SCHUBERT POLYNOMIALS, THE BRUHAT ORDER, AND THE GEOMETRY OF FLAG MANIFOLDS

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To the memory of Marcel Paul Schützenberger

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Extending work of Demazure [14] and of Bernstein, Gelfand, and Gelfand [7], Lascoux and Schützenberger [28] defined remarkable polynomial representatives for Schubert classes in the cohomology of a flag manifold, called Schubert polynomials. For each permutation $w$ in $S_\infty$, there is a Schubert polynomial $\mathfrak{S}_w \in \mathbb{Z}[x_1, x_2, \ldots]$. Schubert polynomials form an additive basis for this ring. Thus the identity

$$\mathfrak{S}_u \cdot \mathfrak{S}_v = \sum_w c_{uw}^v \mathfrak{S}_w$$

defines integral structure constants $c_{uw}^v$ for the ring of polynomials with respect to its Schubert basis. The $c_{uw}^v$ are nonnegative. They enumerate flags in a suitable triple intersection of Schubert varieties. Evaluating a Schubert polynomial at certain Chern classes gives a Schubert class in the cohomology of the flag manifold. This exhibits the cohomology of the flag manifold [10] as

$$\mathbb{Z}[x_1, x_2, \ldots]/\langle \mathfrak{S}_w \mid w \notin S_\infty \rangle.$$ 

It remains an open problem to give a bijective formula for these constants. We expect such a formula will have the form

$$c_{uw}^v = \#\left\{ \text{(saturated) chains in the Bruhat order on } S_\infty \text{ from } u \text{ to } w \text{ satisfying some condition imposed by } v \right\}. \quad (1)$$

Since every Schur symmetric polynomial $S_i(x_1, \ldots, x_k)$ is a Schubert polynomial, this would generalize the Littlewood-Richardson rule [34] (cf. Section 6.1), as Young tableaux are chains in Young’s lattice, a suborder of the Bruhat order. A new proof of Pieri’s formula for Grassmannians [51] suggests a geometric rationale for such “chain-theoretic” formulas. Finally, known formulas for the $c_{uw}^v$ are all of this form. These include Monk’s formula [37], Pieri’s formulas [12], [25], [28], [39], [50], [55], and other formulas of [50].

Here, we illuminate this relation between the Bruhat order and the $c_{uw}^v$, refining (1) and proving many new identities among the $c_{uw}^v$. This enables us to give a description of the form (1) for some $c_{uw}^v$, to compute many more, and to obtain new results about the enumeration of chains in the Bruhat order. Many of these identities have a companion result about the Bruhat order that should imply the identity, were such a formula as (1) known. In fact, they and the Pieri-type formula imply the identities (see [6]). Our combinatorial analysis leads to a new partial order on $S_\infty$ that contains Young’s lattice. We also compute the effect of many specializations of the variables in Schubert polynomials.

Algebraic structures in the cohomology of a flag manifold yield identities among the $c_{uw}^v$, such as $c_{uw}^v = c_{uw}^{w^\circ}$ (commutativity) or $c_{uw}^v = c_{u\circ w v}^{\omega_0 w}$, where $\bar{w} := \omega_0 w \omega_0$ (Poincaré duality). Similar identities for the Littlewood-Richardson
coefficients have been studied combinatorially (see [1], [2], [21], [22], [56]). We expect the identities established here will lead to some beautiful combinatorics, once a combinatorial interpretation for the $c_{w}^{u}$ is known. These identities impose stringent conditions on the form of any combinatorial interpretation and should be useful in finding such an interpretation.

This paper is organized as follows. In Section 1 we describe our results. Section 2 contains necessary background. Section 3 contains most of our combinatorial analysis. In Section 4, we study the effect on cohomology of certain maps between flag manifolds and compute specializations of the variables in a Schubert polynomial. In Section 5, we prove the identities when $\mathfrak{S}_{v}$ is a Schur polynomial. In Section 6, we use these identities to compute many of the $c_{w}^{v}$.

1. Summary

1.1. Suborders of the Bruhat order and the $c_{w}^{v}$. The identity

$$\mathfrak{S}_{u} \cdot S_{\lambda}(x_1, \ldots, x_k) = \sum_{w} c_{u,v(\lambda,k)}^{w} \mathfrak{S}_{w}$$

(1.1.1)

defines integer constants $c_{u,v(\lambda,k)}^{w}$, which share many properties with the Littlewood-Richardson coefficients. They are related to chains in the $k$-Bruhat order, $\leq_{k}$, a suborder of the Bruhat order. Its covers coincide with the index of summation in Monk’s formula [37]:

$$\mathfrak{S}_{u} \cdot \mathfrak{S}_{(k,k+1)} = \mathfrak{S}_{u} \cdot (x_1 + \cdots + x_k) = \sum \mathfrak{S}_{u(a,b)},$$

where the sum is over those $a \leq k < b$ with $\ell(u(a,b)) = \ell(u) + 1$. Young’s lattice of partitions with at most $k$ parts is isomorphic to those permutations comparable to the identity in the $k$-Bruhat order. These are the Grassmannian permutations with descent $k$, whose Schubert polynomials are Schur polynomials in $x_1, \ldots, x_k$. If $f^{\lambda}$ counts the standard Young tableaux of shape $\lambda$, then [36, I.5, Example 2]

$$(x_1 + \cdots + x_k)^m = \sum_{\lambda \vdash m} f^{\lambda} S_{\lambda}(x_1, \ldots, x_k).$$

Considering the coefficient of $\mathfrak{S}_{w}$ in the product $\mathfrak{S}_{u} \cdot (x_1 + \cdots + x_k)^m$ and the definition (1.1.1) of $c_{u,v(\lambda,k)}^{w}$, we obtain the following proposition.

PROPOSITION 1.1.1. The number of chains in the $k$-Bruhat order from $u$ to $w$ is

$$\sum_{\lambda} f^{\lambda} c_{u,v(\lambda,k)}^{w}.$$

In particular, $c_{u,v(\lambda,k)}^{w} = 0$ unless $u \leq_{k} w$. A chain-theoretic description of the constants $c_{u,v(\lambda,k)}^{w}$ should provide a bijective proof of Proposition 1.1.1. By this we
mean a function $\tau$ from the set of chains in $[u, w]_k$ to the set of standard Young tableaux $T$ whose shape is a partition $\lambda$ of $\ell(w) - \ell(u)$ such that $\# \tau^{-1}(T) = c_{w(u)}^{w}(\lambda, k)$. Schensted insertion [47] furnishes a proof [53] for the Littlewood-Richardson coefficients (cf. Section 6.1), as does Schützenberger’s *jeu de taquin* [49]. We show (Theorem 6.3.1) that if $\tau$ is a function where $\# \tau^{-1}(T)$ depends only upon the shape of $T$ and satisfies a condition of compatibility with the Pieri formula, then $\# \tau^{-1}(T) = c_{w(u)}^{w}(\lambda, k)$.

In Section 3.1 we give a nonrecursive description of the $k$-Bruhat order.

**Theorem A.** Let $u, w \in S_\infty$. Then $u \leq_k w$ if and only if

1. $a < k < b$ implies $u(a) < w(a)$ and $u(b) > w(b)$;
2. if $a < b$, $u(a) < u(b)$, and $w(a) > w(b)$, then $a < k < b$.

We generalize Proposition 1.1.1 and refine (1). Let $P$ be a *parabolic subgroup* of $S_\infty$, so $P$ is generated by some adjacent transpositions, $(i, i+1)$. Define the *$P$-Bruhat order* by its covers. A cover $u <_P w$ in the $P$-Bruhat order is a cover in the Bruhat order where $u^{-1}w \not\in P$. When $P$ is generated by all adjacent transpositions except $(k, k+1)$, this is the $k$-Bruhat order.

Let $I \subset \{1, 2, \ldots, n-1\}$ index the adjacent transpositions not in $P$. A *coloured chain* in the $P$-Bruhat order is a chain together with an element of $I \subset \{a, a+1, \ldots, b-1\}$ for each cover $u <_P u(a, b)$ in the chain [30]. Iterating Monk’s rule, we obtain

$$
(\sum_{i \in I} \mathbb{S}_{(i, i+1)})^m = \sum_{v: \ell(v) = m} f_{v}^{u}(P) \mathbb{S}_{v},
$$

where $f_{v}^{u}(P)$ counts the coloured chains in the $P$-Bruhat order from $e$ to $v$. Necessarily, $f_{v}^{u}(P) \neq 0$, only if $v$ is minimal in $vP$. More generally, let $f_{w}^{v}(P)$ count the coloured chains in the $P$-Bruhat order from $u$ to $w$. Multiplying (1.1.2) by $\mathbb{S}_{u}$ and equating coefficients of $\mathbb{S}_{w}$ gives a generalization of Proposition 1.1.1, as follows.

**Theorem B.** Let $u, w \in S_\infty$ and $P \subset S_\infty$ be a parabolic subgroup. Then

$$
f_{w}^{u}(P) = \sum_{v} c_{uvw}^{w} f_{v}^{u}(P).$$

Hence if $v$ is minimal in $vP$, then $c_{uvw}^{w} = 0$ unless $u \leq_P w$. This suggests a refinement of (1). Let $u, v, w \in S_\infty$, and let $P$ be any parabolic subgroup such that $v$ is minimal in $vP$. Then, for every coloured chain $\gamma$ in the $P$-Bruhat order from $e$ to $v$, we expect that

$$
c_{uvw}^{w} = \# \left\{ \text{coloured chains in the } P\text{-Bruhat order on } S_\infty \text{ from } u \text{ to } w \text{ that satisfy some condition imposed by } \gamma \right\}.
$$

Moreover, this rule should give a bijective proof of Theorem B.
This $P$-Bruhat order may be defined for parabolic subgroups of any Coxeter group. Likewise, the problem of determining the structure constants for a Schubert basis also generalizes. For Weyl groups, this is the Schubert basis of cohomology for a generalized flag manifold $G/B$ or the analogues of Schubert polynomials (see [8], [17], [20], [45]). For finite Coxeter groups, this is the basis $\Delta_w$ in the coinvariant algebra [23]. Likewise, Theorem B and the expectation (1.1.3) have analogues. Of the known formulas (see [11], [24], [40], [42], [43], [44], [52] and the survey [41]), few have been expressed in a chain-theoretic manner (see [11], [24], [40], [52]).

1.2. Substitutions and the Schubert basis. In Sections 4.3 and 4.4, we study the $c_{w,v}^w$ when $w(p) = u(p)$ for some $p$. For $w \in S_{n+1}$ and $1 \leq p \leq n+1$, let $w/p \in S_n$ be defined by deleting the $p$th row and $w(p)$th column from the permutation matrix of $w$. If $y \in S_n$ and $1 \leq q \leq n+1$, then $e_{p,q}(y) \in S_{n+1}$ is the permutation such that $e_{p,q}(y)/p = y$ and $e_{p,q}(y)(p) = q$. The index of summation in a particular case of the Pieri formula (see [4], [28], [50]),

$$\mathbb{S}_v \cdot (x_1 \cdots x_{p-1}) = \sum \mathbb{S}_w,$$

defines the relation $v \overset{e_{p,q}}{\rightarrow} w$. Define $\Psi_p : \mathbb{Z}[x_1, x_2, \ldots] \rightarrow \mathbb{Z}[x_1, x_2, \ldots]$ by

$$\Psi_p(x_j) = \begin{cases} x_j & \text{if } j < p, \\ 0 & \text{if } j = p, \\ x_{j-1} & \text{if } j > p. \end{cases}$$

**Theorem C.** Let $u, w \in S_\infty$ and $p \in \mathbb{N}$.

(i) Suppose $w(p) = u(p)$ and $\ell(w) - \ell(u) = \ell(w/p) - \ell(u/p)$. Then

(a) $e_{p,u(p)} : [u/p, w/p] \overset{e_{p,q}}{\rightarrow} [u, w]$;

(b) for every $v \in S_\infty$,

$$c_{w,v}^w = \sum_{y \in S_\infty, v \overset{e_{p,q}}{\rightarrow} e_{p,1}(y) \circ y} c_{u/y}^{w/p};$$

(ii) For $v \in S_\infty$,

$$\Psi_p(\mathbb{S}_v) = \sum_{y \in S_\infty, v \overset{e_{p,q}}{\rightarrow} e_{p,1}(y)} \mathbb{S}_y.$$

We prove the first assertion (Lemma 4.1.1(ii)) using combinatorial arguments. The second (in Section 4.4) and third (in Section 4.3) are proven by computing
certain maps on cohomology. Since $c_{wv}^w = c_{wv}^w = \omega_v \omega_w\omega_v$, Theorem C(i)(b) gives
a recursion for $c_{wv}^w$ when one of $wv^{-1}, wv^{-1},$ or $\omega_v wv^{-1}$ has a fixed point and the
condition on lengths is satisfied.

We compute other substitutions of the variables. Let $P \subseteq \mathbb{N}$, and list the elements of $P$ and $\mathbb{N} - P$ in order as
\[ P : p_1 < p_2 < \cdots, \quad \mathbb{N} - P : p_1^* < p_2^* < \cdots. \]
Define $\Psi_P : \mathbb{Z}[x_1, x_2, \ldots] \to \mathbb{Z}[y_1, y_2, \ldots, z_1, z_2, \ldots]$ by
\[ \Psi_P(x_{p_j}) = y_j, \quad \Psi_P(x_{p_j}^*) = z_j. \]
In Remark 4.6.1, we define an infinite set $I_P$ of permutations with the following property.

**Theorem D.** For every $w \in \mathcal{S}_\infty$, there exists an integer $N$ such that if $\pi \in I_P$ and
$\pi \not\in \mathcal{S}_N$, then
\[ \Psi_P(\mathbb{S}_w) = \sum_{u, v} c_{wv}^{(u \times v)} \cdot \mathbb{S}_u(y) \mathbb{S}_v(z). \]

We prove this in Section 4.6. Theorem D gives infinitely many identities of the form $c_{wv}^{(u \times v)} = c_{v}^{(u \times v)} \sigma$ for $\pi, \sigma \in \mathcal{S}_P$. Moreover, for these $u, v, \pi$ with $c_{wv}^{(u \times v)} \neq 0$, we have $[\pi, (u \times v) \cdot \pi] \cong [e, u] \times [e, v]$, which suggests a chain-theoretic basis for
these identities.

A combinatorial proof of Theorem D may provide insight into the problem of
determining the $c_{wv}^w$. In particular, it would be interesting to find a proof using one of the combinatorial constructions of Schubert polynomials (see [3], [4],
[9], [18], [27], [54]). Theorem D extends 1.5 of [29], which shows that $\Psi_{[n]} \mathbb{S}_w$ is a nonnegative sum of $\mathbb{S}_u(y) \mathbb{S}_v(z)$. The special case of Theorem D when
$P = [n]$, together with the formula $c_{wv}^{(u \times v)} = c_{wv}^u \cdot c_{v}^x$ for $u, v, w \in \mathcal{S}_n$ and
t $x, y, z \in \mathcal{S}_\infty$, was established by Patras [38] using methods similar to ours. Also,
Lascoux and Schützenberger [29] give the special case when $P = \{1\}$.

We consider more general substitutions. Let $P_i := (P_0, P_1, \ldots)$ be any partition of $\mathbb{N}$. For $i > 0$, let $x^{(i)} := x_1^{(i)}, x_2^{(i)}, \ldots$ be variables in bijection with $P_i$. Define $\Psi_{P_i} : \mathbb{Z}[x_1, x_2, \ldots] \to \mathbb{Z}[x^{(1)}, x^{(2)}]$ by
\[ \Psi_{P_i}(x_j) = \begin{cases} 0 & \text{if } j \in P_0, \\ x_j^{(i)} & \text{if } j \text{ is the } i\text{th element of } P_i. \end{cases} \]

**Corollary 1.2.1.** For every partition $P_i$ of $\mathbb{N}$ and $w \in \mathcal{S}_\infty$,
\[ \Psi_{P_i}(\mathbb{S}_w(x)) = \sum_{u_1, u_2, \ldots} d_{w}^{u_1, u_2, \ldots}(P_i) \mathbb{S}_{u_1}(x^{(1)}) \mathbb{S}_{u_2}(x^{(2)}) \cdots, \]
where each $d_{w}^{u_1, u_2, \ldots}(P_i)$ is an (explicit) sum of products of the $c_{v}^{\pi}$.
A ballot sequence \( A = (a_1, a_2, \ldots) \) is a sequence of nonnegative integers where, for each \( i, j \geq 1 \),
\[
\# \{ k \leq j \mid a_k = i \} \geq \# \{ k \leq j \mid a_k = i + 1 \}.
\]
(Consider \( a_i = 0 \) as a vote for "none of the above." ) Given a ballot sequence \( A \), define \( \Psi_A : \mathbb{Z}[x_1, x_2, \ldots] \to \mathbb{Z}[x_1, x_2, \ldots] \) by
\[
\Psi_A(x_i) = \begin{cases} 
0 & a_i = 0, \\
x_{a_i} & a_i \neq 0.
\end{cases}
\]

**Corollary 1.2.2.** For every ballot sequence \( A \) and \( w \in S_n \), there exist nonnegative integers \( d_w^u(A) \) for \( u, w \in S_\infty \) such that
\[
\Psi_A(\mathbb{E}_w(x)) = \sum_u d_w^u(A) \mathbb{E}_u(x).
\]
Moreover, each \( d_w^u(A) \) is an (explicit) sum of products of the \( c_{v'y}^x \).

**Proof.** If \( P_0 := \{ i \mid a_i = 0 \} \) and for \( j > 0 \)
\[
P_j := \{ i \mid a_i \text{ is the } j\text{th occurrence of some integer in } A \},
\]
then \( \Psi_A = \Delta \circ \Psi_{(P_0, P_1, \ldots)} \), where \( \Delta(x_j^{(i)}) = x_j \).

**1.3. Identities when \( w \) is a Schur polynomial.** If \( \lambda, \mu, \) and \( \nu \) are partitions with at most \( k \) parts, then the Littlewood-Richardson coefficients \( c_{\nu\lambda}^\mu \) are defined by the identity
\[
S_\mu(x_1, \ldots, x_k) \cdot S_\lambda(x_1, \ldots, x_k) = \sum_{\nu} c_{\nu\lambda}^\mu S_\nu(x_1, \ldots, x_k).
\]
They depend only on \( \lambda \) and the skew partition \( \nu/\mu \). That is, if \( \kappa \) and \( \rho \) are partitions with \( \kappa/\rho = \nu/\mu \), then for all \( \lambda \),
\[
c_{\nu\lambda}^\mu = c_{\rho\lambda}^\kappa,
\]
and the coefficient of \( S_{\kappa}(x_1, \ldots, x_l) \) in \( S_\mu(x_1, \ldots, x_l) \cdot S_\lambda(x_1, \ldots, x_l) \) is \( c_{\rho\lambda}^\kappa \). The order type of the interval in Young’s lattice from \( \mu \) to \( \nu \) is determined by \( \nu/\mu \).

These facts hold also for the \( e_{\nu\lambda}^\mu \).

If \( u \leq k \), let \([u, w]_k \) be the interval between \( u \) and \( w \) in the \( k \)-Bruhat order. Permutations \( \zeta \) and \( \eta \) are shape equivalent if there exist sets of integers \( P = \{ p_1 < \cdots < p_n \} \) and \( Q = \{ q_1 < \cdots < q_n \} \), where \( \zeta \) (respectively, \( \eta \)) acts as the identity on \( \mathbb{N} - P \) (respectively, \( \mathbb{N} - Q \)), and
\[
\zeta(p_i) = p_j \iff \eta(q_i) = q_j.
\]
THEOREM E. Suppose \( u \leq_k w \) and \( x \leq_I z \), where \( wu^{-1} \) is shape equivalent to \( zx^{-1} \). Then the following statements hold.

(i) We have \([u, w]_k \simeq [x, z]_I\). When \( wu^{-1} = zx^{-1} \), this isomorphism is given by \( v \mapsto vu^{-1}x\).

(ii) For all partitions \( \lambda \), \( c^w_{uv(\lambda, k)} = c^x_{uv(\lambda, l)} \).

