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DISCRETE GROUPS GENERATED BY REFLECTIONS

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CONTENTS

Introduction

It is known that, in ordinary space, the only finite groups generated by reflections are

$[k]$ $(k \geqq 1)$, of order $2k$, with abstract definition

$$R_1^2 = R_2^2 = (R_1R_2)^k = 1,$$

and

$[k_1, k_2]$ $(k_1 \geqq 2,\ k_2 \geqq 2,\ 1/k_1 + 1/k_1 > \frac{1}{2})$, of order $\dfrac{4}{1/k_1 + 1/k_2 - \frac{1}{2}}$, with abstract definition

$$\mathrm{R}_1^2 = R_2^2 = R_3^2 = (R_1 R_2)^{k_1} = (R_1 R_3)^2 = (R_2 R_3)^{k_2} = 1.$$

$[1]$ is the group of order 2 generated by a single reflection. Since this is the symmetry group of the one-dimensional polytope[1] $\{\}$, we write

$$[1] = [\,].$$

$[k] (k \geqq 3)$ and $[k_1, k_2]$ $(k_1 = 3, k_2 = 3, 4, 5)$ are the symmetry groups of the ordinary regular polygons $\{k\}$ and polyhedra $\{k_1, k_2\}$. The rest of the groups can be written as direct products, thus:

$$[2] = [\,] \times [\,],$$
$$[2, 2] = [\,] \times [\,] \times [\,],$$
$$[2, k] = [\,] \times [k].$$

[1] Coxeter **1**, 344.

These groups can be made vividly comprehensible by using actual mirrors for the generating reflections. It is found that a candle makes an excellent object to reflect. By hinging two vertical mirrors at an angle π/k ($k = 2, 3, 4, \ldots$), we easily see $2k$ candle flames, in accordance with the group $[k]$. To illustrate the groups $[k_1, k_2]$, we hold a third mirror in the appropriate positions.

In the present paper, we generalize the above results to space of any number of dimensions,[2] and to groups which, though infinite, are still free from infinitesimal operations. (One such infinite group is called $\boxed{3^3}$, and may be illustrated by means of three vertical mirrors, erected on the sides of an equilateral triangle so as to form a prism, open at the top. A candle placed within this prism gives rise to an unlimited number of images.)

In order to enumerate these groups, we observe that the fundamental region[3] must be a polytope whose dihedral angles, being submultiples of π, are never obtuse. It happens that such polytopes are of a particularly simple form.

Perhaps it is worth while to point out what contact this investigation has with crystallography. Each of the 32 crystal systems corresponds to a finite group of orthogonal transformations. Of those groups, the eleven which Schoenflies[4] calls *Holoedrie* and *Hemimorphe Hemiedrie* are generated by reflections; in fact,

$$\begin{aligned}
C_1^h &= [\,],\\
C_2^v &= [\,] \times [\,],\\
V^h &= [\,] \times [\,] \times [\,],\\
C_k^v &= [k] \quad (k = 3, 4, 6),\\
D_k^h &= [\,] \times [k],\\
T^d &= [3, 3],\\
O^h &= [3, 4],
\end{aligned}$$

Of the 230 *space groups*, the following seven are generated by reflections:

$$\begin{aligned}
\mathfrak{B}_h^1 &= [\infty] \times [\infty] \times [\infty],\\
\mathfrak{D}_{6,h}^1 &= [\infty] \times [3, 6],\\
\mathfrak{D}_{4,h}^1 &= [\infty] \times [4, 4],\\
\mathfrak{D}_{3,h}^1 &= [\infty] \times \boxed{3^3},^5\\
\mathfrak{T}_h^3 &= \boxed{3^4},\\
\mathfrak{O}_h^5 &= \begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix},^6\\
\mathfrak{O}_h^1 &= [4, 3, 4].^7
\end{aligned}$$

[2] For the case of four dimensions, see Goursat **1**, 80–93.

[3] Bieberbach **1**, 312.

[4] Schoenflies **1**.

[5] Coxeter **2**, 147.

[6] *Ibid.*, 148.

[7] *Ibid.*, 150.

([∞] is the group generated by reflections in two points of a line. Its abstract definition is simply
$$R_1^2 = R_2^2 = 1.)$$

THEOREM 1. *In spherical space, every polytope free from obtuse dihedral angles is a simplex.*

LEMMA 1.1. *Every spherical polygon free from obtuse angles is a triangle.*

Since the angle-sum of an n-gon is greater than $(n - 2)\pi$, at least one angle must be greater than $(n - 2)\pi/n$.

LEMMA 1.2. *If two angles of a spherical triangle are non-obtuse, then the third angle is not less than its opposite side.*

By the well-known formula, if A and B are non-obtuse,
$$\cos C = \sin A \sin B \cos c - \cos A \cos B \leqq \cos c.$$

DEFINITION. A *simplex-polycorypha*[8] is a polytope in which the number of edges at any vertex is equal to the number of dimensions of the space. *E.g.*, the measure polytope is a simplex-polycorypha.

LEMMA 1.3. *If a simplex-polycorypha is free from obtuse dihedral angles, so is any bounding figure.*

Consider the section of the simplex-polycorypha Π_m by a 3-space perpendicular to any element Π_{m-3}, meeting that element at an internal point O. The three Π_{m-1}'s which meet at the Π_{m-3} will give, in the section, a trihedral angle with vertex O; and the edges of this trihedral angle will be sections of the three Π_{m-2}'s which separate the Π_{m-1}'s in pairs. Further, on account of the orthogonality, the dihedral angles of the trihedral angle are dihedral angles of Π_m, while its face-angles are dihedral angles of the three Π_{m-1}'s. On a sphere with center O, the trihedral angle cuts out a spherical triangle whose angles A, B, C are dihedral angles of Π_m, and so non-obtuse, by hypothesis. Lemma 1.2 gives
$$c \leqq C \leqq \tfrac{1}{2}\pi.$$

But c can be any dihedral angle of any bounding figure. Thus Lemma 1.3 is proved.

(If $m = 3$, there is no need to take a section. If $m < 3$, the Lemma is meaningless.)

LEMMA 1.4. *In three or more dimensions, every simplex-polycorypha bounded entirely by simplexes is itself a simplex.*

In an m-dimensional simplex-polycorypha, $m - 1$ plane faces pass through each edge. If all the bounding figures are simplexes, these plane faces are triangles. Let $A_1, A_2, \ldots, A_m$ be the m vertices that are joined to the vertex A_0 by edges. Then the $m - 1$ triangles through A_0A_1 must be
$$A_0A_1A_2, \ \ldots, \ A_0A_1A_m.$$

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[8] Sommerville **1**. For the *polytopes* considered here, the definition given in Coxeter **1**, 331 (§1.1) is appropriate. Sommerville calls these *simple convex* polytopes.

By hypothesis, just m edges pass through A_1. Therefore $A_1A_0, A_1A_2, \ldots, A_1A_m$ are the *only* edges through A_1. Similarly for any other A_i. Thus the polycorypha has only $m + 1$ vertices.

Theorem 1 can now be proved by induction. By Lemma 1.1, it is true in two dimensions. Let us then assume it true in $m - 1$ dimensions. Consider an m-dimensional polytope free from obtuse dihedral angles. A small sphere[9] drawn round any vertex cuts out an $(m - 1)$-dimensional spherical polytope whose dihedral angles all occur among those of the m-dimensional polytope. By the inductive assumption, this $(m - 1)$-dimensional polytope is a simplex; *i.e.*, the m-dimensional polytope is a simplex-polycorypha. By Lemma 1.3 and the inductive assumption, every bounding figure is a simplex. The theorem follows by Lemma 1.4.

COROLLARY. *In Euclidean space, every acute-angled polytope is a simplex.*

If possible, consider an acute-angled polytope that is not a simplex. By projection on to a spherical space of sufficiently large radius, we can obtain a spherical polytope whose dihedral angles differ by as little as we please from those of the given polytope. If we make the radius so large that the angles remain acute, Theorem 1 is contradicted.

