TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 370, Number 4, April 2018, Pages 2971–2999
http://dx.doi.org/10.1090/tran/7304
Article electronically published on November 28, 2017

GENERALIZED NIL-COXETER ALGEBRAS OVER DISCRETE COMPLEX REFLECTION GROUPS

APOORVA KHARE

Abstract. We define and study generalized nil-Coxeter algebras associated to Coxeter groups. Motivated by a question of Coxeter (1957), we construct the first examples of such finite-dimensional algebras that are not the "usual" nil-Coxeter algebras: a novel 2-parameter type A family that we call $NC_A(n,d)$. We explore several combinatorial properties of $NC_A(n,d)$, including its Coxeter word basis, length function, and Hilbert–Poincaré series, and show that the corresponding generalized Coxeter group is not a flat deformation of $NC_A(n,d)$. These algebras yield symmetric semigroup module categories that are necessarily not monoidal; we write down their Tannaka–Krein duality.

Further motivated by the Broué–Malle–Rouquier (BMR) freeness conjecture [J. Reine Angew. Math. 1998], we define generalized nil-Coxeter algebras NC_W over all discrete real or complex reflection groups W, finite or infinite. We provide a complete classification of all such algebras that are finite dimensional. Remarkably, these turn out to be either the usual nil-Coxeter algebras or the algebras $NC_A(n,d)$. This proves as a special case—and strengthens—the lack of equidimensional nil-Coxeter analogues for finite complex reflection groups. In particular, generic Hecke algebras are not flat deformations of NC_W for W complex.

Contents

1. Introduction and main results

Throughout this paper, $\Bbbk$ will denote a fixed unital commutative ground ring.

In this paper we define and study generalized nil-Coxeter algebras associated to Coxeter groups and more generally to all discrete complex reflection groups,

Received by the editors April 6, 2017, and, in revised form, May 30, 2017.
2010 *Mathematics Subject Classification.* Primary 20F55; Secondary 20F05, 20C08.
Key words and phrases. Complex reflection group, generalized Coxeter group, generalized nil-Coxeter algebra, length function.

finite or infinite. These are algebras that map onto the associated graded algebras of (generic) Hecke algebras over complex reflection groups and of Iwahori–Hecke algebras over Coxeter groups. As we discuss, working with these algebras allows for a broader class than the corresponding reflection groups.

We begin with real groups. Coxeter groups and their associated Hecke algebras play an important role in representation theory, combinatorics, and mathematical physics. Each such group is defined by a Coxeter matrix, i.e., a symmetric "integer" matrix $M := (m_{ij})_{i,j \in I}$ with I finite and $m_{ii} = 2 \leqslant m_{ij} \leqslant \infty \ \forall i \neq j$. The *Artin monoid* $\mathcal{B}_M^{\geqslant 0}$ associated to M is generated by $\{T_i : i \in I\}$ modulo the braid relations $T_i T_j T_i \cdots = T_j T_i T_j \cdots$ for all $i \neq j$ with $m_{ij} < \infty$, with precisely m_{ij} factors on either side. The Artin group (or generalized braid group) $\mathcal{B}_M$ is the group generated by these relations; typically we use $\{s_i : i \in I\}$ to denote its generators. There are three well-studied algebras associated to the matrix M: the group algebra $\Bbbk W(M)$ of the Coxeter group, the 0-Hecke algebra [Ca, No], and the nil-Coxeter algebra $NC(M)$ [FS] (also called the nilCoxeter algebra, nil Coxeter algebra, and nil Hecke ring in the literature). These are all free $\Bbbk$-modules, with a "Coxeter word basis" $\{T_w : w \in W(M)\}$ and length function $\ell(T_w) := \ell(w)$ in $W(M)$; in each of them the T_i satisfy a quadratic relation.

In a sense, the usual nil-Coxeter algebras $NC(M)$ are better behaved than all other generic Hecke algebras (in which $T_i^2 = a_i T_i + b_i$ for scalars a_i, b_i; see [Hum, Chapter 7]): the words T_w have unique lengths and form a monoid together with 0. Said differently, the algebras $NC(M)$ are the only generic Hecke algebras that are graded with T_i homogeneous of positive degree. Indeed, if $\deg T_i = 1 \ \forall i$, then $NC(M)$ has Hilbert–Poincaré polynomial $\prod_{i \in I} [d_i]_q$, where $[d]_q := \frac{q^d - 1}{q - 1}$ and d_i are the exponents of $W(M)$.

We now introduce the main objects of interest in the present work: generalized Coxeter matrices and their associated nil-Coxeter algebras (which are always $\mathbb{Z}^{\geqslant 0}$-graded).

Definition 1.1. Define a *generalized Coxeter matrix* to be a symmetric "integer" matrix $M := (m_{ij})_{i,j \in I}$ with I finite, $2 \leqslant m_{ij} \leqslant \infty \ \forall i \neq j$, and $m_{ii} < \infty \ \forall i$. Now fix such a matrix M.

(1) Given an integer vector $\mathbf{d} = (d_i)_{i \in I}$ with $d_i \geqslant 2 \ \forall i$, define $M(\mathbf{d})$ to be the matrix replacing the diagonal in M by $\mathbf{d}$.
(2) The *generalized Coxeter group* $W(M)$ is the quotient of the braid group $\mathcal{B}_{M_2}$ by the *order relations* $s_i^{m_{ii}} = 1 \ \forall i$, where M_2 is the Coxeter matrix $M((2, \ldots, 2))$. We remark that we used the familiar Artin group $\mathcal{B}_{M_2}$ associated to the Coxeter group $W(M_2)$. We could just as well have written $\mathcal{B}_M$, since the diagonals of the matrices M_2 and M play no role in $\mathcal{B}_{M_2} = \mathcal{B}_M$.
(3) The *braid diagram* or *Coxeter graph* of M (or of $W(M)$) has vertices indexed by I, and for each pair $i \neq j$ of vertices, $m_{ij} - 2$ edges between them.
(4) Define the corresponding *generalized nil-Coxeter algebra* as follows (with M_2 as above):

(1.2)
$$NC(M) := \frac{\Bbbk \langle T_i, i \in I \rangle}{(\underbrace{T_i T_j T_i \cdots}_{m_{ij} \text{ times}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij} \text{ times}}, \ T_i^{m_{ii}} = 0 \ \forall i \neq j \in I)} = \frac{\Bbbk \mathcal{B}_{M_2}^{\geqslant 0}}{(T_i^{m_{ii}} = 0 \ \forall i)},$$

where we omit the braid relation $T_i T_j T_i \cdots = T_j T_i T_j \cdots$ if $m_{ij} = \infty$.

(5) Given $\mathbf{d} = (d_i)_{i\in I}$ as above, define $W_M(\mathbf{d}) := W(M(\mathbf{d}))$ and $NC_M(\mathbf{d}) := NC(M(\mathbf{d}))$.

We are interested in the family of (generalized) nil-Coxeter algebras for multiple reasons: category theory, real reflection groups, complex reflection groups, and deformation theory. We elaborate on these motivations in this section and the next.

1.1. **Tannaka–Krein duality for semigroup categories.** In [Kho], the representation categories $\operatorname{Rep} NC(A_n)$ were used to categorify the Weyl algebra. For now we highlight two properties of generalized nil-Coxeter algebras $NC(M)$ which also have categorical content: (i) for *no* choice of coproduct on $NC(M)$ can it be a bialgebra (shown below); and (ii) every algebra $NC(M)$ is equipped with a cocommutative coproduct $\Delta : T_i \mapsto T_i \otimes T_i$ for all $i \in I$.

Viewed through the prism of representation categories, the coproduct in (ii) equips $\operatorname{Rep} NC(M)$ with the structure of a *symmetric* semigroup category [ES, §13,14]. Note by (i) that the simple module $\Bbbk$ does not serve as a unit object, whence $\operatorname{Rep} NC(M)$ is necessarily not monoidal. It is natural to apply the Tannakian formalism to such categories with "tensor" structure. We record the answer which, while not surprising once formulated, we were unable to find in the literature.

Definition 1.3. A *semigroup-tensor category* is a semigroup category $(\mathcal{C}, \otimes)$ which is also additive and such that $\otimes$ is bi-additive.

Theorem A. *Let A be an associative unital algebra over a field $\Bbbk$, let $\mathcal{C} := \operatorname{Rep} A$, and let $F : \mathcal{C} \to \operatorname{Vec}_\Bbbk$ be the forgetful functor. Then:*

(1) *Any semigroup-tensor structure on $\mathcal{C}$ together with a tensor structure on F equips A with a coproduct $\Delta : A \to A \otimes A$ that is an algebra map.*
(2) *If the semigroup-tensor structure on $\mathcal{C}$ is braided (respectively, symmetric), then (A, Δ) is a quasi-triangular (respectively, triangular) algebra with coproduct. This simply means there exists an invertible element $R \in A \otimes A$ satisfying the "hexagon relations"*

$$(1 \otimes \Delta)R = R_{13}R_{12}, \qquad (\Delta \otimes 1)R = R_{13}R_{23}$$

and such that $\Delta^{op} = R\Delta R^{-1}$. Triangularity means further that $RR_{21} = 1 \otimes 1$.

Notice that generalized nil-Coxeter algebras are indeed examples of such triangular algebras with a (cocommutative) coproduct but no counit. Such algebras are interesting in the theory of PBW deformations of smash product algebras; see Section 2. We also show below how to obtain an "honest" symmetric tensor category from each algebra $NC(M)$ via a central extension.

As noted above, Theorem A is in a sense "expected" and serves to act more as motivation. That the algebras $NC(M)$ provide concrete examples of symmetric, nonmonoidal semigroup-tensor categories is novel. The main results below now focus on the algebras $NC(M)$ themselves.

1.2. **Real reflection groups and a novel family of finite-dimensional nil-Coxeter algebras.** Our next result constructs a novel family of generalized nil-Coxeter algebras of type A which are finite dimensional. Classifying the finite-dimensional objects in Coxeter-type settings in algebra and combinatorics has been a subject of tremendous classical and modern interest, including Weyl, Coxeter, and

complex reflection groups, their nil-Coxeter and associated Hecke algebras, as well as finite type quivers, Kleinian singularities, the McKay–Slodowy correspondence, simple Lie algebras, etc. A very recent setting involves the classification of finite-dimensional Nichols algebras. Some of the prominent ingredients in the study of these algebras are common to the present work. See [GHV, HV1, HV2] for more details. Another famous recent classification is that of finite-dimensional pointed Hopf algebras [AnSc], which turn out to arise from generalized small quantum groups. With these motivations, our goal is to similarly classify all generalized nil-Coxeter algebras; our next result presents the first novel family of such examples.

We remark that in equation (1.2), in generalizing the "order relations" from $T_i^2 = 0$ to $T_i^{m_{ii}} = 0$, we were also motivated by another such setting: the classical work of Coxeter [Cox2], which investigated generalized Coxeter matrices M for which the group $W(M)$ is finite. Specifically, Coxeter considered the (type A) Artin braid group $\mathcal{B}_{A_{n-1}}$, and instead of quotienting by the relations $s_i^2 = 1$ to obtain the symmetric group S_n, he worked with $s_i^p = 1\ \forall i \in I$. Coxeter was interested in computing for which (n, p) the quotient group $W_{A_{n-1}}((p, \dots, p))$ is a finite group and in determining its order. He showed (see also [As]) that $W_{A_{n-1}}((p, \dots, p))$ is finite if and only if $\frac{1}{n} + \frac{1}{p} > \frac{1}{2}$, in which case the size of the quotient group is $\left(\frac{1}{n} + \frac{1}{p} - \frac{1}{2}\right)^{1-n} \cdot n!/n^{n-1}$. Coxeter's result was extended by Koster in his thesis [Ko] to completely classify the "generalized Coxeter groups" $W(M)$ that are finite. These turn out to be precisely the finite Coxeter groups and the Shephard groups.

Parallel to the above classical works, we wish to understand for which matrices M is the algebra $NC(M)$ finitely generated as a $\Bbbk$-module. If $W(M)$ is a Coxeter group, then $\dim NC(M) = |W(M)|$. Few other answers are known. For instance, Marin [Mar] has shown that the algebra $NC_{A_2}((m, n))$ is not finitely generated when $m, n \geqslant 3$. However, apart from the usual nil-Coxeter algebras, to our knowledge no other finitely generated algebras $NC(M)$ were known to date.

In the following result, following Coxeter's construction in type A above, we exhibit the first such finite-dimensional family of algebras $NC(M)$.

Theorem B. *Given integers $n \geqslant 1$ and $d \geqslant 2$, define the $\Bbbk$-algebra*

$$NC_A(n, d) := NC_{A_n}((2, \dots, 2, d)). \tag{1.4}$$

In other words, $NC_A(n, d)$ is generated by $T_1, \dots, T_n$ with relations

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \qquad \forall\ 0 < i < n; \tag{1.5}$$

$$T_i T_j = T_j T_i \qquad \forall\ |i - j| > 1; \tag{1.6}$$

$$T_1^2 = \dots = T_{n-1}^2 = T_n^d = 0. \tag{1.7}$$

Then $NC_A(n, d)$ is a free $\Bbbk$-module with $\Bbbk$-basis of $n!(1 + n(d-1))$ generators

$$\{T_w :\ w \in S_n\} \sqcup \{T_w T_n^k T_{n-1} T_{n-2} \cdots T_{m+1} T_m :\ w \in S_n,\ k \in [1, d-1],\ m \in [1, n]\}.$$

In particular, for all $l \in [1, n-1]$, the subalgebra R_l generated by $T_1, \dots, T_l$ is isomorphic to the usual nil-Coxeter algebra $NC_{A_l}((2, \dots, 2))$.

Remark 1.8. We adopt the following notation in the sequel without further reference: let

(1.9)

$w_\circ \in S_{n+1}, \quad w'_\circ \in S_n, \quad w''_\circ \in S_{n-1}$ denote the respective longest elements,

where the symmetric group S_{l+1} corresponds to the $\Bbbk$-basis of the algebra R_l for $l = n-2, n-1, n$.

The algebras $NC_A(n,d)$ have not been studied previously for $d > 2$, and we begin to explore their properties. When $d = 2$, $NC_A(n,d)$ specializes to the usual nil-Coxeter algebra of type A_n. In this vein, we present three properties of $NC_A(n,d)$ akin to the usual nil-Coxeter algebras.