Part (i) follows from Theorems 3.1.3 and 3.2.3, which are proven using combinatorial arguments. Part (ii) is proven in Section 5.1 using geometric arguments.

By Theorem E, we may define the skew coefficient \( c^c_{\lambda} \) for \( \zeta \in \mathcal{S}_\infty \) and \( \lambda \) a partition by \( c^c_{\lambda} := c^u_{uv(\lambda, k)} \) and also define \( |\zeta| := \ell(\zeta u) - \ell(u) \) for any \( u \in \mathcal{S}_\infty \) with \( u \leq_k \zeta u \).

This leads (in Section 3.2) to a partial order \( \preceq \) on \( \mathcal{S}_\infty \) graded by \( |\zeta| \) with the following defining property. Let \([e, \zeta]_{\preceq} \) be the interval in the \( \preceq \)-order from the identity to \( \zeta \). If \( u \leq_k \zeta u \), then the map \([e, \zeta]_{\preceq} \to [u, \zeta u]_k \) defined by

\[ \eta \mapsto \eta u \]

is an order isomorphism. Then Proposition 1.1.1 states that \( \sum_{\lambda} f^\lambda c^c_{\lambda} \) counts the chains in \([e, \zeta]_{\preceq} \). This order is studied further in [5].

We express some of the \( c^c_{\lambda} \) in terms of chains in the Bruhat order. If \( u <_k u(a, b) \) is a cover in the \( k \)-Bruhat order, label that edge of the Hasse diagram with the integer \( u(b) \). The word of a chain in the \( k \)-Bruhat order is its sequence of edge labels.

THEOREM F. Suppose \( u \leq_k w \), and \( wu^{-1} \) is shape equivalent to \( v(\mu, l) \cdot v(v, l)^{-1} \) for some \( l \) and partitions \( \mu, v \). Then, for all partitions \( \lambda \) and standard Young tableaux \( T \) of shape \( \lambda \),

\[ c^w_{uv(\lambda, k)} = \#\left\{ \text{chains in } k\text{-Bruhat order from } u \text{ to } w \text{ whose word has recording tableau } T \text{ for Schensted insertion} \right\}. \]

Theorem F gives a combinatorial proof of Proposition 1.1.1 for many \( u, w \). It is proven in Section 6.1.

If a skew partition \( \theta = \rho \bigsqcup \sigma \) is the union of incomparable skew partitions \( \rho \) and \( \sigma \), then

\[ \rho \bigsqcup \sigma \simeq \rho \times \sigma \]

as graded posets. The skew Schur function \( S_\theta \) is defined [36, 1.5] to be \( \sum_{\lambda} c^\rho_{\lambda} S_\lambda \) and \( S_\rho \bigsqcup \sigma = S_\rho \cdot S_\sigma \) [36, 1.5.7]. Thus

\[ c^\rho_{\lambda} \bigsqcup \sigma = \sum_{\mu, \nu} c^\rho_{\mu} c^\nu_{\nu} c^{\sigma}_{\nu}. \quad (1.3.1) \]

Permutations \( \zeta \) and \( \eta \) are disjoint if \( \zeta \) and \( \eta \) have disjoint supports and \( |\zeta \eta| = |\zeta| + |\eta| \).
THEOREM G. Let \( \zeta \) and \( \eta \) be disjoint permutations. Then

(i) the map \( (\zeta', \eta') \mapsto \zeta'\eta' \) induces an isomorphism

\[
[e, \zeta] \times [e, \eta] \xrightarrow{\sim} [e, \zeta\eta] \;.
\]

(ii) for every partition \( \lambda \), \( c_{\lambda}^{\zeta, \eta} = \sum_{\mu, \nu} c_{\mu}^{\zeta} c_{\nu}^{\eta} \).

The first statement is proven in Section 3.3 using a characterization of disjointness (Lemma 3.3.1), and the second in Section 5.2 using geometry.

Our last identity has no analogy with the Littlewood-Richardson coefficients. The \( n \)-cycle \((12 \ldots n)\) cyclicly permutes \([n]\).

THEOREM H. Suppose \( \zeta \in S_n \) and \( \eta = \zeta(12 \ldots n) \). Then, for every partition \( \lambda \),

\[ c_{\lambda}^{\zeta} = c_{\lambda}^{\eta} \;.
\]

This is proven in Section 5.3 using geometry. Combined with Proposition 1.1.1, we obtain the following corollary.

COROLLARY 1.3.1. If \( u \leq_k w \) and \( x \leq_k z \) with \( wu^{-1}, zx^{-1} \in S_n \), and

\( (wu^{-1})^{(12 \ldots n)} = zx^{-1} \), then each of the two intervals \([u, w]_k \) and \([x, z]_k \) have the same number of chains.

These intervals \([u, w]_k \) and \([x, z]_k \) are typically nonisomorphic. For example, in \( S_4 \) let \( u = 1234 \), \( x = 2134 \), and \( v = 1324 \). If \( \zeta = (1243) \), \( \eta = (1423) = \zeta(1234) \), and \( \xi = (1342) = \eta(1234) \), then

\[ u \leq_2 \zeta u, \quad x \leq_2 \eta x, \quad \text{and} \quad v \leq_2 \xi v. \]

Figure 1 shows the intervals \([u, \zeta u]_2 \), \([x, \eta x]_2 \), and \([v, \xi v]_2 \).

\[ \begin{array}{c}
2413 \\
2134 \\
1324 \\
1234 \\
\hline
2413 \\
2134 \\
1324 \\
1234 \\
\hline
2413 \\
2134 \\
1324 \\
1234 \\
\hline
2413 \\
2134 \\
1324 \\
1234
\end{array} \]

FIGURE 1. Effect of cyclic shift on intervals

The theorems of this section, together with the “algebraic” identities \( c_{\lambda}^{w} = c_{\lambda}^{w(\zeta, k)} = c_{\lambda}^{w(\eta, k)} \), greatly reduce the number of distinct coefficients \( c_{u \zeta}^{w(\lambda, k)} \) from which all others may be determined. We indicate this for some small symmetric groups in the table below. The first row counts the number of \( c_{u \zeta}^{w(\lambda, k)} \) with \( u \leq k w \) and \( |\lambda| = \ell'(w) - \ell'(u) \) in \( S_n \), and the second counts those \( c_{\lambda}^{\zeta} \) from which all the \( c_{u \zeta}^{w(\lambda, k)} \)
may be determined using the results of this paper. For a discussion of this table, see http://www.math.yorku.ca/bergeron/coefficients.html.

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<th>$# c_{u(k,l)}^w$</th>
<th>$\mathcal{P}_4$</th>
<th>$\mathcal{P}_5$</th>
<th>$\mathcal{P}_6$</th>
<th>$\mathcal{P}_7$</th>
<th>$\mathcal{P}_8$</th>
</tr>
</thead>
<tbody>
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<td>208</td>
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<td>81669</td>
<td>2285414</td>
<td>79860923</td>
<td></td>
</tr>
<tr>
<td>$# c_{1}$</td>
<td>5</td>
<td>12</td>
<td>62</td>
<td>332</td>
<td>3267</td>
</tr>
</tbody>
</table>

2. Preliminaries

2.1. Permutations. Let $\mathcal{S}_n$ be the group of permutations of $[n] := \{1, 2, \ldots, n\}$. Let $(a, b)$ be the transposition interchanging $a < b$. The length $\ell(w)$ of $w \in \mathcal{S}_n$ counts the inversions $\{i < j \mid w(i) > w(j)\}$ of $w$. The Bruhat order $\leq$ on $\mathcal{S}_n$ is the partial order whose cover relation is $w \prec w(a, b)$ if $w(a) < w(b)$ and $\ell(w) + 1 = \ell(w(a, b))$. If $u \leq w$, let $[u, w] := \{v \mid u \leq v \leq w\}$ be the interval between $u$ and $w$ in $\mathcal{S}_n$, a poset graded by $\ell(v) - \ell(u)$. The longest element $\omega_0 \in \mathcal{S}_n$ is defined by $\omega_0(j) = n + 1 - j$. When it is necessary to consider the longest elements in several symmetric groups, we write $\omega_n$ for $\omega_0 \in \mathcal{S}_n$.

A permutation $w \in \mathcal{S}_n$ acts on $[n+1]$, fixing $n+1$. Thus $\mathcal{S}_n \subseteq \mathcal{S}_{n+1}$. Define $\mathcal{S}_\infty := \bigcup_1^n \mathcal{S}_n$. For $P = \{p_1 < p_2 < \cdots\} \subseteq \mathbb{N}$, define $\phi_P : \mathcal{S}_\# \rightarrow \mathcal{S}_\infty$ by requiring that $\phi_P$ act as the identity on $\mathbb{N} - P$ and $\phi_P(\zeta)(p_i) = p_\zeta(i)$. This injection does not preserve length unless $P = \{n + 1, n + 2, \ldots\}$. For this $P$, set $1^n \times w := \phi_P(w)$. If there exist permutations $\xi, \zeta, \eta$ and sets of positive integers $P, Q$ such that $\phi_P(\xi) = \zeta$ and $\phi_Q(\zeta) = \eta$, then $\xi$ and $\eta$ are shape equivalent.

2.2. Schubert polynomials. Lascoucx and Schützenberger invented and then developed the elementary theory of Schubert polynomials in a series of papers [28]–[33]. For a self-contained exposition of some of this elegant theory see [35].

$\mathcal{S}_n$ acts on polynomials in $x_1, \ldots, x_n$ by permuting the variables. For a polynomial $f, f - (i, i+1)f$ is antisymmetric in $x_i$ and $x_{i+1}$, and hence is divisible by $x_i - x_{i+1}$. Define the divided difference operator

$$\partial_i := (x_i - x_{i+1})^{-1}(e - (i, i+1)).$$

If $a_1, a_2, \ldots, a_{\ell(w)}$ is a reduced word for $w$, then $\partial_{a_1} \circ \cdots \circ \partial_{a_{\ell(w)}}$ depends only upon $w$, defining the operator $\partial_w$. For $w \in \mathcal{S}_n$, Lascoucx and Schützenberger [28] defined the Schubert polynomial $\mathfrak{S}_w$ by

$$\mathfrak{S}_w := \partial_{w^{-1}\omega_0}(x_1^{n-1}x_2^{n-2} \cdots x_{n-1}).$$
The polynomial $\mathcal{G}_w$ is homogeneous of degree $\ell(w)$, and it is independent of the choice of $n$. The set of all Schubert polynomials $\{\mathcal{G}_w \mid w \in S_n\}$ is an integral basis for $\mathbb{Z}[x_1, x_2, \ldots]$. 

A partition $\lambda$ is a decreasing sequence $\lambda_1 \geq \cdots \geq \lambda_k > 0$ of integers. Young's lattice is the set of partitions ordered by $\subseteq$, where $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all $i$. Write $m^l$ for the partition with $l$ parts, each of size $m$. If $\lambda_{k+1} = 0$, the Schur polynomial $S_{\lambda}(x_1, \ldots, x_k)$ is

$$S_{\lambda}(x_1, \ldots, x_k) := \frac{\det[x_j^{k-i+\lambda_i}]_{i,j=1}^k}{\det[x_j^{k-l}]_{i,j=1}^k},$$

which is symmetric and homogeneous of degree $|\lambda| := \lambda_1 + \cdots + \lambda_k$.

A permutation $w \in S_n$ is Grassmannian of descent $k$ if $j \neq k \Rightarrow w(j) < w(j+1)$. Then $w$ defines and is defined by a partition $\lambda$ with $\lambda_{k+1} = 0$:

$$\lambda_{k+1-j} = w(j) - j \quad j = 1, \ldots, k.$$  

(The condition $w(k+1) < w(k+2) < \cdots$ determines the remaining values of $w$.) In this case, write $w = v(\lambda, k)$. The raison d'être for this definition is that $\mathcal{G}_{v(\lambda, k)} = S_{\lambda}(x_1, \ldots, x_k)$. Thus the Schubert polynomials form a basis for $\mathbb{Z}[x_1, x_2, \ldots]$ that contains all Schur symmetric polynomials $S(\lambda_1, \ldots, \lambda_k)$ for all $\lambda$ and $k$.

2.3. The flag manifold. Let $V \cong \mathbb{C}^n$. A flag $F$, in $V$ is a sequence

$$\{0\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset F_n = V$$

of subspaces with $\dim F_i = i$. Flags $F$, and $F'$ are opposite if $F_{n-j} \cap F'_j = \{0\}$ for all $j$. The set of all flags is an $(\binom{n}{2})$-dimensional complex manifold $\mathbb{F}\ell V$ (or $\mathbb{F}\ell_n$), called the flag manifold. There is a tautological flag $\mathcal{F}$ of bundles over $\mathbb{F}\ell V$ whose fibre at $F$ is $F$. Let $-x_i$ be the first Chern class of the line bundle $\mathcal{F}_i / \mathcal{F}_{i-1}$. Borel [10] showed the cohomology ring of $\mathbb{F}\ell V$ is

$$\mathbb{Z}[x_1, \ldots, x_n] / \langle e_i(x_1, \ldots, x_n) \mid i = 1, \ldots, n \rangle,$$

where $e_i(x_1, \ldots, x_n)$ is the $i$th elementary symmetric polynomial in $x_1, \ldots, x_n$.

Let $\langle S \rangle$ be the linear span of $S \subset V$, and $U - W$ be the set-theoretic difference of subspaces $W \subset U$. An ordered basis $f_1, f_2, \ldots, f_n$ for $V$ determines a flag $E := \langle f_1, \ldots, f_n \rangle$, where $E_i = \langle f_1, \ldots, f_i \rangle$ for $1 \leq i \leq n$. A fixed flag $F$ gives a decomposition due to Ehresmann [16] of $\mathbb{F}\ell V$ into affine cells indexed by permutations $w$ of $S_n$. The cell determined by $w$ is

$$X_w^*: = \{ E_i = \langle f_1, \ldots, f_n \rangle \mid f_i \in F_{n+1-w(i)} - F_{n-w(i)}, 1 \leq i \leq n \}.$$

Its closure is the Schubert subvariety $X_w F$, which has codimension $\ell(w)$. Also,
$u \leq w \iff X_u F. \supset X_w F.$. The Schubert class $\mathcal{S}_w$ is the cohomology class Poincaré dual to the fundamental cycle of $X_w F.$.

These classes form a basis for cohomology. Schubert polynomials were defined so that $w(X_1, \ldots, X_n) = \mathcal{S}_w$.

If $F.$ and $F'$ are opposite flags, then $X_u F. \cap X_v F'$ is an irreducible, generically transverse intersection, a consequence of [15] (cf. [50, Section 5]). Thus its codimension is $\ell(u) + \ell(v)$, and the fundamental cycle of $X_u F. \cap X_v F'$ is Poincaré dual to $\mathcal{S}_u \cdot \mathcal{S}_v$. Since

$$\mathbb{Z}[x_1, \ldots, x_n] \longrightarrow \mathbb{Z}[x_1, \ldots, x_{n+m}] / \langle e_i(x_{n+m}) \rangle$$

is an isomorphism on $\mathbb{Z}<x_1^{a_1} \cdots x_n^{a_n} | a_i < m>$, identities of Schubert polynomials follow from product formulas for Schubert classes. The Schubert basis is self-dual. If $\ell'(w) + \ell'(v) = \binom{n}{2}$, then

$$\mathcal{S}_w \cdot \mathcal{S}_v = \begin{cases} \mathcal{S}_{\omega_0} & \text{if } v = \omega_0 w, \\ 0 & \text{otherwise}. \end{cases} \quad (2.3.1)$$

Let $\text{Grass}_k V$ be the Grassmannian of $k$-dimensional subspaces of $V$, a $(n-k)$-dimensional manifold. A flag $F.$ induces a cellular decomposition indexed by partitions $\lambda \subseteq (n-k)^k$. The closure of the cell indexed by $\lambda$ is the Schubert variety $\Omega_\lambda F.$:

$$\Omega_\lambda F. := \{ H \in \text{Grass}_k V \mid \dim H \cap F_{n+j-k-\lambda_j} \geq j, j = 1, \ldots, k \}.$$ 

The cohomology class Poincaré dual to the fundamental cycle of $\Omega_\lambda F.$ is $S_\lambda(x_1, \ldots, x_k)$, where $x_1, \ldots, x_k$ are negative Chern roots of the tautological $k$-plane bundle on $\text{Grass}_k V$. Write $S_\lambda$ for $S_\lambda(x_1, \ldots, x_k)$, if $k$ is understood. These Schubert classes form a basis for cohomology, $\mu \subseteq \lambda \iff \Omega_\mu F. \supset \Omega_\lambda F.$, and if $F., F'$ are opposite flags, then

$$[\Omega_\mu F. \cap \Omega_\nu F'] = [\Omega_\mu F.] \cdot [\Omega_\nu F'] = \sum_{\lambda \subseteq (n-k)^k} c^\lambda_{\mu, \nu} S_\lambda,$$

where the $c^\lambda_{\mu, \nu}$ are the Littlewood-Richardson coefficients [21].

This Schubert basis is self-dual. If $\lambda \subseteq (n-k)^k$, then let $\lambda^c$, the complement of $\lambda$, be the partition $(n-k-\lambda_k, \ldots, n-k-\lambda_1)$. Suppose $|\lambda| + |\mu| = k(n-k)$, then

$$S_\lambda(x_1, \ldots, x_k) \cdot S_\mu(x_1, \ldots, x_k) = \begin{cases} S_{(n-k)^k} & \text{if } \mu = \lambda^c, \\ 0 & \text{otherwise.} \end{cases}$$

We suppress the dependence of $\lambda^c$ on $n$ and $k$.

A map $f : X \rightarrow Y$ between manifolds induces a group homomorphism $f_* : H^* X \rightarrow H^* Y$ via Poincaré duality and the functorial map on homology.
This map satisfies the projection formula (cf. [19, 8.1.7]). Let \( \alpha \in H^*X \) and \( \beta \in H^*Y \), then
\[
f_*(f^* \alpha \cap \beta) = \alpha \cap f_* \beta.
\] (2.3.2)

For an (oriented) manifold \( X \) of dimension \( d \), \( H^dX = \mathbb{Z} \cdot [pt] \) is generated by the class of a point. Let \( \deg : H^*X \to \mathbb{Z} \) be the map that selects the coefficient of \([pt] \). Then \( \deg(f_* \beta) = \deg(\beta) \).

Let \( \pi_k : \mathbb{F} \ell V \to \text{Grass}_k V \) be defined by \( \pi_k(E_\cdot) = E_k \). Then \( \pi_k^{-1} \Omega_k F. = X_{\omega_0^0(\lambda^\ell, k)} F. \) and \( \pi_k : X_{\omega_0^0(\lambda^\ell, k)} F. \to \Omega_k F. \) is generically one-to-one. Thus
\[
\pi_k^* S_\lambda = \mathbb{S}_v(\lambda, k)
\]
\[
(\pi_k)_* \mathbb{S}_w = \begin{cases} 
S_\lambda & \text{if } w = \omega_0^0(\lambda^\ell, k), \\
0 & \text{otherwise}.
\end{cases}
\]

The cohomology of \( \mathbb{F} \ell V \times \mathbb{F} \ell W \) (\( \dim W = m \)) has an integral basis of classes \( \mathbb{S}_u \otimes \mathbb{S}_x \) for \( u \in \mathcal{S}_n \) and \( x \in \mathcal{S}_m \). Likewise the cohomology of \( \text{Grass}_k V \times \text{Grass}_l W \) has a basis \( S_\lambda \otimes S_\mu \) for \( \lambda \subset (n-k)^k \) and \( \mu \subset (m-l)^l \).

While we use the cohomology rings of complex varieties, our results and methods are valid for the Chow rings [19] and \( l \)-adic (étale) cohomology [13] of these same varieties over any field.

