \(d\)-dimensional hypercubic matrix \(A\) of size \(n\), the ideal generated by all minors of size \(r_w(i_1, \ldots, i_d) + 1\) taken from the lowest (in poset sense) \(i_1 \times \cdots \times i_d\) submatrix of \(A\), for all \((i_1, \ldots, i_d) \in [1, n]^d\), is generated by these same minors using only those \((i_1, \ldots, i_d)\) that are in \(\mathcal{E}(w)\).

**Proof.** We will show that the minors that come from any square that is not a white corner lie in the ideal generated by the minors that come from white corners. This is easily seen to be true for a white square \(c\): if \(c\) is not a white corner, then there is some white corner \(c' \geq c\) in the same connected component of white squares, hence with the same rank \(r\), and of course all minors of size \(r + 1\) in the submatrix bounded by \(c\) are included among the minors of size \(r + 1\) in the submatrix bounded by \(c'\).

For a shaded square \(c\), there is a dot shading it somewhere, so if we regard as neighbors also neighboring positions outside the matrix (and let them have rank zero), there must always be at least one direction in which the neighbor \(c' \leq c\) has rank one less, \(r = r_w(c') = r_w(c) - 1\). Since \(c'\) is covered by \(c\), any minor of size \(r + 2\) in the submatrix bounded by \(c\) is spanned by minors of size \(r + 1\) in the submatrix bounded by \(c'\). If \(c'\) is a white square, we are done. If \(c'\) is outside the matrix, then \(c\) makes no contribution to the minors, so we are done. And if \(c'\) is shaded, we proceed in the same manner to find a neighbor \(c''\), etc., and eventually we must reach either a white square or a position outside the matrix.

In the two-dimensional case, Fulton is able to verify nice properties of this variety by reducing to Schubert varieties via a natural smooth surjection from \(\text{GL}_n\) to the flag manifold \(Fl(n)\). We do not know if anything analogous can be done in higher dimensions.

In higher dimensions, the permutation matrices do not realize all rank distributions (with the above definition of determinant) possible for general matrices. There is some intriguing mathematics going on here, which is studied in another paper by the same authors [6].

**6.2. The retrieval algorithm in higher dimensions.** With the definitions of higher-dimensional concepts above, we can take the description of the algorithm \textit{verbatim} from Section 3. Indeed, also the definitions of Viennot layer, white fill-
ing, and core generalize to higher dimensions in an obvious way. Hence, all arguments carry through for the higher-dimensional case.

**Proposition 6.2.** (a) For a d-dimensional permutation matrix, the core is contained in the essential set.

(b) Suppose the algorithm produces a dotting \( w \) for a given input d-dimensional shape \( B \) with labeled subset \( E \), such that the ranks of squares in \( E \) agree with the labels. Then \( w \) is the unique such proper dotting with its core contained in \( E \). If the algorithm fails to produce such a dotting, then there is no such dotting.

(c) A d-dimensional permutation matrix is determined by the restriction of its rank function to its core (and hence in particular to its essential set).

7. Remarks and open problems. (1) Billey, Jockusch, and Stanley [2] defined a class of “heroic” permutations, whose Schubert polynomials have a nice combinatorial property. The heroic permutations are known to include all the vexillary ones. Is there possibly a characterization of the heroic permutations in terms of the essential set?

(2) For which permutations is the core equal to the essential set (instead of being a proper subset)? We noted after Lemma 3.1 that all vexillary permutations have this property, but there are also some permutations of this kind, e.g., 2143, that are not vexillary. On the other hand, we know that, for large \( n \), the essential set must be larger than the core for most permutations in \( S_n \), since the average size of the essential set is \( n^2/36 \) while the maximal size of the core is \( n-1 \).

(3) In [7], Fulton used the characterization of the essential set of vexillary permutations to give an expression for the Schubert polynomial of a vexillary permutation. This was done via a formula for the type of the permutation. Is it possible to describe the Schubert polynomial or at least the type of a permutation in terms of its ranked essential set in general, using the characterization given here?

(4) As we saw in Section 6.2, the fact that a permutation is determined by the restriction of its rank function to its core generalizes to higher dimensions. However, the argument for the linear upper bound on the size of the core in dimension two (Lemma 3.4) does not work in higher dimensions. How big can the core be in general?

(5) A remark on the definition of the higher-dimensional determinant: We have chosen the most natural definition (in our opinion) of \( \text{sgn}(w) \). There are other possible definitions and the results in Section 6.1 are, as a matter of fact, true for all definitions of \( \text{sgn}(w) \) that do not take the value zero.

**References**


Eriksson: Department of Mathematics, Stockholm University, S-106 91 Stockholm, Sweden; kimm@nada.kth.se

Linusson: Department of Mathematics, KTH, S-100 44 Stockholm, Sweden; linusson@math.kth.se