THE PRODUCT OF THE GENERATORS OF A FINITE GROUP GENERATED BY REFLECTIONS

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In Euclidean n-space, every finite group generated by reflections leaves at least one point invariant, and thus may be regarded as operating on a sphere. It has for its fundamental region a spherical simplex whose dihedral angles are submultiples of π, say π/p_{ik} [6; 597, 619], [10; 190]. Accordingly we can use as generators the reflections $R_1, R_2, \cdots, R_n$ in the bounding hyperplanes of this simplex. The product $R_1R_2 \cdots R_n$ has already been found useful in various ways [6; 606–617], [7], [11]. The n generators may be taken in any order, since the products in different orders are all conjugate [6; 602]. Most of the applications of $R_1R_2 \cdots R_n$ were concerned with its period, h. In the present paper we consider its characteristic roots

$$\omega^{m_1}, \omega^{m_2}, \cdots, \omega^{m_n},$$

where $\omega = e^{2\pi i/h}$ and the exponents m_i are certain integers which may be taken to lie between 0 and h. They are computed by a trigonometrical formula involving the periods, p_{ik}, of the products of *pairs* of generators. (The product of two reflections is simply a rotation.)

The point of interest is that the same integers occur in a different connection. It turns out that the order of the group is

$$(m_1 + 1)(m_2 + 1) \cdots (m_n + 1),$$

and that these factors $m_i + 1$ are the degrees of n basic invariant forms [2; Chapter XVII]. Moreover, when every p_{ik} is 2, 3, 4 or 6, so that the group is *crystallographic*, there is a corresponding continuous group, and the Betti numbers of the group manifold are the coefficients in the *Poincaré polynomial*

$$(1 + t^{2m_1+1})(1 + t^{2m_2+1}) \cdots (1 + t^{2m_n+1}).$$

Having computed the m's several years earlier [10; 221, 226, 234], I recognized them in the Poincaré polynomials while listening to Chevalley's address at the International Congress in 1950. I am grateful to A. J. Coleman for drawing my attention to the relevant work of Racah [16], which helps to explain the "coincidence"; also, to J. S. Frame for many helpful suggestions (such as his idea of using the matrix T in §1, see [8; 6]), to J. A. Todd for his conjecture that the Jacobian of the basic invariants will always factorize into linear forms

Received June 4, 1951; presented to the American Mathematical Society at the meeting in New Orleans, April, 1951, under the title "Group invariants and Poincaré polynomials."

corresponding to the reflecting hyperplanes, and to Racah himself for his proof of this conjecture and his verification that the degrees of the invariants for the non-crystallographic group [3, 3, 5] are indeed

$$2, \quad 12, \quad 20, \quad 30.$$

1. **The characteristic equation.** Since $R_1R_2 \cdots R_n$ is an orthogonal transformation, its n characteristic roots have unit modulus and may be expressed as

$$e^{\xi_1 i}, e^{\xi_2 i}, \cdots, e^{\xi_n i}, \tag{1.1}$$

where $0 \leq \xi_1 \leq \xi_2 \leq \cdots \leq \xi_n \leq 2\pi$. Since the roots occur in conjugate pairs,

$$\xi_1 + \xi_n = \xi_2 + \xi_{n-1} = \cdots = 2\pi; \tag{1.2}$$

In particular, when n is odd, $\xi_{\frac{1}{2}(n+1)} = \pi$.

In other words, $R_1R_2 \cdots R_n$ is the product of rotations through angles ξ_1, ξ_2, $\cdots$, $\xi_{[\frac{1}{2}n]}$ in completely orthogonal planes, along with a reflection when n is odd [10; 216]. If its period is h, each angle of rotation must be a multiple of $2\pi/h$, say

$$\xi_j = 2m_j\pi/h \qquad (m_1 \leq m_2 \leq \cdots \leq m_n),$$

the m's being n positive integers connected in pairs by the relations

$$m_j + m_{n+1-j} = h. \tag{1.3}$$

Since the characteristic roots are invariants, they can be computed in terms of any system of affine coordinates, rectangular or oblique. In the present case the fundamental region itself provides a convenient system, the coordinates x_1, x_2, $\cdots$, x_n being distances from the reflecting hyperplanes [10; 182]. The reflection R_k is now given by

$$x_j = x_j' - 2a_{jk}x_k',$$

where $-a_{jk}$ is the cosine of the dihedral angle between the j-th and k-th hyperplanes, and $a_{kk} = 1$. Let the partial product $R_1R_2 \cdots R_k$ transform x_j into $x_j^{(k)}$, and set

$$x_j^{(0)} = x_j, \qquad x_j^{(j)} = y_j, \qquad x_j^{(n)} = z_j.$$

Then the reflection R_k is given by the equations

$$x_j^{(k-1)} = x_j^{(k)} - 2a_{jk}y_k \qquad (j = 1, 2, \cdots, n).$$

By summing the differences $x_j^{(k)} - x_j^{(k-1)}$, we obtain

$$y_j - x_j = \sum_{k=1}^{j} 2a_{jk}y_k, \qquad z_j - y_j = \sum_{k=j+1}^{n} 2a_{jk}y_k. \tag{1.4}$$