3. Orders on \( \mathcal{S}_\infty \)

3.1. The \( k \)-Bruhat order. The \( k \)-Bruhat order \( \leq_k \) is a suborder of the Bruhat order on \( \mathcal{S}_\infty \) related to the coefficients \( c_{\omega_0^0(\lambda, k)}^u \). It was called the \( k \)-coloured Ehresmanœdre in [30]. Its covers are given by the index of summation in Monk’s formula [37]
\[
\mathbb{S}_u \cdot (x_1 + \cdots + x_k) = \sum_{u \leq_k w} \mathbb{S}_w.
\]

Thus \( w \) covers \( u \) in the \( k \)-Bruhat order (\( u \leq_k w \)) if \( \ell(w) = \ell(u) + 1 \) and \( w = u(a, b) \) where \( a \leq k < b \). The \( k \)-Bruhat order has a nonrecursive characterization.

**Theorem A.** Let \( u, w \in \mathcal{S}_\infty \). Then \( u \leq_k w \) if and only if
(i) \( a \leq k < b \) implies \( u(a) \leq w(a) \) and \( u(b) \geq w(b) \),
(ii) \( a < b, u(a) < u(b), \) and \( w(a) > w(b) \), together imply \( a \leq k < b \).

**Proof.** We show the \( k \)-Bruhat order is the transitive relation \( u \leq_k w \) defined by (i) and (ii). If \( u \leq_k u(a, b) \) is a cover, then \( u \leq_k u(a, b) \). Thus \( u \leq_k w \) implies \( u \leq_k w \). Algorithm 3.1.1 completes the proof. \( \square \)
ALGORITHM 3.1.1 (Produces a chain in the $k$-Bruhat order).

**Input:** Permutations $u, w \in S_n$ with $u \leq_k w$.

**Output:** A chain in the $k$-Bruhat order from $w$ to $u$.

Output $w$. While $u \neq w$, do

1. Choose $a \leq k$ with $u(a)$ minimal subject to $u(a) < w(a)$.
2. Choose $k < b$ with $u(b)$ maximal subject to $w(b) < w(a) \leq u(b)$.
3. $w := w(a, b)$, output $w$.

At every iteration of 1, $u \leq_k w$. Moreover, this algorithm terminates in $e(w) - e(u)$ iterations and the sequence of permutations produced is a chain in the $k$-Bruhat order from $w$ to $u$.

**Proof.** It suffices to consider a single iteration. We show it is possible to choose $a$ and $b$, then $u \leq_k w(a, b)$, and finally $w(a, b) \leq_k w$.

In (1), $u \neq w$, so one may always choose $a$. Suppose $u \leq_k w \in S_n$ and it is not possible to choose $b$. In that case, if $j > k$ and $w(j) < w(a)$, then also $u(j) < w(a)$. Similarly, if $j < k$ and $w(j) < w(a)$, then $u(j) < u(a) < w(a)$. Thus $u(a) 
 w^{-1}(a) \leq w(a)$, which contradicts $w^{-1}(w(a)) = u(a) < w(a)$.

Let $w' := w(a, b)$. Since $w(b) \geq u(a)$ implies (i) for $(u, w')$, suppose $w(b) < u(a)$. Set $b_1 := u^{-1}(w(b))$. Then $w(b_1) \neq u(b_1)$ and the minimality of $u(a)$ shows that $b_1 > k$ and $w(b_1) < u(b_1)$. Similarly, if $b_2 := u^{-1}(w(b_2))$, then $b_2 > k$ and $w(b_2) < u(b_2)$. Continuing, we obtain a sequence $b_1, b_2, \ldots$ with $u(a) > u(b_1) > u(b_2) > \ldots$, a contradiction.

$(u, w')$ satisfies (ii). Suppose $i < j$ and $u(i) < u(j)$. If $j \leq k$, then $w(i) < w(j)$. To show $w'(i) < w'(j)$, it suffices to consider the case $j = a$. But then $u(i) < u(a)$, and thus $u(i) = w(i) = w'(i)$, by the minimality of $u(a)$. Then $w'(i) < u(a) \leq w(b) = w'(a)$. Similarly, if $k < i$, then $w'(i) < w'(j)$.

Finally, suppose $w$ does not cover $w'$ in the $k$-Bruhat order. Since $w(a) > w(b)$, there exists a $c$ with $a < c < b$ and $w(a) > w(c) > w(b)$. If $k < c$, then (ii) implies $u(c) > u(b)$ and the maximality of $u(b)$ implies $w(a) < w(c)$, a contradiction. The case $c \leq k$ similarly leads to a contradiction.

**Remark 3.1.2.** Algorithm 3.1.1 depends only upon $\zeta = wu^{-1}$.

**Input:** A permutation $\zeta \in S_n$.

**Output:** Permutations $\zeta, \zeta_1, \ldots, \zeta_m = e$ such that if $u \leq_k \zeta u$, then $u \leq_k \zeta_{m-1} u \leq_k \cdots \leq_k \zeta_1 u \leq_k \zeta u \ (= w)$

is a saturated chain in the $k$-Bruhat order.

Output $\zeta$. While $\zeta \neq e$, do

1. Choose $a$ minimal subject to $\alpha < \zeta(a)$.
2. Choose $\beta$ maximal subject to $\zeta(\beta) < \zeta(a) \leq \beta$.
3. $\zeta := \zeta(\alpha, \beta)$, output $\zeta$.

To see this is equivalent to Algorithm 3.1.1, set $\alpha = u(a)$ and $\beta = u(b)$ so that $w(a) = \zeta(a)$ and $w(b) = \zeta(\beta)$. Thus $w(a, b) = \zeta(\alpha, \beta)u$.

More is true, the full interval $[u, w]_k$ depends only upon $wu^{-1}$.
THEOREM 3.1.3. If \( u \leq_k w \) and \( x \leq_k y \) with \( wu^{-1} = zx^{-1} \), then the map \( v \mapsto vu^{-1}x \) induces an isomorphism \([u, w]_k \xrightarrow{\sim} [x, z]_k\).

This is a consequence of the following lemma.

LEMMA 3.1.4. Let \( u \leq_k w \) and \( x \leq_k z \) with \( wu^{-1} = zx^{-1} \). Then \( u \leq_k (\alpha, \beta)u \leq_k w \iff x \leq_k (\alpha, \beta)x \leq_k z \).

Proof. Let \( \zeta = wu^{-1} = zx^{-1} \). The position of \( y \) in \( u \) is \( u^{-1}(y) \).

Suppose \( (\alpha, \beta)x \) does not cover \( x \) in the \( k \)-Bruhat order, so there is a \( y \) with \( y < (\alpha, \beta)x \) and \( x^{-1}() < x^{-1}(\gamma) < x^{-1}(\beta) \). Then we have

\[
x = \cdots \alpha \cdots \gamma \cdots \beta \cdots,
\]
\[
z = \cdots \zeta(\alpha) \cdots \zeta(\gamma) \cdots \zeta(\beta) \cdots.
\]

Since \( u \leq_k (\alpha, \beta)u \), either \( k < u^{-1}(\beta) < u^{-1}(\gamma) \) or else \( u^{-1}(\gamma) < u^{-1}(\alpha) \leq_k k \). We illustrate \( u, (\alpha, \beta)u, \) and \( w \) for each possibility:

\[
k < u^{-1}(\beta) < u^{-1}(\gamma) \quad \text{and} \quad u^{-1}(\gamma) < u^{-1}(\alpha) \leq_k k
\]

\[
u : \cdots \alpha \cdots \beta \cdots \gamma \cdots \cdots \gamma \cdots \alpha \cdots \beta \cdots
\]
\[
(\alpha, \beta)u : \cdots \beta \cdots \alpha \cdots \gamma \cdots \cdots \gamma \cdots \beta \cdots \alpha \cdots
\]
\[
w : \cdots \zeta(\alpha) \cdots \zeta(\beta) \cdots \zeta(\gamma) \cdots \cdots \zeta(\gamma) \cdots \zeta(\alpha) \cdots \zeta(\beta) \cdots.
\]

Assume \( k < u^{-1}(\beta) < u^{-1}(\gamma) \). Then Theorem A and \( (\alpha, \beta)u \leq_k w \) imply \( \gamma \geq \zeta(\gamma) \) and \( \zeta(\beta) < \zeta(\gamma) \), since \( \alpha < \gamma \) and both have positions greater than \( k \) in \( (\alpha, \beta)u \).

Let \( c := x^{-1}(\gamma) \). If \( c \leq k \), then \( x \leq_k z \) implies \( \gamma \leq \zeta(\gamma) \) so \( \gamma = \zeta(\gamma) \). Also, \( \alpha < \gamma \) implies \( \zeta(\alpha) < \zeta(\gamma) \) and thus \( \zeta(\gamma) = \gamma < \beta \leq \zeta(\alpha) \), a contradiction. Similarly, \( c > k \) or \( u^{-1}(\gamma) < u^{-1}(\alpha) \), which leads to a contradiction. Thus \( x \leq_k (\alpha, \beta)x \).

To show \( y := (\alpha, \beta)x \leq_k z \), first note that \( (y, z) \) satisfies (i) of Theorem A, because \( (\alpha, \beta)u \leq_k w \). For (ii), we need only show the following:

(a) If \( \alpha < \gamma < \beta \) and \( x^{-1}(\gamma) < x^{-1}(\alpha) \) so that \( \gamma = yx^{-1}(\gamma) < yx^{-1}(\alpha) = \beta \), then \( xz^{-1}(\gamma) = \zeta(\gamma) < \zeta(\beta) = zx^{-1}(\alpha) \);

(b) If \( \alpha < \gamma < \beta \) and \( x^{-1}(\beta) < x^{-1}(\gamma) \) so that \( \alpha = yx^{-1}(\beta) < yx^{-1}(\gamma) = \gamma \), then \( \zeta(\alpha) < \zeta(\gamma) \).

If \( \alpha < \gamma < \beta \), then one of these does occur, as \( x \leq_k (\alpha, \beta)x = y \). Suppose \( x^{-1}(\gamma) < x^{-1}(\alpha) \), as the other case is similar.

Since \( x^{-1}(\gamma) < k \) and \( x \leq_k z \), we have \( \gamma \leq \zeta(\gamma) \), by condition (i). If \( u^{-1}(\gamma) < u^{-1}(\alpha) \), then \( (\alpha, \beta)u \leq_k w \Rightarrow \zeta(\gamma) < \zeta(\alpha) \). If \( u^{-1}(\beta) < u^{-1}(\gamma) \), then \( \gamma = \zeta(\gamma) \), and so \( \zeta(\gamma) = \gamma < \beta \leq \zeta(\alpha) \). Since \( u \leq_k (\alpha, \beta)u \), we cannot have \( u^{-1}(\alpha) < u^{-1}(\gamma) < u^{-1}(\beta) \).

Define \( \text{up}_{\zeta} := \{ \alpha \mid \alpha < \zeta(\alpha) \} \) and \( \text{down}_{\zeta} := \{ \beta \mid \beta > \zeta(\beta) \} \).
THEOREM 3.1.5. Let $\zeta \in S_\infty$.

(i) For $u \in S_\infty$, $u \leq_k \zeta u$ if and only if the following conditions are satisfied:
   (a) $u^{-1}(\uparrow_\zeta) \subseteq \{1, \ldots, k\}$;
   (b) $u^{-1}(\downarrow_\zeta) \subseteq \{k+1, k+2, \ldots\}$;
   (c) for all $\alpha, \beta \in \uparrow_\zeta$ (respectively, $\alpha, \beta \in \downarrow_\zeta$), $\alpha < \beta$ and $u^{-1}(\alpha) < u^{-1}(\beta)$ together imply $\zeta(\alpha) < \zeta(\beta)$.

(ii) If $\#(\uparrow_\zeta) < k$, then there is a permutation $u$ such that $u \leq_k \zeta u$.

Proof. Statement (i) follows from Theorem A. For (ii), let $\{a_1, \ldots, a_k\}$ contain $\uparrow_\zeta$ and possibly some fixed points of $\zeta$, and let $\{a_{k+1}, a_{k+2}, \ldots\}$ be its complement in $\mathbb{N}$. Index these sets so that $\zeta(a_i) < \zeta(a_{i+1})$ for $i \neq k$. Define $u \in S_\infty$ by $u(i) = a_i$. Then $\zeta u$ is Grassmannian with descent $k$, and Theorem A implies $u \leq_k \zeta u$. □

3.2. A new partial order on $S_\infty$. For $\zeta \in S_\infty$, define $|\zeta|$ to be the difference of

\[ \#\{(a, b) \in \zeta(\uparrow_\zeta) \times \zeta(\downarrow_\zeta) | a > b \} \]

and

\[ \#\{(a, b) \in \uparrow_\zeta \times \downarrow_\zeta | a > b \} \].

LEMMA 3.2.1. If $u \leq_k \zeta u$, then $\ell(u) + |\zeta| = \ell(\zeta u)$.

Proof. By Theorem 3.1.3, $\ell(\zeta u) - \ell(u)$ depends only upon $\zeta$. Using the permutation $u$ in the proof of Theorem 3.1.5 shows that it equals $|\zeta|$. If $c = \zeta(c)$, then the number of inversions involving $c$ is the same for both $u$ and $\zeta u$. The first term above counts the remaining inversions in $\zeta u$ and the last two terms the remaining inversions in $u$. □

By Theorem 3.1.3, $[u, \zeta u]_k$ depends only upon $\zeta$ if $u \leq_k \zeta u$. In fact, it is independent of $k$ as well. That is, if $x \leq_k \zeta x$, then the map $v \mapsto xu^{-1}v$ defines an isomorphism $[u, \zeta u]_k \cong [x, \zeta x]_1$.

Definition 3.2.2. For $\zeta, \eta \in S_\infty$, let $\eta \preceq \zeta$ if there exist $u \in S_\infty$ and a positive integer $k$ such that $u \leq_k \eta u \leq_k \zeta u$. If $u$ is chosen as in the proof of Theorem 3.1.5, then we see that $\eta \preceq \zeta$ if the following occur:

1. if $\alpha < \eta(\alpha)$, then $\eta(\alpha) \leq \zeta(\alpha)$;
2. if $\alpha > \eta(\alpha)$, then $\eta(\alpha) \geq \zeta(\alpha)$;
3. if $\alpha, \beta \in \uparrow_\zeta$ (respectively, $\alpha, \beta \in \downarrow_\zeta$) with $\alpha < \beta$ and $\zeta(\alpha) < \zeta(\beta)$, then $\eta(\alpha) < \eta(\beta)$.

Figure 2 illustrates $\preceq$ on $S_4$. For $\zeta \in S_n$, define $\zeta := \omega_0 \zeta \omega_0$.

THEOREM 3.2.3. Suppose $u, \zeta, \eta, \xi \in S_\infty$.

(i) $(S_\infty, \preceq)$ is a graded poset with rank function $|\zeta|$.

(ii) The map $\lambda \mapsto v(\lambda, k)$ exhibits Young's lattice of partitions with at most $k$ parts as an induced suborder of $(S_\infty, \preceq)$. 
(iii) If \( u \leq_k \zeta u \), then \( \eta \mapsto \eta u \) induces an isomorphism \([e, \zeta] \xrightarrow{\sim} [u, \zeta u]_k\).

(iv) If \( \eta \preceq \zeta \), then \( \zeta \mapsto \zeta^{-1} \) induces an isomorphism \([\eta, \zeta] \xrightarrow{\sim} [e, \zeta^{-1}]_\preceq\).

(v) For every infinite set \( P \subset \mathbb{N} \), \( \phi_P : \mathcal{S}_\infty \to \mathcal{S}_\infty \) is an injection of graded posets. Thus, if \( \zeta, \eta \in \mathcal{S}_\infty \) are shape equivalent, then \([e, \zeta] \preceq [e, \eta]_\preceq\).

(vi) The map \( \eta \mapsto \eta \zeta^{-1} \) induces an order-reversing isomorphism between \([e, \zeta]_\preceq\) and \([e, \zeta^{-1}]_\preceq\).

(vii) The homomorphism \( \zeta \mapsto \zeta \) on \( \mathcal{S}_n \) is an automorphism of \((\mathcal{S}_n, \preceq)\).

Theorem E(i) follows from the definition of \( \preceq \) and Theorem 3.2.3(v).

Proof. Statements (i)–(v) follow from the definitions. Suppose \( u \leq_k \eta u \leq_k \zeta u \) with \( u, \eta u, \zeta u \in \mathcal{S}_n \). If \( w := \zeta u \), then \( w \omega_0 \leq \eta^{-1} \omega_0 \leq \eta \omega_0 \leq n^{-1} \omega_0 \), which proves (vi). Similarly, \( u \leq_k w \iff \bar{u} \leq \leq_k \bar{w} \) implies (vii).

---

**Example 3.2.4.** Let \( \zeta = (24)(153) \) and \( \eta = (35)(174) = \phi_{\{1,3,4,5,7\}}(\zeta) \). Then \( 21345 \leq_2 \zeta \cdot 21345 \) and \( 3215764 \leq_3 \eta \cdot 3215764 \). Figure 3 shows \([21342, \zeta \cdot 21345], [3215764, \eta \cdot 3215764]_3\), and \([e, \zeta]_\leq\).

---

**Figure 2.** \( \leq \) on \( \mathcal{S}_4 \)

**Figure 3.** Isomorphic intervals in \( \leq_2, \leq_3 \), and \( \leq \)
3.3. Disjoint permutations. Let ζ ∈ ℜₙ and 1, ..., n be the vertices of a convex, planar n-gon numbered consecutively. Define the directed geometric graph Γζ to be the union of directed chords ⟨α, ζ(α)⟩ for α in the support suppζ of ζ.

Permutations ζ and η are disjoint if the edge sets of Γζ and Γη (drawn on the same n-gon) are disjoint as subsets of the plane. This implies (but is not equivalent to) suppζ ∩ suppη = ∅.

In Figure 4, the pair of cycles on the left is disjoint and the other pair is not. We relate this definition to that given in Section 1.3.

![Figure 4. Graphs of the permutations (1782)(345) and (13)(24)](image)

**Lemma 3.3.1.** Let ζ, η ∈ ℜₙ. Then the edges of Γζ are disjoint from the edges of Γη if and only if suppζ ∩ suppη = ∅ and |ζ| + |η| = |ζη|.

**Proof.** Suppose ζ and η have disjoint support, and let ⟨α, ζ(α)⟩ be an edge of Γζ and ⟨b, η(b)⟩ be an edge of Γη. The contribution of the endpoints of these edges to |ζη| = |ζ| + |η| is zero if the edges do not cross, which proves the forward implication.

For the reverse, suppose they cross. The contribution is 1 if a < ζ(a) and b > η(b) (or vice versa), and zero otherwise. Since each edge is part of a directed cycle, there are at least four crossings, one of each type shown in Figure 5. There, the numbers increase in a clockwise direction, with the least number in the northeast (↗). Thus |ζη| > |ζ| + |η|.

![Figure 5. Crossings](image)

**Lemma 3.3.2.** Let α < β, ζ ∈ ℜₙ, and suppose ζ ⊆ ⟨α, β⟩ζ. Then

(i) α and β are connected in Γₙ⟨α, β⟩ζ;
(ii) if ⟨c, d⟩ is any chord meeting Γζ, then ⟨c, d⟩ meets Γₙ⟨α, β⟩ζ;
(iii) if p and q are connected in Γζ, then they are connected in Γₙ⟨α, β⟩ζ;
(iv) if ζ and η are disjoint and ζ ⊆ ζ', then ζ' and η are disjoint.
Proof. Suppose \( u \in \mathcal{S}_\infty \) with \( u \leq_k \zeta u \leq_k (\alpha, \beta)\zeta u \). Define \( i \) and \( j \) by \( \zeta u(i) = \alpha \) and \( \zeta u(j) = \beta \), and set \( a = u(i) \) and \( b = u(j) \). Since \( \zeta u \leq_k (\alpha, \beta)\zeta u \) is a cover, \( i \leq k < j \), and thus \( a < \alpha < \beta \leq b \), as \( u \leq_k \zeta u \). Thus the edges \( \langle a, \beta \rangle \) and \( \langle b, \alpha \rangle \) of \( \Gamma(\alpha, \beta) \zeta \) meet, proving (i).