Theorem C. *Fix integers $n \geqslant 1$ and $d \geqslant 2$.*

(1) *The algebra $NC_A(n,d)$ has a length function that restricts to the usual length function $\ell_{A_{n-1}}$ on $R_{n-1} \simeq NC_{A_{n-1}}((2,\dots,2))$ (from Theorem B), and*

$$\ell(T_w T_n^k T_{n-1} \cdots T_m) = \ell_{A_{n-1}}(w) + k + n - m \tag{1.10}$$

for all $w \in S_n$, $k \in [1, d-1]$, and $m \in [1,n]$.

(2) *There is a unique longest word $T_{w'_\circ} T_n^{d-1} T_{n-1} \cdots T_1$ of length*

$$l_{n,d} := \ell_{A_{n-1}}(w'_\circ) + d + n - 2.$$

(3) *If $\Bbbk$ is a field, then $NC_A(n,d)$ is local, with unique maximal ideal $\mathfrak{m}$ generated by $T_1, \dots, T_n$. For all $\Bbbk$, the ideal $\mathfrak{m}$ is nilpotent with $\mathfrak{m}^{1+l_{n,d}} = 0$.*

We also study the algebra $NC_A(n,d)$ in connection to Khovanov's categorification of the Weyl algebra. See Proposition 5.6 below.

1.3. **Complex reflection groups and BMR freeness conjecture.** Determining the finite-dimensionality of the algebras $NC(M)$ is strongly motivated by the study of complex reflection groups and their Hecke algebras. Recall that such groups were enumerated by Shephard–Todd [ST]; see also [Coh, LT]. Subsequently, Popov [Pop1] classified the infinite discrete groups generated by affine unitary reflections; in the following we will term these *infinite complex reflection groups*. For more on these groups, see, e.g., [BS, Hug1, Hug2, Mal, ORS, ReS] and the references therein.

For complex reflection groups, an important program is the study of *generic Hecke algebras* over them, as well as the associated *BMR freeness conjecture* of Broué, Malle, and Rouquier [BMR1, BMR2] (see also the recent publications [Et, Lo, Mar, MP] and the thesis [Ch]). This conjecture connects the dimension of a generic Hecke algebra to the order of the underlying reflection group. Here we will study this connection for the corresponding nil-Coxeter algebras, which we define as follows given [Be, BMR2].

Definition 1.11. Suppose W is a discrete (finite or infinite) complex reflection group, together with a finite generating set of complex reflections $\{s_i : i \in I\}$, the order relations $s_i^{m_{ii}} = 1\ \forall i$, a set of braid relations $\{R_j : j \in J\}$ (each involving words with at least two distinct reflections s_i), and for the infinite non-Coxeter complex reflection groups W listed in [Mal, Tables I, II], one more order relation $R_0^{m_0} = 1$. Now define $I_0 := I \sqcup \{0\}$ for these infinite non-Coxeter complex reflection groups W, and $I_0 := I$ otherwise. Given an integer vector $\mathbf{d} \in \mathbb{N}^{I_0}$ with $d_i \geqslant 2\ \forall i$, define the corresponding *generalized nil-Coxeter algebra* to be

$$NC_W(\mathbf{d}) := \frac{\Bbbk\langle T_i, i \in I\rangle}{(\{R'_j, j \in J\}, T_i^{d_i} = 0\ \forall i \in I,\ (R'_0)^{d_0} = 0)}, \tag{1.12}$$

where the braid relations R_j are replaced by the corresponding relations R'_j in the alphabet $\{T_i : i \in I\}$, and similarly for R'_0 if $R_0^{m_0} = 1$ in W. There is also the notion of the corresponding *braid diagram* as in [BMR2, Tables 1–4] and [Mal, Tables I, II]; this is no longer always a Coxeter graph.

Note by [Pop1, §1.6] that in the above definition one has to work with a specific presentation for complex reflection groups, as there is no canonical (minimal) set of generating reflections. See [Ba] for related work.

There is no known finite-dimensional generalized nil-Coxeter algebra associated to a finite complex reflection group. Indeed, Marin mentions in [Mar] that a key difference between real and complex reflection groups W is the lack of nil-Coxeter algebras of dimension precisely $|W|$ for the latter. This was verified in some cases for complex reflection groups in loc. cit. Our final result shows this assertion—and in fact a stronger statement—for all discrete finite and infinite, real and complex reflection groups. Even stronger (a priori): we provide a complete classification of finite-dimensional generalized nil-Coxeter algebras for all such groups. Notice by [Pop1, Theorem 1.4] that it suffices to consider only the groups whose braid diagram is connected.

Theorem D. *Suppose W is any irreducible discrete real or complex reflection group. In other words, W is a real reflection group with connected braid diagram, or a complex reflection group with connected braid diagram and presentation given as in* [BMR2, Tables 1–4], [Mal, Tables I, II]*, or* [Pop1, Table 2]*. Also fix an integer vector $\mathbf{d}$ with $d_i \geqslant 2$ $\forall i$ (including possibly for the additional order relation as in* [Mal]*). Then the following are equivalent:*

(1) *The generalized nil-Coxeter algebra $NC_W(\mathbf{d})$ is finitely generated as a $\Bbbk$-module.*
(2) *Either W is a finite Coxeter group and $d_i = 2$ $\forall i$, or W is of type A_n and $\mathbf{d} = (2, \dots, 2, d)$ or $(d, 2, \dots, 2)$ for some $d > 2$.*
(3) *The ideal $\mathfrak{m}$ generated by $\{T_i : i \in I\}$ is nilpotent.*

If these assertions hold, there exists a length function and a unique longest element in $NC_W(\mathbf{d})$, say of length l; now $\mathfrak{m}^{1+l} = 0$.

In other words, the only finite-dimensional examples (when $\Bbbk$ is a field) are the usual nil-Coxeter algebras and the algebras $NC_A(n, d)$. Note also that all of the above results are characteristic-free.

A key tool in proving both Theorems B and D is a diagrammatic calculus, which is akin to crystal theory from combinatorics and quantum groups.

1.4. **Further questions and organization of the paper.** To our knowledge, the algebras $NC_A(n, d)$ for $d > 2$ are a novel construction—and in light of Theorem D, the *only* finite-dimensional generalized nil-Coxeter algebras other than the "usual" ones. In particular, a further exploration of their properties is warranted. We conclude this section by discussing some further directions.

(1) Nil-Coxeter algebras are related to flag varieties [BGG, KK], categorification [Kho, KL], and symmetric function theory [BSS]. Also recall that the divided difference operator representation of the usual type A nil-Coxeter algebra $NC_A(n, 2)$ is used to define Schubert polynomials [FS, LS] and that the

polynomials the T_i simultaneously annihilate are precisely the symmetric polynomials. It will be interesting to determine if $NC_A(n,d)$, $d>2$, has a similar "natural" representation as operators on a polynomial ring and, if so, to consider the polynomials one obtains analogously. (See [Mar] for a related calculation.) We observe here that for $d>2$, the algebra $NC_A(n,d)$ does not "come from" a finite reflection group, as it is of larger dimension than the corresponding generalized Coxeter group, by equation (2.1) below.

(2) Given both the connection to Coxeter groups as well as the crystal methods used below, it will be interesting to explore if the algebras $NC_A(n,d)$ are connected to crystals over some Lie (super)algebra.

(3) Our proof of Theorem D below involves a case-by-case argument, running over all discrete complex reflection groups. A type-free proof of this result would be desirable.

The paper is organized as follows. In Section 2 we elaborate on our motivations and make additional remarks. In the following four sections we prove, in turn, the four main theorems above.

2. Background and motivation

In this section we elaborate on some of the aforementioned motivations for studying generalized nil-Coxeter algebras and their finite-dimensionality. First, these algebras are interesting from a categorical perspective, as their module categories are symmetric semigroup-tensor categories (see Definition 1.3) but are not monoidal. We will discuss in the next section a Tannaka–Krein duality for such categories, as well as a central extension to a symmetric tensor category.

The second motivation comes from real reflection groups: we provide a novel family of finite-dimensional algebras $NC_A(n,d)$ of type A (akin to the work of Coxeter [Cox2] and Koster [Ko]). In this context, it is remarkable (by Theorem D) that the algebras $NC_A(n,d)$ and the usual nil-Coxeter algebras $NC_W((2,\dots,2))$ are the only finite-dimensional examples.

As Theorem C shows, the algebras $NC_A(n,d)$ for $d>2$ are similar to their "usual" nil-Coxeter analogues for $d=2$. Note however that these algebras also differ in key aspects. See Theorem 5.2 and Proposition 5.3, which show in particular that for $NC_A(n,d)$ with $d>2$, there are multiple "maximal" words, i.e., words killed by left- and right-multiplication by every generator T_i. A more fundamental difference arises out of considerations of *flat deformations*, which we make precise in the remarks around equation (2.1) below.

Our third motivation comes from complex reflection groups and is of much recent interest: the BMR freeness conjecture, which discusses the equality of dimensions of generic Hecke algebras and (the group algebra of) the underlying finite complex reflection group. In this paper we study the associated graded algebra, i.e., where all deformation parameters are set to zero. As shown by Marin [Mar] in some of the cases, non-Coxeter reflection groups do not come equipped with finite-dimensional nil-Coxeter analogues. We make this precise in a strong way in Theorem D above, for all complex reflection groups W. In particular, Theorem D shows that generic Hecke algebras are not flat deformations of their underlying (associated graded) nil-Coxeter analogues for complex W. This is a property shared by the algebras $NC_A(n,d)$ for $d>2$ (but not by Iwahori–Hecke algebras of Coxeter groups $W=W(M)$, which are flat deformations of $NC(M)$). Indeed, if

$M_{n,d}$ denotes the generalized Coxeter matrix corresponding to $NC_A(n,d)$, then we claim that

(2.1)

$$\dim NC_A(n,d) = n!(1+n(d-1)) \ > \ |W(M_{n,d})| = \begin{cases} (n+1)! & \text{if } d>2 \text{ is even,} \\ 1 & \text{if } d>2 \text{ is odd.} \end{cases}$$

To see (2.1), if m_{ij} is odd for any generalized Coxeter matrix $M = M(\mathbf{d})$, then s_i, s_j are conjugate in $W(M)$, whence $s_i^g = s_j^g = 1$ in $W(M)$ for $g = gcd(d_i, d_j)$. On the other hand, $NC_M(\mathbf{d})$ surjects onto the nil-Coxeter algebra $NC_M((2,\dots,2))$ if $d_i \geqslant 2 \ \forall i$. Now if $d_i, d_j \geqslant 2$ are coprime, say, then s_i generates the trivial subgroup of $W(M)$, while T_i does not vanish in $NC(M)$.

The generic Hecke algebras discussed above fit in a broader framework of *deformation theory*, which provides a fourth motivation behind this paper (in addition to the question of flatness discussed above). The theory of flat/PBW deformations of associative algebras is an area of sustained activity, and subsumes Drinfeld Hecke/orbifold algebras [Dr], graded affine Hecke algebras [Lu], symplectic reflection algebras and rational Cherednik algebras [EG], infinitesimal and other Hecke algebras, and other programs in the literature. We also highlight the program of Shepler and Witherspoon; see [SW1, SW2] and the references therein. In all of these settings, a bialgebra A (usually a Hopf algebra) acts on a vector space V and hence on a quotient S_V of its tensor algebra, and one characterizes the deformations of this smash-product algebra $A \ltimes S_V$ which are flat, also termed the "PBW deformations".

In this regard, the significance of the generalized nil-Coxeter algebras $NC(M)$ is manifold. First, the above bialgebra settings were extended in recent work [Kha] to the framework of "cocommutative algebras" A, which also include the algebras $NC_W(\mathbf{d})$. Moreover, we characterized the PBW deformations of $A \ltimes \mathrm{Sym}(V)$, thereby extending in loc. cit. the PBW theorems in the previously mentioned works. The significance of our framework incorporating $A = NC_W(\mathbf{d})$ along with the previously studied algebras is that the full Hopf/bialgebra structure of A—specifically, the antipode or even counit—is *not* required in order to characterize the flat deformations of $A \ltimes \mathrm{Sym}(V)$.

Coming to finite-dimensionality, it was shown in the program of Shepler and Witherspoon (see, e.g., [SW2]), and then in [Kha], that when the algebra A with coproduct is finite dimensional over a field $\Bbbk$, it is possible to characterize the graded $\Bbbk[t]$-deformations of $A \ltimes \mathrm{Sym}(V)$, whose fiber at $t = 1$ has the PBW property. For $A = NC_W(\mathbf{d})$, this deformation-theoretic consideration directly motivates our classification result in Theorem D.

We conclude with a third connection to the aforementioned active program on PBW deformations. We studied in [Kha] the case when $(A, \mathfrak{m}, \Delta)$ is local with $\Delta(\mathfrak{m}) \subset \mathfrak{m} \otimes \mathfrak{m}$. In this setting, if $\mathfrak{m}$ is a *nilpotent* two-sided ideal, then one obtains a lot of information about the deformations of $A \ltimes \mathrm{Sym}(V)$, including understanding the PBW deformations, as well as their center, abelianization, and modules, especially the simple modules. Now if $A = NC_W(\mathbf{d})$, then $\mathfrak{m}$ is generated by the T_i; this explains the interest above in understanding when $\mathfrak{m}$ is nilpotent. Theorem D shows that this condition is in fact equivalent to the generalized nil-Coxeter algebra being finite dimensional.

3. Proof of Theorem A: Tannakian formalism for semigroup categories

The remainder of this paper is devoted to proving the four main theorems in the opening section. We begin by studying the representation category of $NC(M)$ for a generalized Coxeter matrix M. The first assertion is that this category can never be a monoidal category in characteristic zero, and it follows from the following result.

Proposition 3.1. *Suppose $\Bbbk$ is a field of characteristic zero and M is a generalized Coxeter matrix. Then $NC(M)$ is not a bialgebra.*

The result fails to hold in positive characteristic. Indeed, for any prime $p \geqslant 2$ the algebra $(\mathbb{Z}/p\mathbb{Z})[T]/(T^p)$ is a bialgebra, with coproduct $\Delta(T) := 1 \otimes T + T \otimes 1$ and counit $\varepsilon(T) := 0$.