Let X, Y, Z denote row vectors with respective components x_j , y_j , z_j ; and T the upper triangular matrix with entries

$$(1.5) \qquad t_{jk} = \begin{cases} 2a_{jk} & (j < k), \\ 1 & (j = k), \\ 0 & (j > k), \end{cases}$$

so that the matrix $\| a_{jk} \|$ is $\frac{1}{2}(\mathrm{T} + \mathrm{T}')$. Equations (1.4) may now be expressed as

$$-x_j = \sum_{k=1}^{n} y_k t_{kj} , \qquad z_j = \sum_{k=1}^{n} y_k t_{jk} ,$$

or

$$(1.6) \qquad -\mathrm{X} = \mathrm{YT}, \qquad \mathrm{Z} = \mathrm{YT}',$$

whence

$$\mathrm{Z} = \mathrm{X}(-\mathrm{T}^{-1}\mathrm{T}').$$

Since the product $\mathrm{R}_1\mathrm{R}_2 \cdots \mathrm{R}_n$ defines the mapping obtained by eliminating Y from (1.6), its matrix is

$$(1.7) \qquad \mathrm{R} = -\mathrm{T}^{-1}\mathrm{T}'.$$

Since $\mathrm{T}(\lambda \mathrm{I} - \mathrm{R}) = \lambda\mathrm{T} + \mathrm{T}'$, its characteristic equation (whose roots are (1.1)) is

$$| \lambda\mathrm{T} + \mathrm{T}' | = 0;$$

that is,

$$(1.8) \qquad \begin{vmatrix} \frac{1}{2}(\lambda + 1) & a_{12}\lambda & a_{13}\lambda & \cdots & a_{1n}\lambda \\ a_{21} & \frac{1}{2}(\lambda + 1) & a_{23}\lambda & \cdots & a_{2n}\lambda \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \frac{1}{2}(\lambda + 1) \end{vmatrix} = 0.$$

The determinant, when expanded, has one term for each permutation in the symmetric group of degree n; for example, the permutations

$$(1\ 2)(3\ 4) \qquad \text{and} \qquad (1\ 2\ 3)(4\ 5\ 6\ 7)$$

yield the terms

$$a_{12}a_{21}a_{34}a_{43}\lambda^2\{\tfrac{1}{2}(\lambda + 1)\}^{n-4}$$

and

$$-a_{12}a_{23}a_{31}a_{45}a_{56}a_{67}a_{74}\lambda^5\{\tfrac{1}{2}(\lambda + 1)\}^{n-7}.$$

2. The equation for $\cos m\pi/h$. Let us represent the fundamental region by a graph, in such a way that the nodes represent the n bounding hyperplanes, two nodes being joined by a branch whenever the corresponding hyperplanes are not perpendicular. Since $-a_{jk}$ is the cosine of the dihedral angle between two hyperplanes, it vanishes whenever the j-th and k-th hyperplanes are perpendicular, that is, whenever the j-th and k-th nodes are not directly joined. Since the dihedral angles of the spherical simplex are submultiples of π, the graph is always a tree or a forest [10; 195]; this means that it contains no circuit. Hence every cyclic product of a's, such as

$$a_{12}a_{23}a_{31} \quad \text{or} \quad a_{45}a_{56}a_{67}a_{74} ,$$

must vanish, and the only non-vanishing terms in the expansion of (1.8) are those whose corresponding permutations are of period 1 or 2; namely, the identity, the transpositions, and the products of disjoint transpositions.

Dividing by $\lambda^{\frac{1}{2}n}$, writing

$$X = \tfrac{1}{2}(\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}),$$

and observing that $a_{kj} = a_{jk}$, we obtain the simplified equation

$$X^n - \sum a_{jk}^2 X^{n-2} + \sum a_{jk}^2 a_{lm}^2 X^{n-4} - \cdots = 0, \tag{2.1}$$

where the first summation is over all the branches of the graph, the second is over all pairs of non-adjacent branches, and so on [8; 5]. A more elegant way of expressing it is

$$\begin{vmatrix} X & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & X & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdots & X \end{vmatrix} = 0. \tag{2.2}$$

If the graph is not connected, the determinant is equal to the product of several determinants of lower order corresponding to the various trees in the forest. Thus the ξ's for the direct product of several groups are just all the ξ's for the various irreducible components. Accordingly we restrict consideration to the irreducible groups

$$[3^{n-1}], \quad [3^{n-2}, 4], \quad [3^{n-3,1,1}],$$

$$[3^{n-4,2,1}] \qquad (n = 6, 7, 8),$$

$$[3, 4, 3], \quad [p], \quad [3, 5], \quad [3, 3, 5],$$

whose fundamental regions are represented by trees [10; 200].

Since the roots of (1.8) are (1.1), those of (2.1) or (2.2) are

$$\cos \tfrac{1}{2}\xi_1 \,, \cos \tfrac{1}{2}\xi_2 \,, \cdots , \cos \tfrac{1}{2}\xi_n \,,$$

or, in the other notation,

$$\cos m_j\pi/h \qquad (j = 1, 2, \cdots , n).$$

3. **The particular cases.** In the case of $[3^{n-1}]$, where the non-vanishing a_{jk}'s with $j < k$ are

$$a_{12} = a_{23} = \cdots = a_{n-1n} = -\cos \pi/3 = -\tfrac{1}{2},$$

the number of sets of r non-adjacent branches is $\binom{n-r}{r}$, and the equation is

$$(2X)^n - \binom{n-1}{1}(2X)^{n-2} + \binom{n-2}{2}(2X)^{n-4} - \cdots = 0.$$

In the case of $[3^{n-2}, 4]$, the only change is that $a_{n-1n} = -\cos \pi/4 = -(\tfrac{1}{2})^{\frac{1}{2}}$. Among the $\binom{n-r}{r}$ sets of r non-adjacent branches, $\binom{n-r-1}{r-1}$ contain this last branch, while the remaining $\binom{n-r-1}{r}$ do not. Thus the coefficient of $(2X)^{n-r}$ is now

$$2\binom{n-r-1}{r-1} + \binom{n-r-1}{r} = \frac{n}{n-r}\binom{n-r}{r};$$

in other words, it is equal to the number of sets of r non-adjacent branches in the graph

which is used by Witt [17; 301] to represent this group.