For (ii), note that \( \Gamma(\alpha, \beta) \zeta \) differs from \( \Gamma \zeta \) only by the (possible) deletion of edges \( \langle a, \alpha \rangle \) and \( \langle b, \beta \rangle \) and the addition of the edges \( \langle a, \beta \rangle \) and \( \langle b, \alpha \rangle \). Checking all possibilities for \( <c, d> \), \( \langle a, \alpha \rangle \), and \( \langle b, \beta \rangle \) shows (ii). Statement (iii) follows from (ii) by considering \( \Gamma \zeta - \langle a, \alpha \rangle - \langle b, \beta \rangle \). The contrapositive of (iv) is also a consequence of (ii). If \( \zeta' \) and \( \eta \) are not disjoint and \( \zeta' \preceq \zeta \), then \( \zeta \) and \( \eta \) are not disjoint.

**Lemma 3.3.3.** Suppose \( u, \zeta, \eta \in \mathcal{S}_\infty \) with \( \zeta, \eta \) disjoint. Then

\[
\begin{align*}
\text{if } u \leq_k \zeta u & \iff u \leq_k \zeta u \quad \text{and} \quad u \leq_k \eta u. 
\end{align*}
\]

Proof. Suppose \( u \leq_k \zeta u \). Let \( i \leq k \) so that \( u(i) \leq \zeta u(i) \). Since \( \text{supp}_\zeta \cap \text{supp}_\eta = \emptyset \), \( u(i) \leq \zeta u(i) \). Similarly, if \( k < j \), then \( u(j) \geq \zeta u(j) \), showing that (i) of Theorem A holds for the pair \( (u, \zeta u) \).

For (ii), suppose \( i < j \), \( u(i) < u(j) \), and \( \zeta u(i) > \zeta u(j) \). If \( j < k \), then \( u(i) \in \text{supp}_\zeta \). Since \( u \leq_k \zeta u \), and \( \zeta, \eta \) have disjoint supports, \( \zeta u(i) = \eta \zeta u(i) < \eta \zeta u(j) \), thus \( u(j) \in \text{supp}_\eta \), and so

\[
\begin{align*}
u(i) < u(j) < \zeta u(i) < \eta u(j). \end{align*}
\]

But then the edge \( \langle u(i), \zeta u(i) \rangle \) of \( \Gamma \zeta \) meets the edge \( \langle u(j), \eta u(j) \rangle \) of \( \Gamma \eta \), a contradiction. The assumption that \( k < i \) leads similarly to a contradiction. Thus \( u \leq_k \zeta u \), and similarly \( u \leq_k \eta u \).

Suppose now that \( u \leq_k \zeta u \) and \( u \leq_k \eta u \). Condition (i) of Theorem A holds for \( (u, \zeta u) \) as \( \zeta \) and \( \eta \) have disjoint support. For (ii), let \( i < j \) with \( u(i) < u(j) \), and suppose \( j < k \). If the set \( \{u(i), u(j)\} \) meets at most one of \( \text{supp}_\zeta \) or \( \text{supp}_\eta \), say, \( \text{supp}_\zeta \), then \( u \leq_k \zeta u \) implies \( \eta u(i) < \eta u(j) \). Suppose now that \( u(i) \in \text{supp}_\zeta \) and \( u(j) \in \text{supp}_\eta \). Since \( u \leq_k \zeta u \), we have \( \zeta u(i) < \zeta u(j) = u(j) \). But \( u \leq_k \eta u \) implies \( u(j) \leq \eta u(j) \). Thus \( \eta \zeta u(i) = \zeta u(i) < u(j) \leq \eta u(j) = \eta \zeta u(j) \). Similar arguments suffice when \( k < i \).

**Proof of Theorem G(i).** Suppose \( \zeta \) and \( \eta \) are disjoint. By Lemmas 3.3.2 and 3.3.3, the map \( [e, \zeta] \rightarrow [e, \eta] \rightarrow [e, \zeta \eta] \), defined by \( \langle \zeta', \eta' \rangle \mapsto \zeta' \eta' \), is an injection. For surjectivity, let \( \xi \preceq \zeta \eta \). By Lemma 3.3.2(iii) and downward induction from \( \zeta \eta \) to \( \xi \), \( \Gamma \zeta \) has no edges connecting \( \text{supp}_\zeta \) to \( \text{supp}_\eta \). Set \( \xi' := \xi|_{\text{supp}_\zeta} \) and \( \xi'' := \xi|_{\text{supp}_\eta} \). Then \( \xi = \xi' \xi'' \), and \( \xi' \) and \( \xi'' \) are disjoint. Surjectivity follows by showing \( \xi' \preceq \zeta \) and \( \xi'' \preceq \eta \).

It suffices to consider the case \( \zeta \preceq (\alpha, \beta) \zeta = \zeta \eta \) of a cover. By Lemma 3.3.2(ii), \( \alpha \) and \( \beta \) are connected in \( \Gamma \eta \), so we may assume that \( \alpha, \beta \in \text{supp}_\zeta \). Then \( \xi'' = \eta \) and \( (\alpha, \beta) \xi' = \zeta \). We show that \( \xi' \preceq (\alpha, \beta) \zeta' = \zeta \) is a cover, which completes the proof.
Choose \( u \in S_\infty \) with \( u \leq_k \zeta u \leq_k \zeta \eta u \). Let \( a := (\zeta' u)^{-1}(\alpha) \) and \( b := (\zeta' u)^{-1}(\beta) \). Since \( \zeta' \) and \( \eta \) are disjoint, \( \alpha, \beta \notin \text{supp}_u \) and so \( a, b \notin \text{supp}_u \). Thus \((\alpha, \beta)\zeta' \eta u = \zeta' \eta u(a, b)\), showing \( a < k < b \), as \((\alpha, \beta)\zeta' \eta u \).

Since \( \zeta' \) and \( \eta \) are disjoint and \( \xi = \zeta' \eta \), Lemma 3.3.3 implies \( u \leq_k \zeta' u \). Thus \(|\zeta'| + |\zeta(u)| = |\zeta' \eta|\). But since \( \zeta' \) and \( \eta \) are disjoint and \( \zeta' \eta \rightarrow \zeta \eta \) is a cover, we have

\[
|\zeta| + |\eta| = |\zeta \eta| = 1 + |\zeta'| + |\eta|,
\]

so \( \ell'((\zeta' u) + 1 = \ell'((\zeta' u(a, b)) \). Since \( a \leq k < b \) and \( \zeta u = \zeta' u(a, b) \), this implies \( \zeta' u \leq_k \zeta u \).

Example 3.3.4. Let \( \zeta = (2354) \) and \( \eta = (176) \), which are disjoint. Let \( u = 2316745 \). Then

\[
u \leq_3 \zeta \eta u = 3571624, \quad u \leq_3 \zeta u = 3516724, \quad u \leq_3 \eta u = 2371645.
\]

The intervals \([u, \zeta u]_3\), \([u, \eta u]_3\), and \([u, \zeta \eta u]_3\) are illustrated in Figure 6.

---

4. Cohomological formulas and identities for the \( e_{wv}^w \)

4.1. Maps on \( S_\infty \). For \( p, q \in \mathbb{N} \) and \( w \in S_\infty \), define \( e_{p,q}(w) \in S_\infty \):

\[
e_{p,q}(w)(j) = \begin{cases} 
  w(j) & j < p \text{ and } w(j) < q, \\
  w(j) + 1 & j < p \text{ and } w(j) \geq q, \\
  q & j = p, \\
  w(j - 1) & j > p \text{ and } w(j) < q, \\
  w(j - 1) + 1 & j > p \text{ and } w(j) \geq q.
\end{cases}
\]
Note that $e_{p,q} = \phi_{n-\{p\}}$. If $p \neq q$, then $e_{p,q} : \mathcal{S}_\infty \to \mathcal{S}_\infty$ is not a group homomorphism. The map $e_{p,q}$ has a left inverse $/p$, defined by

$$u/p(j) = \begin{cases} u(j) & j < p \text{ and } u(j) < u(p), \\ u(j) - 1 & j < p \text{ and } u(j) > u(p), \\ u(j + 1) & j \geq p \text{ and } u(j) < u(p), \\ u(j + 1) - 1 & j \geq p \text{ and } u(j) > u(p). \end{cases}$$

Representing permutations as matrices, $u/p$ erases the $p$th row and $u(p)$th column of $u$, and $e_{p,q}$ adds a new $p$th row and $q$th column consisting mostly of zeroes, but with a 1 in the $(p,q)$th position. For example,

$$e_{3,3}(23154) = 243165; \quad 264351/3 = 25341.$$

**Lemma 4.1.1.** Suppose $u \leq w$ and $p,q$ are positive integers. Then we have the following:

(i) $e_{p,q}(u) \leq e_{p,q}(w)$.

(ii) If $\ell(w) - \ell(u) = \ell(e_{p,q}(w)) - \ell(e_{p,q}(u))$, then

$$e_{p,q} : [u,w] \sim [e_{p,q}(u), e_{p,q}(w)].$$

(iii) If $u,w \in \mathcal{S}_n$ and either of $p$ or $q$ is equal to either 1 or $n+1$, then

$$\ell(w) - \ell(u) = \ell(e_{p,q}(w)) - \ell(e_{p,q}(u)).$$

(iv) If $u \leq_k w$ and $u(p) = w(p)$, then $u/p \leq_k w/p$ and $[u,w]_k \simeq [u/p,w/p]_{k'}$, where $k'$ is equal to $k$ if $k < p$, and $k-1$ otherwise. Furthermore, $wu^{-1} = e_{u(p),u(p)}(w/p(u/p)^{-1})$.

**Proof.** Suppose $u \leq (a,b)$ is a cover. Then $e_{p,q}(u) < e_{p,q}(u(a,b))$ is a cover if either $p \leq a$ or $b < p$, or else $a < p \leq b$ and either $q \leq u(a)$ or $u(b) < q$. If, however, $a < p \leq b$ and $u(a) < q \leq u(b)$, then there is a chain of length 3 from $e_{p,q}(u)$ to $e_{p,q}(u(a,b)) = e_{p,q}(u)(a,b+1)$:

$$e_{p,q}(u) < e_{p,q}(u)(a,p) < e_{p,q}(u)(a,b+1,p) < e_{p,q}(u)(a,b+1).$$

The lemma follows from this. For example, under the hypothesis of (ii), $e_{p,q}(w)$ and $e_{p,q}(u)$ each have the same number of inversions involving $q$. Thus, if $e_{p,q}(u) \leq v \leq e_{p,q}(w)$, then $v(p) = q$. \qed

**4.2. An embedding of flag manifolds.** Let $W \subset V$ with $W \simeq \mathbb{C}^n$ and $V \simeq \mathbb{C}^{n+1}$. Suppose $f \in V - W$ so that $V = \langle W, f \rangle$. For $p \in [n+1]$, define the injection $\psi_p : \mathbb{F} \ell W \hookrightarrow \mathbb{F} \ell V$ by

$$\psi_p E_j = \begin{cases} E_j & \text{if } j < p, \\ \langle E_{j-1}, f \rangle & \text{if } j \geq p. \end{cases}$$
Proposition 4.2.1 [50, Lemma 12]. Let $E, w \in \mathcal{S}_n$. Then, for every $p, q \in [n+1]$, 

$$\psi_p X_w E. \subseteq X_{\delta_q(w)} \psi_{n+2-q} E..$$

Recall that $e$ is the identity permutation.

Corollary 4.2.2. Let $w \in \mathcal{S}_n$ and $E, E' \in \mathcal{F}\ell W$ be opposite flags. Then $\psi_1 E.$ and $\psi_{n+1} E.$ are opposite flags in $\mathcal{F}\ell V$, and

$$\psi_p X_w E. = X_{\delta_p(w)} \psi_{n+1} E. \cap X_{\delta_{p+1}(e)} \psi_1 E'. = X_{\delta_p(e)} \psi_{n+1} E'. \cap X_{\delta_{p+1}(w)} \psi_1 E..$$

Proof. Since $X_e E' = \mathcal{F}\ell W$, Proposition 4.2.1 with $q = 1$ or $n + 1$ implies that $\psi_p X_w E.$ is a subset of either intersection

$$X_{\delta_p(w)} \psi_{n+1} E. \cap X_{\delta_{p+1}(e)} \psi_1 E',$$

or

$$X_{\delta_p(e)} \psi_{n+1} E'. \cap X_{\delta_{p+1}(w)} \psi_1 E..$$

Since $E.$ and $E'$ are opposite flags, $\psi_{n+1} E.$ and $\psi_1 E'.$ are opposite flags, so both intersections are generically transverse and irreducible. Since

$$\ell(\delta_p(w)) = \ell(w) + p - 1,$$

$$\ell(\delta_{p+1}(w)) = \ell(w) + n + 1 - p,$$

both intersections have the same dimension as $\psi_p X_w E.$, which proves equality.

Since $\delta_{p+1}(e) = v(n+1-p, p)$, where $n + 1 - p$ is the partition of $n + 1 - p$ into a single part, we see that $\mathcal{S}_{\delta_{p+1}(e)} = h_{n+1-p}(x_1, \ldots, x_p)$, the complete symmetric polynomial of degree $n + 1 - p$ in $x_1, \ldots, x_p$. Similarly, $\mathcal{S}_{\delta_p(e)} = e_{p-1}(x_1, \ldots, x_{p-1}) = x_1 \cdots x_{p-1}$, as $e_{p-1} = v(1^{p-1}, p-1)$, where $1^{p-1}$ is the partition of $p-1$ into $p-1$ equal parts, each of size 1.

Corollary 4.2.3. Let $w \in \mathcal{S}_n$. In $H^* \mathcal{F}\ell V$,

$$\mathcal{S}_{\delta_{p+1}(w)} \cdot h_{n+1-p}(x_1, \ldots, x_p) = \mathcal{S}_{\delta_{p+1}(w)} \cdot x_1 \cdots x_{p-1},$$

and these products are equal to $(\psi_p)_* \mathcal{S}_w$.

We compute $\psi_p^*$. The Pieri formulas of [50] show that if $u \in \mathcal{S}_n$ and $k, m \leq n$
are positive integers, then

\[ \mathbb{S}_u \cdot \mathbb{S}_{\omega \omega w} \cdot e_m(x_1 \cdots x_k) = \begin{cases} 1 & u \xrightarrow{c_{k,m}} w, \\ 0 & \text{otherwise}, \end{cases} \tag{4.2.1} \]

\[ \mathbb{S}_u \cdot \mathbb{S}_{\omega \omega w} \cdot h_{n+1-m}(x_1, \ldots, x_k) = \begin{cases} 1 & u \xrightarrow{r_{k,m}} w, \\ 0 & \text{otherwise}, \end{cases} \tag{4.2.2} \]

where \( u \xrightarrow{c_{k,m}} w \) if there is a chain in the \( k \)-Bruhat order,

\[ u \prec_k (\alpha_1, \beta_1) u \prec_k \cdots \prec_k (\alpha_m, \beta_m) \cdots (\alpha_1, \beta_1) u = w, \]

such that \( \beta_1 > \cdots > \beta_m \). When \( k = m \), it follows that \( \{\alpha_1, \ldots, \alpha_k\} = \{u(1), \ldots, u(k)\} \). When \( k = m = p - 1 \), write \( \xrightarrow{\omega} \) for this relation. Similarly, \( u \xrightarrow{r_{k,m}} w \) if there is a chain in the \( k \)-Bruhat order,

\[ u \prec_k (\alpha_1, \beta_1) u \prec_k \cdots \prec_k (\alpha_{n+1-m}, \beta_{n+1-m}) \cdots (\alpha_1, \beta_1) u = w \]

such that \( \beta_1 < \beta_2 < \cdots < \beta_{n+1-m} \). Recall that \( \omega_n \) is the longest element in \( S_n \).

**Theorem 4.2.4.** Let \( v \in S_{n+1} \). In \( H^*\mathcal{F}_n \),

(i) \( \psi_p^* \mathbb{S}_v = \sum_{y \in S_n} \mathbb{S}_y = \sum_{v \rightarrow \psi_p(y)} \mathbb{S}_y \)

(ii) \( \psi_p^*(x_i) = \begin{cases} x_i & i < p, \\ 0 & i = p, \\ x_{i-1} & i > p. \end{cases} \)

**Proof.** In \( H^*\mathcal{F}_n \),

\[ \psi_p^* \mathbb{S}_v = \sum_{y \in S_n} \deg(\mathbb{S}_{\omega_n y} \cdot \psi_p^* \mathbb{S}_v) \mathbb{S}_y. \]

By the projection formula (2.3.2) and Corollary 4.2.3, we have

\[ \deg(\mathbb{S}_{\omega_n y} \cdot \psi_p^* \mathbb{S}_v) = \deg(\mathbb{S}_v \cdot (\psi_p)_* \mathbb{S}_{\omega_n y}) = \deg(\mathbb{S}_v \cdot \mathbb{S}_{\epsilon_{p,n+1}(\omega_n y)} \cdot x_1 \cdots x_{p-1}). \]

Note that \( \epsilon_{p,n+1}(\omega_n y) = \omega_n \epsilon_{p,1}(y) \). By (4.2.1), the triple product

\[ \mathbb{S}_v \cdot \mathbb{S}_{\epsilon_{p,n+1}(\omega_n y)} \cdot x_1 \cdots x_{p-1} \]
is zero unless \( v \xrightarrow{\epsilon_p} e_{p,1}(y) \), and in this case it equals \( \mathcal{G}_{o^{p+1}} \). This establishes the first equality of (i). For the second, use the other formula for \( (\psi_p)_* \mathcal{G}_y \) from Corollary 4.2.3 and (4.2.2).

For (ii), let \( \mathcal{F} \) be the tautological flag on \( \mathbb{F} \ell_{n+1}, \mathcal{E} \), the tautological flag on \( \mathbb{F} \ell_n \), and \( 1 \) the trivial line bundle. Then

\[
\psi_p^*(\mathcal{F}_1/\mathcal{F}_{i-1}) = \begin{cases} 
\mathcal{E}_i/\mathcal{E}_{i-2} & \text{if } i > p, \\
1 & \text{if } i = p, \\
\mathcal{E}_i/\mathcal{E}_{i-1} & \text{if } i < p.
\end{cases}
\]

But \(-x_i\) is the Chern class of both \( \mathcal{F}_i/\mathcal{F}_{i-1} \) and \( \mathcal{E}_i/\mathcal{E}_{i-1} \). \( \square \)

4.3. The endomorphism \( x_p \mapsto 0 \). For \( p \in \mathbb{N} \) and \( v \in \mathcal{S}_\infty \), define

\[
A_p(v) := \{ y \in \mathcal{S}_\infty | v \xrightarrow{\epsilon_p} e_{p,1}(y) \}.
\]

**Lemma 4.3.1.** If \( v \in \mathcal{S}_n \) and \( p \leq n \), then \( A_p(v) = \{ y \in \mathcal{S}_n | v \xrightarrow{\epsilon_{p,n+1}} e_{p,n+1}(y) \} \).

**Proof.** If \( v \in \mathcal{S}_n, p \leq n \), and \( v \xrightarrow{\epsilon_p} W \), then \( w \in \mathcal{S}_{n+1} \), so \( A_p(v) \subset \mathcal{S}_n \). But then \( A_p(v) \) and \( \{ y \in \mathcal{S}_n | v \xrightarrow{\epsilon_{p,n+1}} e_{p,n+1}(y) \} \) index the two equal sums in Theorem 4.2.4(i). \( \square \)

Let \( \Psi_p : \mathbb{Z}[x_1,x_2,\ldots] \rightarrow \mathbb{Z}[x_1,x_2,\ldots] \) be defined by

\[
\Psi_p(x_i) = \begin{cases} 
x_i & \text{if } i < p, \\
0 & \text{if } i = p, \\
x_{i-1} & \text{if } i > p.
\end{cases}
\]

**Theorem C(ii).** For \( v \in \mathcal{S}_\infty \) and \( p \in \mathbb{N} \), \( \Psi_p \mathcal{G}_v = \sum_{y \in A_p(v)} \mathcal{G}_y \).