Proof. Note there is a unique possible counit, $\varepsilon : T_i \mapsto 0 \ \forall i \in I$. Now suppose $\Delta : NC(M) \to NC(M) \otimes NC(M)$ is such that

$$(\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id} = (\varepsilon \otimes \mathrm{id}) \circ \Delta$$

on $NC(M)$. Setting $\mathfrak{m} := \ker \varepsilon$ to be the ideal generated by $\{T_i : i \in I\}$, it follows that

$$\Delta(T_i) \in 1 \otimes T_i + T_i \otimes 1 + \mathfrak{m} \otimes \mathfrak{m}. \tag{3.2}$$

Note that $\mathfrak{m} \otimes \mathfrak{m}$ constitutes the terms of higher "total degree" in $\Delta(T_i)$, in the $\mathbb{Z}^{\geqslant 0}$-grading on $NC(M)$. Now if Δ is multiplicative, then raising (3.2) to the m_{ii}th power yields

$$0 = \Delta(T_i)^{m_{ii}} = \sum_{k=1}^{m_{ii}-1} \binom{m_{ii}}{k} T_i^k \otimes T_i^{m_{ii}-k} + \text{ higher degree terms.}$$

This is impossible as long as the image of T_i in $NC(M)$ is nonzero. Assuming this, it follows that Δ cannot be multiplicative, hence not a coproduct on $NC(M)$. Finally, $NC(M)$ surjects onto the usual nil-Coxeter algebra $NC(M_2)$ with $M_2 = M((2, \dots, 2))$. As $NC(M_2)$ has a Coxeter word basis indexed by $W(M)$, it follows that T_i is indeed nonzero in $NC(M)$. □

As a consequence of Proposition 3.1 and the Tannakian formalism in [ES, Theorem 18.3], for any generalized Coxeter matrix M the module category $\operatorname{Rep} NC(M)$ is necessarily not a tensor category. That said, the map $\Delta : T_i \mapsto T_i \otimes T_i$ is a coproduct on $NC(M)$, i.e. a coassociative algebra map. The cocommutativity of Δ implies $\operatorname{Rep} NC(M)$ is a symmetric semigroup category. We now outline how to show the first theorem above, which seeks to understand Tannaka–Krein duality for such categories (possibly without unit objects).

Proof of Theorem A. The proof of part (1) follows that of [ES, Theorem 18.3]; one now ignores the last statement in that proof. The additional data required in the two braided versions in part (2) can be deduced from the proof of [ES, Proposition 14.2]. □

We conclude this section by passing from $\operatorname{Rep} NC(M)$ to an "honest" tensor category—say with $\Bbbk$ a field. Alternately, via the Tannakian formalism in [ES, Theorem 18.3], we produce a bialgebra $\widehat{NC}(M)$ that surjects onto $NC(M)$. Namely,

$\widetilde{NC}(M)$ is generated by $\{T_i : i \in I\}$ and an additional generator T_∞, subject to the braid relations on the former set, as well as

$$T_i^{m_{ii}} = T_i T_\infty = T_\infty T_i = T_\infty^2 := T_\infty \ \forall i \in I.$$

Note that $\widetilde{NC}(M)$ is no longer $\mathbb{Z}^{\geqslant 0}$-graded, but it is a central extension:

$$0 \to \Bbbk T_\infty \to \widetilde{NC}(M) \to NC(M) \to 0.$$

Now asking for all T_i and T_∞ to be grouplike yields a unique bialgebra structure on $\widetilde{NC}(M)$:

$$\widetilde{\Delta} : T_i \mapsto T_i \otimes T_i, \quad T_\infty \mapsto T_\infty \otimes T_\infty, \qquad \widetilde{\varepsilon} : T_i, T_\infty \mapsto 1,$$

and hence a monoidal category structure on $\operatorname{Rep} \widetilde{NC}(M)$, as claimed.

4. Proof of Theorem B: Distinguished basis of words

We now prove our main theorems on the algebras $NC(M)$—specifically, the family $NC_A(n,d)$—beginning with Theorem B. Note that if $d = 2$, then the algebra $NC_A(n,d)$ is the usual nil-Coxeter algebra, while if $n = 1$, then the algebra is $\Bbbk[T_1]/(T_1^d)$. Theorems B and C are easily verified for these cases, e.g., using [Hum, Chapter 7]. Thus, we assume throughout their proofs below that $n \geqslant 2$ and $d \geqslant 3$.

We begin by showing that the $\Bbbk$-rank of $NC_A(n,d)$ is at most $n!(1 + n(d-1))$. Notice that $NC_A(n,d)$ is spanned by words in the T_i. We now **claim** that a word in the T_i is either zero in $NC_A(n,d)$ or equal by the braid relations to a word in which all occurrences of T_n are successive, in a monomial T_n^k for some $1 \leqslant k \leqslant d-1$.

To show the claim, consider a word $\mathcal{T} := \cdots T_n^a T_w T_n^b \cdots$, where $a, b > 0$ and $T_w = T_{i_1} \cdots T_{i_k}$ is a word in $T_1, \ldots, T_{n-1}$. Rewrite $\mathcal{T}$ using the braid relations if required, so that $w \in S_n$ has minimal length, say k. We may assume $k > 0$, else we would be done. Now using the braid relations $T_i T_n = T_n T_i$ for $i \leqslant n-2$, further assume that $i_1 = i_k = n-1$ (otherwise the factors may be "moved past" the T_n using the braid relations). Similarly, $i_2 = i_{k-1} = n-2$, and so on. Thus, if $T_w \neq 0$, then assume by the minimality of $\ell(w)$ that

$$T_w = T_{n-1} T_{n-2} \cdots T_{m+1} T_m T_{m+1} \cdots T_{n-2} T_{n-1} \quad \text{for some } 1 \leqslant m \leqslant n-1.$$

We next claim that the following relation holds in the Artin braid group $\mathcal{B}_n$, and hence in $NC_{A_n}(\mathbf{d})$ for any $\mathbf{d}$:

$$T_{n-1} \cdots T_m \cdots T_{n-1} = T_m T_{m+1} \cdots T_{n-2} T_{n-1} T_{n-2} \cdots T_{m+1} T_m. \tag{4.1}$$

This is shown by descending induction on $m \leqslant n-1$. Hence,

$$\begin{aligned} & T_n^a \cdot (T_{n-1} \cdots T_m \cdots T_{n-1}) \cdot T_n^b \\ &= T_n^a \cdot (T_m \cdots T_{n-2} T_{n-1} T_{n-2} \cdots T_m) \cdot T_n^b \\ &= (T_m \cdots T_{n-2}) T_n^{a-1} (T_n T_{n-1} T_n) T_n^{b-1} (T_{n-2} \cdots T_m) \\ &= (T_m \cdots T_{n-2}) T_n^{a-1} (T_{n-1} T_n T_{n-1}) T_n^{b-1} (T_{n-2} \cdots T_m). \end{aligned} \tag{4.2}$$

If $\max(a,b) = 1$, then the claim follows; if $a > 1$, then the last expression contains the substring $(T_n T_{n-1} T_n) T_{n-1} = T_{n-1} T_n T_{n-1}^2 = 0$; similarly if $b > 1$. This shows the claim.

We now prove the upper bound on the $\Bbbk$-rank. Notice that $T_1, \ldots, T_{n-1}$ generate a subalgebra $R_{n-1} \subset NC_A(n,d)$ in which the nil-Coxeter relations for $W_{A_{n-1}} = S_n$

are satisfied. Hence the map : $NC_{A_{n-1}}((2,\dots,2)) \twoheadrightarrow R_{n-1} := \langle T_1,\dots,T_{n-1}\rangle$ is an algebra map.

Now notice by equation (4.2) that every nonzero word in $NC_A(n,d) \setminus R_{n-1}$ is of the form $\mathcal{T} = T_w T_n^k T_{w'}$, where $1 \leqslant k \leqslant d-1$, $w, w' \in W_{A_{n-1}}$, and hence $T_w, T_{w'} \in R_{n-1}$. By a similar reasoning as above, assuming w' of minimal length in S_{n-1}, we may rewrite $\mathcal{T}$ such that $T_{w'} = T_{n-1}\cdots T_m$ for some $1 \leqslant m \leqslant n$. Carrying out this operation yields $T_{w''} T_n^k T_{n-1}\cdots T_m$ for some reduced word $w'' \in W_{A_{n-1}}$ (i.e., such that $T_{w''}$ is nonzero in $NC_{A_{n-1}}((2,\dots,2))$). Thus,

$$NC_A(n,d) = R_{n-1} + \sum_{k=1}^{d-1}\sum_{m=1}^{n} R_{n-1}\cdot T_n^k \cdot (T_{n-1}\cdots T_m).$$

As R_{n-1} has at most $n!$ generators, it follows that $NC_A(n,d)$ has at most $(1+n(d-1))\cdot n!$ generators, which shows the desired upper bound on its $\Bbbk$-rank.

The hard part of the proof involves showing that the words $T_w T_n^k T_{n-1}\cdots T_m$ form a $\Bbbk$-basis of $NC_A(n,d)$. We will require the following technical lemma on the symmetric group and its nil-Coxeter algebra. A proof is included for completeness.

Lemma 4.3. *Suppose $W = W_{A_{n-1}} = S_n$ is the symmetric group, with simple reflections $s_1,\dots,s_{n-1}$ labelled as usual. Then every element w of $W_{A_{n-1}} \setminus W_{A_{n-2}} = S_n \setminus S_{n-1}$ can be written in reduced form as $w = w' s_{n-1}\cdots s_{m'}$, where $w' \in S_{n-1} = W_{A_{n-2}}$ and $m' \in [1, n-1]$ are unique. Given such an element $w \in S_n$, we have in the usual nil-Coxeter algebra $NC_{A_n}((2,\dots,2))$:*

$$T_n \cdot T_w \cdot T_n \cdots T_m = \begin{cases} T_{w'} T_{n-1}\cdots T_{m-1}\cdot T_n\cdots T_{m'} & \text{if } m' < m, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

Note that equation (4.4) can be thought of as a statement on lengths in the symmetric group.

Proof. We first claim that $w \in W_{A_{n-1}} \setminus W_{A_{n-2}}$ has a reduced expression in which s_{n-1} occurs exactly once. The proof is by induction on n; clearly the claim is true for $n = 2$. Now given the claim for $n-2 \geqslant 2$, consider any reduced expression for w that contains a subword $s_{n-1} w'' s_{n-1}$, where $w \in W_{A_{n-2}}$. By the induction hypothesis, $w'' = w' s_{n-2}\cdots s_m$ for some $w' \in W_{A_{n-3}}$ and $m \in [1, n-1]$. Hence if $m \leqslant n-2$, then

$$w = \cdots s_{n-1}\,(w' s_{n-2} s_{n-3}\cdots s_m)\, s_{n-1} = \cdots w'(s_{n-1}s_{n-2}s_{n-1})(s_{n-3}\cdots s_m)\cdots,$$

and by the braid relations, this equals a reduced expression for $w \in W_{A_{n-1}}$, with one less occurrence of s_{n-1}. A similar analysis works if $m = n-1$. Repeatedly carrying out this procedure proves the claim.

We can now prove the uniqueness of w', m' as in the lemma. By the previous paragraph, write $w \in W_{A_{n-1}} \setminus W_{A_{n-2}}$ as $w = w_1 s_{n-1} w_2$, with $w_1, w_2 \in W_{A_{n-2}}$ and w_2 of smallest possible length, say $w_2 = s_{i_1}\cdots s_{i_k}$ for $i_1,\dots,i_k \leqslant n-2$. Using the braid relations, clearly $i_1 = n-2$; hence $i_2 = n-3$ (by minimality of k). Choose the smallest $l \geqslant 3$ such that $i_l \neq n-1-l$. We now produce a contradiction assuming that such an integer l exists. If $i_l < n-1-l$, then we may move s_{i_l} past the preceding terms, contradicting the minimality of k. Clearly $i_l \neq n-l$, else w_2 was not reduced. Thus $i_l > n-1-l$, whence w_2 is of the form $s_{n-2}\cdots s_{i_l} s_{i_l-1} s_{i_l}\cdots$.

Now verify in $W_{A_{n-1}}$ that

$$w = w_1 s_{n-1} s_{n-2} \cdots s_{i_l+1} s_{i_l} s_{i_l-1} s_{i_l} \cdots = w_1 s_{n-1} \cdots s_{i_l+1} s_{i_l-1} s_{i_l} s_{i_l-1} \cdots$$
$$= w_1 s_{i_l-1} \cdot s_{n-1} \cdot s_{n-2} \cdots s_{i_l+1} s_{i_l} s_{i_l-1} \cdots,$$

which contradicts the minimality of k. Thus such an integer l cannot exist, which proves that $w = w_1 s_{n-1} \cdots s_{m'}$ for some $m' \in [1, n-1]$.

We next claim that the integer m' is unique for $w \in W_{A_n} \setminus W_{A_{n-1}}$. We first make the subclaim that if $w \in W_{A_{n-1}}$ is reduced, then so is $w s_n s_{n-1} \cdots s_m$. To see why, first recall [Hum, Lemma 1.6, Corollary 1.7], which together imply that if $w\alpha > 0$ for any finite Coxeter group W, any $w \in W$, and any simple root $\alpha > 0$, then $\ell(ws_\alpha) = \ell(w)+1$. Now the subclaim follows by applying this result successively to $(ws_n \cdots s_{j+1}, \alpha_j)$ for $j = n, n-1, \ldots, m$. Next, define $C_m := W_{A_{n-1}} \cdot s_n s_{n-1} \cdots s_m$, with $C_{n+1} := W_{A_{n-1}}$. It follows by the subclaim above that $|C_m| = |W_{A_{n-1}}| = n!$ for all m. Hence,

$$(n+1)! = |W_{A_n}| \leqslant \sum_{m=1}^{n+1} |C_m| \leqslant \sum_{m=1}^{n+1} n! = (n+1)!.$$

This shows that $W_{A_n} = \bigsqcup_{m=1}^{n+1} C_m$, which proves the uniqueness of m in the above claim. Now write w_1 in reduced form to obtain that $w' = w s_{m'} \cdots s_{n-1}$ is also unique.