In the notation of Chebyshev polynomials [10; 222], the equations for $[3^{n-1}]$ and $[3^{n-2}, 4]$ are

$$U_n(X) = 0 \qquad \text{and} \qquad T_n(X) = 0. \tag{3.1}$$

Hence the values of X are, respectively,

$$\cos j\pi/(n+1) \qquad \text{and} \qquad \cos (2j-1)\pi/2n \qquad (j = 1, 2, \cdots , n).$$

The number of sets of r non-adjacent branches is not altered when the double branch is separated to form a fork. But then the number of nodes, which is the number of dimensions, is increased by 1, and we have the equation $XT_n(X)$ for $[3^{n-2,1,1}]$. Returning to n dimensions we have, for $[3^{n-3,1,1}]$, the equation

$$XT_{n-1}(X) = 0, \tag{3.2}$$

with roots

$$0 \qquad \text{and} \qquad \cos (2j-1)\pi/2(n-1) \qquad (j = 1, 2, \cdots , n-1).$$

The graph for $[3^{n-4,2,1}]$ consists of a row of $n - 4$ branches, a second row of two, and a third that is a single branch, all radiating from one central node. Thus the number of sets of r non-adjacent branches is

$$\binom{(n-1)-r}{r} + \binom{(n-4)-(r-1)}{r-1} + \binom{(n-4)-(r-2)}{r-2}$$

$$= \binom{n-r-1}{r} + \binom{n-r-3}{r-1} + \binom{n-r-2}{r-2} = \binom{n-r}{r} - \binom{n-r-3}{r-2}.$$

Taking this to be the coefficient of $(2X)^{n-2r}$ (with the appropriate sign) we obtain the equation

$$U_n(X) - 2XU_{n-5}(X) = 0. \tag{3.3}$$

The individual cases are shown in Table 1, where $Y = 2X$.

TABLE 1

Group	n	Equation for $2 \cos \frac{1}{2}\xi_i$	Values of $\xi_i/2\pi$
$[3^3]$	4	$Y^4 - 3Y^2 + 1 = 0$	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$
$[3^{2,1,1}]$	5	$Y(Y^4 - 4Y^2 + 2) = 0$	$\frac{1}{8}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{7}{8}$
$[3^{2,2,1}]$	6	$(Y^2 - 1)(Y^4 - 4Y^2 + 1) = 0$	$\frac{1}{12}, \frac{1}{3}, \frac{5}{12}, \frac{7}{12}, \frac{2}{3}, \frac{11}{12}$
$[3^{3,2,1}]$	7	$Y(Y^6 - 6Y^4 + 9Y^2 - 3) = 0$	$\frac{1}{18}, \frac{5}{18}, \frac{7}{18}, \frac{1}{2}, \frac{11}{18}, \frac{13}{18}, \frac{17}{18}$
$[3^{4,2,1}]$	8	$Y^8 - 7Y^6 + 14Y^4 - 8Y^2 + 1 = 0$	$\frac{1}{30}, \frac{7}{30}, \frac{11}{30}, \frac{13}{30}, \frac{17}{30}, \frac{19}{30}, \frac{23}{30}, \frac{29}{30}$
$[3^{5,2,1}]$	9	$Y(Y^2 - 4)(Y^2 - 1)(Y^4 - 3Y^2 + 1) = 0$	$0, \frac{1}{5}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{4}{5}, 1$

The final entry ($n = 9$) is not strictly relevant, as the group is infinite. But it is interesting to observe that $\xi_1 = 0$, corresponding to the fact that the transformation $R_1R_2 \cdots R_9$, of infinite period, is a kind of screw.

These, as well as the other infinite groups $[3^{3,3,1}]$ and $[3^{2,2,2}]$, are special cases of the general "trigonal" group $[3^{n,p,q}]$ in $n + p + q + 1$ dimensions [5; 159]. Frame has investigated the operation

$$R_1R_2 \cdots R_{n+p+q+1}$$

for $[3^{n,p,q}]$, obtaining the characteristic equation

$$\phi_{n+p+q+1} - \lambda^2\phi_{n-1}\phi_{p-1}\phi_{q-1} = 0, \tag{3.4}$$

where $\phi_r = 1 + \lambda + \lambda^2 + \cdots + \lambda^r$. Since

$$\lambda^{-\frac{1}{2}r}\phi_r = \frac{\lambda^{\frac{1}{2}(r+1)} - \lambda^{-\frac{1}{2}(r+1)}}{\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}}} = U_r(X), \qquad \text{where } X = \frac{\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}}{2},$$

the corresponding equation for X is

$$U_{n+p+q+1}(X) - U_{n-1}(X)U_{p-1}(X)U_{q-1}(X) = 0. \tag{3.5}$$

Since $U_{-1}(X) = 0$, this reduces to $U_{n+p+1}(X) = 0$ for the symmetric group $[3^{n,p,0}] = [3^{n+p}]$, in agreement with the first equation (3.1). Since $U_0(X) = 1$, it reduces to

$$U_{n+3}(X) - U_{n-1}(X) = 0$$

for $[3^{n,1,1}]$; this agrees with (3.2) by virtue of the identity

$$U_n(X) - U_{n-4}(X) = 4XT_{n-1}(X).$$

Finally, since $U_1(X) = 2X$, we have

$$U_{n+4}(X) - 2XU_{n-1}(X) = 0$$

for $[3^{n,2,1}]$, in agreement with (3.3).