**Proof.** For \( p \leq n+1 \), the homomorphism \( \Psi_p \) induces the map \( \psi_p^* : H^*\mathbb{F} \ell_{n+1} \rightarrow H^*\mathbb{F} \ell_n \) by Theorem 4.2.4(ii). Choosing \( n \) large enough completes the proof. \( \square \)

**Corollary 4.3.2.** For \( w, x, y \in \mathcal{S}_\infty \) and \( p \in \mathbb{N} \),

\[
\sum_{u \in A_p(x)} \sum_{v \in A_p(y)} c_{uv}^w = \sum_{x \in A_p(z)} c_{xy}^z.
\]

**Proof.** Apply \( \Psi_p \) to the identity \( \mathcal{G}_x \cdot \mathcal{G}_y = \sum_z c_{xy}^z \mathcal{G}_z \) to obtain

\[
\sum_{u \in A_p(x)} \sum_{v \in A_p(y)} \mathcal{G}_u \cdot \mathcal{G}_v = \sum_z c_{xy}^z \sum_{w \in A_p(z)} c_{xy}^w \mathcal{G}_w.
\]
Expanding the product $\mathfrak{S}_u \cdot \mathfrak{S}_v$ and equating the coefficients of $\mathfrak{S}_w$ proves the identity. 

**Example 4.3.3.** We compute $\Psi_3(\mathfrak{S}_{413652})$. The polynomial $\mathfrak{S}_{413652}$ is

$$x_1^4x_2x_3x_5 + x_1^3x_2^2x_4x_5 + x_1^3x_2x_4^2x_5 + x_1^4x_2x_3x_4 + x_1^4x_2x_3x_5 + x_1^4x_3x_4x_5 + x_1^3x_2x_3x_4 + x_1^3x_2x_4x_5 + x_1^3x_2x_3x_4 + x_1^3x_2x_4x_5 + x_1^3x_2x_3x_4 \cdot 2.$$ 

Thus $\Psi_3(\mathfrak{S}_{413652}) = x_1^4x_2x_3x_4 + x_1^3x_2^2x_3x_4 + x_1^3x_2^2x_3x_4$. However,

$$\mathfrak{S}_{52341} = x_1^4x_2x_3x_4$$

and

$$\mathfrak{S}_{42531} = x_1^3x_2^3x_3x_4 + x_1^3x_2x_3^2x_4,$$

which shows $\Psi_3(\mathfrak{S}_{413652}) = \mathfrak{S}_{52341} + \mathfrak{S}_{42531}$. To see that this agrees with Theorem C, we compute the permutations $w$ such that $x \xrightarrow{e_3} w$:

![Figure 7](image)

Of these, only the two underlined permutations are of the form $e_{3,1}(u)$:

$$631452 = e_{3,1}(52341)$$

and

$$531642 = e_{3,1}(42531).$$

**Lemma 4.3.4.** Let $\lambda$ be a partition and $p, k$ be positive integers. Then $A_p(v(\lambda, k)) = (v(\lambda, k'))$, where $k' = k - 1$ if $p < k$, and $k$ otherwise.

**Proof.** By the combinatorial definition of Schur functions [46, Section 4.4], $\Psi_p(\mathfrak{S}_{v(\lambda, k)}) = \mathfrak{S}_{v(\lambda, k')}$. 

Lemma 4.3.4 implies that $v(\lambda, k')$ is the only solution $x$ to $v(\lambda, k) \xrightarrow{e_p} e_{p,1}(x)$, a statement about chains in the Bruhat order.
4.4. Identities for $c_{u,v}^w$ when $u(p) = w(p)$

**Lemma 4.4.1.** Let $u, w \in S_{n+1}$ with $u(p) = w(p)$ for some $p \in [m+1]$, and suppose $\ell(w) - \ell(u) = \ell(w/p) - \ell(u/p)$. Then in $H^*F\ell_{n+1}$,

$$(\psi_p)_* \left( \mathbb{G}_{u,p} \cdot \mathbb{G}_{w,p} \right) = \mathbb{G}_u \cdot \mathbb{G}_{w,p}.\)$$

**Proof.** Let $E, E'$ be opposite flags in $W$. By Proposition 4.2.1,

$$(4.4.1)$$

Note that $C_{\mathbb{G}(W/p)}(\mathbb{G}+W)/$. Since $u(p) = w(p)$, $\psi_w(p) E.$ and $\psi_{n+2-u(p)}E'$ are opposite in $V$. Also, as $\ell(w) - \ell(u) = \ell(w/p) - \ell(u/p)$, both sides of (4.4.1) have the same dimension, so they are equal. 

**Proof of Theorem C (i) (b).** It suffices to compute this in $H^*F\ell_{n+1}$ for $n$ such that $p \leq n$, $v \in S_{n+1}$, and $A_p(v) \subset S_n$. By Lemma 4.4.1,

$$\mathbb{G}_u \cdot \mathbb{G}_{w,p} = (\psi_p)_* \left( \mathbb{G}_{u,p} \cdot \mathbb{G}_{w,p} \right) = (\psi_p)_* \left( \sum_{y \in S_n} c_{u,p}^{w,y} \mathbb{G}_{w,y} \right).$$

Since $c_{x,y}^{w,y} = c_{x,y}^z$ for $x, y, z \in S_n$ and $\epsilon_{p,1}(\omega_n y) = \omega_{n+1} \epsilon_{p,1}(y)$,

$$\mathbb{G}_u \cdot \mathbb{G}_{w,p} = \sum_{y \in S_n} c_{u,p}^{w,y} (\psi_p)_* (\mathbb{G}_{w,y}) = \sum_{y \in S_n} c_{u,p}^{w,y} \mathbb{G}_{w,p,1}(y) \cdot x_1 \cdots x_{p-1},$$

by Corollary 4.2.3. Thus

$$c_{u,v}^w = \deg \left( \mathbb{G}_u \cdot \mathbb{G}_{w,p} \cdot \mathbb{G}_v \right) = \sum_{y \in S_n} c_{u,p}^{w,y} \cdot \deg \left( \mathbb{G}_{w,p,1}(y) \cdot (x_1 \cdots x_{p-1}) \cdot \mathbb{G}_v \right) = \sum_{y \in A_p(v)} c_{u,p}^{w,y}.\)$$

When $p = 1$, this has the following consequence.

**Corollary 4.4.2.** If $u(1) = w(1)$, then $c_{x,y}^w = 0$ unless $v = 1 \times y$. In that case, $c_{u,1x,y}^w = c_{u,y}^{w,1}$. 

4.5. Products of flag manifolds. Let \( P, Q \in \binom{[n+m]}{n} \); that is, \( P, Q \subseteq [n+m] \) and each has order \( n \). List \( P, Q \), and their complements \( P^c, Q^c \) in order
\[
P = p_1 < \cdots < p_n, \quad P^c := [n+m] - P = p_1^c < \cdots < p_m^c,
\]
\[
Q = q_1 < \cdots < q_n, \quad Q^c := [n+m] - Q = q_1^c < \cdots < q_m^c.
\]
Define a function \( \epsilon_{P,Q} : \mathcal{S}_n \times \mathcal{S}_m \rightarrow \mathcal{S}_{m+n} \) by
\[
\epsilon_{P,Q}(v, w)(p_i) = q_{w(i)} \quad i = 1, \ldots, n,
\]
\[
\epsilon_{P,Q}(v, w)(q_j^c) = p_{v(j)} \quad i = 1, \ldots, m.
\]
Here \( \epsilon_{P,Q}(v, w) \) is the permutation matrix obtained by placing the entries of \( v \) in the blocks \( P \times Q \) and those of \( w \) in the blocks \( P^c \times Q^c \). If \( P = [n+1] - \{p\} \) and \( Q = [n+1] - \{q\} \), then \( \epsilon_{P,Q}(v, e) = \epsilon_{P,Q}(v) \).

Suppose \( V \cong \mathbb{C}^n \), \( W \cong \mathbb{C}^m \), and \( P \in \binom{[n+m]}{n} \). Define a map
\[
\psi_P : \mathbb{F}\ell V \times \mathbb{F}\ell W \rightarrow \mathbb{F}\ell (V \oplus W)
\]
by \( \psi_P(E_i, F_j) = \langle E_i, F_j \rangle \). Equivalently, if \( e_1, \ldots, e_n \) is a basis for \( V \) and \( f_1, \ldots, f_m \) a basis for \( W \), then
\[
\psi_P(\langle e_1, \ldots, e_n \rangle, \langle f_1, \ldots, f_m \rangle) = \langle g_1, \ldots, g_{n+m} \rangle,
\]
where \( g_i = e_i \) and \( g_j^c = f_i \). From this, it follows that if \( E_i, E'_i \in \mathbb{F}\ell V \) and \( F_i, F'_i \in \mathbb{F}\ell W \) are pairs of opposite flags, then \( \psi_P(E_i, F_i) \) and \( \psi_{\omega_{n+m}P}(E'_i, F'_i) \) are opposite flags in \( V \oplus W \).

**Lemma 4.5.1.** Let \( P, Q \in \binom{[n+m]}{n} \), \( v \in \mathcal{S}_n \), and \( w \in \mathcal{S}_m \). Then, for \( E \in \mathbb{F}\ell V \) and \( F \in \mathbb{F}\ell W \),
\[
\psi_P(X_{\omega_{n+m}P}E \times X_{\omega_{n+m}Q}F) = X_{\omega_{n+m}P,Q}(v,w)\psi_Q(E, F),
\]
\[
\psi_P(X_{\psi_P}E \times X_{\psi_P}F) = X_{\psi_P,Q}(v,w)\psi_Q(E, F).
\]

**Proof.** For a flag \( G_i \), define \( G_i^0 := G_i - G_{i-1} \). By the definition of \( \psi_Q \), we have \( E_i^0 \subset \psi_Q(E_i, F_i)^0 \), and \( F_i^0 \subset \psi_Q(E_i, F_i)^0 \). Since
\[
\omega_{n+m}Q = n + m + 1 - q_n < \cdots < n + m + 1 - q_1,
\]
\[
E_{n+1-j} \subset \psi_{\omega_{n+m}Q}(E_j, F_j)^{n+m+1-q_j},
\]
\[
F_{n+1-j} \subset \psi_{\omega_{n+m}Q}(E_j, F_j)^{n+m+1-q_j},
\]
the lemma follows from the definitions of Schubert varieties and \( \psi_P \). \( \square \)
Corollary 4.5.2. Let $E_., E'_\in \mathbb{F}\ell V$ and $F_., F'_\in \mathbb{F}\ell W$ be pairs of opposite flags and let $P \in \left\{ m+1, \ldots, m+n \right\}$. Set $Q = \left\{ m + 1, \ldots, m + n \right\}$. Then, for every $v \in \mathcal{S}_n$ and $w \in \mathcal{S}_m$,

$$\psi_P(X_v E_\times X_w F_.) = X_{e_P,|n|}(v,w) \psi_Q(E_., F_.) \cap X_{e_P,|n|}(e_n, F_.)$$

$$= X_{e_P,|n|}(e_n, F_.) \cap X_{e_P,|n|}(e_n, F_.)$$

$$= X_{e_P,|n|}(e_n, F_.) \cap X_{e_P,|n|}(e_n, F_.)$$

$$= X_{e_P,|n|}(e_n, F_.) \cap X_{e_P,|n|}(e_n, F_.).$$

Proof. Since $\omega_{n+m}[n] = Q$, $X_v E_\times \mathbb{F}\ell V$, and $X_v F_\times \mathbb{F}\ell W$, Lemma 4.5.1 shows that $\psi_P(X_v E_\times X_w F_.)$ is a subset of any of the four intersections. Equality follows, as they have the same dimension. Indeed, for $x, z \in \mathcal{S}_n$ and $y, u \in \mathcal{S}_m$,

$$\ell(e_P,|n|)(x, y) = \ell(x) + \ell(y) + \# \{ i \in [n], j \in [m] \mid p_i > p_j \},$$

$$\ell(e_P,|n|)(z, u) = \ell(z) + \ell(u) + \# \{ i \in [n], j \in [m] \mid p_i > p_j \}. $$

Thus, $\ell(e_P,|n|)(x, y) + \ell(e_P,|n|)(z, u) = \ell(x) + \ell(y) + \ell(z) + \ell(u) + n \cdot m$, and so

$$\left( \begin{array}{c} n + m \\ 2 \end{array} \right) - \ell(e_P,|n|)(x, y) - \ell(e_P,|n|)(z, u) = \left( \begin{array}{c} n \\ 2 \end{array} \right) - \ell(x) - \ell(y) - \ell(z) - \ell(u).$$

If $(x, y, z, u)$ is one of $(v, w, e, e)$, $(v, e, e, w)$, $(e, w, v, e)$, and $(e, e, v, w)$, then the left-hand side is the dimension of the corresponding intersection, and the right-hand side is the dimension of $X_v E_\times X_w F_..$

Corollary 4.5.3. Let $Q = \left\{ m + 1, \ldots, m + n \right\} = \omega_{n+m}[n]$. For every $v \in \mathcal{S}_n$, $w \in \mathcal{S}_m$, and $P \in \left\{ m+1, \ldots, m+n \right\}$, the identities hold in $H^1(\mathbb{F}\ell^{n+m})$:

$$\mathbb{S}_{e_P,|n|}(v, w) \cdot \mathbb{S}_{e_P,|n|}(e, e) = \mathbb{S}_{e_P,|n|}(v, e) \cdot \mathbb{S}_{e_P,|n|}(e, w)$$

$$= \mathbb{S}_{e_P,|n|}(e, w) \cdot \mathbb{S}_{e_P,|n|}(e, e) = \mathbb{S}_{e_P,|n|}(e, e) \cdot \mathbb{S}_{e_P,|n|}(w, w),$$

and this common cohomology class is $(\psi_P)_*(\mathbb{S}_v \otimes \mathbb{S}_w)$.

Theorem 4.5.4. Let $x \in \mathcal{S}_{n+m}$ and $P \in \left\{ m+1, \ldots, m+n \right\}$. Then we have the following.

(i) Let

$$\psi_P \mathbb{S}_x = \sum_{e \in \mathcal{S}_n, w \in \mathcal{S}_m} c_{e_P,|n|}(e, w) \mathbb{S}_v \otimes \mathbb{S}_w$$

$$= \sum_{e \in \mathcal{S}_n, w \in \mathcal{S}_m} c_{e_P,|n|}(e, w) \mathbb{S}_v \otimes \mathbb{S}_w.$$
(ii) Let $Q = \{m + 1, \ldots, m + n\}$. For every $v \in S_n$ and $w \in S_m$, we have

$$
\sum_{v \in S_n, w \in S_m} \epsilon_{P, \omega_n, \omega_m}^{v, w} \otimes \mathbb{G}_v \otimes \mathbb{G}_w,
$$

and

$$
\sum_{v \in S_n, w \in S_m} \epsilon_{P, \omega_n, \omega_m}^{v, w} \otimes \mathbb{G}_v \otimes \mathbb{G}_w.
$$

**Remark 4.5.5.** Each structure constant in (ii) is of the form $c_{y, z}^x$, where $\zeta$ is one of $v \times w, v \times w^{-1}, v^{-1} \times w, \text{ or } v^{-1} \times w^{-1}$. Each interval $[y, z]$ is isomorphic to $[e, v] \times [e, w]$. This is consistent with the expectation that the $c_{y, z}^x$ should only depend upon $y, z$ and $x$.

**Proof.** In (ii), the second row is a consequence of the first, as $c_{y, z}^x = c_{\omega_n, \omega_m}^{\omega_n, \omega_m}$ for $x, y, z \in S_{n+m}$, and the first row is a consequence of the identities in (i). For (i), there exist constants $d_{x, w}^{w, w}$ defined by

$$
\psi^* \mathbb{G}_x = \sum d_{x, w}^{w, w} \otimes \mathbb{G}_w.
$$

Since the Schubert basis is self-dual (2.3.1), we have

$$
d_{x, w}^{w, w} = \deg \left( \psi^* \mathbb{G}_x \cdot (\mathbb{G}_{\omega_n} \otimes \mathbb{G}_{\omega_m}) \right) = \deg \left( \mathbb{G}_x \cdot (\psi^*_x \mathbb{G}_{\omega_n} \otimes \mathbb{G}_{\omega_m}) \right).
$$

Each expression for $(\psi^*_x \mathbb{G}_{\omega_n} \otimes \mathbb{G}_{\omega_m})$ of Corollary 4.5.3 yields one of the sums in (i). For example, the last expression yields

$$
d_{x, w}^{w, w} = \deg \left( \mathbb{G}_x \cdot e_{P, \omega_n, \omega_m}^{v, w} \cdot \mathbb{G}_{\omega_n + m e_{P, n}} \right) = c_{P, \omega_n, \omega_m}^{v, w},
$$

since $\omega_n + m e_{P, n} (v, w) = e_{P, \omega_n + m} (\omega_n v, \omega_m w)$.

**4.6. Maps** $\mathbb{Z}[x_1, x_2, \ldots] \to \mathbb{Z}[y_1, y_2, \ldots, z_1, z_2, \ldots]$. Let $P \subseteq \mathbb{N}$, define $P^c := \mathbb{N} - P$, and suppose $P^c$ is infinite. List $P$ and $P^c$:

$$
P : p_1 < p_2 < \left\{ \ldots < p_s \text{ if } \# P = s \right\}, \text{ otherwise,}
$$

$$
P^c : p_1^c < p_2^c < \ldots.
$$
Define \( \Psi_P : \mathbb{Z}[x_1, x_2, \ldots] \rightarrow \mathbb{Z}[y_1, y_2, \ldots, z_1, z_2, \ldots] \) by
\[
x_{p_1} \mapsto y_1, \quad x_{p_i}^c \mapsto z_i.
\]
Then there exist \( d_{w}^{uv}(P) \in \mathbb{Z} \) for \( u, v, w \in \mathcal{S}_\infty \) defined by
\[
\Psi_P(\mathcal{S}_w(x)) = \sum_{u,v} d_{w}^{uv}(P) \mathcal{S}_u(y) \mathcal{S}_v(z).
\]

For \( l, d \in \mathbb{N} \) and \( R \subset \{d + 1, \ldots, d + 2l\} \) with \( \# R = l \), define \( \tilde{P}(l, d, R) := (P \cap [d]) \cup R \).

**Theorem D'.** Let \( P \in \mathbb{N} \) and \( w \in \mathcal{S}_\infty \). For any integers \( l > \ell(w) \) and \( d \) exceeding the last descent of \( w \) and any subset \( R \) of \( \{d + 1, \ldots, d + 2l\} \) of cardinality \( l \), set \( n := \# \tilde{P}(l, d, R) \), \( m := d + 2l - n \), and \( \pi := e_{\tilde{P}(l,d,R)}[n] e_{e} \). Then \( d_{w}^{uv}(P) = 0 \), unless \( u \in \mathcal{S}_n \) and \( v \in \mathcal{S}_m \), and in that case,
\[
d_{w}^{uv}(P) = c_{\pi}^{(u \times v)\pi}.
\]
Moreover, \( d_{w}^{uv}(P) \neq 0 \) implies that \( a := \# P \cap [d] \) exceeds the last descent of \( u \), and \( d - a \) exceeds the last descent of \( v \).

**Remark 4.6.1.** Theorem D' generalizes [29, 1.5] (see also [35, 4.19]) where it is shown that \( d_{0}(\mathcal{S}_3) > 0 \). Define \( I_P \) to be
\[
\{e_{\tilde{P}(l,d,R)}[n] e_{e}, l \in \mathbb{N}, n = l + \#(P \cap [l]), R \subset \{l + 1, \ldots, 3l\}, \# R = l\}.
\]
For \( w \in \mathcal{S}_n \), choose \( N \in \mathbb{N} \) so that \( N/3 \) exceeds both the \( \ell(w) \) and the last descent of \( w \). If \( \pi \in I_P \) with \( \pi \notin \mathcal{S}_N \), then \( \pi = e_{\tilde{P}(l,d,R)}[n] e_{e} \) for \( l, d, R \) satisfying the conditions of Theorem D'. So \( d_{w}^{uv}(P) = c_{\pi w}^{(u \times v)\pi} \) for \( \pi \in I_P - \mathcal{S}_N \), which establishes Theorem D.