It remains to show equation (4.4) in $NC_{A_n}((2, \ldots, 2))$. Using the above analysis, write $T_w = T_{w'} T_{n-1} \cdots T_{m'}$; since T_n commutes with $T_{w'}$, we may assume $w' = 1$. First suppose $m' \geqslant m$. Then it suffices to prove that $(T_n \cdots T_{m'})^2 = 0$ for all $1 \leqslant m' \leqslant n$. Without loss of generality we may work in the subalgebra generated by $T_{m'}, \ldots, T_n$, and hence suppose $m' = 1$. We now prove by induction that $(T_n \cdots T_1)^2 = 0$. This is clear if $n = 1, 2$, and for $n > 2$,

$$(T_n \cdots T_1)^2 = T_n T_{n-1} T_n \cdot T_{n-2} \cdots T_1 \cdot T_{n-1} \cdots T_1 = T_{n-1} T_n \cdot (T_{n-1} \cdots T_1)^2 = 0.$$

Next suppose $m' < m$; once again we may suppose $m' = 1$. We prove the result by induction on n, the base case of $n = 2$ (and $m = 2$) being easy. Thus, for $1 < m \leqslant n$, we compute

$$\begin{aligned} T_n \cdots T_1 \cdot T_n \cdots T_m &= T_n T_{n-1} T_n \cdot T_{n-2} \cdots T_1 \cdot T_{n-1} \cdots T_m \\ &= T_{n-1} T_n \cdot (T_{n-1} \cdots T_1) \cdot (T_{n-1} \cdots T_m) \\ &= T_{n-1} T_n \cdot (T_{n-2} \cdots T_{m-1}) \cdot (T_{n-1} \cdots T_1) \\ &= T_{n-1} T_{n-2} \cdots T_{m-1} \cdot T_n T_{n-1} \cdots T_1. \end{aligned}$$

□

Remark 4.5. Notice that equation (4.4) holds in any algebra containing elements $T_1, \ldots, T_n$ that satisfy the braid relations and $T_1^2 = 0$. In particular, (4.4) holds in $NC_A(n, d)$ for $n > 1$.

Returning to the proof of Theorem B, we now introduce a diagrammatic calculus akin to crystal theory. We first write out the $n = 2$ case in order to provide intuition for the case of general n. Let $\mathscr{M}$ be a free $\Bbbk$-module, with basis given by the nodes in the graph in Figure 1.

In the figure, the node 12^21 should be thought of as $T_1 T_2^2 T_1$ (applied to the unit $1_{NC_A(2,d)}$, i.e., to the generating basis vector corresponding to $\emptyset$), and similarly for the other nodes. The arrows denote the action of T_1 and T_2; all remaining generator actions on nodes yield zero. Now one verifies by inspection that the

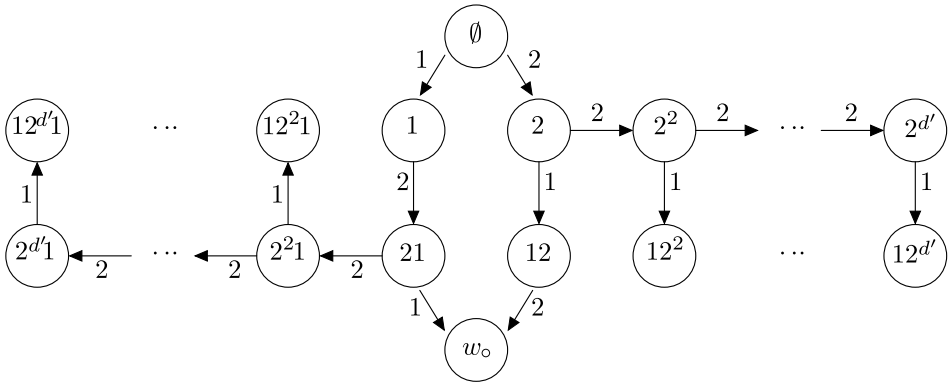

FIGURE 1. Regular representation for $NC_A(2,d)$, with $d' = d-1$

defining relations in $NC_A(2,d)$ are satisfied by this action on $\mathscr{M}$. Therefore $\mathscr{M}$ is an $NC_A(2,d)$-module of $\Bbbk$-rank $4d-2 = 2!(1+2(d-1))$. Since $\mathscr{M}$ is generated by the basis vector corresponding to the node $\emptyset$, we have a surjection $: NC_A(2,d) \twoheadrightarrow \mathscr{M}$ that sends $T_1, T_2^k, T_1T_2^k, T_2^kT_1, T_1T_2^kT_1$ to the corresponding basis vectors in the free $\Bbbk$-module $\mathscr{M}$. Now the result for $n=2$ follows by the upper bound on the $\Bbbk$-rank, proved above.

The strategy is similar for general n, but uses the following more detailed notation. For each $w \in S_l$ with $l \leqslant n$, let T_w denote the corresponding (well-defined) word in the alphabet $\{T_1, \dots, T_{l-1}\}$ and let R_{l-1} denote the subalgebra of $NC_A(n,d)$ generated by these letters. Now define a free $\Bbbk$-module $\mathscr{M}$ of $\Bbbk$-rank $n!(1+n(d-1))$, with basis elements the set of words

$$\mathcal{B} := \{B(w,k,m) : w \in S_n,\ k \in [1,d-1],\ m \in [1,n]\} \sqcup \{B(w) : w \in S_n\}. \tag{4.6}$$

We observe here that the basis vectors $B(w,k,m), B(w)$ are to be thought of as corresponding respectively to the words

$$T_wT_n^kT_{n-1}\cdots T_m, \quad T_w, \qquad w \in S_n,\ k \in [1,d-1],\ m \in [1,n]. \tag{4.7}$$

Definition 4.8. An expression for a word in $NC_A(n,d)$ of the form (4.7) will be said to be in *standard form.*

We now define an $NC_A(n,d)$-module structure on $\mathscr{M}$, via defining a directed graph structure on $\mathcal{B}$ (or more precisely, on $\mathcal{B} \sqcup \{0\}$) that we now describe. Figure 2 may help in visualizing the structure. The figure should be thought of as analogous to the central hexagon and either of the two "arms" in Figure 1.

We begin by explaining the figure. Each node (wkm) (or (w)) corresponds to the basis vector $B(w,k,m)$ (or $B(w)$). Notice that the vectors $\{B(w,1,m)\} \sqcup \{B(w)\}$ are in bijection with the Coxeter word basis of the usual nil-Coxeter algebra $NC_{A_n}((2,\dots,2))$. Let V_1 denote their span of $\Bbbk$-rank $(n+1)!$. Now given $1 \leqslant m \leqslant n$ and $1 \leqslant k \leqslant d-1 =: d'$, define $V_{k,m}$ to be the span of the basis elements $\{B(w,k,m) : w \in S_n\}$, of $\Bbbk$-rank $n!$. Then $\mathscr{M} = V_1 \oplus \bigoplus_{k>1,m} V_{k,m}$. Note as a special case that in Figure 1 the central hexagon spans V_1, the nodes $2^k1, 12^k1$ span $V_{k,1}$, and $2^k, 12^k$ span $V_{k,2}$. We now define the $NC_A(n,d)$-action:

- Let $V_{1,n+1}$ denote the $\Bbbk$-span of $\{B(w) : w \in S_n\}$. Then for $1 \leqslant m \leqslant n+1$, each $V_{1,m}$ has a distinguished basis in bijection with S_n; the same holds

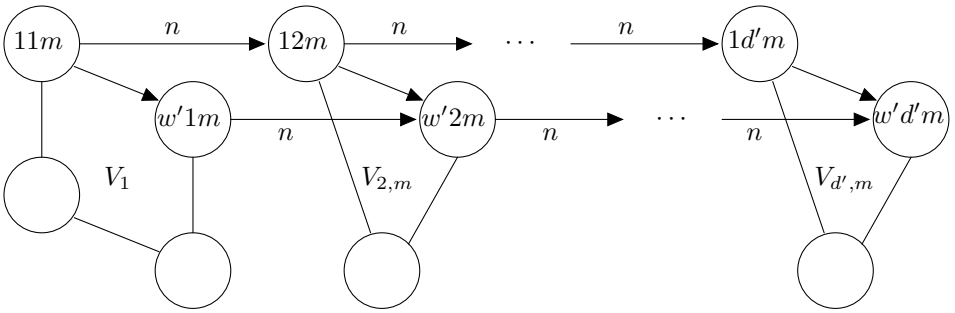

FIGURE 2. Regular representation for $NC_A(n,d)$, with $d' = d-1$

for each $V_{k,m}$ with $k \in [2, d-1]$ and $m \in [1,n]$. Now equip all of the above spaces $V_{k,m}$ with the corresponding module structure over the usual nil-Coxeter algebra of type A_{n-1}. Such a structure is uniquely determined if given $w = s_{i_1} \cdots s_{i_l} \in S_n$ with all $i_j < n$, we set $T_w \cdot B(1,k,m) := B(w,k,m)$ and $T_w \cdot B(1) := B(w)$.

- We next define the action of T_n on $\mathscr{M}$. Via Lemma 4.3, write $w \in S_n$ as $w' s_{n-1} \cdots s_{m'}$ with w', m' unique. Now using the previous paragraph, it follows that $B(w,k,m) = T_{w'} T_{n-1} \cdots T_{m'} \cdot B(1,k,m)$. Correspondingly, if $w \in S_{n-1}$ (i.e., $m' = n$), define

$$T_n \cdot B(w,k,m) := \mathbf{1}(k \leqslant d-2) B(w,k+1,m), \qquad T_n \cdot B(w) := B(w,1,n).$$

- On the other hand, suppose $m' \leqslant n-1$. If $k \geqslant 2$, then define $T_n \cdot B(w,k,m) := 0$. Otherwise define $T_n \cdot B(w) := B(w',1,m')$ with w', m' as in Lemma 4.3, and (see equation 4.4):

$$T_n \cdot B(w,1,m) := \begin{cases} B(w' s_{n-1} \cdots s_{m-1}, 1, m') & \text{if } m' < m, \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

It remains to show that the above graph structure indeed defines an $NC_A(n,d)$-module structure on $\mathscr{M}$; then a similar argument as above (in the $n=2$ case) completes the proof. In the following argument, we will occasionally use Lemma 4.3 (as well as Remark 4.5) without reference. First notice that the algebra relations involving only $T_1, \ldots, T_{n-1}$ are clearly satisfied on $\mathscr{M}$ as it is an R_{n-1}-free module by construction. To verify that the relations involving T_n hold on $\mathscr{M}$, notice (e.g., via Figure 2) that the $\Bbbk$-basis $\mathcal{B}$ of $\mathscr{M}$ can be partitioned into three subsets:

$$\begin{aligned} \mathcal{B}_1 &:= \{B(w,k,m) : k \geqslant 1, m \in [1,n], w \in S_{n-1}\} \sqcup \{B(w) : w \in S_{n-1}\}, \\ \mathcal{B}_2 &:= \{B(w,k,m) : k \geqslant 2, m \in [1,n], w \in S_n \setminus S_{n-1}\}, \\ \mathcal{B}_3 &:= \{B(w,1,m) : m \in [1,n], w \in S_n \setminus S_{n-1}\} \sqcup \{B(w) : w \in S_n \setminus S_{n-1}\}. \end{aligned} \tag{4.10}$$

Recall by the opening remarks in Section 4 that $n \geqslant 2$ and $d \geqslant 3$. We first show that the relation $T_n^d = 0$ holds as an equality of linear operators on each vector $b \in \mathcal{B}$, and hence on the $\Bbbk$-module $\mathscr{M}$. We separately consider the cases $b \in \mathcal{B}_i$ for $i = 1,2,3$, as in (4.10).

(1) If $b = B(w,k,m) \in \mathcal{B}_1$, then b lies in the "top rows" of Figure 2. It is easily verified that $T_n^{d-k-1} \cdot B(w,k,m) = B(w,d-1,m)$, and this is killed by T_n, as desired. The same reasoning shows that T_n^d kills $b = B(w)$ for $w \in S_{n-1}$.

(2) Let $b = B(w,k,m) \in \mathcal{B}_2$. Then the relation holds on b since $T_n \cdot B(w,k,m) = 0$. (These correspond to vectors in $V_{k,m}$ for $k \geqslant 2$, which do not lie in the "top rows" in Figure 2.)
(3) Finally, let $b \in \mathcal{B}_3$; thus $w \in S_n \setminus S_{n-1}$, and we write $w = w's_{n-1}\cdots s_{m'}$ by Lemma 4.3. It follows from Remark 4.5 that $T_n^2 \cdot B(w,1,m) = 0$ and $T_n^d \cdot B(w) = T_n^{d-1} \cdot B(w',1,m') = 0$.

We next show that the relation $T_iT_n = T_nT_i$ holds on $\mathcal{B}$ for all $i \leqslant n-2$. We consider the same three cases as in (4.10).

(1) Fix $w \in S_{n-1}$. If $b = B(w,k,m)$ with $k \geqslant 1$, then verify using the aforementioned action that both $T_iT_n \cdot B(w,k,m)$ and $T_nT_i \cdot B(w,k,m)$ equal $B(s_iw, k+1, m)$ if $\ell(s_iw) > \ell(w)$ and $k \leqslant d-2$, and 0 otherwise. Similarly,

$$T_iT_n \cdot B(w) = \mathbf{1}(\ell(s_iw) > \ell(w))B(s_iw,1,n) = T_nT_i \cdot B(w).$$

(2) Let $b = B(w,k,m)$ with $w \in S_n \setminus S_{n-1}$ and $k \geqslant 2$. Then $T_iT_n \cdot B(w,k,m) = 0$. To compute $T_nT_i \cdot B(w,k,m)$, since $i \leqslant n-2$, it follows that $s_iw \in S_n \setminus S_{n-1}$. If $T_iT_w = 0$, then we are done since $B(w,k,m) = T_w \cdot B(1,k,m)$. Else note that $s_iw \in S_n \setminus S_{n-1}$, whence $T_nT_i \cdot B(w,k,m) = T_n \cdot B(s_iw,k,m) = 0$ from above.
(3) Finally, let $w \in S_n \setminus S_{n-1}$ and write $w = w's_{n-1}\cdots s_{m'}$ by Lemma 4.3. First suppose $b = B(w,1,m)$. If $\ell(s_iw) < \ell(w)$, then it is not hard to show that both $T_iT_n \cdot B(w,1,m)$ and $T_nT_i \cdot B(w,1,m)$ vanish. Otherwise both terms are equal to $B(s_iw's_{n-1}\cdots s_{m-1}, 1, m')$. A similar analysis shows that if $\ell(s_iw) < \ell(w)$, then $T_iT_n \cdot B(w) = T_nT_i \cdot B(w) = 0$, otherwise $T_iT_n \cdot B(w) = T_nT_i \cdot B(w) = B(s_iw', 1, m')$.

Next, we show that the braid relation $T_{n-1}T_nT_{n-1} = T_nT_{n-1}T_n$ holds on $\mathcal{B}$. This is the most involved computation to carry out. We consider the same three cases as above.