DEFINITION. A *prism*[10] is the topological product of a number of polytopes lying in absolutely perpendicular spaces. Regarding a polytope as a closed set of points (*viz.* all the points within it and on its boundary), we can define the prism $[\Pi_p, \Pi_q]$ as follows.

In $p + q$ dimensions, suppose Π_p to be fixed, while Π_q is constrained to lie in an absolutely perpendicular space and to have a constant orientation. Suppose further that the common point of the spaces of Π_p and Π_q belongs to both polytopes, and is a *definite* point of Π_q (*e.g.* a vertex). Then the prism is the totality of points that belong to the possible positions of Π_q.

Topological products being commutative and associative, a prism may have any number of constituents $\Pi_p, \Pi_q, \ldots$, and the order of their arrangement is immaterial. It is convenient to admit the trivial case when there is only one constituent.

A *simplicial prism* is a "rectangular product" of (one or more) simplexes. We use the symbol Σ_m for a general simplex in m dimensions. Thus Σ_0 denotes a point, and Σ_1 a straight segment (of unspecified length).

THEOREM 2. *In Euclidean space, every polytope free from obtuse dihedral angles is a simplicial prism.*

LEMMA 2.1. *Every polygon free from obtuse angles is either a triangle Σ_2 or a rectangle $[\Sigma_1, \Sigma_1]$.*

This follows from the fact that the angle-sum of an n-gon is $(n - 2)\pi$.

LEMMA 2.2. *Every polytope free from obtuse dihedral angles is a simplex-polycorypha.*

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[9] We shall use the word *sphere* for the analogue in any number of dimensions.

[10] Called *prismotope* by Schoute **1**. *Cf.* Coxeter **1, 351**.

A small sphere drawn round any vertex cuts out a spherical polytope whose dihedral angles all occur among those of the whole polytope. By Theorem 1, this spherical polytope is a simplex.

LEMMA 2.3. *A prism whose constituents are simplex-polycoryphas is itself a simplex-polycorypha.*

Since every vertex of $[\Pi_p, \Pi_q]$ is a vertex of one Π_p and of one Π_q, the edges which meet there are the covertical edges of Π_p and of Π_q. If, in particular, Π_p and Π_q have respectively p and q covertical edges, then $[\Pi_p, \Pi_q]$ has $p + q$; *i.e.* the prism is a simplex-polycorypha. This result, having been proved for two constituents, at once extends to the case of any number.

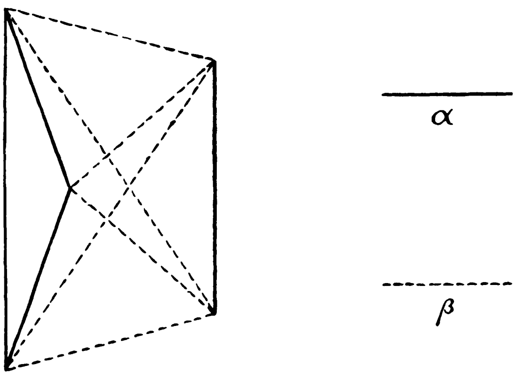
LEMMA 2.4. *Every simplicial prism can be characterized by a "vertex diagram" consisting of a simplex with edges of two types.*

Consider again the construction for $[\Pi_p, \Pi_q]$. When Π_q moves so that each point of it describes an edge of Π_p, its edges describe rectangles. Thus, if AB and AD are edges of Π_p and Π_q respectively, the rectangle $ABCD$ is an element of $[\Pi_p, \Pi_q]$.

In the simplicial prism $[\Sigma_p, \Sigma_q, \ldots]$, the edges that meet at any vertex fall into sets of p, q, etc., such that the first set belong to Σ_p, the second set to Σ_q, and so on. Two edges occurring in any one set are sides of a triangle belonging to the corresponding simplex. Two edges occurring in separate sets are sides of a rectangle.

We can now represent the $p + q + \ldots$ covertical edges as vertices of a topological simplex, putting the mark "α" against those edges of the simplex which correspond to triangles, and "β" against those which correspond to rectangles. The vertices of this *vertex diagram* fall into sets of $p, q, \ldots$, such that every pair occurring in the same set are joined by α-edges, while every pair occurring in different sets are joined by β-edges. (In the special case of $[\alpha_p, \alpha_q, \ldots]$,[11] the vertex diagram can be identified with the *vertex figure*, and then the α-edges and β-edges are α_1's and β_1's;[12] hence the notation.)

Since the form of the vertex diagram depends only on the numbers $p, q, \ldots$, we shall not obtain a different diagram by beginning with a different vertex. (The following illustration is the vertex diagram for $[\Sigma_3, \Sigma_2]$.)



LEMMA 2.5. *Every trihedral*[13] *polyhedron bounded by triangles* Σ_2 *and rectangles* $[\Sigma_1, \Sigma_1]$ *is either a tetrahedron* Σ_3 *or a triangular prism* $[\Sigma_2, \Sigma_1]$ *or a rectangular solid* $[\Sigma_1, \Sigma_1, \Sigma_1]$.

If a trihedral polyhedron is bounded by f_3 triangles and f_4 quadrangles, it must have $\frac{1}{2}(3f_3 + 4f_4)$ edges and $\frac{1}{3}(3f_3 + 4f_4)$ vertices. Hence $\frac{1}{2}f_3$, $\frac{1}{3}f_4$ must be (non-negative) integers, and, by Euler's Theorem,

$$\tfrac{1}{2}f_3 + \tfrac{1}{3}f_4 = 2.$$

The solutions $2 + 0$, $1 + 1$, $0 + 2$ give the three possible kinds of polyhedron.

LEMMA 2.6. *If a simplex-polycorypha in more than three dimensions is bounded solely by simplicial prisms, then the types of prisms occurring at any vertex are the same as at any other vertex.*

If a simplex-polycorypha in m dimensions is bounded solely by simplicial prisms, we can construct a vertex diagram at any vertex, and we have to show that this will be the same for all vertices. Any edge AB of the simplex-polycorypha will be represented by a vertex B' in the vertex diagram at A and by a vertex A' in the vertex diagram at B. The edge AB belongs to $m - 1$ bounding prisms; in the vertex diagrams considered, these are represented by the $m - 1$ bounding simplexes that meet at B' or A' respectively. Hence the two vertex diagrams are simplexes, alike in $m - 1$ of their m bounding figures. If $m > 3$, this is sufficient to make them identical.

Since any vertex of the polycorypha can be reached from any other by a chain of edges, repeated application of this result proves the lemma.

LEMMA 2.7. *In three or more dimensions, every simplex-polycorypha bounded solely by simplicial prisms is itself a simplicial prism.*

The case of three dimensions is covered by Lemma 2.5, so we may suppose the number of dimensions to exceed three. By Lemma 2.6, the polycorypha is characterized by a vertex diagram, *viz.* a simplex with α-edges and β-edges. Let P be any vertex of this vertex diagram. The remaining vertices form a simplex which, being the vertex diagram of one of the bounding prisms of the polycorypha, has sets of vertices joined among themselves by α-edges and joined to one another by β-edges.

We have to prove that the vertices of the whole simplex fall into sets in the same manner. This is certainly true if all the edges through P are of type β. On the other hand, if P is joined to one of the other vertices by an α-edge, it must be joined by α-edges to all the other vertices of the same set, since otherwise there would be a triangle of sides α, α, β, which is impossible by Lemma 2.5. For the same reason, P cannot be joined by α-edges to vertices of two distinct sets. Thus the vertices of the whole simplex are distributed in the desired manner, the vertex P either forming a new set by itself or becoming attached to one of the old sets.

LEMMA 2.8. *Every simplex-polycorypha free from obtuse dihedral angles is a simplicial prism.*

By Lemma 2.1, this is true in two dimensions. Let us assume it true in $m - 1$ dimensions, and use induction. By Lemma 1.3 and the inductive assumption,

[13] *I.e.*, a polyhedron with trihedral vertices.

the m-dimensional polycorypha is bounded by simplicial prisms. Hence, by Lemma 2.7, it is itself a simplicial prism.