Apply the ring homomorphism \( \Psi_P \) to both sides of the product
\[
\mathcal{S}_w(x) \mathcal{S}_y(x) = \sum_{\zeta} c_{\pi w}^{\zeta} \mathcal{S}_\zeta(x).
\]
Expand this in terms of \( \mathcal{S}_\eta(y) \mathcal{S}_\xi(z) \), and equate coefficients to obtain the following corollary.

**Corollary 4.6.2.** Let \( w, \gamma, \eta, \xi \in \mathcal{S}_\infty \), and \( P \subset \mathbb{N} \). Then there exists an integer \( N \in \mathbb{N} \) such that if \( \pi \in I_P - \mathcal{S}_N \), then
\[
\sum_{\zeta} c_{\pi \xi}^{(\eta \times \zeta)\pi} c_{\pi \gamma}^{\zeta} = \sum_{u,v,\alpha,\beta} c_{\pi \pi}^{(u \times v)\pi} c_{\eta \gamma}^{(\alpha \times \beta)\pi} c_{\xi}^{u \alpha} c_{\xi}^{v \beta}.
\]
Proof of Theorem D’. First, $\Sigma_n(x) \in \mathbb{Z}[x_1, \ldots, x_d]$ whenever $s$ exceeds the last descent of $\pi$ (see [28]; see also [35, 4.13]). Thus, $\Sigma_n(x) \in \mathbb{Z}[x_1, \ldots, x_d]$. So if $d^w(P) \neq 0$, then $\Sigma_n(y) \in \mathbb{Z}[y_1, \ldots, y_d]$ and $\Sigma_n(z) \in \mathbb{Z}[z_1, \ldots, z_b]$. Hence $a$, respectively, $b$, exceeds the last descent of $u$, respectively, $v$. Since $\deg \Sigma_n(x) < l$, both $\deg \Sigma_u(y)$ and $\deg \Sigma_v(z)$ are at most $l$. Consider the commutative diagram

\[
\begin{array}{ccc}
\mathbb{Z}[x_1, \ldots, x_d] & \xrightarrow{\iota} & \mathbb{Z}[x_1, \ldots, x_{n+m}] \\
\downarrow & & \downarrow \\
H^*F_{\ell_{n+m}} & \xrightarrow{\Psi_p^*} & H^*F_{\ell_{n}} \otimes H^*F_{\ell_{m}}.
\end{array}
\]

Here, $\Psi_p^*$ is the restriction of $\Psi_p$ to $\mathbb{Z}[x_1, \ldots, x_{n+m}]$. The vertical arrows are injective on the module $\mathbb{Z} \langle x_1^{a_1} \cdots x_d^{a_d} | x_i < l \rangle$ and its image $\mathbb{Z} \langle y_1^{b_1} \cdots y_d^{b_d} \cdots z_b^{c} | y_j < l \rangle \subset \mathbb{Z}[y_1, \ldots, y_d, z_1, \ldots, z_m]$. Moreover, as $P \cap [d] = \mathbb{P} \cap [d]$, the composition $\Psi_p \circ \iota$ of the top row coincides with $\Psi_p \circ \iota$. Since $\Sigma_n(x) \in \mathbb{Z} \langle x_1^{a_1} \cdots x_d^{a_d} | x_i < l \rangle$, the formula for $\Psi_p^*(\Sigma_n)$ in Theorem 4.5.4 computes $\Psi_p^*(\Sigma_n)$.

4.7. Products of Grassmannians. Let $k \leq n$ and $l \leq m$ be integers, $V \simeq \mathbb{C}^n$ and $W \simeq \mathbb{C}^m$. Define $\varphi_{k,l} : \text{Grass}_k V \times \text{Grass}_l W \rightarrow \text{Grass}_{k+l}(V \oplus W)$ by

$\varphi_{k,l} : (H, K) \mapsto H \oplus K$.

**Theorem 4.7.1**

(i) For every Schubert class $S_\lambda \in H^* \text{Grass}_{k+l} V \oplus W$,

$\varphi^*_{k,l}(S_\lambda) = \sum_{\mu \nu} c^\lambda_{\mu \nu} S_\mu \otimes S_\nu$.

(ii) If $S_\mu \otimes S_\nu \in H^* \text{Grass}_k V \otimes H^* \text{Grass}_l W$, then

$$(\varphi_{k,l})_*(S_\mu \otimes S_\nu) = \sum_\lambda c^\lambda_{\mu \nu} S_\lambda,$$

where $\lambda^c$, $\mu^c$, and $\nu^c$ are defined by $\mu^c_{i} = n - k - \mu_k+1-1$, $\nu^c_{i} = m - l - \nu_l+1-1$, and $\lambda^c_i = m + n - k - l - \lambda_k+1+1-1$.

**Remark 4.7.2.** Suppose $-x_1, \ldots, -x_k$ are Chern roots of the tautological bundle over $\text{Grass}_k V$, $-y_1, \ldots, -y_l$ those of the tautological bundle over $\text{Grass}_l W$, and $f \in H^* \text{Grass}_{k+l} V \oplus W$ (which is a symmetric polynomial in the
negative Chern roots of the tautological bundle over $\text{Grass}_{k+l} V \oplus W$). Then

$$\varphi_{k,l}^* f = f(x_1, \ldots, x_k, y_1, \ldots, y_l).$$

Let $\Lambda = \Lambda(\mathbf{z})$ be the ring of symmetric functions, which is the inverse limit (in the category of graded rings) of the rings of symmetric polynomials in the variables $z_1, \ldots, z_n$. Fixing $\lambda$ and choosing $k, l, n,$ and $m$ large enough gives a new proof of the following proposition.

**Proposition 4.7.3** [36, I.5.9]. Let $\lambda$ be a partition and $x, y$ be infinite sets of variables. Then

$$S_{\lambda}(x, y) = \sum_{\mu, \nu} c_{\mu \nu}^\lambda S_{\mu}(x) \cdot S_{\nu}(y),$$

where $S_{\mu}$ are Schur functions in the ring $\Lambda$ of symmetric functions.

If we define a linear map $\Delta : \Lambda(\mathbf{z}) \rightarrow \Lambda(\mathbf{x}) \otimes \Lambda(\mathbf{y})$ by $\Delta(f(\mathbf{z})) = f(x, y)$, then $\Delta$ is induced by the maps $\varphi_{k,l}^*$. Moreover, the obvious commutative diagrams of spaces give a new proof of [36, I.5.25]: $\Lambda$ is a cocommutative Hopf algebra with comultiplication $\Delta$.

**Proof of Theorem 4.7.1.** The first statement is a consequence of the second. Schubert classes form a basis for the cohomology ring, so there exist integral constants $d_{\lambda}^{\mu \nu}$ such that

$$\varphi_{k,l}^*(S_{\lambda}) = \sum_{\mu, \nu} d_{\lambda}^{\mu \nu} S_{\mu} \otimes S_{\nu}.$$  

Since the Schubert basis diagonalizes the intersection pairing,

$$d_{\lambda}^{\mu \nu} = \deg(\varphi_{k,l}^*(S_{\lambda}) \cdot (S_{\mu} \otimes S_{\nu})).$$

Apply $(\varphi_{k,l})_*$ and use the second assertion to obtain

$$d_{\lambda}^{\mu \nu} = \deg(S_{\lambda} \cdot (\varphi_{k,l})_*(S_{\mu} \otimes S_{\nu})).$$

$$= S_{\lambda} \cdot \sum_{\kappa} c_{\mu \nu}^\kappa S_{\kappa}$$

$$= c_{\mu \nu}^\lambda.$$

The second assertion is a consequence of the following lemma.

**Lemma 4.7.4.** Suppose $\mu, \nu$ are partitions with $\mu \subseteq (n-k)^k$ and $\nu \subseteq (m-l)^l$. Let $E, F \in \mathcal{F} \ell V$ and $F, E \in \mathcal{F} \ell W$, and let $G'$ be any flag opposite to $\psi_{[n]}(E, F)$ with
\(G_m' = W. \) Then
\[
\varphi_{k,l}(\Omega_{\mu^c}E_\times \Omega_{\nu^c}F_\times) = \Omega_{\rho^c}\psi_{[n]}(E_\times F_\times) \cap \Omega_{(n-k)}G',
\]
(4.7.1)
where \(\rho\) is the partition
\[
v_1 + (n-k) \geq \cdots \geq v_l + (n-k) \geq \mu_1 \geq \cdots \geq \mu_k.
\]
We finish the proof of Theorem 4.7.1. Lemma 4.7.4 implies
\[
(g_{k,l})_*(S_{\mu^c} \otimes S_{\nu^c}) = \left[\Omega_{\rho^c}\psi_{[n]}(E_\times F_\times) \cap \Omega_{(n-k)}G'\right] = \sum_{\lambda} c_{\rho^c(n-k)}^\lambda S_\lambda^c.
\]
Since \(\text{deg}(S_\alpha \cdot S_\beta \cdot S_{\nu}) = c_{\mu \nu}^\lambda\), we see that
\[
c_{\rho^c(n-k)}^\lambda = c_{\mu \nu}^\lambda = c_{\mu \nu}^\lambda = c_{\mu \nu}^\lambda.
\]
Here, \(\mu \bigcup_{\nu} \nu\) is a skew partition with two components \(\mu\) and \(\nu\), and the last equality is a special case of (1.3.1) in Section 1.3.

Proof of Lemma 4.7.4. Since
\[
\Omega_{(n-k)}G' = \{M \in \text{Grass}_{k+1}V \oplus W | \dim M \cap G_m' \geq l\}
\]
and \(G_m' = W\), we see that \(\varphi_{k,l}(\text{Grass}_kV \times \text{Grass}_lW) \subset \Omega_{(n-k)}G'\). The inclusion in (4.7.1) follows, as the definitions imply
\[
\varphi_{k,l}(\Omega_{\mu^c}E_\times \Omega_{\nu^c}F_\times) = \Omega_{\rho^c}\psi_{[n]}(E_\times F_\times).
\]
Equality follows, as the cycles have the same dimension. The intersection has dimension \(|\mu| - |(n-k)| = |\mu| + |\nu| = \dim \Omega_{\mu^c}E_\times \Omega_{\nu^c}F_\times\). \(\square\)

5. Identities among the \(c_{\mu \nu}^\lambda\).

5.1. Proof of Theorem E (ii). Combining Lemma 4.3.4 with Theorem C(i)(b), we deduce the following lemma.

Lemma 5.1.1. Suppose \(x \leq_k z\) and \(x(p) = z(p)\). Let \(k' = k - 1\) if \(p < k\) and \(k' = k\), otherwise. Then for all partitions \(\lambda\), we have
\[
c_{x \nu(\lambda,k)} = c_{x \nu(\lambda,k')}.\]
By Lemma 4.1.1(iv), \(zx^{-1}\) and \(z/\nu(x/p)^{-1}\) are shape-equivalent.
**Lemma 5.1.2.** Let \( x, z, u, w \in \mathcal{S}_n \). Suppose \( x \leq_k z \), \( u \leq_k w \), and \( zx^{-1} = wu^{-1} \). Further suppose that \( w \) is Grassmannian with descent \( k \), the permutation \( wu^{-1} \) has no fixed points, and, for \( k < i \leq n \), \( u(i) = x(i) \). Then, for all partitions \( \lambda \) with at most \( k \) parts,

\[
c^w_{u(\lambda, k)} = c^x_{v(\lambda, k)}.\]

**Proof of Theorem E(ii) using Lemma 5.1.2.** We reduce Theorem E(ii) to Lemma 5.1.2. First, by Lemma 5.1.1, it suffices to prove Theorem E(ii) when \( x, z, u, w \in \mathcal{S}_n \), \( k = 1 \), with \( wu^{-1} = zx^{-1} \) and the permutation \( wu^{-1} \) has no fixed points.

Define \( s \in \mathcal{S}_n \) by

\[
s(i) := \begin{cases} u(i) & 1 \leq i \leq k, \\ x(i) & k < i \leq n, \end{cases}
\]

and set \( t := wu^{-1}s \). Then \( s \leq_k t \) and

\[
t(i) = \begin{cases} w(i) & 1 \leq i \leq k, \\ z(i) & k < i \leq n. \end{cases}
\]

It suffices to show that \( c^w_{u(\lambda, k)} \) and \( c^t_{v(\lambda, k)} \) each equal \( c^x_{v(\lambda, k)} \). Thus we may further assume \( u(i) = x(i) \) for \( 1 \leq i \leq k \) or \( u(i) = x(i) \) for \( k < i \leq n \).

Suppose that \( u(i) = x(i) \) for \( 1 \leq i \leq k \). If for \( \nu \in \mathcal{S}_n \), \( \nu := \omega_0 \nu \omega_0 \),

\[
c^x_{v(\lambda, k)} = c^w_{u(\lambda, k)} \iff c^x_{s(\lambda, k)} = c^w_{u(\lambda, k)}.\]

Set \( l = n - k \) and \( \lambda^t \) the partition conjugate to \( \lambda \). Then \( \tilde{x} \leq_k \tilde{z} \), \( \tilde{u} \leq_k \tilde{w} \), \( \tilde{z}(\tilde{x}^{-1}) = \tilde{w}\tilde{u}^{-1} \), \( \tilde{v}(\lambda, k) = v(\lambda^t, l) \), and \( \tilde{x}(i) = \tilde{u}(i) \) for \( l < i \leq n \). Thus we may assume \( x(i) = u(i) \) for \( 1 \leq i \leq k \).

Finally, there is a permutation \( s \in \mathcal{S}_n \) such that \( wu^{-1}s \) is Grassmannian of descent \( k \). Thus it suffices to further assume that \( w \) is Grassmannian with descent \( k \), the situation of Lemma 5.1.2. \( \square \)

We prove Lemma 5.1.2 by studying two intersections of Schubert varieties and their image under the projection \( F_{\mathcal{E}} V \to \text{Grass}_k V \). Let \( e_1, \ldots, e_n \) be a basis for \( V \), and set \( F. = \langle\langle e_1, \ldots, e_n\rangle\rangle \). Let \( M(w) \subset M_{nxn} \mathbb{C} \) be the set of matrices satisfying the conditions

(a) \( M(w)_{i, w(i)} = 1 \),

(b) \( M(w)_{i, j} = 0 \) if either \( w(i) < j \) or else \( w^{-1}(j) < i \).

Then \( M(w) \simeq \mathbb{C}^{|\ell'(w)|} \) because the only unconstrained entries of \( M(w) \) are \( M(w)_{i, j} \) when \( j < w(i) \) and \( i < w^{-1}(j) \), and there are \( \ell'(w) \) such entries.
Example 5.1.3. \( M(25134) \) is the set of matrices

\[
\begin{bmatrix}
  a & 1 & 0 & 0 & 0 \\
  b & 0 & c & d & 1 \\
  1 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0
\end{bmatrix}
\]

Fix a basis \( e_1, \ldots, e_n \) for \( V \). For \( \alpha \in M(w) \) and \( 1 \leq i \leq n \), define the vector \( f_i(\alpha) := \sum_j \alpha_{i,j} e_j \). Then \( f_1(\alpha), \ldots, f_n(\alpha) \) are the “row vectors” of the matrix \( \alpha \), and they form a basis for \( V \) as \( \alpha \) has determinant \((-1)^e(\alpha)\). Set \( E_\alpha(\alpha) := \langle f_1(\alpha), \ldots, f_n(\alpha) \rangle \). Since \( f_1(\alpha) \in F(W) - F(W) - 1 \), we see that \( E_\alpha(\alpha) \in X_{\alpha \circ \alpha}(F) \). In fact, \( M(w) \) parameterizes the Schubert cell \( X_{\alpha \circ \alpha}(F) \). When \( w \) is Grassmannian with descent \( k \), matrices in \( M(w) \) have a simple form: if \( k < i \), then \( f_i(\alpha) = e_{w(i)} \).

For opposite flags \( F, F' \), \( \mathcal{G}_{\alpha \circ \alpha} \cdot \mathcal{G}_u \) is the class Poincaré dual to the fundamental cycle of \( X_{\alpha \circ \alpha}(F) \cap X_u F' \). We use the projection formula (2.3.2) to compute the coefficient \( c_{u \circ \alpha}(\lambda, k) \) as

\[
c_{u \circ \alpha}(\lambda, k) = \deg(S_{\lambda}(x_1, \ldots, x_k) \cdot \mathcal{G}_{\alpha \circ \alpha} \cdot \mathcal{G}_u)
= \deg(\pi_k)_*(S_{\lambda}(x_1, \ldots, x_k) \cdot \mathcal{G}_{\alpha \circ \alpha} \cdot \mathcal{G}_u)
= \deg(S_{\lambda} \cdot (\pi_k)_*(\mathcal{G}_{\alpha \circ \alpha} \cdot \mathcal{G}_u)).
\]

Thus Lemma 5.1.2 is a consequence of Lemma 5.1.4, which shows

\[
(\pi_k)_*(\mathcal{G}_{\alpha \circ \alpha} \cdot \mathcal{G}_u) = (\pi_k)_*(\mathcal{G}_{\alpha \circ \alpha} \cdot \mathcal{G}_x).
\]

**Lemma 5.1.4** Let \( u, w, x, z \) satisfy the hypotheses of Lemma 5.1.2. Then, if \( F \) and \( F' \) are opposite flags in \( V \),

\[
\pi_k(X_{\alpha \circ \alpha}(F) \cap X_u F') = \pi_k(X_{\alpha \circ \alpha}(F) \cap X_x F')
\]

and the projections \( \pi_k \) onto the image cycle have the same degree.

**Proof.** Let \( e_1, \ldots, e_n \) be a basis for \( V \) such that \( F = \langle \langle e_1, \ldots, e_n \rangle \rangle \) and \( F' = \langle \langle e_n, \ldots, e_1 \rangle \rangle \), and define \( M(w) \) as before. Let \( A \subset M(w) \) consist of those matrices \( \alpha \) such that \( E_\alpha(\alpha) \in X_{\circ \alpha}(F) \). If \( j > k \), set \( g_j(\alpha) = \epsilon_{w(j)} \). For \( j < k \) construct \( g_j(\alpha) \) inductively, setting \( g_j(\alpha) \) to be the intersection of \( F_{n+1-u(j)} \) and the affine space \( f_j(\alpha) + \langle g_i(\alpha) | i < j \text{ and } u(i) < u(j) \rangle \). Since \( E_\alpha(\alpha) \in X_{u \circ \alpha}(F) \) and \( \dim E_\alpha(\alpha) \cap F_{n+1-u(j)} = \# \{ i \leq j \text{ and } u(i) > u(j) \} \), this intersection consists of a single, nonzero vector \( g_j(\alpha) \).

The algebraic map \( A \ni \alpha \mapsto (g_1(\alpha), \ldots, g_n(\alpha)) \in V^n \) parameterizes a basis of \( V \). Moreover, for \( \alpha \in A \), \( E_\alpha(\alpha) = \langle \langle g_1(\alpha), \ldots, g_n(\alpha) \rangle \rangle \), and if \( 1 \leq j \leq k \), then
\( g_j(x) \in F'_{n+1-u(f)} \cap F_{w(f)} \). For \( x \in A \),

\[
G_\alpha := \langle g_{u^{-1}x(1)}(\alpha), \ldots, g_{u^{-1}x(n)}(\alpha) \rangle \in X_{o_{\alpha+2}} F \cap X_{x} F',
\]  

(5.1.1)

and thus \( A \) parameterizes a subset of \( X_{o_{\alpha+2}} F \cap X_x F' \). Indeed, for \( 1 \leq j \leq k \),

\( g_{u^{-1}x(j)} \in F'_{n+1-x(j)} \cap F_{y(j)} \). Also for \( j > k \), we have \( u^{-1}x(j) = j = w^{-1}y(j) \); thus \( g_j(x) = f_j(x) = e_j(x) \) and \( G_j(\alpha) = E_j(\alpha) \). Then the definition of Schubert varieties in Section 2.3 implies (5.1.1).