(1) Fix $w \in S_{n-1}$. If $b = B(w,k,m)$ with $k \geqslant 2$, then it is easily verified that both sides of the braid relation kill $B(w,k,m)$. If instead $k = 1$, then

$$T_nT_{n-1}T_n \cdot B(w,1,m) = T_nT_{n-1} \cdot B(w,2,m) = 0.$$

To compute the other side, first notice $B(w,1,m) = T_n \cdot B(ws_{n-1}\cdots s_m)$. Hence,

$$T_nT_{n-1} \cdot B(w,1,m) = T_nT_{n-1}T_n \cdot B(ws_{n-1}\cdots s_m).$$

Now if the braid relation holds on $B(w)$ for all $w \in S_n$, then

$$\begin{aligned} T_{n-1}T_nT_{n-1} \cdot B(w,1,m) &= T_{n-1} \cdot T_nT_{n-1}T_n \cdot B(ws_{n-1}\cdots s_m) \\ &= T_{n-1}T_{n-1}T_nT_{n-1} \cdot B(ws_{n-1}\cdots s_m) \in T_{n-1}^2 \cdot \mathscr{M} = 0, \end{aligned}$$

where the last equality follows from the definition of $\mathscr{M}$ as an R_{n-1}-module. It thus suffices for this case to verify that the braid relation holds on $B(w)$ for $w \in S_n$. This is done by considering the following four subcases.
(a) If $w \in S_{n-2}$ commutes with s_{n-1}, s_n, then both $T_nT_{n-1}T_n \cdot B(w)$ and $T_{n-1}T_nT_{n-1} \cdot B(w)$ are easily seen to equal $B(s_{n-1}w, 1, n-1)$.

(b) Suppose $w = w's_{n-2}\cdots s_{m'} \in S_{n-1} \setminus S_{n-2}$, with $w' \in S_{n-2}$ and $m' \in [1, n-2]$. Then using Remark 4.5 and the R_{n-1}-module structure of $\mathscr{M}$, we compute

$$\begin{aligned} T_nT_{n-1}T_n \cdot B(w) &= T_nT_{n-1} \cdot B(w,1,n) = T_n \cdot B(w's_{n-1}\cdots s_{m'},1,n) \\ &= B(w's_{n-1},1,m'), \\ T_{n-1}T_nT_{n-1} \cdot B(w) &= T_{n-1}T_n \cdot B(w's_{n-1}\cdots s_{m'}) = T_{n-1} \cdot B(w',1,m') \\ &= B(s_{n-1}w',1,m'), \end{aligned}$$

whence we are done since s_{n-1} commutes with $w' \in S_{n-2}$.

(c) In the last two subcases, $w = w's_{n-1}\cdots s_{m'} \in S_n \setminus S_{n-1}$ with $m' \in [1, n-1]$. As in the previous two subcases, first suppose $w' \in S_{n-2}$. Similar to the above computations, one verifies that both $T_nT_{n-1}T_n \cdot B(w)$ and $T_{n-1}T_nT_{n-1} \cdot B(w)$ vanish.

(d) Finally, suppose $w = w''s_{n-2}\cdots s_m \cdot s_{n-1}\cdots s_{m'}$ with $m \in [1, n-2]$, $m' \in [1, n-1]$, and $w'' \in S_{n-2}$. Then one verifies that

$$T_nT_{n-1}T_n \cdot B(w) = \mathbf{1}(m < m')B(w''s_{n-2}\cdots s_{m'-1},1,m) = T_{n-1}T_nT_{n-1} \cdot B(w).$$

(2) Next suppose $b \in \mathcal{B}_2$ is of the form $B(w,k,m)$ with $k \geqslant 2$ and $w = w's_{n-1}\cdots s_{m'} \in S_n \setminus S_{n-1}$. Then $T_n \cdot B(w,k,m) = 0$ by definition, so $T_nT_{n-1}T_n \cdot B(w,k,m) = 0$. To show that $T_{n-1}T_nT_{n-1}$ kills $B(w,k,m)$, we consider two subcases. If $w' \in S_{n-2}$, then $T_{n-1} \cdot B(w,k,m) = 0$ and we are done. Otherwise suppose $w' = w''s_{n-2}\cdots s_{m''} \in S_{n-1} \setminus S_{n-2}$, with $m'' \in [1, n-2]$. Now compute using Remark 4.5 and the relations verified above:

$$\begin{aligned} &T_{n-1}T_nT_{n-1} \cdot B(w,k,m) \\ &= T_{n-1}T_n \cdot B(w''s_{n-1}\cdots s_{m''}s_{n-1}\cdots s_{m'},k,m) \\ &= \mathbf{1}(m'' < m')T_{n-1}T_n \cdot B(w''s_{n-2}\cdots s_{m'-1}s_{n-1}\cdots s_{m''},k,m) \\ &= \mathbf{1}(m'' < m')T_{n-1} \cdot 0 = 0, \end{aligned}$$

where the penultimate equality uses that $k = 2$.

(3) Finally, suppose $b \in \mathcal{B}_3$. By the analysis in the first case above, we only need to consider $b = B(w,1,m)$ with $w = w's_{n-1}\cdots s_{m'} \in S_n \setminus S_{n-1}$. It is now not hard to show that both $T_nT_{n-1}T_n \cdot B(w,1,m)$ and $T_{n-1}T_nT_{n-1} \cdot B(w,1,m)$ vanish if $w' \in S_{n-2}$. On the other hand, if $w' = w''s_{n-2}\cdots s_{m''} \in S_{n-1} \setminus S_{n-2}$, then repeated use of Remark 4.5 (and equation (4.4)) shows that

$$\begin{aligned} &T_{n-1}T_nT_{n-1} \cdot B(w,1,m) \\ &= T_{n-1}T_nT_{n-1} \cdot B(w'' \cdot s_{n-2}\cdots s_{m''} \cdot s_{n-1}\cdots s_{m'},1,m) \\ &= \mathbf{1}(m'' < m')T_{n-1}T_n \cdot B(w'' \cdot s_{n-2}\cdots s_{m'-1} \cdot s_{n-1}\cdots s_{m''},1,m) \\ &= \mathbf{1}(m'' < m')\mathbf{1}(m'' < m)T_{n-1} \cdot B(w'' \cdot s_{n-2}\cdots s_{m'-1} \cdot s_{n-1}\cdots s_{m-1},1,m'') \\ &= \mathbf{1}(m'' < m')\mathbf{1}(m'' < m)\mathbf{1}(m' < m)B(w''s_{n-2}\cdots s_{m-2}s_{n-1}\cdots s_{m'-1},1,m) \\ &= \mathbf{1}(m'' < m' < m)B(w''s_{n-2}\cdots s_{m-2}s_{n-1}\cdots s_{m'-1},1,m). \end{aligned}$$

Notice this calculation shows the "braid-like" action of T_n, T_{n-1} on strings of the type

$$T_{n-2}\cdots T_{m''}, \qquad T_{n-1}\cdots T_{m'}, \qquad T_n\cdots T_m.$$

Similarly, one shows that

$$T_nT_{n-1}T_n \cdot B(w,1,m) = \mathbf{1}(m'' < m' < m)B(w''s_{n-2}\cdots s_{m-2}s_{n-1}\cdots s_{m'-1},1,m),$$

which verifies that the last braid relation holds in the last case.

Thus the algebra relations hold on all of $\mathscr{M}$, making it an $NC_A(n,d)$-module generated by $B(1)$, as claimed. In particular, $NC_A(n,d) \simeq \mathscr{M}$ as $\Bbbk$-modules, by the analysis in the first part of this proof. This completes the proof of all but the last assertion in Theorem B. Finally, the nil-Coxeter algebra $NC_{A_l}((2,\dots,2))$ surjects onto R_l, and $R_l \simeq R_l \cdot B(\emptyset) \subset V_1$ is free of $\Bbbk$-rank $(l+1)!$ from above. Hence $R_l \simeq NC_{A_l}((2,\dots,2))$, as desired. □

5. Proof of Theorem C, primitive elements, and categorification

In this section we continue our study of the algebras $NC_A(n,d)$, starting with Theorem C.

Proof of Theorem C. We retain the notation of Theorem B. Via the $\Bbbk$-module isomorphism $\mathscr{M} \simeq NC_A(n,d)$, we identify the basis element $B(w,k,m)$ with $T_wT_n^kT_{n-1}\cdots T_m$ and $B(w)$ with T_w, where $w \in S_n$, $k \in [1,d-1]$, and $m \in [1,n]$. Let $\ell : \mathcal{B} \to \mathbb{Z}^{\geqslant 0}$ be as in equation (1.10).

We now **claim** that if $\mathcal{T} = T_{i_1}\cdots T_{i_l}$ is *any* nonzero word in $NC_A(n,d)$, then l is precisely the length of $\mathcal{T}$ when expressed (uniquely) in standard form (4.7). The proof is by induction on l. For $l=1$, T_i is already in standard form (and nonzero). Now given a word $\mathcal{T} = T_i\mathcal{T}'$ of length $l+1$ (so $\mathcal{T}'$ has length l and satisfies the claim), write $\mathcal{T}'$ via the induction hypothesis as a word in standard form of length l. Now the proof of Theorem B shows that applying any T_i to this standard form for $\mathcal{T}'$ either yields zero or has length precisely $l+1$. This proves the claim.

The above analysis shows (1) and (2). Now suppose $\Bbbk$ is a field. Then the algebra $NC_A(n,d)$ has a maximal ideal $\mathfrak{m} = \langle\{T_i : i \in I\}\rangle$; in fact, $\mathfrak{m}$ has $\Bbbk$-corank 1 by the proof of Theorem B. Moreover, $\mathfrak{m}$ is local because any element of $A\setminus\mathfrak{m}$ is invertible. (In particular, one understands representations of the algebra $NC_A(n,d)$, e.g., by [Kha, §6.1].)

The aforementioned claim also proves that $\mathfrak{m}^{l+1} = 0$, where $l := \ell_{A_{n-1}}(w_\circ') + d + n - 2$. This is because any nonzero word can be expressed in standard form without changing the length. □

As an immediate consequence, we have the following corollary.

Corollary 5.1. *If $\Bbbk$ is a field and $T_1,\dots,T_n$ all have graded degree 1, the Hilbert–Poincaré series of $NC_A(n,d)$ is the polynomial*

$$[n]_q!\,(1+[n]_q\,[d-1]_q), \qquad \textit{where } [n]_q := \frac{q^n-1}{q-1},\ [n]_q! := \prod_{j=1}^{n}[j]_q.$$

The proof also uses the standard result that the Hilbert–Poincaré series of the usual nil-Coxeter algebra $NC_A(n,2)$ is $[n]_q!$ (see, e.g., [Hum, §3.12, 3.15]).

Next, we discuss a property that was explored in [Kho] for the usual nil-Coxeter algebras $NC_A(n,2)$: these algebras are always Frobenius. We now study when the

algebras $NC_A(n,d)$ are also Frobenius for $d \geqslant 3$. As the following result shows, this only happens in the degenerate case of $n = 1$, i.e., $\Bbbk[T_1]/(T_1^d)$.

Theorem 5.2. *Suppose $\Bbbk$ is a field. Given $n \geqslant 1$ and $d \geqslant 2$, the algebra $NC_A(n,d)$ is Frobenius if and only if $n = 1$ or $d = 2$.*

One checks via equation (2.1) that these conditions are further equivalent to (the group algebra of) the "generalized Coxeter group" $W(M_{n,d})$ being a flat deformation of $NC_A(n,d)$.

The proof of Theorem 5.2 crucially uses the knowledge of "maximal", i.e., primitive words in the algebra $NC_A(n,d)$. Formally, given a generalized Coxeter matrix M, say that an element $x \in NC(M)$ is *left* (respectively, *right*) *primitive* if $\mathfrak{m}x = 0$ (respectively, $x\mathfrak{m} = 0$); cf. Theorem C(3). Now x is *primitive* if it is both left- and right-primitive. Denote these sets of elements respectively by

$$\mathrm{Prim}_L(NC(M)), \quad \mathrm{Prim}_R(NC(M)), \quad \mathrm{Prim}(NC(M)).$$

Proposition 5.3. *Every generalized nil-Coxeter algebra $NC(M)$ is equipped with an anti-involution θ that fixes each generator T_i. Now θ is an isomorphism: $\mathrm{Prim}_L(NC(M)) \longleftrightarrow \mathrm{Prim}_R(NC(M))$. Moreover, the following hold:*

(1) *If $W(M)$ is a finite Coxeter group with unique longest word $w_\circ$, then*
$$\mathrm{Prim}_L(NC(M)) = \mathrm{Prim}_R(NC(M)) = \mathrm{Prim}(NC(M)) = \Bbbk T_{w_\circ}.$$
(2) *If $NC(M) = NC_A(1,d)$, then*
$$\mathrm{Prim}_L(NC(M)) = \mathrm{Prim}_R(NC(M)) = \mathrm{Prim}(NC(M)) = \Bbbk \cdot T_1^{d-1}.$$
(3) *If $NC(M) = NC_A(n,d)$ with $n \geqslant 2$ and $d \geqslant 3$, then:*
   (a) $\mathrm{Prim}_L(NC(M))$ *is spanned by* $T_{w_\circ} := T_{w'_\circ} T_n T_{n-1} \cdots T_1$ *and the* $n(d-2)$ *words*
   $$\{T_{w'_\circ} T_n^k T_{n-1} \cdots T_m : \ k \in [2, d-1], \ m \in [1,n]\}.$$
   (b) $\mathrm{Prim}(NC(M))$ *is spanned by the* $d-1$ *words* $T_{w'_\circ} T_n^k T_{n-1} \cdots T_1$, *where* $1 \leqslant k \leqslant d-1$.

In all cases, the map θ fixes both $\mathrm{Prim}(NC(M))$ *as well as the lengths of all nonzero words.*

Proof. The first two statements are obvious since θ preserves the defining relations in $\Bbbk\langle\{T_i : i \in I\}\rangle$. The assertion in (1) is standard (see, e.g., [Hum, Chapter 7]) and (2) is easily verified.