Combining all these with the known results for the groups of the regular polytopes [10; 221], we obtain the complete list shown in Table 2.

TABLE 2

Group	n	h	$m_1, m_2, \cdots, m_n$	$\sum m = \frac{1}{2}nh$	$\prod (m+1)$
$[3^{n-1}]$		$n+1$	$1, 2, 3, \cdots, n$	$\frac{1}{2}n(n+1)$	$(n+1)!$
$[3^{n-2}, 4]$		$2n$	$1, 3, 5, \cdots, 2n-1$	n^2	$2^n \cdot n!$
$[3^{n-3,1,1}]$		$2(n-1)$	$1, 3, \cdots, n-1, \cdots, 2n-5, 2n-3$	$n(n-1)$	$2^{n-1} \cdot n!$
$[3^{2,2,1}]$	6	12	1, 4, 5, 7, 8, 11	36	$72 \cdot 6!$
$[3^{3,2,1}]$	7	18	1, 5, 7, 9, 11, 13, 17	63	$8 \cdot 9!$
$[3^{4,2,1}]$	8	30	1, 7, 11, 13, 17, 19, 23, 29	120	$192 \cdot 10!$
[3, 4, 3]	4	12	1, 5, 7, 11	24	1152
$[p]$	2	p	$1, p-1$	p	$2p$
[3, 5]	3	10	1, 5, 9	15	120
[3, 3, 5]	4	30	1, 11, 19, 29	60	14400

Among the m's, for $[3^{n-3,1,1}]$, the value $n-1$ breaks the rhythm of the sequence of odd numbers. When n is odd, it comes between $n-2$ and n. When n is even, it is repeated: the equation has a double root.

The group contains the central inversion if and only if the m's are all odd [10; 225–226].

4. **The number of reflections.** By (1.3), the sum of all the m's is $\frac{1}{2}nh$, which we recognize as the number of reflections in the group [6; 610], though the underlying reasons for this remain obscure when $n > 3$. It is also very remarkable that the product $\prod(m + 1)$ is equal to the order of the group. To seek a possible hint towards explaining these mysteries, let us re-examine the small values of n.

When $n = 2$, we have the simple kaleidoscope consisting of two mirrors inclined at π/p. The product R_1R_2 is a p-gonal rotation, the number of reflections is $p = h$, and the order is

$$2p = (m_1 + 1)(m_2 + 1).$$

When $n = 3$, the situation is already non-trivial. The group $[p, q]$ is generated by the three mirrors of Möbius's polyhedral kaleidoscope, which cut out a spherical triangle of angles

$$\pi/p, \ \pi/q, \ \pi/2.$$

The order, being equal to the number of such triangles that fit together to fill up the whole surface of the sphere, is

$$g = 8pq/\{4 - (p - 2)(q - 2)\}$$

[10; 82]. The sides of the triangles are arcs of certain great circles which we call *lines of symmetry*. (They lie in the planes of actual or virtual mirrors.) They cross one another orthogonally at $\frac{1}{4}g$ points lying also on another set of great circles which we call *equators* [10; 67, 73]. The lines of symmetry dissect the equators into g arcs: altitudes of the g spherical triangles.

The product $R_1R_2R_3$ is a rotatory-reflection along an equator [10; 90]; that is, reflection in the equator combined with an h-gonal rotation. With this definition for h, the length of the altitude is π/h, whence, by spherical trigonometry,

$$\cos^2\frac{\pi}{h} = \cos^2\frac{\pi}{p} + \cos^2\frac{\pi}{q}. \tag{4.1}$$

Thus the values of h for [3, 3], [3, 4], [3, 5] are 4, 6, 10. Postponing consideration of the exceptional case where p or $q = 2$, we observe that h is even; consequently, h is the period not only of the rotatory component of $R_1R_2R_3$ but of this operation itself. The equation (2.1) becomes

$$X^3 - \left(\cos^2\frac{\pi}{p} + \cos^2\frac{\pi}{q}\right)X = 0,$$

whose roots are $\pm\cos \pi/h$ and 0. Thus

$$m_1 = 1, \qquad m_2 = \tfrac{1}{2}h, \qquad m_3 = h - 1.$$

Since each equator contains $2h$ of the g altitudes, the number of equators is

$g/2h$. A given one of the $g/2h$ equators meets each of the others at a pair of antipodal points. But the number of such points on each equator is h. Hence,

$$\frac{g}{2h} - 1 = \frac{h}{2}; \tag{4.2}$$

that is [10; 19, 91],

$$g = (h + 2)h = (m_1 + 1)(m_2 + 1)(m_3 + 1).$$

Each line of symmetry consists of a certain number of arcs, which are sides of characteristic triangles. Every point where a right angle occurs is a common end of two arcs on each of two perpendicular lines of symmetry; it also lies on two intersecting equators. Every arc not of this kind is a common hypotenuse of two of the triangles, and is crossed by an equator (joining the other vertices of these triangles). A given line of symmetry meets each of the $g/2h$ equators at a pair of antipodal points, which are either right-angle points or points on hypotenuses. Hence, the number of arcs into which the line of symmetry is divided (by other lines of symmetry) is twice the number of equators, namely, g/h. But the total number of arcs (being the $3g$ sides of the g triangles, each used twice) is $\frac{3}{2}g$. Hence, the number of lines of symmetry, or of reflecting planes [10; 68] is

$$\frac{3}{2}h = m_1 + m_2 + m_3 .$$

The above theory breaks down when we try to apply it to the reducible group $[p, 2]$, whose single equator coincides with a line of symmetry. Here (4.1) yields $h = p$; but the period of $R_1R_2R_3$ is p or $2p$ according as p is even or odd. Since $g = 4p$, (4.2) is no longer valid; but the number of lines of symmetry is still

$$h + \frac{g}{2h} - 1. \tag{4.3}$$

(This supersedes the more complicated expression 4.51 of [10; 68].)