Both cycles \( X_{o_{\alpha+2}} F \cap X_x F' \) and \( X_{o_{\alpha+2}} F \cap X_x F' \) are irreducible and have the same dimension, \( \ell(w) - \ell(u) = |w^{-1}| \). Since \( G(\alpha) = G(\beta) \) if and only if \( \alpha = \beta \), the loci of flags \( \{ G(\alpha) \mid \alpha \in A \} \) is dense in \( X_{x} F' \cap X_{o_{\alpha+2}} F \). Also, for \( \alpha \in A \), we have \( G_k(\alpha) = E_k(\alpha) \), as \( u^{-1}x \) permutes \( \{1, \ldots, k\} \), showing the equality of the image cycles. Since the association \( E(\alpha) \mapsto G(\alpha) \) induces a rational map

\[
X_{o_{\alpha+2}} F \cap X_x F' \rightrightarrows X_{o_{\alpha+2}} F \cap X_x F'.
\]

covering the projections \( \pi_k \), these projections have the same degree. This completes the proof. \( \square \)

5.2. Proof of Theorem G (ii). We show that if \( \zeta \) and \( \eta \) are disjoint permutations and \( \lambda \) is any partition, then

\[
c_{\lambda}^{\zeta \eta} = \sum_{\mu, \nu} c_{\mu}^{\zeta} c_{\nu}^{\eta}.
\]

**Lemma 5.2.1.** Let \( \zeta, \eta \in \mathscr{S}_{n+m} \) be disjoint permutations. Suppose \( k \geq \# \text{up} \zeta, l \geq \# \text{up} \eta, n \geq \# \text{supp} \zeta \), and \( m \geq \# \text{supp} \eta \). Let \( u \in \mathscr{S}_{n+m} \) be a permutation such that \( u \leq k+1 \zeta l \). Let \( \mathcal{Q} \) be any element of \( \binom{[n+m]}{\text{supp} \zeta} \) that contains \( \text{supp} \zeta \) for which \( k \# u^{-1}(\mathcal{Q}) \cap [k+1] \). Set \( \mathcal{Q}^{c} := [n+m] - \mathcal{Q} \).

Define \( \zeta' \in \mathscr{S}_n \) and \( \eta' \in \mathscr{S}_m \) by \( \phi_\mathcal{Q}(\zeta') = \zeta \) and \( \phi_\mathcal{Q}(\eta') = \eta \). Set \( P = u^{-1}(\mathcal{Q}) \), \( P^{c} = u^{-1}(\mathcal{Q}^{c}) \), and define \( v \in \mathscr{S}_n \) and \( w \in \mathscr{S}_m \) by \( u(p_i) = q_v(i) \) and \( u(p^{c}_i) = q_w(i) \), where

\[
P = p_1 < p_2 < \cdots < p_n, \quad P^{c} = p_1^{c} < p_2^{c} < \cdots < p_m^{c},
\]

\[
Q = q_1 < q_2 < \cdots < q_n, \quad Q^{c} = q_1^{c} < q_2^{c} < \cdots < q_m^{c}.
\]

Then we have the following:

(i) \( v \leq_k \zeta' v \) and \( w \leq \eta'w \);

(ii) \( u = e_{P,Q}(v, w) \) and \( \zeta \eta u = e_{P,Q}(\zeta' v, \eta'w) \);

(iii) for all pairs of opposite flags \( E, E' \in \mathbb{F} \ell_n \) and \( F, F' \in \mathbb{F} \ell_m \),

\[
\psi_F \left[ (X_{o_{\alpha+2}} E_v \cap X_v E'_v) \times (X_{o_{\alpha+2}} \eta' w F \cap X_w F'_w) \right]
\]

\[
= X_{o_{n+m} \zeta \eta \mu} \psi_Q(E, F) \cap X_{u} \psi_{o_{n+m} \mathcal{Q}}(E', F').
\]
Proof. Since $\zeta \leq k+1 \zeta u$, (i) follows from Theorem A. Statement (ii) is also immediate. For (iii), Lemma 4.5.1 shows the inclusion $\subset$. Since $\xi'$ is shape-equivalent to $\xi$, as is $\eta'$ to $\eta$, and $\xi$ and $\eta$ are disjoint, $|\xi\eta| = |\xi'| + |\eta'|$. This shows that both cycles have the same dimension, and hence are equal, as $\psi^n Q(E_*, F_*)$ and $\psi_{\omega(\zeta \eta)} Q(E'_*, F'_*)$ are opposite.

Note that if $u \leq k \zeta u$, then
\[ c^t_\lambda = \deg(S_{\lambda} \cdot (\pi_k)_* (\mathbb{S}_{\omega_0 \zeta u} \cdot \mathbb{S}_u)). \]
Thus the skew coefficients $c^t_\lambda$ are defined by the identity in $H^* Grass_k V$,
\[ (\pi_k)_* (\mathbb{S}_{\omega_0 \zeta u} \cdot \mathbb{S}_u) = \sum_{\lambda \in (n-k)^k} c^t_\lambda S_{\lambda^t}. \tag{5.2.1} \]

Proof of Theorem G (ii). We use the notation of Lemma 5.2.1. The following diagram commutes, since $[k+1] = \{p_1, \ldots, p_k, p'_1, \ldots, p'_l\}$:
\[
\begin{array}{ccc}
\mathbb{F}^{l+n} \times \mathbb{F}^{l+m} & \xrightarrow{\psi_p} & \mathbb{F}^{l+n+m} \\
\pi_k \times \pi_l \downarrow & & \downarrow \pi_{k+l} \\
Grass_k \mathbb{C}^n \times Grass_l \mathbb{C}^m & \xrightarrow{\varphi_{k,l}} & Grass_{k+l} \mathbb{C}^{n+m}.
\end{array}
\]
From this and Lemma 5.2.1, we see that
\[ \pi_{k+l} \left( X_{\omega_{n+m} \zeta u} \psi^n Q(E_*, F_*) \cap X_u \psi^{\omega_0 (m+n)} Q(E'_*, F'_*) \right) \]
is equal to
\[ \varphi_{k,l} \left( (\pi_k)_* \left( X_{\omega_{n} \zeta u} E_* \cap X_u E'_* \right) \times \pi_l \left( X_{\omega_{n+m} \eta u} F_* \cap X_{\omega_{n} \eta' u} F'_* \right) \right). \]
Thus $(\pi_{k+l})_* (\mathbb{S}_{\omega_{n+m} \zeta u} \cdot \mathbb{S}_u)$ is equal to
\[ (\varphi_{k,l})_* ((\pi_k)_* (\mathbb{S}_{\omega_{n} \zeta u} \cdot \mathbb{S}_u) \otimes (\pi_l)_* (\mathbb{S}_{\omega_{n+m} \eta u} \cdot \mathbb{S}_u)). \]
This, together with (5.2.1) and Theorem 4.7.1(ii), gives
\[ \sum_{\lambda} c^t_{\lambda} S_{\lambda^t} = (\pi_{k+l})_* (\mathbb{S}_{\omega_{n+m} \zeta u} \cdot \mathbb{S}_u) \]
\[ = (\varphi_{k,l})_* \left( \sum_{\mu} c^t_{\mu} S_{\mu^t} \otimes \sum_{v} c^v_{\nu} S_{\nu^v} \right) \]
\[ \sum_{\mu, \nu} \binom{\lambda}{\mu} \binom{\nu}{\lambda} (\varphi_{k, l})_{\mu} S_{\mu} \otimes S_{\nu} \]

\[ = \sum_{\mu, \nu} \binom{\lambda}{\mu} \binom{\nu}{\lambda} \sum_{\lambda} \binom{\lambda}{\mu} \binom{\lambda}{\nu} S_{\lambda}. \]

We are done, as \zeta', \zeta and \eta', \eta are shape-equivalent pairs. \(\square\)

5.3. Cyclic shift

**Theorem H' (Cyclic shift).** Let \(u, w, x, z \in \mathcal{S}_n\) with \(u \leq_k w\) and \(x \leq_1 z\). Suppose \(wu^{-1} \in \mathcal{S}_n\) and \(zx^{-1}\) is shape-equivalent to \((wu^{-1})^{(12...n)}\) for some \(t\). Then, for every partition \(\lambda\),

\[ c_{u,w}(\lambda,k) = c_{x,w}(\lambda,l). \]

**Proof.** It suffices to prove a restricted case. Suppose \(u, w \in \mathcal{S}_n\), \(u \leq_k w\), and \(w\) is Grassmannian with descent \(k\). We construct permutations \(x, z \in \mathcal{S}_n\) with \(x \leq_k z\) and \(zx^{-1} = (wu^{-1})^{(12...n)}\) for which

\[ \pi_k(X_{00}wF_r \cap X_uF'_r) = \pi_k(X_{00}xG_s \cap X_xG'_s). \quad (5.3.1) \]

Here \(e_1, \ldots, e_n\) is a basis for \(V\), and the flags \(F_r, F'_r, G_s,\) and \(G'_s\) are

\[ F_r = \langle \langle e_1, \ldots, e_n \rangle \rangle, \quad F'_r = \langle \langle e_n, \ldots, e_1 \rangle \rangle, \]

\[ G_s = \langle \langle e_n, e_{n-1}, \ldots, e_1 \rangle \rangle, \quad G'_s = \langle \langle e_{n-1}, \ldots, e_1 \rangle \rangle. \]

Then (5.3.1) implies \(c_{u,w}(\lambda,k) = c_{x,w}(\lambda,l)\), which completes the proof.

If \(wu^{-1}(n) = n\), then \(zx^{-1} = 1 \times wu^{-1}\), which is shape-equivalent to \(wu^{-1}\), and the result follows by Theorem E(ii). Assume \(wu^{-1}(n) \neq n\). Then \(w(k) = n\) and \(u(k) < n\), as \(w\) is Grassmannian with descent \(k\). Set \(m := u(k), p := u^{-1}(n)(> k),\) and \(l := w(p)\). Define \(x \in \mathcal{S}_n\) by

\[ x(j) = \begin{cases} u(j) + 1 & \text{if } 1 \leq j < k \text{ or } p < j, \\ 1 & j = k, \\ m + 1 & j = k + 1, \\ u(j - 1) + 1 & k + 1 < j \leq p. \end{cases} \]

Then \(x \leq_k z := (wu^{-1})^{(12...n)}x\), where

\[ z(j) = \begin{cases} w(j) + 1 & \text{if } 1 \leq j < k \text{ or } p < j, \\ l + 1 & j = k, \\ 1 & j = k + 1, \\ w(j - 1) + 1 & k + 1 < j \leq p. \end{cases} \]
To show (5.3.1), let \( g_\alpha(\alpha), \ldots, g_n(\alpha) \) for \( \alpha \in A \) be the parameterized basis for flags \( E(\alpha) \in X^\nu_{u} F^\nu_{u} \cap X^\omega_{\omega(w)} F \) constructed in the proof of Lemma 5.1.4. Since \( g_k(\alpha) \in F_{n+1-\nu(k)} \cap F_{w(k)} \), \( u(k) = m \), and \( w(k) = n \), there exist regular functions \( \beta_j(\alpha) \) on \( A \) such that

\[
g_k(\alpha) = e_n + \sum_{j=m}^{n-1} \beta_j(\alpha)e_j.
\]

Since \( F_1 = \langle e_n \rangle \subset E_p(\alpha) - E_{p-1}(\alpha) \) and \( g_p(\alpha) = e_i \), there exist regular functions \( \delta_j(\alpha) \) on \( A \) with \( \delta_p(\alpha) \) nowhere vanishing such that

\[
\begin{align*}
\delta_j(\alpha) &= \sum_{j=m}^{k-1} \delta_j(\alpha)g_j(\alpha) + \sum_{j=k+1}^{p} \delta_j(\alpha)e_{w(j)} ,
\end{align*}
\]

as \( g_k(\alpha) \) is the only \( g_j(\alpha) \) whose \( e_n \)-coefficient is nonzero. Thus

\[
e_n - \sum_{j=k+1}^{p} \delta_j(\alpha)e_{w(j)} = g_k(\alpha) + \sum_{j=1}^{k-1} \delta_j(\alpha)g_j(\alpha) \in E_k(\alpha) - E_{k-1}(\alpha).
\]

Define a basis \( h_1(\alpha), \ldots, h_n(\alpha) \) for \( V \) by

\[
h_j(\alpha) = \begin{cases} 
g_j(\alpha) & 1 \leq j < k \text{ or } p < j, 
e_n - \left( \sum_{j=k+1}^{p} \delta_j(\alpha)e_{w(j)} \right) & j = k, 
e_n & j = k + 1, 
g_{j-1}(\alpha) & k + 1 < j \leq p. \end{cases}
\]

We claim \( E'(\alpha) := \langle h_1(\alpha), \ldots, h_n(\alpha) \rangle \) is a flag in \( X^\omega_{\omega_G} G \cap X^\omega_{\omega} G' \), which implies (5.3.1). Since \( h_k(\alpha) \in E_k(\alpha) - E_{k-1}(\alpha) \) and \( h_j(\alpha) = g_j(\alpha) \) for \( j < k \), we have

\[E'_k(\alpha) = \langle h_1(\alpha), \ldots, h_k(\alpha) \rangle = E_k(\alpha).
\]

Thus if \( \alpha \neq \alpha' \), then \( E'(\alpha) \neq E'(\alpha') \) and so \( \{ E'(\alpha) \mid \alpha \in A \} \) is a subset of the intersection \( X^\omega_{\omega_G} G \cap X^\omega_{\omega} G' \) of dimension equal to \( \dim A = \ell(w) - \ell(u) = \ell(z) - \ell(x) \), the dimension of \( X^\omega_{\omega_G} G \cap X^\omega_{\omega} G' \). Thus \( \{ E'(\alpha) \mid \alpha \in A \} \) is dense, and so \( E'_k(\alpha) = E_k(\alpha) \) implies (5.3.1).

For notational convenience, set \( G_j^\nu := G_j - G_{j-1} \), and similarly for \( F_j^\nu \). To establish this claim, we first show that \( h_j(\alpha) \in G_j^\omega(\alpha) \) for \( j = 1, \ldots, n \), which shows \( h_1(\alpha), \ldots, h_n(\alpha) \) is a parameterized basis for \( V \) and \( E'(\alpha) \in X^\omega_{\omega_G} G \). Then, for a
fixed $\alpha \in A$, we construct $h'_1, \ldots, h'_n$ that satisfy $E'_1(\alpha) = \langle h'_1, \ldots, h'_n \rangle$ and $h'_j \in G'_{n+1-x(j)}$ for $j = 1, \ldots, n$, showing $E'_1(\alpha) \in X G'_i$.

Note that if $i < n$, then $G_{i+1} = \langle e_n, F_i \rangle$. Thus $h_j(\alpha) \in F_{w(j)}^\circ \subset G'_{n+1-x(j)}$ for $1 \leq j < k$ and $p < j$, and if $k + 1 < j \leq p$, then $h_j(\alpha) \in F_{w(j-1)}^\circ \subset G'_{n+1-x(j-1)}$. Then, since $G_1 = \langle e_n \rangle$, we see that $h_{k+1}(\alpha) = e_n \in G'_{1} = G'_{n+1-x(k+1)}$. Finally, since $w$ is Grassmannian of descent $k$, if $k + 1 \leq i \leq p$, then $w(i) \leq w(p) = l$, which shows $h_k(\alpha) \in G'_{l+1} = G'_{2(k)}$. Thus $E'_1(\alpha) \in X^\circ G_i$.

We now show that $E'_i(\alpha) \in X G'_i$. Note that if $a < b < n$, then $F_{n+1-a} \cap F_b = G_{n+1-x(j)} \cap G_{b+1}$. Thus if $1 \leq j < k$, $h_j(\alpha) = g_j(\alpha) \in F_{n+1-u(j)}' \cap F_{w(j)} \subset G_{n+1-x(j)}'. $ Since $x(k) = 1$, we see that $h_k(\alpha) \in G_{n+1-x(k)} = V$. Fix $\alpha \in A$ and set $h'_j = h_j(\alpha)$ for $1 \leq j \leq k$. Define

$$h'_{k+1} := g_k(\alpha) - e_n = \sum_{j=m}^{n-1} \beta_j(\alpha) e_j \in G'_{n+1-(m+1)} = G'_{n+1-x(k+1)}.$$ 

Since $h'_{k+1} + h_{k+1}(\alpha) = g_k(\alpha)$, we see that $E'_k(\alpha) = \langle E'_k(\alpha), H'_{k+1} \rangle$.

Finally, since $E_\alpha(\alpha) \in X F'_i$, if $k < j$, there exists a vector

$$g'_j := \sum_{i < j} \gamma_{i,j} g_i(\alpha) \in F_{n+1-u(j)}'$$

such that $\langle E_{j-1}(\alpha), g'_j \rangle = E_j(\alpha)$. For $k + 1 < j \leq p$, set

$$h'_j = g'_{j-1} - \gamma_{k,j-1} e_n \in \langle e_n - \ldots, e_{n+1-u(j-1)} \rangle = G'_{n+1-x(j)},$$

as $g_k(\alpha)$ is the only vector among $\{g_1(\alpha), \ldots, g_n(\alpha)\}$ that is not in the span of $e_1, \ldots, e_{n-1}$. If $p < j$, set $h'_j = g'_j - \gamma_{k,j} e_n \in G'_{n+1-x(j)}$. Then $\langle h'_1, \ldots, h'_n \rangle = E'_1(\alpha)$, completing the proof. 

6. Formulas for some $c^n_{\nu(\lambda,k)}$

6.1. A chain-theoretic interpretation. We give a chain-theoretic interpretation for some coefficients $c^n_{\lambda}$ which is similar to the results of [50]. If either $u \lessdot_k (\alpha, \beta) u$ or $\zeta \lessdot (\alpha, \beta) \zeta$ is a cover, label that edge in the Hasse diagram with the integer $\beta = \max\{\alpha, \beta\}$. Given a saturated chain in the $k$-Bruhat order from $u$ to $\zeta u$ (or, equivalently, a saturated $\preceq$-chain from $e$ to $\zeta$), the word of that chain is its sequence of edge labels. Given a word $\omega = a_1 \cdot a_2 \cdot \ldots \cdot a_m$, Schensted insertion (see [47] or [46, Section 3.3]) of $\omega$ into the empty tableau gives a pair $(S, T)$ of Young tableaux, where $S$ is the insertion tableau and $T$ the recording tableau of $\omega$.

Let $\mu \subset \lambda$ be partitions. A permutation $\zeta$ is shape-equivalent to a skew Young diagram $\lambda/\mu$ if there is a $k$ such that $\zeta$ is shape-equivalent to $v(\lambda, k) \cdot v(\mu, k)^{-1}$. It follows that $\zeta$ is shape-equivalent to some skew partition $\lambda/\mu$ if and only if
(whenever $\alpha, \beta \in \text{up}_c$ or $\alpha, \beta \in \text{down}_c$),

$$\alpha < \beta \iff \zeta(\alpha) < \zeta(\beta).$$

**Theorem F'**. Let $\mu \subset \lambda$ be partitions, and suppose $\zeta \in \mathcal{S}_\omega$ is shape-equivalent to $\lambda/\mu$. Then for every partition $v$,

(i) $c^\zeta_v = c^\lambda_\mu$, and

(ii) for every standard Young tableau $T$ of shape $v$,

$$c^\zeta_v = \# \{ \text{\$\zeta$-chains from $e$ to $\zeta$ whose word has recording tableau $T$} \}.$$  

Equivalently, if $u \leq_k w$ and $wu^{-1} = \zeta$, then

$$c^w_{u;v(v,k)} = \# \{ \text{chains in $k$-Bruhat order from $u$ to $w$ whose word has recording tableau $T$} \}.$$  

**Remark 6.1.1.** Theorem F'(ii) gives a combinatorial proof of Proposition 1.1, when $wu^{-1}$ is shape-equivalent to a skew partition. Theorem F'(ii) is deceptively similar to Theorem 8 of [50].