We next classify the left-primitive elements as in (3)(a). Suppose for some $k \geqslant 2$ and $1 \leqslant m \leqslant n$ that $\mathcal{T} = T_{w'_\circ} T_n^k T_{n-1} \cdots T_m$. Then clearly $T_i \mathcal{T} = 0$ for all $i < n$, and $T_n \mathcal{T} = 0$ since $k \geqslant 2$, as discussed in the proof of Theorem B. Similarly, if $\mathcal{T} = T_{w_\circ}$, then $T_i \mathcal{T} = 0$ for $i < n$, and we also computed in the proof of Theorem B that $T_{w_\circ} = T_1 \cdots T_{n-1} T_n T_{w'_\circ}$. Hence,

$$T_n T_{w_\circ} = T_1 \cdots T_{n-2} (T_n T_{n-1} T_n) T_{w'_\circ} = T_1 \cdots T_{n-1} T_n (T_{n-1} T_{w'_\circ}) = 0.$$

To complete the proof of (3)(a), it suffices to show that no nonzero linear combination of the remaining words of the form $T_w T_n^k T_{n-1} \cdots T_m$ is left-primitive. Suppose first that there is a word $w \in W_{A_{n-1}}$ such that the coefficient of T_w is nonzero. In that case, choose such an element w of smallest length and left-multiply the linear combination by $T_n^{d-1} T_{w'_\circ w^{-1}}$. As discussed in the proof of Theorem B, this kills all terms $T_{w'} T_n^k T_{w''}$ with $w', w'' \in W_{A_{n-1}}$ and $k \geqslant 1$. Moreover, by [Hum, Chapter 7],

left-multiplication by $T_{w'_\circ w^{-1}}$ also kills all terms of the same length that are not T_w. Thus we are left with $T_n^{d-1}T_{w'_\circ w^{-1}}T_w = T_n^{d-1}T_{w'_\circ} \neq 0$, so the linear combination was not left-primitive.

The other case is that all words in the linear combination are of the form $T_w T_n^k T_{n-1}\cdots T_m$ with $k \geqslant 1$. Once again, choose $w \in W_{A_{n-1}}$ of smallest length for which the corresponding word has nonzero coefficient, and left-multiply by $T_{w'_\circ w^{-1}}$. This yields a nonzero linear combination by the analysis in Theorem B, which proves the assertion about left-primitivity.

We next identify the primitive elements in $NC(M) = NC_A(n,d)$. The first claim is that $\mathcal{T}_k := T_{w'_\circ}T_n^k T_{n-1}\cdots T_1$ is fixed by θ. Indeed, we compute using the braid relations in type A that θ fixes $T_{w'_\circ} \in R_{n-1}$ and $T_{w''_\circ} \in R_{n-2}$. Hence,

$$\theta(\mathcal{T}_k) = T_1\cdots T_{n-1}T_n^k T_{w''_\circ}T_{n-1}\cdots T_1 = T_1\cdots T_{n-1}T_{w''_\circ}T_n^k T_{n-1}\cdots T_1 = \mathcal{T}_k. \tag{5.4}$$

Using this we claim that $\mathcal{T}_k$ is right-primitive. Indeed, if $i < n$, then

$$\mathcal{T}_k T_i = T_1\cdots T_{n-1}T_n^k T_{w'_\circ}T_i = 0,$$

while for $i = n$, we compute

$$\mathcal{T}_k T_n = T_{w'_\circ}T_n^k T_{n-1}T_n \cdot T_{n-2}\cdots T_1 = T_n^{k-1}T_{w'_\circ}T_{n-1}\cdot T_n T_{n-1}T_{n-2}\cdots T_1 = 0.$$

We now claim that no linear combination of the remaining left-primitive elements listed in (3)(a) is right-primitive. Indeed, let m_0 denote the minimum of the m-values in words with nonzero coefficients; then $m_0 > 1$ by the above analysis. Now right-multiply by $T_{m_0-1}\cdots T_1$. This kills all elements with m-value $> m_0+1$, since T_{m_0-1} commutes with $T_nT_{n-1}\cdots T_m$, and hence can be taken past them to multiply against $T_{w'_\circ}$ and be killed. The terms with m-value equal to m_0 are not killed by the analysis in Theorem B. It follows that such a linear combination is not right-primitive, which completes the classification of the primitive elements in (3)(b).

Next, that $\mathrm{Prim}(NC(M))$ is fixed by θ was shown in equation (5.4). Moreover, if $NC(M)$ equals $NC_A(n,d)$ or $\Bbbk W(M)$ with $W(M)$ finite, then it is equipped with a suitable length function ℓ. Now θ preserves the length because the algebra relations are ℓ-homogeneous and preserved by θ. □

Remark 5.5. In light of Proposition 5.3, it is natural to ask how to write right-primitive words in standard form. More generally, given $w = w's_{n-1}\cdots s_{m'}$ for unique $w' \in S_{n-1}$ and $m' \in [1,n]$ (via Lemma 4.3), we have: $T_m\cdots T_{n-1}T_n^k T_w = T_{\widetilde{w}}T_n^k T_{n-1}\cdots T_{m'}$, where $\widetilde{w} = s_m\cdots s_{n-1}w'$.

With Proposition 5.3 in hand, we turn to the Frobenius property of $NC_A(n,d)$. The following proof reveals that $NC_A(n,d)$ is Frobenius if and only if $\mathrm{Prim}(NC(M))$ is one dimensional.

Proof of Theorem 5.2. For finite Coxeter groups $W(M)$, the corresponding nil-Coxeter algebras $NC(M)$ are indeed Frobenius; see, e.g., [Kho, §2.2]. It is also easy to verify that $NC_A(1,d) = \Bbbk[T_1]/(T_1^d)$ is Frobenius by using the symmetric bilinear form uniquely specified by $\sigma(T_1^i, T_1^j) = \mathbf{1}(i+j = d-1)$. Thus, it remains to show that for $n \geqslant 2$ and $d \geqslant 3$, the algebra $NC_A(n,d)$ is not Frobenius. Indeed, if $NC_A(n,d)$ is Frobenius with nondegenerate invariant bilinear form σ, then for each nonzero primitive p there exists a vector a_p such that $0 \neq \sigma(p,a_p) = \sigma(pa_p,1)$. It follows that we may take $a_p = 1$ for all p.

Now the linear functional $\sigma(-,1) : \mathrm{Prim}(NC_A(n,d)) \to \Bbbk$ is nonsingular, whence $\dim_{\Bbbk} \mathrm{Prim}(NC_A(n,d)) = 1$. Thus $n = 1$ or $d = 2$ by Proposition 5.3. □

We conclude this section by discussing the connection of $NC_A(n,d)$ to the categorification by Khovanov [Kho] of the Weyl algebra $W_n := \mathbb{Z}\langle x, \partial\rangle/(\partial x = 1 + x\partial)$. Namely, the usual type A nil-Coxeter algebra $\mathcal{A}_n := NC_A(n,2)$ is a bimodule over $\mathcal{A}_{n-1}$, and this structure was studied in loc. cit., leading to the construction of tensor functors categorifying the operators x, ∂.

We now explain how the algebra $NC_A(n,d)$ fits into this framework.

Proposition 5.6. *For all $n \geqslant 1$ and $d \geqslant 2$, we have an isomorphism of $\mathcal{A}_{n-1}$-bimodules:*

$$NC_A(n,d) \simeq \mathcal{A}_{n-1} \oplus \bigoplus_{k=1}^{d-1} \left(\mathcal{A}_{n-1} \otimes_{\mathcal{A}_{n-2}} \mathcal{A}_{n-1}\right).$$

When $d = 2$, this result was shown in [Kho, Proposition 5]. For general $d \geqslant 2$, using the notation of [Kho], this result implies in the category of $\mathcal{A}_{n-1}$-bimodules that the algebra $NC_A(n,d)$ corresponds to $1 + (d-1)x\partial$ (including the previously known case of $d = 2$). In particular, Proposition 5.6 strengthens Theorems B and C, which explained a left $\mathcal{A}_{n-1}$-module structure on $NC_A(n,d)$ (namely, that $NC_A(n,d)$ is free of rank $1 + n(d-1)$).

Proof of Proposition 5.6. From the proof of Theorem B, the algebra $NC_A(n,d)$ has a "regular representation" $\varphi : \mathscr{M} \xrightarrow{\sim} NC_A(n,d)$, sending $B(w) \mapsto T_w$ and $B(w,k,m) \mapsto T_w T_n^k T_{n-1} \cdots T_m$ for $w \in S_n$, $k \in [1, d-1]$, and $m \in [1,n]$. Also recall the subspaces $V_{k,m}$ defined in the discussion following equation (4.7): $V_{k,m} = \bigoplus_{w \in S_n} \Bbbk B(w,k,m)$.

By Theorem B, $M_k := \bigoplus_{m=1}^{n} \varphi(V_{k,m})$ is a free left $\mathcal{A}_{n-1}$-module of rank one. It is also a free right $\mathcal{A}_{n-1}$-module of rank one, using the anti-involution θ from Proposition 5.3 and Remark 5.5. In fact, the uniqueness of the standard form (4.7) shown in the proof of Theorem B implies that for all $1 \leqslant k \leqslant d-1$ the map

$$\varphi_k : \mathcal{A}_{n-1} \otimes_{\mathcal{A}_{n-2}} \mathcal{A}_{n-1} \to M_k, \qquad a \otimes a' \mapsto a T_n^k a'$$

is an isomorphism of $\mathcal{A}_{n-1}$-bimodules. Now the result follows from (the proof of) Theorem B. □

Remark 5.7. Notice that the proof of Proposition 5.6 also categorifies Corollary 5.1.

6. Proof of Theorem D: Finite-dimensional generalized nil-Coxeter algebras

We now prove Theorem D, which classifies the generalized nil-Coxeter algebras of finite $\Bbbk$-rank. The bulk of the proof involves showing (1) $\implies$ (2). We again employ the diagrammatic calculus used to show Theorem B, now applied to the five diagrams in Figure 3.

We begin by **assuming** that $W = W(M)$ is a generalized Coxeter group and then by classifying the algebras $NC(M)$ that have finite $\Bbbk$-rank. Following this classification, we address the remaining finite complex reflection groups W (and all **d**), followed by the infinite discrete complex reflection groups with their Coxeter-type presentations.

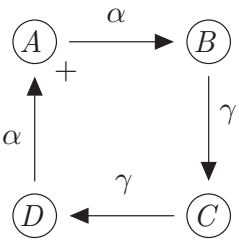

Figure 3.1

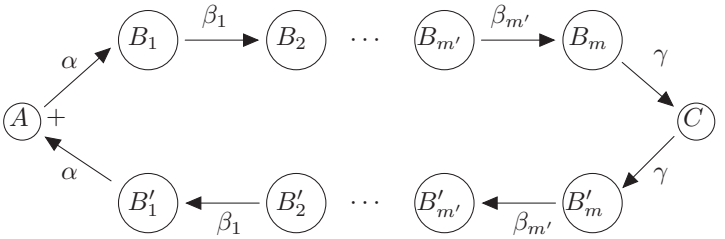

Figure 3.2 ($m' = m - 1$)

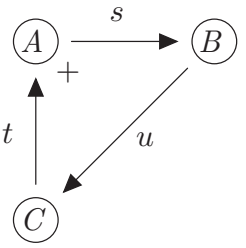

Figure 3.3

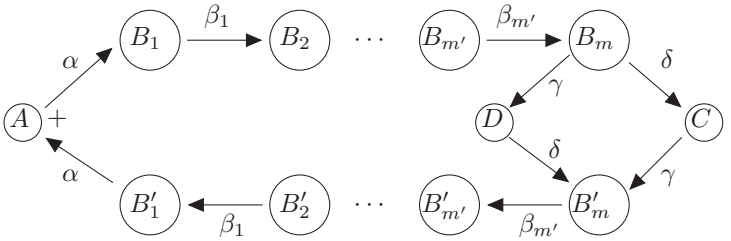

Figure 3.4 ($m' = m - 1$)

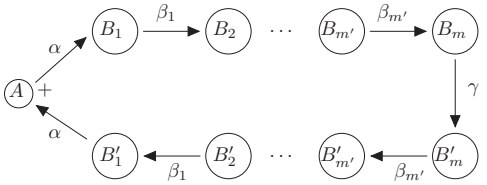

Figure 3.5 ($m' = m - 1$)

FIGURE 3. Modules for the infinite-dimensional generalized nil-Coxeter algebras

Case 1. Suppose $m_{ii} = 2$ for all $i \in I$. In this case $W(M)$ is a Coxeter group, so by, e.g., [Hum, Chapter 7], $NC(M)$ has a $\Bbbk$-basis in bijection with $W(M)$, which must therefore be finite.

Case 2. Suppose $m_{\alpha\alpha}, m_{\gamma\gamma} \geqslant 3$ for some $\alpha, \gamma \in I$ with $m_{\alpha\gamma} \geqslant 3$. In this case we appeal to Figure 3.1 and work as in the proof of [Mar, Proposition 3.2]. Thus, fix a free $\Bbbk$-module $\mathscr{M}$ with basis given by the countable set $\{A_r, B_r, C_r, D_r : r \geqslant 1\}$, and define an $NC(M)$-action via Figure 3.1. Namely, T_i kills all basis vectors for all $i \in I$, with the following exceptions:

$$T_\alpha(A_r) := B_r, \quad T_\gamma(B_r) := C_r, \quad T_\gamma(C_r) := D_r, \quad T_\alpha(D_r) := A_{r+1} \quad \forall r \geqslant 1.$$

(The "+" at the head of an arrow refers precisely to the index increasing by 1.) It is easy to verify that the defining relations of $NC(M)$ hold in $\mathrm{End}_\Bbbk(\mathscr{M})$, as they hold on each A_r, B_r, C_r, D_r. Therefore $\mathscr{M}$ is a module over $NC(M)$ that is generated by A_1 but is not finitely generated as a $\Bbbk$-module. As $NC(M) \twoheadrightarrow \mathscr{M}$, $NC(M)$ is also not a finitely generated $\Bbbk$-module.

This approach is used in the remainder of the proof, to obtain a $\Bbbk$-basis and the $NC(M)$-action on it, from the diagrams in Figure 3. Thus we only mention the figure corresponding to each of the cases below.