5. **Laporte's theorem and its extension.** Another expression for the number of lines of symmetry (valid for both reducible and irreducible groups, operating on a sphere in ordinary space) has been discovered by Laporte [14; 455]. He finds it equal to *the quotient of the perimeter and area of the characteristic triangle.*

In fact, the combined perimeters of the g triangles amount to twice the total length of the lines of symmetry, that is, 4π times their number. Hence, the number is

$$g(\varphi + \chi + \psi)/4\pi;$$

namely, the perimeter $\varphi + \chi + \psi$ divided by the area $4\pi/g$.

Combining this with (4.3), we obtain

$$\varphi + \chi + \psi = \frac{4\pi}{g}\left(h + \frac{g}{2h} - 1\right) = \frac{2\pi}{h} + \frac{4\pi}{g}(h - 1).$$

When the group is irreducible [10; 228] so that $g = h(h + 2)$, this formula for the perimeter becomes simply

$$\varphi + \chi + \psi = 6\pi/(h + 2),$$

in agreement with [10; 74].

Analogously, in four dimensions, the reflecting hyperplanes meet the unit hypersphere around their common point in an equal number of great spheres, forming g spherical tetrahedra of surface S and volume V, say. Clearly $gV = 2\pi^2$. Since gS is twice the total surface of the great spheres, their number is

$$\frac{gS}{8\pi} = \frac{\pi}{4}\frac{S}{V}.$$

Thus, Laporte's theorem extends as follows:

For a finite group generated by reflections in four dimensions, the number of reflecting hyperplanes is $\frac{1}{4}\pi$ *times the quotient of the surface and volume of the characteristic tetrahedron.*

The surface S, being the sum of the spherical excesses of the four faces, is equal to the sum of the twelve face angles minus 4π. In the case of the group $[p, q, r]$ with p, q, r all greater than 2 [10; 139], we have

$$S = \varphi_{p,q} + \chi_{p,q} + \psi_{p,q} + \varphi_{q,r} + \chi_{q,r} + \psi_{q,r} + \frac{\pi}{p} + \frac{\pi}{r} + 4\frac{\pi}{2} - 4\pi$$

$$= \left(\frac{6}{h_{p,q} + 2} + \frac{6}{h_{q,r} + 2} + \frac{1}{p} + \frac{1}{r} - 2\right)\pi.$$

Since $V = 2\pi^2/g_{p,q,r}$, this agrees with the formula 12.81 of [10; 232], provided we accept the expression $2h_{p,q,r}$ for the number of reflecting hyperplanes.

Laporte's theorem extends likewise to more than four dimensions, but is then less useful because the hyper-surface of the simplex is a difficult collection of Schläfli functions [10; 142].

6. **Invariant forms.** It is well known that every finite group of homogeneous linear transformations on n real variables has a set of n algebraically independent invariants, such that every invariant is algebraically expressible in terms of them [2; 357]. When these are polynomials of smallest possible degrees, we call them a set of *basic invariant forms.* One of them, of course, is a positive definite quadratic form, which can be used to express the group as a group of orthogonal transformations. The Jacobian of any set of basic invariant forms is independent of the particular forms chosen, and is a relative invariant (that

is, every transformation of the group either leaves it unchanged or reverses its sign).

By equating each of a given set of n invariants to a constant, say

$$I_k(x_1, \cdots, x_n) = c_k \qquad (k = 1, \cdots, n), \tag{6.1}$$

we obtain n equations for n unknown x's. These equations can be solved if the invariants are independent, that is, if the Jacobian does not vanish identically. Then the values of the x's for which the Jacobian does vanish are multiple solutions of the equations (with appropriate c's), and if the group is generated by reflections, the locus of such points is the set of reflecting hyperplanes. Hence,

6.2. *For any finite group generated by reflections, the Jacobian of a set of basic invariant forms breaks up into the product of the linear forms whose vanishing determines the reflecting hyperplanes.*

This theorem was conjectured by Todd (see [13; 239, second footnote] for a special case). The above proof was kindly supplied by Racah. Racah and Todd also made (independently) the following observation.

By eliminating all but one of the variables in (6.1) we obtain an equation whose degree is the product of the degrees of the invariants. If the constants c_k are sufficiently general, the solution will separate into (say) M systems of g points [2; 358]. Hence, the product of the degrees is equal to Mg. *If the n independent invariant forms have the property that the product of their degrees is exactly g* (so that $M = 1$), *then they are a basic set, and every other invariant is not merely algebraically but rationally expressible in terms of them.*

Since the Jacobian of n forms of various degrees k has degree $\sum (k - 1) = \sum k - n$, and the number of reflecting hyperplanes [6; 610] is $\frac{1}{2}nh$, we seek invariants I_k such that

$$\sum k = \tfrac{1}{2}n(h + 2), \qquad \prod k = g.$$

Moreover, the k's will be all even when the group contains the central inversion, and only then. Let us now try some special cases.

In terms of Cartesian coordinates, the symmetric group $[3^{n-1}]$, of order $(n + 1)!$, is generated by reflections in the n hyperplanes

$$x_1 - x_2 = 0, \qquad x_2 - x_3 = 0, \qquad \cdots, \qquad x_n - x_{n+1} = 0,$$

all orthogonal to $\sum x = 0$ in Euclidean $(n + 1)$-space [5; 163], [10; 226]. Hence a set of basic invariant forms is provided by the elementary symmetric functions

$$\sum x_1x_2, \qquad \sum x_1x_2x_3, \qquad \cdots, \qquad x_1x_2 \cdots x_{n+1},$$

of degrees $2, 3, \cdots, n + 1$.