Theorem 2 follows from Lemmas 2.2 and 2.8.

THEOREM 3.[14] *If all the dihedral angles of an m-dimensional polytope are less than or equal to* θ $(0 < \theta < \pi)$, *then the number of bounding figures is less than some number depending only on m and* θ.

Take first the case when $m = 3$. In the spherical image by parallel normals ("Gaussian" image for surfaces of continuous curvature), to each face of the polyhedron corresponds to a point on the unit sphere, to an edge where the dihedral angle is θ_1 corresponds a great-circular arc of length $\pi - \theta_1$, and to an n-hedral vertex corresponds a spherical n-gon. By hypothesis, the spherical distances between the points on the sphere are greater than or equal to $\pi - \theta$. Consequently, circles of spherical radius $\frac{1}{2}(\pi - \theta)$, described round the points, will not overlap. If N is the number of faces of the polyhedron, comparison of areas gives

$$2\pi(1 - \sin \tfrac{1}{2}\theta)N < 4\pi,$$

$$N < 2/(1 - \sin \tfrac{1}{2}\theta).$$

When $\theta = \frac{1}{2}\pi$, this gives the "best possible" result $N \leqq 6$. (The actual polyhedra are those considered in Lemma 2.5.)

The analogous result for general m is easily seen to be

$$N < 2 \int_0^{\frac{1}{2}\pi} \sin^{m-2} \varphi \, d\varphi \bigg/ \int_0^{\frac{1}{2}(\pi-\theta)} \sin^{m-2} \varphi \, d\varphi.$$

E.g., for $m = 4$, $N < 2\pi/(\pi - \theta - \sin \theta)$.

DEFINITIONS. A group of congruent transformations (of Euclidean space into itself) is said to be *discrete* if the totality of transforms of a point never has a limit point. *E.g.*, the group generated by a single rotation, through an angle incommensurable with π, is not discrete; but every finite group is *a fortiori* discrete. Bieberbach[15] has proved that a group is discrete if and only if it is free from infinitesimal operations.

Consider a point of such general position that it has distinct transforms under all the different operations of the group. Sufficiently small neighborhoods of these transforms can be taken so as to be equivalent under the group, and not overlap. By gradually increasing these neighborhoods, we eventually obtain a set of congruent[16] regions which together fill up the whole space. Any one of these regions is, by definition, a *fundamental region*; and the group can be generated by the operations which transform this particular region into its neighbors.

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[14] I am indebted to Prof. G. Pólya for this extension of Theorem 2. It will not be applied in the sequel, but is inserted for its intrinsic interest.

[15] Bieberbach **1**, 313–314.

[16] In the wide sense; *i.e.*, directly congruent, or enantiomorphous.

A point of general position on the boundary of the fundamental region is also on the boundary of a neighboring region, and may or may not be transformed into itself by the relevant generator. If not, a sufficiently small neighborhood of this point, in the neighboring region, will be equivalent to a certain portion of the original region. The fundamental region can then be modified by subtracting this portion and adding the aforesaid neighborhood. Thus the only case in which the fundamental region is uniquely determined is when every point of the boundary is transformed into itself. It is easily seen that this can only happen when all the generators are reflections.

A group of congruent transformations is said to be *reducible*[17] if it can be regarded as the direct product of groups of congruent transformations in two absolutely perpendicular complementary subspaces $\tilde{\omega}$, $\tilde{\omega}'$. Every operation is then of the form QQ' $(=Q'Q)$, where Q transforms $\tilde{\omega}$ into itself and Q' transforms $\tilde{\omega}'$ into itself. The groups in the subspaces are called *components*, and may themselves be reducible; but we can eventually analyse the original group into a number of *irreducible* components.

A group is said to be *trivially reducible* if it has identity for a component. In this case the complementary component operates in a subspace $\tilde{\omega}$ which is transformed into itself by every operation of the group, and we regard the fundamental region as lying in $\tilde{\omega}$ instead of in the whole space. In other words, we consider the section of the true fundamental region by $\tilde{\omega}$. Thus we regard a trivially reducible group and its non-trivial component as having the same fundamental region.

If a group leaves one point invariant, it transforms into itself the unit sphere around this point. In this case we regard the fundamental region as lying on the sphere. In other words, we replace the angular fundamental region by its spherical section. Thus we confuse a group in spherical space with the corresponding Euclidean group which leaves the center invariant. (A group which leaves more than one point invariant is always trivially reducible.)

Consider, for example, the group generated by reflections in two perpendicular planes in ordinary space. This is trivially reducible, its non-trivial component is generated by reflections in two perpendicular lines of a plane, and its fundamental region is an arc of length $\frac{1}{2}\pi$.

We use the word *prime* for a space of one fewer than the current number of dimensions, and *secundum* for the next lower space (*i.e.* the intersection of two primes).

The Fricke-Klein construction for a fundamental region.[18] Given a point P which is not invariant under any operation, and its transforms P_1, P_2, ... , we can construct a fundamental region by drawing the perpendicularly bisecting

[17] More strictly, "completely reducible." When the group is finite, our omission of the word "completely" is justified by Burnside **1**.

[18] Fricke-Klein **1**, 108, 216.

primes of PP_1, PP_2, ... A finite[19] number of these primes cut off a fundamental region around P, and the rest of them are irrelevant.

ABBREVIATION. For "group generated by reflections" we shall write "g.g.r.".

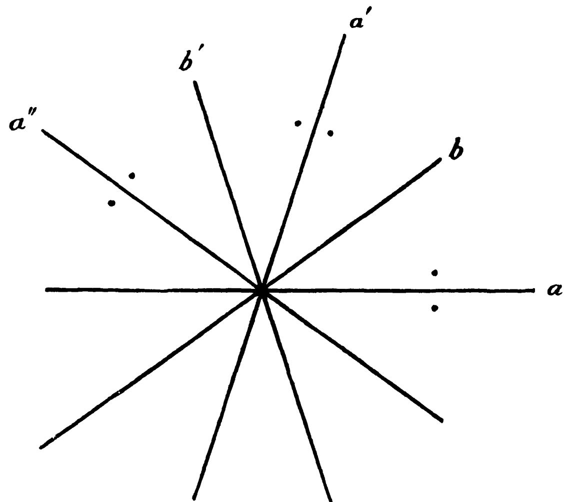
THEOREM 4. *The general finite g.g.r. has for fundamental region a spherical simplex, whose vertices fall into sets which belong to the fundamental regions of the irreducible components, vertices in different sets being joined by edges of length* $\frac{1}{2}\pi$.

LEMMA 4.1.[20] *Every finite group of congruent transformations leaves at least one point invariant.*

Consider the set of transforms of a point of general position. Every operation of the group permutes these points among themselves, and leaves their centroid invariant.

LEMMA 4.2. *If a discrete group is generated by reflections in certain primes, then these primes and their transforms are so distributed that, whenever n of the primes meet in a secundum, they are inclined to one another at angle π/n in cyclic succession.*

Let a, b be any two of the primes that are not parallel. Then the group must contain reflections in new primes a', b', a'', b'', ... which are the reflected images of a, b, a', b', ... in b, a', b', a'', ... respectively. These are inclined to a at all multiples of the angle $(a\,b)$.



If this angle were incommensurable with π, the sequence of primes would be infinite, a point of general position would have an infinite number of transforms lying on a finite circle, and the group would not be discrete. Hence we can write

$$(a\,b) = d\,\pi/n,$$

[19] Cartan **1**.

[20] Bieberbach **1**, 327 (XI). Coxeter **2**, 182 (§20.2) is an immediate corollary of this lemma.

where n and d are co-prime integers. If integers ν and δ are chosen so that $\nu d - \delta n = 1$, we shall have

$$\nu (a b) = \delta \pi + \pi/n.$$

Thus there is a prime inclined to a at angle π/n, and the primes can be re-ordered so as to make $d = 1$.