Case 3. Figure 3.1 is actually a special case of Figure 3.2 and was included to demonstrate a simpler case. Now suppose more generally that there are two nodes $\alpha, \gamma \in I$ such that $m_{\alpha\alpha}, m_{\gamma\gamma} \geqslant 3$. Since the Coxeter graph is connected, there exist nodes $\beta_1, \ldots, \beta_{m-1}$ for some $m \geqslant 1$ (in the figure we write $m' := m - 1$) such that

$$\alpha \quad \longleftrightarrow \quad \beta_1 \quad \longleftrightarrow \quad \cdots \quad \longleftrightarrow \quad \beta_{m-1} \quad \longleftrightarrow \quad \gamma$$

is a path (so each successive pair of nodes is connected by at least a single edge). Now appeal to Figure 3.2, i.e., define an $NC(M)$-module structure on the free $\Bbbk$-module

$$\mathscr{M} := \mathrm{span}_\Bbbk\{A_r, B_{1r}, \ldots, B_{mr}, C_r, B'_{1r}, \ldots, B'_{mr} : r \geqslant 1\},$$

where each T_i kills all basis vectors above, except for the actions obtained from Figure 3.2. Once again, $\mathscr{M}$ is generated by A_1, so proceed as above to show that $NC(M)$ is not finitely generated.

Case 4. The previous cases reduce the situation to a unique vertex α in the Coxeter graph of M for which $m_{\alpha\alpha} \geqslant 3$. The next two steps show that α is adjacent to a unique node γ and that $m_{\alpha\gamma} = 3$. First suppose α is adjacent to γ with $m_{\alpha\gamma} \geqslant 4$. Now appeal to Figure 3.3, setting $(s, t, u) \rightsquigarrow (\alpha, \alpha, \gamma)$, and define an $NC(M)$-module structure on $\mathscr{M} := \mathrm{span}_\Bbbk\{A_r, B_r, C_r : r \geqslant 1\}$. Then proceed as above.

Case 5. Next suppose α is adjacent in the Coxeter graph to two nodes γ, δ. By the previous case, $m_{\alpha\gamma} = m_{\alpha\delta} = 3$. Now appeal to Figure 3.4 with $m = 1$ to define an $NC(M)$-module structure on $\mathscr{M} := \mathrm{span}_\Bbbk\{A_r, B_{1r}, B'_{1r}, C_r, D_r : r \geqslant 1\}$ and then proceed as in the previous cases.

We now observe that if $NC(M)$ is finitely generated, then so is $NC_M((2, \ldots, 2))$, which corresponds to the Coxeter group $W_M((2, \ldots, 2))$. Hence the Coxeter graph of M is of finite type. These graphs were classified by Coxeter [Cox1]. We now rule out all cases other than type A, in which case the above analysis shows that $\mathbf{d} = (2, \ldots, 2, d)$ or $(d, 2, \ldots, 2)$.

Case 6. First notice that dihedral types (i.e., types G_2, H_2, I) are already ruled out by the above cases. The same cases also rule out one possibility in types B, C, H, where we may now set $n \geqslant 3$. For the remaining cases of types B, C, H, assume that the Coxeter graph is labelled

$$\alpha \quad \longleftrightarrow \quad \beta_1 \quad \longleftrightarrow \quad \cdots \quad \longleftrightarrow \quad \beta_{m-1} \quad \longleftrightarrow \quad \gamma,$$

with $m_{\alpha\alpha} \geqslant 3, m_{\gamma\gamma} = 2, m_{\beta_{m-1}\gamma} \geqslant 4$. In this case we construct the $NC(M)$-module $\mathscr{M}$ by appealing to Figure 3.5; now proceed as above.

Case 7. The next case is of type D_n, with $n \geqslant 4$. Notice that α is an extremal (i.e., pendant) vertex by the above analysis. First assume α is the extremal node on the "long arm" of the Coxeter graph. Now appeal to Figure 3.4 with $m = n-2$ to construct an $NC(M)$-module $\mathscr{M}$.

The other subcase is when α is one of the other two extremal nodes in the D_n-graph. Define the quotient algebra $NC'(M)$ whose Coxeter graph is of type D_4 (i.e., where we kill the $n-4$ generators T_i in the long arm that are the furthest away from α). Now repeat the construction in the previous paragraph, using Figure 3.4 with $m = 2$. It is easy to verify that the space $\mathscr{M}$ is a module for $NC'(M)$ and hence for the algebra $NC_{D_4}((2,2,2,m_{\alpha\alpha}))$. This allows us to proceed as in the previous subcase and show that $NC'(M)$ is not finitely generated, whence neither is $NC(M)$.

Case 8. If the Coxeter graph is of type E, then we may reduce to the D_n-case by the analysis in the previous case. Hence it follows using Figure 3.4 that $NC(M)$ is not finitely generated.

Case 9. If the Coxeter graph is of type F_4, then we may reduce to the B_n-case by the analysis in Case 7. It now follows from Case 6 that $NC(M)$ is not finitely generated.

This completes the classification for generalized Coxeter groups $W(M)$. We now appeal to the classification and presentation of all finite complex reflection groups whose Coxeter graph is connected. These groups and their presentations are listed in [BMR2, Tables 1–4]. In what follows, we adopt the following notation: if $W = G_m$ for $4 \leqslant m \leqslant 37$, then the corresponding generalized nil-Coxeter algebras will be denoted by $NC_m(\mathbf{d})$. Similarly if $W = G(de, e, r)$, then we work with $NC_{(de,e,r)}(\mathbf{d})$. In what follows, we will often claim that $NC_W(\mathbf{d})$ is not finitely generated (over $\Bbbk$), omitting the phrase "unless it is the usual nil-Coxeter algebra over a finite Coxeter group".

Case 10 (Exceptional types with finite Coxeter graph). If $W = G_m$ for $m = 4, 8, 16, 25, 32$, then its Coxeter graph is of type A. This case has been addressed above; thus the only possibility that $NC_W(\mathbf{d})$ has finite rank is that it equals $NC_{A_n}((2,\ldots,2,d))$ for $d \geqslant 2$, as desired.

Next if $W = G_m$ for $m = 5, 10, 18, 26$, then its Coxeter graph is of type B, which was also addressed above and never yields an algebra of finite $\Bbbk$-rank. Now suppose $W = G_{29}$. Then s, t, u form a subdiagram of type B_3, whence the quotient algebra NC'_{29} generated by T_s, T_t, T_u is not finitely generated by arguments as in Case 7 above. It follows that $NC_{29}(\mathbf{d})$ is also not finitely generated.

The next case is if $W = G_m$ for $m = 6, 9, 14, 17, 20, 21$. In this case the Coxeter graph is of dihedral type, which was also addressed above.

Case 11 (All other exceptional types). For the remaining exceptional values of $m \in [4, 37]$, with W not a finite Coxeter group, we will appeal to Figure 3.3. There are three cases: first, suppose $m = 31$. In this case, set $(s, t, u) \rightsquigarrow (s, u, t)$ in Figure 3.3 and define an $NC_m(\mathbf{d})$-module $\mathscr{M}$ that is $\Bbbk$-free with basis $\{A_r, B_r, C_r : r \geqslant 1\}$. Now proceed as above.

Next if $m = 33, 34$, then set $(s, t, u) \rightsquigarrow (w, t, u)$ in Figure 3.3 to define an $NC_m(\mathbf{d})$-module $\mathscr{M}$ and proceed as above.

Finally, fix any other m, i.e., $m = 7, 11, 12, 13, 15, 19, 22, 24, 27$. In this case, use Figure 3.3 to define an $NC_m(\mathbf{d})$-module $\mathscr{M}$ and proceed as above.

Case 12 (The infinite families). It remains to consider the six infinite families enumerated in [BMR2] which make up the family $G(de, e, r)$. Three of the families consist of finite Coxeter groups of types A, B, I, which were considered above. We now consider the other three families:

(a) Suppose $W = G(de, e, r)$ with $e \geqslant 3$. Then by [BMR2, Table 1], consider the quotient algebra $NC'_{(de,e,r)}$ of $NC_{(de,e,r)}(\mathbf{d})$ which is generated by $s, t := t'_2, u := t_2$, by killing all other generators T_i. The generators of $NC'_{(de,e,r)}$ now satisfy the relations

$$T_s^{d_s} = T_t^{d_t} = T_u^{d_u} = 0, \qquad T_sT_tT_u = T_tT_uT_s, \qquad \underbrace{T_uT_sT_tT_u\cdots}_{(e+1)\text{ times}} = \underbrace{T_sT_tT_uT_s\cdots}_{(e+1)\text{ times}}.$$

Thus, use Figure 3.3 to define an $NC'_{(de,e,r)}$-module structure on $\mathscr{M}$ and proceed as above to show that $NC_{(de,e,r)}(\mathbf{d})$ is not finitely generated.

(b) Suppose $W = G(2d, 2, r)$ with $d \geqslant 2$; see [BMR2, Table 2]. Apply a similar argument as in the previous subcase, using the same generators and the same figure.

(c) Suppose $W = G(e, e, r)$ with $e \geqslant 2$ and $r > 2$. If $e = 2$, then $G(2, 2, r)$ is a finite Coxeter group, and hence was addressed above. Next, if $r > 3$, then killing T_s reduces to (a quotient of) the D_n-case, which was once again addressed above. Finally, suppose $r = 3 \leqslant e$. Setting $s := t_3, t := t'_2, u := t_2$, the generators of $NC_{(e,e,3)}$ satisfy

$$T_s^{d_s} = T_t^{d_t} = T_u^{d_u} = 0, \qquad T_sT_tT_s = T_tT_sT_t, \qquad T_sT_uT_s = T_uT_sT_u,$$
$$(T_sT_tT_u)^2 = (T_tT_uT_s)^2, \qquad \underbrace{T_uT_tT_uT_t\cdots}_{e\text{ times}} = \underbrace{T_tT_uT_tT_u\cdots}_{e\text{ times}}.$$

Once again, use Figure 3.3 to define an $NC_{(e,e,3)}$-module structure on $\mathscr{M}$ and proceed as above.

This completes the proof of (1) $\implies$ (2) for finite complex reflection groups. Next, by, e.g., [Hum, Chapter 7], for no infinite Coxeter group W is $NC_W((2, \ldots, 2))$ a finitely generated $\Bbbk$-module; hence the same result holds for $NC_W(\mathbf{d})$ when all $d_i \geqslant 2$. We now use the classification of the (remaining) infinite complex reflection groups W associated to a connected braid diagram. These groups were described in [Pop1] and subsequently in [Mal]. Thus, there exists a complex affine space E with group of translations V; choosing a basepoint $v_0 \in E$, we can identify the semidirect product $GL(V) \ltimes V$ with the group $A(E)$ of *affine transformations* of E. Moreover, $W \subset A(E)$. Define $\mathrm{Lin}(W)$ to be the image of W in the factor group $GL(V)$ and define $\mathrm{Tran}(W)$ to be the subset of W in V, i.e.,

$$\mathrm{Tran}(W) := W \cap V, \qquad \mathrm{Lin}(W) := W/\mathrm{Tran}(W). \tag{6.1}$$

It remains to consider three cases for irreducible infinite complex reflection groups W.

Case 13. The group W is *noncrystallographic*, i.e., E/W is not compact. Then by [Pop1, Theorem 2.2], there exists a real form $E_{\mathbb{R}} \subset E$ whose complexification is E, i.e., $E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = E$. Moreover, by the same theorem, restricting the elements of W to $E_{\mathbb{R}}$ yields an affine Weyl group $W_{\mathbb{R}}$. Hence if $NC_W(\mathbf{d})$ is a finitely generated $\Bbbk$-module, then so is $NC_{W_{\mathbb{R}}}((2,\dots,2))$, which is impossible.

Case 14. The group W is a *genuine crystallographic group*, i.e., E/W is compact and $\mathrm{Lin}(W)$ is not the complexification of a real reflection group. Such groups were studied by Malle in [Mal], and Coxeter-type presentations for these groups were provided in Tables I, II in loc. cit. Specifically, Malle showed that these groups are quotients of a free monoid by a set of braid relations and order relations, together with one additional order relation $R_0^{m_0} = 1$. We now show that for none of these groups W is the algebra $NC_W(\mathbf{d})$ (defined in Definition 1.11) a finitely generated $\Bbbk$-module. To do so, we proceed as above by specifying the subfigure in Figure 3 that corresponds to each of these groups. There are three subcases:

(1) Suppose W is the group $[G(3,1,1)]$ in [Mal, Table I] or $[K_m]$ in [Mal, Table II] for $m = 4, 8, 25, 32$. For these groups we appeal to Figure 3.1 and proceed as in Case 2 above.

(2) For $W = [K_{33}], [K_{34}]$, notice that it suffices to show the **claim** that given the $\widetilde{A_3}$ Coxeter graph (i.e., a 4-cycle) with nodes labelled $\alpha_1, \dots, \alpha_4$ in clockwise fashion, the corresponding algebra $NC_{\widetilde{A_3}}(\mathbf{d})$ is not a finitely generated $\Bbbk$-module. For this we construct a module $\mathscr{M}$, using Figure 4 with $n = 4$. Now proceeding as above shows the claim, and hence the result for $[K_{33}], [K_{34}]$.

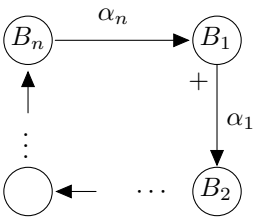

FIGURE 4. Module $\mathscr{M}$ for $NC_{\widetilde{A_n}}(\mathbf{d})$

(3) For the remaining cases in [Mal, Tables I, II], we appeal to Figure 3.3 as in Case 11 above, with three suitably chosen generators in each case.

Case 15. Finally, we consider the remaining "nongenuine, crystallographic" cases as in [Pop1, Table 2]. Thus, E/W is compact and $\mathrm{Lin}(W)$ is the complexification of a real reflection group. In these cases, verify by inspection from [Pop1, Table 2] that the cocycle c is always trivial. Thus $W = \mathrm{Lin}(W) \ltimes \mathrm{Tran}(W)$, with $W' := \mathrm{Lin}(W)$ a finite Weyl group and $\mathrm{Tran}(W)$ a lattice of rank $2|I'|$, where I' indexes the simple reflections in the Weyl group W'.