The "hyper-octahedral" group $[3^{n-2}, 4]$, of order $2^n n!$, is generated by reflections in the n hyperplanes

$$x_1 - x_2 = 0, \qquad x_2 - x_3 = 0, \qquad \cdots, \qquad x_{n-1} - x_n = 0, \qquad x_n = 0$$

of Euclidean n-space, or by all the sign-changes and permutations of the n coordinates. Hence, a set of basic invariant forms is provided by the elementary symmetric functions of $x_1^2, x_2^2, \cdots, x_n^2$, of degrees

$$2, \quad 4, \quad 6, \quad \cdots, \quad 2n.$$

The subgroup $[3^{n-3,1,1}]$, of index 2 [10; 200], generated by reflections in the hyperplanes

$$x_1 - x_2 = 0, \quad x_2 - x_3 = 0, \quad \cdots, \quad x_{n-1} - x_n = 0, \quad x_{n-1} + x_n = 0,$$

differs in that it admits only even numbers of sign-changes. Thus the expression $(x_1^2 x_2^2 \cdots x_n^2)^{\frac{1}{2}} = \prod x$ is an invariant (instead of a relative invariant), and the degree $2n$ is replaced by n. (It follows that, when n is even, there are two independent invariants of the same degree n.)

In terms of complex coordinates $x = x_1 + ix_2$ and $y = x_1 - ix_2$, the dihedral group $[p]$ is generated by

$$x' = y, \qquad y' = x$$

and

$$x' = \omega y, \qquad y' = \omega^{-1} x,$$

where $\omega = e^{2\pi i/p}$; and we find the basic invariants

$$xy, \qquad x^p + y^p$$

[2; 362], of degrees 2 and p. Clearly, their Jacobian is a numerical multiple of $x^p - y^p = \prod (x - \omega^j y)$.

Klein [13; 237, 242] obtained invariants A, B, C, D, of degrees 2, 6, 10, 15, for the icosahedral group. He observed that D (the Jacobian of A, B, C) is the square root of a polynomial in A, B, C. It follows that A, B, C alone form a basic set of invariants for the extended icosahedral group [3, 5].

This and the other three-dimensional results (including the dihedral group) may be summarized by saying that the degrees of the basic invariants for $[p, q]$ are

$$2, \qquad g/2h, \qquad h,$$

where

$$g = \frac{8pq}{2p + 2q - pq}, \qquad \cos^2 \frac{\pi}{h} = \cos^2 \frac{\pi}{p} + \cos^2 \frac{\pi}{q}.$$

The invariant of degree $g/2h$ may be taken to be the product of the $g/2h$ linear forms whose vanishing determines the $g/2h$ equatorial planes of the polyhedron $\{{p \atop q}\}$ [10; 18–19 ($g = 4N_1$)].

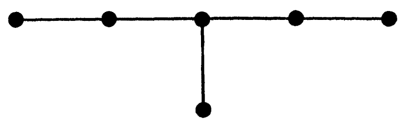
$[3^{2,2,1}]$, of order $72 \cdot 6!$, is the group of Gosset's six-dimensional semi-regular polytope 2_{21}, whose 27 vertices correspond to the 27 lines on the general cubic surface [9; 465], [4; 388]. It is generated by the following six linear transformations of seven variables, $x_1, \cdots, x_6, y$, satisfying the relation

$$x_1 + \cdots + x_6 = 0:$$

the five transpositions (x_1x_2), (x_2x_3), (x_3x_4), (x_4x_5), (x_5x_6) and the substitution

$$Q\colon\ (x_1, x_2, x_3, x_4, x_5, x_6, y)$$
$$\to (x_1 - u, x_2 - u, x_3 - u, x_4 + u, x_5 + u, x_6 + u, y - u),$$

where $u = \frac{1}{2}(x_1 + x_2 + x_3 + y) = \frac{1}{2}(-x_4 - x_5 - x_6 + y)$. These six transformations are easily seen to be involutory and to correspond to the nodes of the tree



in such a way that the product of two transformations is of period 3 or 2 according as the corresponding nodes are adjacent or non-adjacent. Moreover, they permute the 27 linear forms

$$a_i = x_i + y, \qquad b_i = x_i - y \qquad (i = 1, \cdots, 6),$$
$$c_{ij} = -x_i - x_j \qquad (i < j);$$

for example, Q interchanges them in pairs thus:

$$(a_1\, c_{23})(a_2\, c_{13})(a_3\, c_{12})(b_4\, c_{56})(b_5\, c_{46})(b_6\, c_{45}).$$