LEMMA 4.3. *If a finite group is reducible, there exist two absolutely perpendicular complementary subspaces $\tilde{\omega}$, $\tilde{\omega}'$, such that every operation is of the form QQ', where Q leaves invariant every point of $\tilde{\omega}'$ and Q' leaves invariant every point of $\tilde{\omega}$.*

In the notation of matrices,

$$\begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & A' \end{pmatrix}.$$

LEMMA 4.4. *If a reducible finite group is generated by reflections, its components are likewise generated by reflections.*

Consider any reflection which belongs to a reducible finite group. By Lemma 4.3, it transforms $\tilde{\omega}$ and $\tilde{\omega}'$ into themselves. Therefore $\tilde{\omega}$ (or $\tilde{\omega}'$) must lie in the reflecting prime, while $\tilde{\omega}'$ (or $\tilde{\omega}$) is perpendicular to it. Hence Q (or Q') is identity, while Q' (or Q) is a reflection.

LEMMA 4.5. *If a finite g.g.r. is not trivially reducible, it leaves invariant just one point of the Euclidean space, and its fundamental region is one of the parts into which the unit sphere around this point is divided by the reflecting primes and their transforms.*

A finite g.g.r., operating in Euclidean space, leaves at least one point invariant, by Lemma 4.1. If it leaves more than one point invariant, it must leave all the points of a subspace of one or more dimensions; this makes it trivially reducible. The group can be regarded as operating in a spherical space, and its fundamental region is then given by the Fricke-Klein construction.

LEMMA 4.6. *If a finite g.g.r. is not trivially reducible, it is generated by reflections in the bounding primes of its fundamental region.*

These are the operations which transform the fundamental region into the neighboring regions.

LEMMA 4.7. *The fundamental region of a finite g.g.r. is a spherical simplex.*

If the fundamental region has a pair of antipodal vertices, all its bounding primes must pass through the join of these vertices, since any other prime would divide the region into two parts. There are thus two invariant points, the group is trivially reducible, and we construct the fundamental region in a subspace.

If the fundamental region has no vertices (*e.g.* a hemisphere), suppose it has a k-dimensional element but no $(k - 1)$-dimensional element. Then all the bounding primes must pass through this k-dimensional element, since any other prime would give a $(k - 1)$-dimensional element by intersection.

Thus the fundamental region, which we have agreed to derive from the non-trivial component, has vertices, but no antipodal vertices. This makes it a polytope. By Lemma 4.2, all the dihedral angles are submultiples of π; therefore none of them can be obtuse. By Theorem 1, the polytope is a simplex.

The rest of Theorem 4 follows by means of Lemma 4.4. The sets of vertices lie in absolutely perpendicular spaces.[21]

THEOREM 5. *If an infinite discrete g.g.r. has a finite fundamental region, this fundamental region is a simplicial prism whose constituents are the fundamental regions of the irreducible components.*

LEMMA 5.1. *If a reducible discrete group is generated by reflections, its components are likewise generated by reflections.*

We proceed as in Lemma 4.4, bearing in mind the fact that the subspaces $\tilde{\omega}$, $\tilde{\omega}'$ may be at infinity.

LEMMA 5.2. *If an infinite discrete g.g.r. is not trivially reducible, its fundamental region is one of the parts into which space is divided by the reflecting primes and their transforms. It is generated by reflections in the bounding primes of the fundamental region.*

Cf. Lemmas 4.5, 4.6.

The fundamental region of an infinite discrete g.g.r. may be finite or infinite. If finite, Lemma 4.2 shows that it is a Euclidean polytope whose dihedral angles are submultiples of π. Hence, by Theorem 2, it is a simplicial prism.

The bounding figures of the simplicial prism

$$[\Sigma_{m_1}, \Sigma_{m_2}, \Sigma_{m_3}, \ldots]$$

fall into sets of the form

$$[\Sigma_{m_1-1}^{(i)}, \Sigma_{m_2}, \Sigma_{m_3}, \cdots], \qquad [\Sigma_{m_1}, \Sigma_{m_2-1}^{(j)}, \Sigma_{m_3}, \cdots], \text{ etc.}$$

Two bounding figures belonging to different sets lie in perpendicular primes, so that the reflections in them are permutable. The group generated by reflections in the bounding primes of the prism is thus the direct product of groups generated by reflections in the bounding primes of Σ_{m_1}, Σ_{m_2}, etc. In this manner we analyse the group into its irreducible components.

COROLLARY. *All the components are then infinite.*

THEOREM 6. *Every irreducible discrete g.g.r. has a simplicial fundamental region; and the general discrete g.g.r. is simply isomorphic with a direct product of such groups.*

If the group is finite, the fundamental region is a simplex (whether the group is irreducible or not). If the group is infinite while the fundamental region is finite, the fundamental region is a simplicial prism; but the group is reducible if this prism has more than one constituent. Bieberbach[22] has proved that groups

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[21] Cf. Coxeter **2,** 146 (§16.4). The fundamental region of $[m] \times [n]$ is described in Goursat **1,** 82.

[22] Bieberbach **1,** 327.

with infinite fundamental regions are always reducible. By Lemma 5.1, the irreducible components are generated by reflections.

The words "*simply isomorphic with*" are inserted in order to admit trivially reducible groups.

THEOREM 7. *If a discrete g.g.r. has both finite and infinite components, its fundamental region can be regarded as a generalized pyramid which joins a simplicial prism to a simplex situated at infinity in a space absolutely perpendicular to the space of the prism.*

Such a group is the direct product of a finite group whose fundamental region is a spherical simplex, and an infinite group whose fundamental region is a simplicial prism. Let us imagine the spherical simplex to be drawn (with its proper angles) on a sphere of infinite radius, lying in a Euclidean $(q + 1)$-space absolutely perpendicular to the p-space of the prism. Consider the infinite $(p + q + 1)$-dimensional pyramidal region obtained by joining every point of the prism to every point of the simplex at infinity. This region has bounding primes of two types:

(1) joining a bounding $(p - 1)$-space of the prism to the q-space (at infinity) of the simplex;

(2) joining a bounding $(q - 1)$-space of the simplex to the p-space of the prism. Since the p-space is absolutely perpendicular to the q-space at infinity, every prime of the first type is perpendicular to every prime of the second. Reflections in the primes of the first type generate the infinite component of our group, while reflections in primes of the second type generate the finite component. Hence reflections in all the primes must generate the direct product, as required.

For example, the fundamental region of

$$[k] \times [\infty]$$

is an infinite "wedge" (in ordinary space), bounded by four planes: two parallel; and two, perpendicular to these, mutually inclined at π/k.

THEOREM 8. *Every discrete g.g.r. has an abstract definition of the form*

$$R_i^2 = (R_i R_j)^{k_{ij}} = 1. \tag{8.1}$$

Since the direct product of several groups having such abstract definitions is another group of the same kind, it will be sufficient, by Theorem 6, to consider the cases when the fundamental region is a spherical or Euclidean simplex. Let R_i denote the reflection in the i^{th} bounding prime of this simplex, and π/k_{ij} the angle between the i^{th} and j^{th} bounding primes. Then the relations (8.1) are certainly satisfied. Clearly, also, the R's suffice to generate the whole group. It remains to be proved that every relation satisfied by the R's is a consequence of (8.1).

Let[23]

$$R_a R_b R_c \dots R_z = 1 \tag{8.2}$$

[23] This proof is the "obvious extension of an argument used by Burnside" to which reference was made in Coxeter **2**, 146.

be any relation satisfied by the R's. Consider a point, moving continuously in the spherical or Euclidean space in which the given group operates, beginning within a particular fundamental region, proceeding through the a-face into a neighboring region, then through the b-face of this region into a third (which may be the first over again), ... and finally through the z-face of some region into another region (which we shall identify with the first).

The first region is transformed into the second by the operation R_a, into the third by

$$R_a\, R_b\, R_a^{-1} . R_a = R_a\, R_b,$$

into the fourth by

$$(R_a\, R_b) R_c (R_a\, R_b)^{-1} \cdot R_a\, R_b = R_a\, R_b\, R_c,$$

... and into the last by $R_a\, R_b\, R_c \ldots R_z$. By hypothesis, this operation is identity. Hence the last region is the same as the first, and we may regard the moving point as describing a *closed* path.