We now **claim** that the corresponding family of generalized nil-Coxeter algebras $NC_W(\mathbf{d})$ are not finitely generated as $\Bbbk$-modules. To show the claim requires

a presentation of W in terms of generating reflections. The following recipe for such a presentation was communicated to us by Popov [Pop2]. Notice from, e.g., [Pop1, Table 2] that $\mathrm{Tran}(W)$ is a direct sum of two $\mathrm{Lin}(W)$-stable lattices Λ_1 and $\Lambda_2 = \alpha\Lambda_1$ (with $\alpha \notin \mathbb{R}$), each of rank $|I'|$. Thus, $\Lambda_1 \cong \Lambda_2$ as $\mathbb{Z}W'$-modules, with $W' = \mathrm{Lin}(W)$ a finite real reflection group as above. Moreover, for $j = 1, 2$, the semidirect product $S_j := W' \ltimes \Lambda_j$ is a real crystallographic reflection group whose fundamental domain is a simplex; this yields a presentation of S_j via $|I'|+1$ generating reflections in the codimension-one faces of this simplex. One now combines these presentations for S_1, S_2 to obtain a system of $|I'|+2$ generators for W; see in this context the remarks following [Mal, Theorem 3.1]. In this setting, it follows by [Pop1, Theorem 4.5] that each S_j is isomorphic, as a real reflection group, to the affine Weyl group $\widetilde{W'}$ over W', since the Coxeter type of S_j is determined by the Coxeter types of W' and Λ_j. Thus W is in some sense a "double affine Weyl group". (For simply laced W', it is also easy to verify by inspection from [Pop1, Table 2] that Λ_j is isomorphic as a $\mathbb{Z}W'$-module to the root lattice for W', whence $S_j \cong \widetilde{W'}$ for $j = 1, 2$.)

Equipped with this presentation of W from [Pop2], we analyze $NC_W(\mathbf{d})$ as follows. Fix a $\mathbb{Z}W'$-module isomorphism $\varphi : \Lambda_1 \to \Lambda_2$ and choose affine reflections $s_{0j} \in S_j$, corresponding to μ_1 and $\mu_2 = \varphi(\mu_1)$ respectively, which together with the simple reflections $\{s_i : i \in I'\} \subset W$ generate $S_j \cong \widetilde{W'}$. Then $W \twoheadrightarrow \widetilde{W'}$ upon quotienting by the relation $s_{01} = s_{02}$. Using the presentation of $NC_W(\mathbf{d})$ via the corresponding $|I'| + 2$ generators $\{T_i : i \in I'\} \sqcup \{T_{01}, T_{02}\}$,

$$NC_W(\mathbf{d}) \twoheadrightarrow NC_W((2, \dots, 2)) \twoheadrightarrow NC_W((2, \dots, 2))/(T_{01} - T_{02}) \cong NC_{\widetilde{W'}}((2, \dots, 2)),$$

and this last term is an affine Weyl nil-Coxeter algebra, hence is not finitely generated as a $\Bbbk$-module. Therefore neither is $NC_W(\mathbf{d})$, as desired.

This shows that (1) $\implies$ (2); the converse follows by [Hum, Chapter 7] and Theorem B.

We now show that (2) and (3) are equivalent. Note from the above case-by-case analysis that if $NC_W(\mathbf{d})$ is not finitely generated, then either it surjects onto an affine Weyl nil-Coxeter algebra $NC_{\widetilde{W'}}((2, \dots, 2))$ or one can define a module $\mathscr{M}$ as above, and for each $r \geqslant 1$ there exists a word $\mathcal{T}_{w_r} \in NC_W(\mathbf{d})$, expressed using $O(r)$ generators, which sends the $\Bbbk$-basis vector $A_1 \in \mathscr{M}$ to A_r. It follows in both cases that $\mathfrak{m}$ is not nilpotent. Next, if $W = W(M)$ is a finite Coxeter group, then it is well known (see, e.g., [Hum, Chapter 7]) that $\mathfrak{m}$ is nilpotent. Finally, if $NC(M) = NC_A(n, d)$, then $\mathfrak{m}$ is nilpotent by Theorem C. This shows (2) $\iff$ (3).

The final statement on the length function ℓ and the longest element also follows from [Hum, Chapter 7] and Theorem C. □

Remark 6.2. If M is a generalized Coxeter matrix with some $m_{ij} = \infty$, then we can similarly work with $\mathscr{M}$ the $\Bbbk$-span of $\{a_r, b_r : r \geqslant 1\}$, where

$$T_i : a_r \mapsto b_r,\ b_r \mapsto 0, \qquad T_j : b_r \mapsto a_{r+1},\ a_r \mapsto 0,$$

and all other T_k kill $\mathscr{M}$. It follows that $NC(M)$ once again has infinite $\Bbbk$-rank.

Acknowledgments

The author is grateful to Ivan Marin, Vladimir Popov, and Victor Reiner for informative and stimulating correspondences, and to James Humphreys for carefully going through an earlier draft and providing valuable feedback. The author also

thanks Daniel Bump, Gunter Malle, Eric Rowell, Travis Scrimshaw, and Bruce Westbury for useful references and discussions. The author is partially supported by a Young Investigator Award from the Infosys Foundation.

References

[AnSc] Nicolás Andruskiewitsch and Hans-Jürgen Schneider, *On the classification of finite-dimensional pointed Hopf algebras*, Ann. of Math. (2) **171** (2010), no. 1, 375–417, DOI 10.4007/annals.2010.171.375. MR2630042

[As] Joachim Assion, *A proof of a theorem of Coxeter*, C. R. Math. Rep. Acad. Sci. Canada **1** (1978/79), no. 1, 41–44. MR511837

[Ba] Tathagata Basak, *On Coxeter diagrams of complex reflection groups*, Trans. Amer. Math. Soc. **364** (2012), no. 9, 4909–4936, DOI 10.1090/S0002-9947-2012-05517-6. MR2922614

[BSS] Chris Berg, Franco Saliola, and Luis Serrano, *Pieri operators on the affine nilCoxeter algebra*, Trans. Amer. Math. Soc. **366** (2014), no. 1, 531–546, DOI 10.1090/S0002-9947-2013-05895-3. MR3118405

[BGG] Joseph N. Bernstein, Israel M. Gelfand, and Sergei I. Gelfand, *Schubert cells, and the cohomology of the spaces G/P*, Uspehi Mat. Nauk **28** (1973), no. 3(171), 3–26. MR0429933

[BS] Joseph Bernstein and Ossip Schwarzman, *Complex crystallographic Coxeter groups and affine root systems*, J. Nonlinear Math. Phys. **13** (2006), no. 2, 163–182, DOI 10.2991/jnmp.2006.13.2.2. MR2238956

[Be] David Bessis, *Zariski theorems and diagrams for braid groups*, Invent. Math. **145** (2001), no. 3, 487–507, DOI 10.1007/s002220100155. MR1856398

[BMR1] Michel Broué, Gunter Malle, and Raphaël Rouquier, *On complex reflection groups and their associated braid groups*, Representations of groups (Banff, AB, 1994), CMS Conf. Proc., vol. 16, Amer. Math. Soc., Providence, RI, 1995, pp. 1–13. MR1357192

[BMR2] Michel Broué, Gunter Malle, and Raphaël Rouquier, *Complex reflection groups, braid groups, Hecke algebras*, J. Reine Angew. Math. **500** (1998), 127–190. MR1637497

[Ca] Roger W. Carter, *Representation theory of the* 0*-Hecke algebra*, J. Algebra **104** (1986), no. 1, 89–103, DOI 10.1016/0021-8693(86)90238-3. MR865891

[Ch] Eirini Chavli, *The Broué-Malle-Rouquier conjecture for the exceptional groups of rank 2*, Ph.D. thesis, 2016.

[Coh] Arjeh M. Cohen, *Finite complex reflection groups*, Ann. Sci. École Norm. Sup. (4) **9** (1976), no. 3, 379–436. MR0422448

[Cox1] Harold S. M. Coxeter, *Discrete groups generated by reflections*, Ann. of Math. (2) **35** (1934), no. 3, 588–621, DOI 10.2307/1968753. MR1503182

[Cox2] Harold S. M. Coxeter,*Factor groups of the braid group*, Proceedings of the 4th Canadian Mathematical Congress (Banff, 1957), University of Toronto Press, 95–122, 1959.

[Dr] Vladimir G. Drinfeld, *Degenerate affine Hecke algebras and Yangians*, Funktsional. Anal. i Prilozhen. **20** (1986), no. 1, 69–70. MR831053

[Et] Pavel Etingof,*Proof of the Broué–Malle–Rouquier conjecture in characteristic zero (after I. Losev and I. Marin–G. Pfeiffer)*, Arnold Mathematical Journal, **3** (2017), no. 3, 445–449.

[EG] Pavel Etingof and Victor Ginzburg, *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism*, Invent. Math. **147** (2002), no. 2, 243–348, DOI 10.1007/s002220100171. MR1881922

[ES] Pavel Etingof and Olivier Schiffmann, *Lectures on quantum groups*, Lectures in Mathematical Physics, International Press, Boston, MA, 1998. MR1698405

[FS] Sergey Fomin and Richard P. Stanley, *Schubert polynomials and the nil-Coxeter algebra*, Adv. Math. **103** (1994), no. 2, 196–207, DOI 10.1006/aima.1994.1009. MR1265793

[GHV] Matias Graña, István Heckenberger, and Leandro Vendramin, *Nichols algebras of group type with many quadratic relations*, Adv. Math. **227** (2011), no. 5, 1956–1989, DOI 10.1016/j.aim.2011.04.006. MR2803792

[HV1] István Heckenberger and Leandro Vendramin, *A classification of Nichols algebras of semisimple Yetter-Drinfeld modules over non-abelian groups*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 2, 299–356, DOI 10.4171/JEMS/667. MR3605018

[HV2] István Heckenberger and Leandro Vendramin, *The classification of Nichols algebras over groups with finite root system of rank two*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 7, 1977–2017, DOI 10.4171/JEMS/711. MR3656477

[Hug1] Mervyn C. Hughes, *Complex reflection groups*, Comm. Algebra **18** (1990), no. 12, 3999–4029, DOI 10.1080/00927879008824120. MR1084439

[Hug2] Mervyn C. Hughes, *Extended root graphs for complex reflection groups*, Comm. Algebra **27** (1999), no. 1, 119–148, DOI 10.1080/00927879908826424. MR1668224

[Hum] James E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990. MR1066460

[Kha] Apoorva Khare, *Generalized nil-Coxeter algebras, cocommutative algebras, and the PBW property*, Groups, rings, group rings, and Hopf algebras, Contemp. Math., vol. 688, Amer. Math. Soc., Providence, RI, 2017, pp. 139-168. MR3649174

[Kho] Mikhail Khovanov, *Nilcoxeter algebras categorify the Weyl algebra*, Comm. Algebra **29** (2001), no. 11, 5033–5052, DOI 10.1081/AGB-100106800. MR1856929

[KL] Mikhail Khovanov and Aaron D. Lauda, *A diagrammatic approach to categorification of quantum groups. I*, Represent. Theory **13** (2009), 309–347, DOI 10.1090/S1088-4165-09-00346-X. MR2525917

[KK] Bertram Kostant and Shrawan Kumar, *The nil Hecke ring and cohomology of G/P for a Kac-Moody group G*, Adv. Math. **62** (1986), no. 3, 187–237, DOI 10.1016/0001-8708(86)90101-5. MR866159

[Ko] David W. Koster, *Complex reflection groups*, ProQuest LLC, Ann Arbor, MI, 1975. Thesis (Ph.D.)–The University of Wisconsin - Madison. MR2625485

[LS] Alain Lascoux and Marcel-Paul Schützenberger, *Fonctorialité des polynômes de Schubert*, Invariant theory (Denton, TX, 1986), Contemp. Math., vol. 88, Amer. Math. Soc., Providence, RI, 1989, pp. 585–598, DOI 10.1090/conm/088/1000001. MR1000001

[LT] Gustav I. Lehrer and Donald E. Taylor, *Unitary reflection groups*, Australian Mathematical Society Lecture Series, vol. 20, Cambridge University Press, Cambridge, 2009. MR2542964

[Lo] Ivan Losev, *Finite-dimensional quotients of Hecke algebras*, Algebra Number Theory **9** (2015), no. 2, 493–502, DOI 10.2140/ant.2015.9.493. MR3320850

[Lu] George Lusztig, *Affine Hecke algebras and their graded version*, J. Amer. Math. Soc. **2** (1989), no. 3, 599–635, DOI 10.2307/1990945. MR991016

[Mal] Gunter Malle, *Presentations for crystallographic complex reflection groups*, Transform. Groups **1** (1996), no. 3, 259–277, DOI 10.1007/BF02549209. MR1417713

[Mar] Ivan Marin, *The freeness conjecture for Hecke algebras of complex reflection groups, and the case of the Hessian group G_{26}*, J. Pure Appl. Algebra **218** (2014), no. 4, 704–720, DOI 10.1016/j.jpaa.2013.08.009. MR3133700

[MP] Ivan Marin and Götz Pfeiffer, *The BMR freeness conjecture for the 2-reflection groups*, Math. Comp. **86** (2017), no. 306, 2005–2023, DOI 10.1090/mcom/3142. MR3626546

[No] P. N. Norton, 0-*Hecke algebras*, J. Austral. Math. Soc. Ser. A **27** (1979), no. 3, 337–357. MR532754

[ORS] Peter Orlik, Victor Reiner, and Anne V. Shepler, *The sign representation for Shephard groups*, Math. Ann. **322** (2002), no. 3, 477–492, DOI 10.1007/s002080200001. MR1895703

[Pop1] Vladimir L. Popov, *Discrete complex reflection groups*, Communications of the Mathematical Institute, Rijksuniversiteit Utrecht, vol. 15, Rijksuniversiteit Utrecht, Mathematical Institute, Utrecht, 1982. MR645542

[Pop2] Vladimir L. Popov, personal communication, Feb. 2016.

[ReS] Victor Reiner and Anne V. Shepler, *Invariant derivations and differential forms for reflection groups*, preprint, arXiv:1612.01031.

[ST] Geoffrey C. Shephard and John A. Todd, *Finite unitary reflection groups*, Canadian J. Math. **6** (1954), 274–304. MR0059914

[SW1] Anne V. Shepler and Sarah Witherspoon, *A Poincaré-Birkhoff-Witt theorem for quadratic algebras with group actions*, Trans. Amer. Math. Soc. **366** (2014), no. 12, 6483–6506, DOI 10.1090/S0002-9947-2014-06118-7. MR3267016

[SW2] Anne V. Shepler and Sarah Witherspoon, *Poincaré-Birkhoff-Witt theorems*, Commutative algebra and noncommutative algebraic geometry. Vol. I, Math. Sci. Res. Inst. Publ., vol. 67, Cambridge Univ. Press, New York, 2015, pp. 259–290. MR3525474

Department of Mathematics, Indian Institute of Science, Bangalore – 560012, India
E-mail address: khare@iisc.ac.in