(Burnside [2; 487] used equivalent linear forms, only he wrote a, b, c, d, e, f, s instead of $a_1, a_2, a_3, a_4, a_5, a_6, 6y$.) In this manner we obtain the invariants

$$I_k = \sum a_i^k + \sum b_i^k + \sum c_{ij}^k$$
$$= 2 \sum_{j=0}^{[k/2]} \binom{k}{2j} s_{k-2j} y^{2j} + (-1)^k \left\{ (6 - 2^{k-1}) s_k + \frac{1}{2} \sum_{r=2}^{k-2} \binom{k}{r} s_r s_{k-r} \right\},$$

where $s_k = x_1^k + \cdots + x_6^k$ (for example, $s_0 = 6$, $s_1 = 0$). The only I's that vanish identically are I_1 and I_3. The quadratic invariant is

$$I_2 = 6(s_2 + 2y^2).$$

The relations $\sum k = 42$ and $\prod k = 72 \cdot 6!$ suggest the values $k = 2, 5, 6, 8, 9, 12$. In fact, we can prove that the six invariants $I_2, I_5, I_6, I_8, I_9, I_{12}$ are independent, by verifying that the Jacobian

$$J = \frac{\partial(I_2, I_5, I_6, I_8, I_9, I_{12})}{\partial(x_2, x_3, x_4, x_5, x_6, y)} = \frac{\partial(s_1, I_2, I_5, I_6, I_8, I_9, I_{12})}{\partial(x_1, x_2, x_3, x_4, x_5, x_6, y)}$$

does not vanish identically. For this purpose we use special values of the variables, namely,

$$(x_1, x_2, x_3, x_4, x_5, x_6, y) = (1, \epsilon, \epsilon^2, \epsilon^3, \epsilon^4, 0, 0),$$

where $\epsilon = e^{2\pi i/5}$. Actually these values would make $J = 0$, since y is a factor of every $\partial I_k/\partial y$. (In fact, $y = 0$ is one of the 36 reflecting hyperplanes; the others consist of 15 such as $x_1 - x_2 = 0$ and 20 such as $x_1 + x_2 + x_3 + y = 0$.) But we can first remove the factor y and then set $y = 0$, obtaining

$$\frac{\partial I_k}{\partial x_i} = \begin{cases} (8 - 2^{k-1})kx_i^{k-1} + \sum_{r=2}^{k-2} (k - r)\binom{k}{r} s_r x_i^{k-r-1} & (k \text{ even}), \\ (2^{k-1} - 4)kx_i^{k-1} - \sum_{r=2}^{k-2} (k - r)\binom{k}{r} s_r x_i^{k-r-1} & (k \text{ odd}), \end{cases}$$

$$\frac{1}{y}\frac{\partial I_k}{\partial y} = 2k(k - 1)s_{k-2} .$$

With the special values chosen above, we have $s_k = 0$ when $k \not\equiv 0 \pmod 5$, and $s_5 = s_{10} = 5$. Thus

$$\frac{\partial I_2}{\partial x_i} = 12x_i , \qquad \frac{\partial I_5}{\partial x_i} = 60x_i^4 , \qquad \frac{\partial I_6}{\partial x_i} = -144x_i^5 ,$$

$$\frac{\partial I_8}{\partial x_i} = -960x_i^7 + 840x_i^2 = -120x_i^2 ,$$

$$\frac{\partial I_9}{\partial x_i} = 2268x_i^8 - 2520x_i^3 = -252x_i^3 ,$$

$$\frac{\partial I_{12}}{\partial x_i} = -24480x_i^{11} + 27720x_i^6 + 660x_i = 3900x_i ,$$

$$\frac{1}{y}\frac{\partial I_k}{\partial y} = \begin{cases} 24 & (k = 2), \\ 0 & (k = 5, 6, 8, 9), \\ 1320 & (k = 12). \end{cases}$$

It follows that

$$\frac{J}{y} = 12\cdot 60\cdot 144\cdot 120\cdot 252(7800 - 1320) \begin{vmatrix} x_1 & \cdots & x_5 \\ x_1^4 & \cdots & x_5^4 \\ 1 & \cdots & 1 \\ x_1^2 & \cdots & x_5^2 \\ x_1^3 & \cdots & x_5^3 \end{vmatrix} .$$

This differs from zero since the Vandermonde determinant is the product of differences of $x_1, \cdots, x_5$, which are different fifth roots of unity. Hence J does not vanish identically, and the six chosen invariants are indeed independent. Since

$$2 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdot 12 = 72 \cdot 6!,$$

they are a basic set.

Frame [12] has obtained the same degrees by using a different coordinate system (with six variables instead of seven).

Racah [16] has proceeded along similar lines for the remaining irreducible groups:

$[3^{3,2,1}]$, of order $8 \cdot 9!$, the group of Gosset's seven-dimensional polytope 3_{21}, whose 56 vertices are

$$(3, 3, -1, -1, -1, -1, -1, -1) \text{ and } (1, 1, 1, 1, 1, 1, -3, -3) \text{ permuted,}$$

in the 7-space $x_1 + \cdots + x_8 = 0$ [4; 387];

$[3^{4,2,1}]$, of order $192 \cdot 10!$, the group of Gosset's eight-dimensional polytope 4_{21}, whose 240 vertices are

$$(2, 2, 2, -1, -1, -1, -1, -1, -1), \qquad (1, 1, 1, 1, 1, 1, -2, -2, -2)$$

$$(3, 0, 0, 0, 0, 0, 0, 0, -3),$$

permuted, in the 8-space $x_1 + \cdots + x_9 = 0$ [4; 395];

[3, 4, 3], of order $2(4!)^2$, the group of the regular 24-cell {3, 4, 3}, whose 24 vertices are the permutations of $(\pm 1, \pm 1, 0, 0)$ [10; 156]; and finally

[3, 3, 5], of order $(5!)^2$, the group of the regular 600-cell {3, 3, 5}, whose 120 vertices are the permutations of

$$(\pm 2, 0, 0, 0), \qquad (\pm 1, \pm 1, \pm 1, \pm 1)$$

along with the even permutations of $(\pm\tau, \pm 1, \pm\tau^{-1}, 0)$, where

$$\tau = \tfrac{1}{2}(5^{\frac{1}{2}} + 1) = 2 \cos \pi/5$$

[10; 157].

Racah's results enable us to complete Table 3.