We can immediately reduce the general sequence

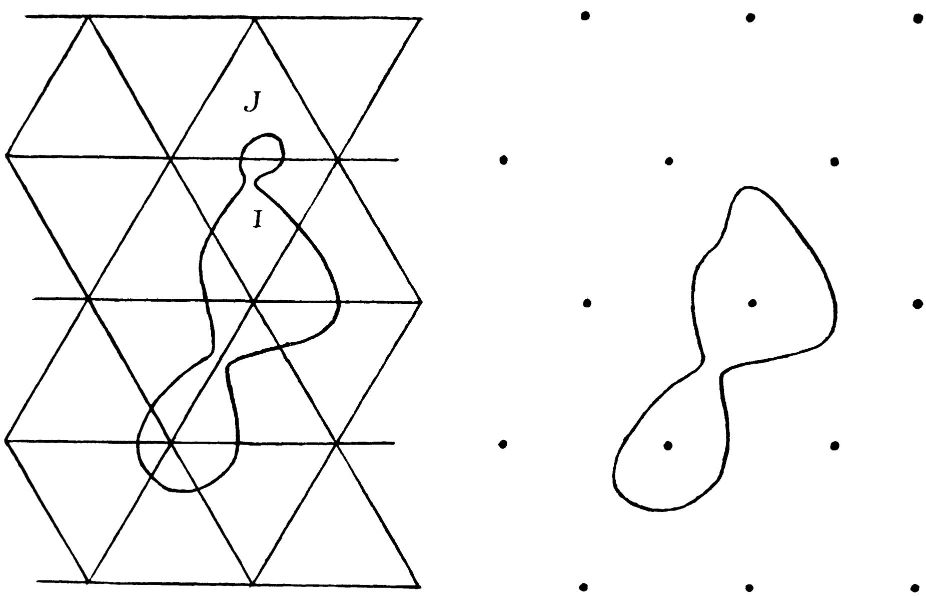
$$R_a, R_b, \ldots, R_z$$

to one in which no two consecutive R's are the same. For, since

$$R_i^2 = 1,$$

$$R_a \ldots R_h\, R_i\, R_j\, R_k \ldots R_z = R_a \ldots R_h\, R_k \ldots R_z \qquad \text{if} \qquad i = j.$$

In terms of the path, the moving point passes through the i-face of some region I into J (say), and then back through the same interface into I. This path can be decomposed into one which misses J altogether (unless it enters on a different occasion) along with a loop passing twice through the interface. The loop itself is a path which corresponds, in the manner described, to the relation $R_i^2 = 1$.



We shall now reduce the net of fundamental regions to a "skeleton," by discarding the primes but retaining the secunda. The possible paths, of the kind considered, constitute the *Poincaré group* of the space residual to these secunda. Since the whole space is simply connected, any such path can be decomposed into loops around single secunda. This decomposition of the path corresponds to a factorization of $R_a R_b \dots R_z$. By reinserting the primes, we see that each factor is of the form

$$(R_p \dots R_q)\,(R_i R_j)^{k_{ij}}\,(R_p \dots R_q)^{-1}.$$

Hence, finally, (8.2) is an algebraic consequence of

$$(R_i R_j)^{k_{ij}} = 1.$$

THEOREM 9. *The following list*[24] *comprises all irreducible discrete groups generated by reflections.*

Finite groups: $[3^n]\ (n \geqq 0)$, $[3^n, 4]\ (n \geqq 0)$, $\begin{bmatrix}3^n\\3\\3\end{bmatrix}(n \geqq 1)$,[25] $[k]\ (k \geqq 5)$,

$[3,5]$, $[3,4,3]$, $[3,3,5]$, $\begin{bmatrix}3,3\\3,3\\3\end{bmatrix}$, $\begin{bmatrix}3,3,3\\3,3\\3\end{bmatrix}$, $\begin{bmatrix}3,3,3,3\\3,3\\3\end{bmatrix}$.

Infinite groups: $\boxed{3^n}\ (n \geqq 3)$, $[4,3^n,4]\ (n \geqq 0)$, $\begin{bmatrix}3^n,4\\3\\3\end{bmatrix}(n \geqq 0)$,

$\begin{bmatrix}3^n,{}^{3}_{3}\\3\\3\end{bmatrix}(n \geqq 0)$, $[\infty]$, $[3,6]$, $[3,3,4,3]$, $\begin{bmatrix}3,3\\3,3\\3,3\end{bmatrix}$, $\begin{bmatrix}3,3,3\\3,3,3\\3\end{bmatrix}$, $\begin{bmatrix}3,3,3,3,3\\3,3\\3\end{bmatrix}$.

LEMMA 9.1.[26] *If all diagonal minors of a determinant are positive, while all non-diagonal elements are negative or zero, then all algebraic first minors are positive or zero.*

Let A denote the m-rowed determinant $|\,a^i_j\,|$, A^i_j the minor derived by omitting the i^{th} row and j^{th} column, $A^{h\,i}_{j\,k}$ that derived by omitting two rows and columns, and so on. We are given

$$A^j_j > 0, \quad A^{jk}_{jk} > 0, \quad A^{jk\cdots}_{jk\cdots} > 0, \cdots, a^i_i > 0, \quad a^i_j \leqq 0 \quad (i \neq j),$$

and we have to prove that $(-)^{j-i} A^i_j \geqq 0$.

[24] For the notation, see Todd **1**, 214; Coxeter **2**, 151, 147, 162. The number of dimensions of the space in which the group operates (spherical space, in the finite case) is equal to the number of digits in the symbol, except in the case of $\boxed{3^n}$, when it is $n - 1$. For a more systematic notation, see p. 618, below.

For an independent investigation of the groups $[3^n]$, $[3^n, 4]$, $[k]$, $[3, 5]$, $[3, 4, 3]$, $[3, 3, 5]$, see Motzok **1**.

[25] To avoid repetition.

[26] This lemma, being of unnecessary generality, could be replaced by Stieltjes **1**.

Since we can transpose the j^{th} and m^{th} columns (provided we also transpose the j^{th} and m^{th} rows), it will be sufficient to prove that

$$(-)^{m-i} A_m^i \geqq 0 \qquad (i < m).$$

This is true when $m = 2$, since $A_2^1 = a_1^2 \leqq 0$. Using induction, we assume the lemma as applied to A_m^m, so that

$$(-)^{j-i} A_{jm}^{im} \geqq 0.$$

We now have

$$A_m^i = -\sum_{j=1}^{m-1} (-)^{j-m} a_j^m A_{mj}^{im},$$

whence

$$(-)^{m-i} A_m^i = \sum_{j=1}^{m-1} (-a_j^m)(-)^{j-i} A_{jm}^{im} \geqq 0.$$

LEMMA 9.2. *A determinant of the kind considered in Lemma* 9.1 *is definitely diminished by simultaneously increasing* $-a_j^i$ *and* $-a_i^j$ $(i \neq j)$.

Writing $A = f(a_2^1, a_1^2)$, we have

$$f(a_2^1 - \delta, a_1^2 - \epsilon) - f(a_2^1, a_1^2) = \delta A_2^1 + \epsilon A_1^2 - \delta\epsilon A_{12}^{12}.$$

If $\delta > 0$ and $\epsilon > 0$, this expression is definitely negative, since $A_2^1 \leqq 0$, $A_1^2 \leqq 0$ and $A_{12}^{12} > 0$.

The enumeration of our irreducible groups merely involves the enumeration of simplexes whose dihedral angles are submultiples of π, for which see Coxeter **2,** 136–144 (§§15.4–15.9). The only step left obscure[27] was the "Note" in the middle of §15.2. This can be derived from Lemma 9.2 by putting $a_i^i = 1$ and $a_j^i = -\cos(i\,j)$.

THEOREM 10. *In spherical or Euclidean space, the continued product of the reflections in the bounding primes of a simplex, taken in any order, is an operation which leaves no point invariant.*

The reader will have no difficulty in proving this.