**Theorem 8 [50].** Suppose $v = (p, 1^{q-1})$. Then for every $u, w \in \mathcal{S}_\omega$ and $k \in \mathbb{N}$, the constant $c^w_{u;v(v,k)}$ counts either

(i) $\# \{ \text{chains in $k$-Bruhat order from $u$ to $w$ with word $a_1 < \cdots < a_p > a_{p+1} > \cdots > a_p + q - 1$} \}$

or

(ii) $\# \{ \text{chains in $k$-Bruhat order from $u$ to $w$ with word $a_1 > \cdots > a_q < a_q + 1 < \cdots < a_p + q - 1$} \}.$

The recording tableaux of words in (i) have $1, 2, \ldots, p$ in the first row and $1, p+1, \ldots, p+q-1$ in the first column. These are the only words with this recording tableau. Similarly, the recording tableaux of words in (ii) have $1, 2, \ldots, q$ in the first column and $1, q+1, \ldots, p+q-1$ in the first row. Theorem F, however, is not a generalization of this result: The permutation $\zeta := (143562)$ is not shape-equivalent to any skew partition as $4, 5 \in \text{down}_c$ but $\zeta(4) > \zeta(5)$. Nevertheless, $c^\zeta_{(4,1)} = 1$. Interestingly, $\zeta$ does satisfy the conclusions of Theorem F'.

While the hypothesis of Theorem F' is not necessary for the conclusion to hold, some hypotheses are necessary. Let $\zeta = (162)(354)$, a product of two disjoint 3-cycles. Then $\zeta^{(1 \ldots 6)} = (132)(465) = v(\ominus, 2) \cdot v(\ominus, 2)^{-1}$. Hence, by Theorem H, we have

$$c^\zeta_{\ominus (1,2)} = c^\zeta_\ominus = c^\zeta_\ominus = 1.$$
This is also a consequence of Theorem G and the form of the Pieri formula in [28] or of [50, Theorem 5]. If \( u = 312645 \), then \( \zeta u = 561234 \) and the labeled Hasse diagram of \([u, \zeta u]_2\) is

![Hasse diagram]

The labels of the six chains are

\[ 2456, 2465, 2645, 4526, 4256, 4265, \]

and these have (respective) recording tableaux

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 2 & 4 & 3 \\
1 & 2 & 4 & 3 \\
1 & 3 & 4 & 2 \\
1 & 3 & 4 & 2 \\
1 & 3 & 4 & 2 \\
\end{array}
\]

This list omits \( \begin{array}{ll} 3 & 4 \\
1 & 2 
\end{array} \), and the third and fourth tableaux are identical.

Proof of Theorem F'. Suppose \( v(\lambda, k) = v(\mu, k) \). Then

\[
[e, \zeta] \simeq [v(\mu, k), v(\lambda, k)] \simeq [\mu, \lambda] \simeq [\mu, \lambda].
\]

The first isomorphism preserves the edge labels; in the second, these labels correspond to diagonals in a Young diagram. If \( v \prec v' \) is a cover in Young's lattice, there is a unique row \( i \) such that \( v_i \neq v'_i \). In that case, \( v_i + 1 = v'_i \) and the label of the corresponding edge in the \( k \)-Bruhat order is \( k - i + v'_i \), the diagonal on which the new box of \( v' \) lies.

A chain in Young's lattice from \( \mu \) to \( \lambda \) is a standard skew tableau \( R \) of shape \( \lambda/\mu \). Consider its word \( a_1 \cdots a_m \) as a two-rowed array

\[
w = \begin{pmatrix} 1 & 2 & \cdots & m \\ a_1 & a_2 & \cdots & a_m \end{pmatrix}.
\]

Then the entry \( i \) of \( R \) is in the \( a_i \)th diagonal.
Let $S$ and $T$ be, respectively, the insertion and recording tableaux of $w$. Consider the two-rowed array consisting of the columns $(^n_w)$ arranged in lexicographic order. Then the insertion and recording tableaux of this new array are $T$ and $S$, respectively (see [26], [48]).

The second row of this new array, the word inserted to obtain $T$, is the “diagonal” word of $R$; the entries of $R$ are ordered lexicographically by diagonal. By Lemma 6.1.2, the diagonal word is Knuth-equivalent to the original word. Thus $T$ is the unique tableau of partition shape Knuth-equivalent to $R$. This gives a combinatorial bijection

$$\begin{array}{c}
\{ \prec\text{-chains from } e \text{ to } \zeta \text{ whose } \\
\text{word has recording tableau } T \} \\
\{ \text{skew tableaux } R \text{ of shape } \\
\lambda/\mu \text{ Knuth-equivalent to } T \}
\end{array} \iff 
\begin{array}{c}
\{ \text{skew tableaux } R \text{ of shape } \\
\lambda/\mu \text{ Knuth-equivalent to } T \}
\end{array},$$

proving the theorem in this case, as it is well-known that $\zeta$ is shape-equivalent to $v(\lambda, k) \cdot v(\mu, k)^{-1}$. By Theorem E(ii), $c^\lambda/\mu_\nu = c_\nu^\lambda/\mu$, proving (i). Assume $\lambda, \mu, k$ have been chosen so that $\zeta = \phi_P(v(\lambda, k) \cdot v(\mu, k)^{-1})$, for some $P$. By Theorem 3.2.3(iii), $\phi_P$ induces an isomorphism

$$\phi_P : [e, v(\lambda, k) \cdot v(\mu, k)^{-1}]_\prec \sim [e, \zeta]_\preceq.$$

If $\eta \prec (\alpha, \beta)\eta$ is a cover in $[e, v(\lambda, k) \cdot v(\mu, k)^{-1}]_\prec$, then $\phi_P\eta \prec \phi_P((\alpha, \beta)\eta)$ is a cover in $[e, \zeta]_\preceq$ with label $p\beta$, where $P = p_1 < p_2 < \cdots$. Thus, if $\gamma$ is a chain in $[e, v(\lambda, k) \cdot v(\mu, k)^{-1}]_\preceq$ whose word $a_1, \ldots, a_m$ has recording tableau $T$, then $\phi_p(\gamma)$ is a chain in $[e, \zeta]_\preceq$ with word $p_{a_1}, \ldots, p_{a_m}$, which also has recording tableau $T$.

Order the diagonals of a skew Young tableau $R$ beginning with the diagonal incident to the end of the first column of $R$. The diagonal word of $R$ is the entries of $R$ listed in lexicographic order by diagonal, with magnitude-breaking ties. The tableau on the left below has diagonal word 7 5 8 1 2 3 6 4 9 7 5 8. Schensted insertion of the initial segment 7 5 8 1 2 3 6 4 9 (those diagonals incident upon the first column) gives the tableau on the right, whose row word is the initial segment.

\[
\begin{array}{c|c}
7 & 8 & 9 \\
5 & 7 & 8 \\
3 & 4 & 6 & 6 \\
1 & 2 & 2 & 5 & 8 \\
\end{array}
\quad
\begin{array}{c|c}
7 \\
5 & 8 \\
3 & 7 & 9 \\
1 & 4 & 8 \\
\end{array}
\]

This observation is the key to the proof of the following lemma.
**Lemma 6.1.2.** The diagonal word of a skew tableau is Knuth-equivalent to its column word.

**Proof.** Let $d(R)$ be the diagonal word of a skew tableau $R$. We show that $d(R)$ is Knuth-equivalent to the word $c \cdot d(R')$, where $c$ is the first column of $R$ and $R'$ is $R$ with $c$ removed. An induction completes the proof.

Suppose the first column of $R$ has length $b$ and $R$ has $r$ diagonals. For $1 \leq j \leq b$, let $w_j := a_{ij} \cdots a_{ij}$ be the subword of $d(R)$ consisting of the $j$th diagonal. Then $a_{ij} \cdots a_{ij} < a_{ij} \cdots a_{ij}$, $s_1 \leq s_2 \leq \cdots \leq s_b$, and if $k \leq s_j$, then $a_{ij} > a_{ij} > \cdots > a_{ij}$, as these are consecutive entries in the $k$th column of $R$.

For $1 \leq l \leq b$, let $T_l$ be the insertion tableau of the word $w_1, w_2, \ldots, w_l$. Then the $k$th column of $T_l$ is $a_{ij} > a_{ij} > \cdots > a_{ij}$, where $s_{j-1} < k < s_j$. Hence $c \cdot d(R') = c \cdot \text{row}(T') \cdot w_{b+1} \cdots w_r$, where $\text{row}(T')$ is the row word of $T_b$ with its first column $c$ removed. Since the column word of a tableau is Knuth-equivalent to its row word, we have the Knuth-equivalences $c \cdot \text{row}(T') \equiv_k c \cdot \text{col}(T') = \text{col}(T_b) \equiv_k \text{row}(T_b)$, and this completes the proof.

6.2. Skew permutations. Define the set of skew permutations to be the smallest set of permutations containing all skew partitions $\nu(\lambda, k) \cdot \nu(\mu, k)^{-1}$ which is closed under the following conditions.

1. Shape equivalence. If $\eta$ is shape-equivalent to a skew permutation $\zeta$, then $\eta$ is skew.
2. Cyclic shift. If $\zeta \in S_n$ is skew, then so is $\zeta(12\ldots n)$.
3. Products of disjoint permutations. If $\zeta, \eta$ are disjoint and skew, then $\zeta \eta$ is skew.

A shape of a skew permutation $\zeta$ is a (nonunique) skew partition $\theta$, which is defined inductively. If $\zeta$ is shape-equivalent to $\lambda/\mu$, then $\zeta$ has shape $\lambda/\mu$. If $\zeta \in S_n$ is a skew permutation with shape $\theta$, then $\zeta(12\ldots n)$ has shape $\theta$. If $\zeta$ and $\eta$ are disjoint skew permutations with respective shapes $\rho$ and $\sigma$, then $\zeta \eta$ has skew shape $\rho \coprod \sigma$.

**Theorem 6.2.1.** Let $\zeta$ be a skew permutation with shape $\theta$.

(i) For all partitions $\nu$,

$$c_{\nu}^\zeta = c_{\nu}^\theta.$$

(ii) The number of chains in the interval $[e, \zeta]_{\leq}$ is equal to the number of standard Young tableaux of shape $\theta$.

**Proof.** The number of standard skew tableaux of shape $\theta$ is $\sum_{\lambda} f^\lambda c_\lambda^\theta$, hence (ii) is consequence of (i) and Proposition 1.1. To show (i), we need only consider the last part (3) of the recursive definition of skew permutations, by Theorems E(ii) and H. Suppose $\zeta$ and $\eta$ are disjoint skew permutations with respective shapes $\rho$ and $\sigma$, and for all partitions $\nu$, $c_{\nu}^\zeta = c_{\nu}^\rho$ and $c_{\nu}^\eta = c_{\nu}^\sigma$. Then by Theorem G(ii),
Example 6.2.2. Consider the graph of \((1978)(26354)\):

![Diagram](image)

Thus the two cycles \(\zeta = (1978)\) and \(\eta = (26354)\) are disjoint. Note that \(\zeta\) is shape-equivalent to \((1423)\) and \((1423)(1234) = (1342)\). Similarly, \(\eta\) is shape-equivalent to \((15243)\) and \((15243)(12345) = (13542)\). Both of these cycles, \((1423)\) and \((15243)\), are skew partitions. Let \(\lambda = \square\) and \(\mu = \bigotimes\), and \(\nu = \bigotimes\). Then

\[
\nu(\lambda, 2) = 13245, \quad \nu(\mu, 2) = 34125, \quad \nu(\nu, 2) = 35124
\]

and

\[
\nu(\lambda, 2) \leq_{\Sigma} (1342) \cdot \nu(\lambda, 2) = \nu(\mu, 2),
\]

\[
\nu(\lambda, 2) \leq_{\Sigma} (13542) \cdot \nu(\lambda, 2) = \nu(\nu, 2).
\]

Hence, for every partition \(\kappa\), \(c^\zeta_\kappa = c^{\mu/\lambda}_\kappa\) and \(c^\eta_\kappa = c^{\nu/\lambda}_\kappa\). Thus it follows that \(c^\zeta_\kappa \cdot \eta = c^\nu_\kappa\), where \(\rho\) is any of the four skew partitions:

\[
\begin{align*}
\square & \bigotimes, & \square & \bigotimes, & \square & \bigotimes, & \square & \bigotimes.
\end{align*}
\]

### 6.3. Further remarks.

For small symmetric groups, it is instructive to examine all permutations and determine to which class they belong. Here, we enumerate each class in \(S_4\), \(S_5\), and \(S_6\).
If $\zeta$ is one of the forty-two permutations in $\mathcal{S}_6$ that is not a skew permutation, and $\zeta$ is not one of

$$(125634), (145236), (143652), (163254), (153)(246), \text{ or } (135)(264), \quad (6.3.1)$$

then there is a skew partition $\theta$ such that $c_\theta^\zeta = c_\theta^\nu$ for all partitions $\nu$. It would be interesting to understand why this occurs for all but these six permutations. Can one characterize those permutations $\zeta$ such that there exists a skew partition $\theta$ with $c_\theta^\zeta = c_\theta^\nu$ for all partitions $\nu$?

For each of these six "exceptional" permutations $\zeta$ in (6.3.1), there is a skew partition $\theta$ for which $c_\theta^\zeta = c_\theta^\nu$ for all $\nu \subset a^b$, where $a = \#\text{up}_\zeta$ and $b = \#\text{down}_\zeta$. For these we have $\theta \not\subset a^b$. For example, let $\zeta = (153)(246)$. If $u = 214365$, then $u \leq_3 \zeta u$ and there are forty-two chains in $[u, \zeta u]_3$. Also

$$c_\zeta^{\zeta} = 1, \quad c_\zeta^{\zeta} = 2, \quad c_\zeta^{\zeta} = 1,$$

which verifies Proposition 1.1 as $f^{\zeta \zeta} = 5$, $f^{\zeta \zeta} = 16$, and $f^{\zeta \zeta} = 5$. In this case, $\theta = \zeta$. Since $\up_\zeta = \{1, 2, 4\}$ and $\down_\zeta = \{6, 5, 3\}$, we see that $a = b = 3$. However, $\theta \not\subset a^b$.

A bijective interpretation of the $c_{u(w,\lambda,k)}^w$ should also give a bijective proof of Proposition 1.1. We show that a function $s$ from chains to standard tableaux satisfying some extra conditions provides a bijective interpretation of the $c_{u(w,\lambda,k)}^w$. Let $\text{ch}[u,w]_k$ denote the set of (saturated) chains in the interval $[u,w]_k$. For a partition $\mu$ and integer $m$, let $\mu * m$ be the set of partitions $\lambda$ with $\lambda - \mu$ a horizontal strip of length $m$. These arise in the classical Pieri formula

$$S_\mu(x_1, \ldots, x_k) \cdot h_m(x_1, \ldots, x_k) = \sum_{\lambda \in \mu \cup m} S_\lambda(x_1, \ldots, x_k).$$

If $T$ is a standard tableau of shape $\mu$ and $m$ any integer, let $T * m$ be the set of tableaux $U$ that contain $T$ as an initial segment such that $U-T$ is a horizontal strip whose entries increase from left to right.
THEOREM 6.3.1. Suppose that for every $u \leq_W w$, there is a map

$$ch[u, w]_k \rightarrow \left\{ \text{standard Young tableau } T \text{ whose } \begin{array}{l} \text{shape is a partition of } \ell(w) - \ell(u) \end{array} \right\}$$

such that we have the following.

1. $d^w_{u,v(\lambda,k)} := \# \{ \gamma \in ch[u, w]_k \mid \tau(\gamma) = T \}$ depends only upon the shape $\lambda$ of the standard tableau $T$.

2. If $\gamma = \delta \cdot e$ is the concatenation of two chains $\delta$ and $e$, then $\tau(\delta)$ is a subtableau of $\tau(\gamma)$. (This means that $\tau(\gamma)$ is a recording tableau.)

3. Suppose $\gamma = \delta \cdot e$ with $\delta \in ch[u, x]_k$, and hence $e \in ch[x, w]_k$. Then $\tau(\delta) \cdot e$ is a subtableau of $\tau(\delta)$ if and only if $x \leq_W w$; $e(\delta) := e \in ch[x, w]_k$ must be unique for this to occur.

Then, for every standard tableau $T$ of shape $\lambda$ and $u \leq_W w$,

$$c^w_{u,v(\lambda,k)} = d^w_{u,v(\lambda,k)}.$$

Such a map $\tau$ is a generalization of Schensted insertion. In that respect, the existence of such a map would generalize Theorem F'.

**Proof.** We induct on $\lambda$. Assume the theorem holds for all $u, w$, and partitions $\pi$ with fewer rows than $\lambda$, or if $\lambda$ and $\pi$ have the same number of rows, then the last row of $\lambda$ exceeds the last row of $\pi$.

The form of the Pieri formulas expressed in [50], [55] (see also Section 4.2), and condition (3) prove the theorem when $\lambda$ consists of a single row. Assume that $\lambda$ has more than one row, and set $\mu$ to be $\lambda$ with its last row removed. Let $m$ be the length of the last row of $\lambda$, and let $T$ be any tableau of shape $\mu$. Recall that $U \mapsto \text{shape}(U)$ gives a one-to-one correspondence between $T \cdot m$ and $\mu \cdot m$.

By the definition of $c^x_{u,v(\mu,k)}$, we have

$$\mathcal{S}_u \cdot S_{\mu}(x_1, \ldots, x_k) = \sum_{u \leq_W x} c^x_{u,v(\mu,k)} \mathcal{S}_y.$$

By the Pieri formula for Schubert polynomials,

$$\mathcal{S}_u \cdot S_{\mu}(x_1, \ldots, x_k) \cdot h_m(x_1, \ldots, x_k) = \sum_{w} \sum_{u \leq_W x} c^x_{u,v(\mu,k)} \mathcal{S}_w.$$

By the classical Pieri formula, this also equals

$$\mathcal{S}_u \cdot \sum_{\pi \in \mu \cdot m} S_{\pi}(x_1, \ldots, x_k) = \sum_{w} \sum_{\pi \in \mu \cdot m} c^w_{u,v(\pi,k)} \mathcal{S}_w.$$
Hence
\[
\sum_{\pi \in \mu \ast m} c_{uv(\pi, k)}^w = \sum_{u \leq k \ y \rightarrow w} c_{uv(\mu, k)}^y.
\]

We exhibit a bijection between the two sets
\[
M_{T, k, m} := \coprod_{\stackrel{u \leq k \ y \rightarrow w}{r_{k, m}}} \{ \delta \in \text{ch}[u, y]_k | \tau(\delta) = T \}
\]
and \(\coprod_{\pi \in \mu \ast m} L_{\pi}\), where
\[
L_{\pi} := \{ \gamma \in \text{ch}[u, w]_k | \tau(\gamma) \in T \ast m \text{ and } \tau(\gamma) \text{ has shape } \pi \}.
\]

This completes the proof. Indeed, by the induction hypothesis
\[
\# M_{T, k, m} = \sum_{\stackrel{u \leq k \ y \rightarrow w}{r_{k, m}}} c_{uv(\mu, k)}^y
\]
and for \(\pi \in \mu \ast m\) with \(\pi \neq \lambda\),
\[
\# L_{\pi} = c_{uv(\pi, k)}^w.
\]

Thus the bijection shows
\[
c_{uv(\lambda, k)}^w = \sum_{\stackrel{u \leq k \ y \rightarrow w}{r_{k, m}}} c_{uv(\mu, k)}^y - \sum_{\pi \in \mu \ast m} c_{uv(\pi, k)}^w = \# L_{\lambda},
\]
which is \(\# \tau^{-1}(U)\), for any \(U\) of shape \(\lambda\).

To construct the desired bijection, consider first the map
\[
M_{T, k, m} \longrightarrow \coprod_{\pi \in \mu \ast m} L_{\pi}
\]
defined by \(\delta \in \text{ch}[u, y]_k \mapsto \delta.\varepsilon(\delta)\). By property (3), \(\tau(\delta.\varepsilon(\delta)) \in T \ast m\), so this injective map has the stated range. To see that it is surjective, let \(\pi \in \mu \ast m\) and \(\gamma \in L_{\pi}\). Let \(\delta\) be the first \(|\mu|\) steps in the chain \(\gamma\), so that \(\gamma = \delta.\varepsilon\), and suppose \(\delta \in \text{ch}[u, y]_k\). Then \(\tau(\delta) = T\), so \(\tau(\delta.\varepsilon) = \tau(\delta) \ast m\). By (3), this implies \(y \rightarrow w\), and hence \(\delta \in M_{T, k, m}\). \qed
References


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