TABLE 3

Group	n	Degrees	Sum	Product
$[3^{n-1}]$		$2, 3, 4, \cdots, n+1$	$\frac{1}{2}n(n+3)$	$(n+1)!$
$[3^{n-2}, 4]$		$2, 4, 6, \cdots, 2n$	$n(n+1)$	$2^n\, n!$
$[3^{n-3,1,1}]$		$2, 4, \cdots, n, \cdots, 2n-4, 2n-2$	n^2	$2^{n-1}\, n!$
$[3^{2,2,1}]$	6	2, 5, 6, 8, 9, 12	42	$72 \cdot 6!$
$[3^{3,2,1}]$	7	2, 6, 8, 10, 12, 14, 18	70	$8 \cdot 9!$
$[3^{4,2,1}]$	8	2, 8, 12, 14, 18, 20, 24, 30	128	$192 \cdot 10!$
[3, 4, 3]	4	2, 6, 8, 12	28	1152
$[p]$	2	$2, p$	$p+2$	$2p$
[3, 5]	3	2, 6, 10	18	120
[3, 3, 5]	4	2, 12, 20, 30	64	14400

Comparison with Table 2 reveals an extremely remarkable fact:
The degrees of the basic invariant forms are

$$m_1 + 1, \qquad m_2 + 1, \qquad \cdots, \qquad m_n + 1.$$

In other words, the degrees, which were found with great labor for each group separately, can actually be derived from the graphical symbols by means of the equation (2.1) or (2.2).

If this could be explained in general terms, the other mysteries would be cleared up at once. From (1.3) we have $\sum m = \frac{1}{2}nh$, and Theorem 6.2 shows that this is the number of reflecting hyperplanes. Also $\prod (m+1)$, being the product of the degrees, is the order of the group.

By (1.3) again, since the smallest degree is $m_1 + 1 = 2$, the greatest is $m_n + 1 = h$. When the group is of the form $[p, q, \cdots]$, namely, the group of a *regular* polytope $\{p, q, \cdots\}$, h is the number of sides of the Petrie polygon [10; 225]. This suggests that there may be some significance in the fact that the degrees 12 and 20 for [3, 3, 5] are the numbers of sides of the Petrie polygons of the regular star polytopes

$$\left\{5, 3, \frac{5}{2}\right\} \quad \text{and} \quad \left\{3, 5, \frac{5}{2}\right\}$$

which have the same vertices and edges as $\{3, 3, 5\}$ [10; 266, 267, 278]. Similarly, 6 is the number of sides of the Petrie polygon of the great dodecahedron $\{5, \frac{5}{2}\}$, which has the same vertices and edges as the icosahedron $\{3, 5\}$ [10; 102].

7. The application to simple Lie groups. Among the finite groups generated by reflections, those in which the only angles occurring between pairs of reflecting hyperplanes are multiples of $\pi/4$ and $\pi/6$ are said to be *crystallographic*. For, each is a subgroup of an infinite discrete group generated by reflections, and leaves a lattice invariant [10; 191, 205]. These crystallographic groups play a vital role in the theory of simple Lie groups. Table 4 gives in each case the finite group, the corresponding Lie group, and the Poincaré polynomial for the group manifold as computed by Pontrjagin [15], Brauer [1], and Chevalley. (Our n is the l of Cartan [3].)

TABLE 4

Finite group	Lie group	Poincaré polynomial
$[3^{n-1}]$	A_n	$(1 + t^3)(1 + t^5) \cdots (1 + t^{2n+1})$
$[3^{n-2}, 4]$	B_n or C_n	$(1 + t^3)(1 + t^7) \cdots (1 + t^{4n-1})$
$[3^{n-3,1,1}]$	D_n	$(1 + t^3)(1 + t^7) \cdots (1 + t^{4n-5})(1 + t^{2n-1})$
$[3^{2,2,1}]$	E_6	$(1 + t^3)(1 + t^9)(1 + t^{11})(1 + t^{15})(1 + t^{17})(1 + t^{23})$
$[3^{3,2,1}]$	E_7	$(1 + t^3)(1 + t^{11})(1 + t^{15})(1 + t^{19})(1 + t^{23})(1 + t^{27})(1 + t^{35})$
$[3^{4,2,1}]$	E_8	$(1 + t^3)(1 + t^{15})(1 + t^{23})(1 + t^{27}) \cdot (1 + t^{35})(1 + t^{39})(1 + t^{47})(1 + t^{59})$
$[3, 4, 3]$	F_4	$(1 + t^3)(1 + t^{11})(1 + t^{15})(1 + t^{23})$
$[6]$	G_2	$(1 + t^3)(1 + t^{11})$

Comparing this with Tables 2 and 3, we see that the Poincaré polynomial is

$$(1 + t^{2m_1+1})(1 + t^{2m_2+1}) \cdots (1 + t^{2m_n+1}).$$

Chevalley observed that the differences of consecutive exponents form a palindromic sequence; for example, for E_6, where the exponents are 3, 9, 11, 15, 17, 23, the differences are 6, 2, 4, 2, 6. This property is an immediate consequence of the fact that, by (1.3),

$$(2m_1 + 1) + (2m_n + 1) = (2m_2 + 1) + (2m_{n-1} + 1) = \cdots = 2h + 2.$$

The dimension of the group manifold, being the degree of the Poincaré polynomial, $1 + B_1 t + \cdots + B_{r-1}t^{r-1} + t^r$, is

$$r = (2m_1 + 1) + (2m_2 + 1) + \cdots + (2m_n + 1) = n(h + 1).$$

Cartan showed that $r - n$ is twice the number of reflections in the finite group, in agreement with our remark that the number of reflections is $\frac{1}{2}nh$.

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