DEFINITION. By the *generators* of a g.g.r., we shall mean the reflections in the bounding primes of its fundamental region.

THEOREM 11. *Of the continued products of the generators of a finite g.g.r., arranged in various orders, any two are conjugate.*

LEMMA 11.1. *If the vertices of an m-gon are numbered from* 1 *to m in any order, we can arrange the numbers in natural order by repeated application of the following operation: whenever a side of the m-gon has inconsecutive numbers at its ends, we are allowed to transpose those numbers.*

We prove this by showing that the numbers at the ends of a side can be transposed even when they are consecutive.

[27] I am indebted to Prof. Pólya for pointing this out.

Suppose first that the numbers in order are

$$1,\ 2,\ p_3,\ p_4,\ \dots,\ p_m,$$

so that every $p_i > 2$. Interchanging 1 with p_m, then with p_{m-1}, and so on, we obtain the new order

$$p_m,\ 2,\ 1,\ p_3,\ \dots,\ p_{m-1},$$

which is the same as the old except that 1 and 2 have been transposed. We may describe this process as "making 1 run round the polygon."

Using induction, let us assume that the numbers $l - 1$ and l can be transposed (when associated with consecutive vertices of the polygon). Then we can transpose l and $l + 1$ by making l run round the polygon. Hence, finally, we can transpose the numbers at the ends of any side whatever, and the lemma is reduced to the familiar generation of the symmetric group by transpositions.

LEMMA 11.2. *If m vertices of an $(m + 1)$-gon are numbered from 1 to m in any order, while the remaining vertex is marked l' $(1 \leqq l \leqq m)$, we can derive any rearrangement of the numbers by repeated application of the operation of Lemma* 11.1, *regarding l' as consecutive only to l.*

We can transpose l and l' by making l' run round the polygon; and then the presence of l' will not affect the argument used in proving Lemma 11.1.

By Theorem 9, every irreducible finite g.g.r. is of the form

$$[k_1, k_2, \dots, k_{m-1}] \qquad \text{or} \qquad \begin{bmatrix} 3^n \\ 3^p \\ 3 \end{bmatrix}.$$

In the former case[28] the abstract definition is

$$R_i^2 = (R_i\, R_{i+1})^{k_i} = (R_i\, R_j)^2 = 1 \qquad (i < j - 1).$$

Thus inconsecutive R's are permutable.

Consider any ordering of the m R's. By numbering the vertices of an m-gon in accordance with the suffixes, we obtain a representation, not only of the continued product corresponding to the chosen ordering, but also of the obvious conjugates of this product, derived by cyclic permutation of the factors. The operation described in Lemma 11.1 will not affect this set of m conjugate products. Hence every such product is conjugate to $R_1\, R_2 \dots R_m$.

$\begin{bmatrix} 3^n \\ 3^p \\ 3 \end{bmatrix}$ has an abstract definition conveniently represented by the diagram

$$\begin{array}{ccccccccccccc} N_{n-1} & \text{———} \cdots \text{———} & N_1 & \text{———} & N & \text{———} & O & \text{———} & P & \text{———} & P_1 & \text{———} \cdots \text{———} & P_{p-1} \\ & & & & & & | & & & & & & \\ & & & & & & Q & & & & & & \end{array}$$

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[28] Todd **1**, 224 (4).

of which the case $n = p = 2$ is explained in Coxeter **2,** 164.[29]

By associating the generators

$$N_{n-1}, \ldots, N_1, N, O, P, P_1, \ldots, P_{p-1}, Q$$

with the numbers

$$1, \ldots, l-2, l-1, l, l+1, l+2, \ldots, m, l',$$

we are able to apply Lemma 11.2, and to conclude that all continued products of these generators are conjugate.

Theorem 11 now follows, by Lemma 4.4.

These arguments can be extended to cover most of our *infinite* groups; but it is easily seen that the operations $R_1 R_2 R_3 R_4$ and $R_1 R_3 R_2 R_4$ of $\left|\underline{3^4}\right|$ are *not* conjugate.

THEOREM 12. *The continued product of the generators of an infinite g.g.r., arranged in any order, is of infinite period.*

This follows easily from Lemma 4.1, Theorem 10 and Theorem 6.

DEFINITION. R (without a suffix) will denote a particular continued product of the generators of an irreducible finite g.g.r., *viz.*

$$R = R_1 R_2 \ldots R_m \quad \text{for} \quad [k_1, k_2, \ldots, k_{m-1}]$$

and

$$R = N_{n-1} \ldots N_1 N O P P_1 \ldots P_{p-1} Q \quad \text{for} \quad \begin{bmatrix} 3^n \\ 3^p \\ 3 \end{bmatrix}.$$

(We now make a brief digression into the theory of regular polytopes.)

DEFINITION. The *Petrie polygon* of a regular polygon is the regular polygon itself. The Petrie polygon of a regular polyhedron is a skew polygon of which every two consecutive sides, but no three, belong to one face.[30] The Petrie polygon of a regular polytope in m dimensions is a skew polygon of which every $m-1$ consecutive sides, but no m consecutive sides, belong to the Petrie polygon of one bounding figure. Thus, if

$$\ldots A_0 A_1 \ldots A_{m-1} A_m \ldots$$

is a Petrie polygon of the regular polytope Π_m, then

$$A_0 A_1 \ldots A_{m-1} \text{ and } A_1 \ldots A_{m-1} A_m$$

occur in Petrie polygons of two adjacent Π_{m-1}'s.

Remark. Every finite regular polytope can be orthogonally projected on to a plane in such a way that one Petrie polygon appears as an ordinary regular poly-

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[29] All the generators are involutory. Pairs not directly linked are permutable. If X, Y are linked, $XYX = YXY$.

[30] Coxeter **3.**

gon whose interior is the projection of the rest of the polytope. The square aspect of the regular tetrahedron, the hexagonal aspects of the octahedron and cube, and the decagonal aspects of the icosahedron and dodecahedron, are familiar. In the projections of the four-dimensional polytopes, a second Petrie polygon (really congruent to the first, of course) appears as a star polygon,[31] *viz.*

$$\{\tfrac{5}{2}\} \text{ for } \{3, 3, 3\}, \qquad \{\tfrac{8}{3}\} \text{ for } \{3, 3, 4\} \text{ and } \{4, 3, 3\},^{32}$$

$$\{\tfrac{12}{5}\} \text{ for } \{3, 4, 3\},^{33} \qquad \{\tfrac{30}{11}\} \text{ for } \{3, 3, 5\}^{34} \text{ and } \{5, 3, 3\}.$$

THEOREM 13. *Of the cycles in which the operation R of $[k_1, k_2, \ldots, k_{m-1}]$ permutes the vertices of $\{k_1, k_2, \ldots, k_{m-1}\}$, one is the cycle of vertices of a Petrie polygon.*

Let X_0 be an alternative symbol for the vertex A_0, and let X_1 denote the mid-point of the edge A_0A_1, X_2 the center of the plane face $\ldots A_0A_1A_2$, X_3 the center of the solid whose Petrie polygon involves $A_0A_1A_2A_3, \ldots$ and X_m the center of the whole polytope.[35] We take R_i $(i \leqq m)$ to be the reflection in the prime determined by the m points that we obtain by omitting X_{i-1} from the set of X's.

When $m = 2$, R_1 and R_2 are the reflections in an in-radius X_1X_2 and a circum-radius A_0X_2 of the polygon $\{k\}$, X_1 being the mid-point of the side A_0A_1. Since the angle between these radii is π/k, the product $R \equiv R_1R_2$ is a rotation through $2\pi/k$, which effects a cyclic permutation of the vertices of $\{k\}$.

We thus have a basis for induction, and can assume that

$$r \equiv R_1R_2 \ldots R_{m-1}$$

effects a cyclic permutation of the vertices of the Petrie polygon

$$A_0A_1 \ldots A_{m-1}A_m' \ldots$$

of the Π_{m-1} whose center is X_{m-1}. In virtue of this inductive assumption,

$$A_{i+1} = rA_i \qquad (i < m - 1)$$

and

$$A_m' = rA_{m-1}.$$

Also, since R_m is the reflection in $A_0A_1 \ldots A_{m-2}X_m$,

$$A_i = R_mA_i \qquad (i < m - 1).$$

Hence

$$A_{i+1} = rR_mA_i$$

$$= RA_i \qquad (i < m - 1).$$

[31] Called $\{h'\}$ in Coxeter **3**.

[32] Van Oss **1**, Tafel I.

[33] *Ibid.*, Tafel II.

[34] *Ibid.*, Tafel VIII^a. The projections given in Tafeln VI^a, VII^a exhibit Petrie polygons of $\{5, 3, \tfrac{5}{2}\}$, $\{3, 5, \tfrac{5}{2}\}$.

[35] Coxeter **2**, 155.

Finally, since the reflection in the prime $A_1 \ldots A_{m-1}X_m$ is rR_mr^{-1},

$$\begin{aligned} A_m &= rR_mr^{-1}A'_m \\ &= rR_mA_{m-1} \\ &= RA_{m-1}. \end{aligned}$$

DEFINITION. S will denote the *central inversion*, *i.e.* that transformation in spherical space which replaces every point by its antipodes. S has the peculiarity of being permutable with every transformation in the spherical space.

THEOREM 14. *If the fundamental region of a finite g.g.r. has no internal symmetry, the central inversion is an operation of the group.*

Consider a particular fundamental region F. By producing its bounding primes, we see that one of the congruent regions is antipodal to F. If S' is that operation of the group which transforms F into this antipodal region, it is clear that the product $S'^{-1}S$ leaves F invariant. If F has no internal symmetry, this operation must be identity, and so $S' = S$.

COROLLARY. *If the fundamental region has internal symmetries, we may augment the g.g.r. by adjoining these symmetries. The augmented group will contain the central inversion* (*although the g.g.r. may or may not*).

DEFINITION. By Theorems 11 and 12, (in any discrete g.g.r.) the period of a continued product of the generators is independent of their order of arrangement. We shall call this period h. In particular, R is of period h.

THEOREM 15. *If any irreducible finite g.g.r. contains the central inversion S, then h is even and*

$$S = R^{\frac{1}{2}h}.$$

(By Theorem 11, this result would not be affected by rearranging the factors of R.)

We shall prove this theorem by examining, in each of the ten cases, the effect of the operation R on the simplest polytope whose group of symmetries includes the irreducible group under consideration. This method has the advantage that it enables us to find the actual values of h, and to see which of our groups do not contain the central inversion. (The results of this investigation are summarized in the Table on p. 618, below.)

(i) In the case of $[3^n]$, the polytope is the *regular simplex* $\{3^n\}$ or α_{n+1}. Since its vertices do not occur in opposite pairs, the group cannot contain the central inversion.

If $a_1, a_2, \ldots, a_{n+2}$ are the vertices, the generators are the transpositions

$$R_i = (a_i a_{i+1}),$$

and $R \equiv R_1R_2 \ldots R_{n+1}$ permutes the a's cyclicly.[36] Therefore

$$h = n + 2.$$

[36] Cf. Todd **1**, 229.

An $\{n+2\}$ with all pairs of vertices joined can be regarded as a projection (of the vertices and edges) of α_{n+1}. The $\{n+2\}$ itself is then the projection of a Petrie polygon.

(ii) In the case of $[3^n, 4]$, we consider the *cross polytope* $\{3^n, 4\}$ or β_{n+2}, whose vertices

$$a_1, a_2, \ldots, a_{n+2}, c_1, c_2, \ldots, c_{n+2}$$

occur in opposite pairs a_i, c_i. The generators can be taken to be reflections in the primes

$$x_1 = x_2, \quad x_2 = x_3, \ldots, \quad x_{n+1} = x_{n+2}, \quad x_{n+2} = 0,$$

viz.[37]

$$R_1 = (a_1a_2)(c_1c_2), \quad R_2 = (a_2a_3)(c_2c_3), \ldots, \quad R_{n+1} = (a_{n+1}a_{n+2})(c_{n+1}c_{n+2}),$$
$$R_{n+2} = (a_{n+2}c_{n+2}).$$

We now have

$$\begin{aligned} R = R_1R_2 \ldots R_{n+2} &= (a_1a_2 \ldots a_{n+2})(c_1c_2 \ldots c_{n+2}a_{n+2}) \\ &= (a_1a_2 \ldots a_{n+2}c_1c_2 \ldots c_{n+2}). \end{aligned}$$

Therefore
$$h = 2(n+2)$$
and
$$R^{\frac{1}{2}h} = (a_1c_1)(a_2c_2) \ldots (a_{n+2}c_{n+2}) = S.$$

A $\{2(n+2)\}$ with all pairs of vertices joined, except those that are diametrically opposite, can be regarded as a projection (of the vertices and edges) of β_{n+2}. The $\{2(n+2)\}$ itself is then the projection of a Petrie polygon.

(iii) In the case of $\begin{bmatrix} 3^n \\ 3 \\ 3 \end{bmatrix}$, we consider the cross polytope $\left\{3^n, \begin{matrix} 3 \\ 3 \end{matrix}\right\}$ or β_{n+3}. The generators, being reflections in the primes

$$x_1 = x_2, \quad x_2 = x_3, \ldots, \quad x_n = x_{n+1}, \quad x_{n+1} = x_{n+2}, \quad x_{n+2} = x_{n+3},$$
$$x_{n+2} + x_{n+3} = 0,$$

are

$$N_{n-1} = (a_1a_2)(c_1c_2), \quad N_{n-2} = (a_2a_3)(c_2c_3), \ldots, \quad N = (a_na_{n+1})(c_nc_{n+1}),$$
$$O = (a_{n+1}a_{n+2})(c_{n+1}c_{n+2}), \quad P = (a_{n+2}a_{n+3})(c_{n+2}c_{n+3}),$$
$$Q = (a_{n+2}c_{n+3})(c_{n+2}a_{n+3}).$$

Since $PQ = (a_{n+2}c_{n+2})(a_{n+3}c_{n+3})$, we have

$$R \equiv N_{n-1}N_{n-2} \ldots NOPQ = (a_1a_2 \ldots a_{n+2}c_1c_2 \ldots c_{n+2})(a_{n+3}c_{n+3}).$$

[37] Todd's R's are in the reverse order.

Therefore $h = 2(n+2)$ again.

If n is even, the central inversion does not belong to the group, since the polytope $\left\{3, \begin{matrix} 3^n \\ 3 \end{matrix}\right\}$ or $h\gamma_{n+3}$ then has no opposite vertices. But if n is odd,

$$R^{n+2} = (a_1c_1)(a_2c_2) \ldots (a_{n+3}c_{n+3}) = S.$$

(iv) In the case of $[k]$, $R = R_1R_2$ and $h = k$. If k is odd, the central inversion does not belong to the group, since $\{k\}$ then has no opposite vertices. But if k is even,

$$R^{\frac{1}{2}k}$$

is the rotation through π, which is S.

(v) In the decagonal projection of the icosahedron $\{3, 5\}$, the decagon clearly represents a Petrie polygon, and opposite vertices of the Petrie polygon are opposite vertices of $\{3, 5\}$. Therefore,[38] for $[3, 5]$,

$$h = 10 \quad \text{and} \quad R^5 = S.$$

(vi) In Van Oss's dodecagonal projection of $\{3, 4, 3\}$, the peripheral $\{12\}$ represents a Petrie polygon (since four consecutive vertices belong to the Petrie polygon of a bounding octahedron), and opposite vertices of the Petrie polygon are opposite vertices of $\{3, 4, 3\}$. Therefore,[39] for $[3, 4, 3]$,

$$h = 12 \quad \text{and} \quad R^6 = S.$$

(vii) In Van Oss's triacontagonal projection of $\{3, 3, 5\}$, the peripheral $\{30\}$ represents a Petrie polygon (since four consecutive vertices belong to a bounding tetrahedron), and opposite vertices of the Petrie polygon are opposite vertices of $\{3, 3, 5\}$. Therefore,[38] for $[3, 3, 5]$,

$$h = 30 \quad \text{and} \quad R^{15} = S.$$

(viii) In the case of $\begin{bmatrix} 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}$, we consider the vertices[40]

$$a_1, \cdots, a_6; \qquad b_1, \cdots, b_6; \qquad c_{12}, \cdots, c_{56}$$

of 2_{21}, and observe[41] that

$$R \equiv N_1NOPP_1Q = (654321)\,[456.123]$$

$$= (a_1a_6c_{34}c_{23}b_6b_5b_4b_3c_{16}c_{56}a_3a_2)\ (a_5c_{35}c_{24}c_{13}b_1c_{12}b_2c_{26}c_{15}c_{46}a_4c_{45})\ (c_{36}c_{25}c_{14}),$$

whence $$h = 12.$$

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[38] Todd **1**, 230.

[39] *Ibid.*, 227.

[40] Coxeter **1**, 388 (§9.4). Todd **2** provided the most satisfactory proof of the correspondence between these vertices and the lines on the general cubic surface (Schläfli **1**, Dickson **1**).

[41] Coxeter **2**, 165 (18.31).

Since 2_{21} has no opposite vertices, the central inversion is absent.

(ix) In the case of $\begin{bmatrix} 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}$, we consider the vertices[42]

$$c_{12}, \cdots, c_{78}; \qquad C_{12}, \cdots, C_{78}$$

of 3_{21}, and observe[43] that

$$R \equiv N_2N_1NOPP_1Q = (7654321)\,[4567.1238]$$

$$\begin{aligned} = \; & (C_{17}C_{67}c_{34}c_{23}C_{78}C_{68}C_{58}C_{48}C_{38}c_{17}c_{67}C_{34}C_{23}c_{78}c_{68}c_{58}c_{48}c_{38}) \\ & (C_{57}c_{35}c_{24}c_{13}C_{18}c_{12}C_{28}c_{27}c_{16}c_{57}C_{35}C_{24}C_{13}c_{18}C_{12}c_{28}C_{27}C_{16}) \\ & (c_{36}c_{25}c_{14}c_{37}c_{26}c_{15}c_{47}C_{45}c_{56}C_{36}C_{25}C_{14}C_{37}C_{26}C_{15}C_{47}c_{45}C_{56}) \\ & (c_{46}C_{46}), \end{aligned}$$

whence $$h = 18 \quad \text{and} \quad R^9 = (cC) = S.$$

(x) In the case of $\begin{bmatrix} 3, 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}$, we consider the vertices[44]

$$a_{123}, \cdots, a_{789}; \qquad b_{123}, \cdots, b_{789}; \qquad c_{12}, \cdots, c_{98}$$

of 4_{21}, and observe[45] that

$$R \equiv N_3N_2N_1NOPP_1Q = (87654321)\,V_{123}$$

$$\begin{aligned} = \; & (a_{178}a_{678}b_{349}b_{239}c_{98}c_{97}c_{96}c_{95}c_{94}c_{93}b_{189}b_{789}a_{345}a_{234}c_{83} \\ & \; b_{178}b_{678}a_{349}a_{239}c_{89}c_{79}c_{69}c_{59}c_{49}c_{39}a_{189}a_{789}b_{345}b_{234}c_{38}) \\ & (a_{578}b_{359}b_{249}b_{139}c_{91}b_{129}c_{92}b_{289}b_{179}b_{689}a_{346}a_{235}c_{84}c_{73}b_{168} \\ & \; b_{578}a_{359}a_{249}a_{139}c_{19}a_{129}c_{29}a_{289}a_{179}a_{689}b_{346}b_{235}c_{48}c_{37}a_{168}) \\ & (a_{369}a_{259}a_{149}a_{389}a_{279}a_{169}a_{589}b_{356}b_{245}b_{134}c_{31}c_{28}a_{278}a_{167}a_{568} \\ & \; b_{369}b_{259}b_{149}b_{389}b_{279}b_{169}b_{589}a_{356}a_{245}a_{134}c_{13}c_{82}b_{278}b_{167}b_{568}) \\ & (a_{379}a_{269}a_{159}a_{489}b_{456}a_{679}b_{347}b_{236}c_{58}c_{47}c_{36}a_{158}a_{478}b_{459}a_{567} \\ & \; b_{379}b_{269}b_{159}b_{489}a_{456}b_{679}a_{347}a_{236}c_{85}c_{74}c_{63}b_{158}b_{478}a_{459}b_{567}) \\ & (a_{148}a_{378}a_{267}a_{156}a_{458}b_{569}a_{367}a_{256}a_{145}a_{348}a_{237}c_{86}c_{75}c_{64}c_{53} \\ & \; b_{148}b_{378}b_{267}b_{156}b_{458}a_{569}b_{367}b_{256}b_{145}b_{348}b_{237}c_{68}c_{57}c_{46}c_{35}) \\ & (a_{468}b_{469}a_{467}b_{479}a_{457}b_{579}a_{357}a_{246}a_{135}c_{14}a_{123}b_{128}c_{72}b_{268}b_{157} \\ & \; b_{468}a_{469}b_{467}a_{479}b_{457}a_{579}b_{357}b_{246}b_{135}c_{41}b_{123}a_{128}c_{27}a_{268}a_{157}) \\ & (a_{368}a_{257}a_{146}a_{358}a_{247}a_{136}c_{15}a_{124}c_{23}c_{12}c_{81}b_{127}c_{62}b_{258}b_{147} \\ & \; b_{368}b_{257}b_{146}b_{358}b_{247}b_{136}c_{51}b_{124}c_{32}c_{21}c_{18}a_{127}c_{26}a_{258}a_{147}) \\ & (c_{17}a_{126}c_{25}a_{248}a_{137}c_{16}a_{125}c_{24}a_{238}c_{87}c_{76}c_{65}c_{54}c_{43}b_{138} \\ & \; c_{71}b_{126}c_{52}b_{248}b_{137}c_{61}b_{125}c_{42}b_{238}c_{78}c_{67}c_{56}c_{45}c_{34}a_{138}), \end{aligned}$$

whence $$h = 30 \quad \text{and} \quad R^{15} = (ab)(c_{ij}c_{ji}) = S.$$

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[42] Coxeter **1**, 387.
[43] Coxeter **2**, 168.
[44] *Ibid.*, 170.
[45] *Ibid.*, 172.

THEOREM 16. *If an irreducible discrete g.g.r. operates in Euclidean space of m dimensions (or spherical space of $m - 1$ dimensions), the total number of reflections is $\frac{1}{2}mh$.*

LEMMA 16.1. *In an irreducible discrete g.g.r. whose symbol contains no even digits, all reflections are conjugate.*

Since every reflection in the group is conjugate to some generator, it will be sufficient to prove that the generators are conjugate. If U, Z are any two generators, our assumption implies that we can find a sequence of generators U, V, W, $\cdots$, Y, Z, such that the products UV, VW, $\cdots$, YZ are all of odd period. Suppose

$$(UV)^{2j+1} = 1.$$

Then
$$(UV)^j U (VU)^j = V,$$

whence U and V are conjugate. Proceeding similarly throughout the sequence, we conclude that U and Z are conjugate.

For infinite groups, Theorem 16 is obvious, since h is infinite. For finite groups, we consider the actual cases (as in Theorem 15).

(i)[46] $[3^n]$. $\quad m = n + 1, \quad h = m + 1.$

The transposition $(a_i a_j)$ is the reflection that reverses the edge $a_i a_j$ of α_m, leaving the opposite α_{m-2} invariant. By Lemma 16.1, all the reflections are of this kind. Hence their number is $\frac{1}{2}m(m + 1)$.

(ii)[47] $[3^n, 4]$. $\quad m = n + 2, \quad h = 2m.$

Since
$$(R_1R_2)^3 = (R_2R_3)^3 = \cdots = (R_{m-2}R_{m-1})^3 = 1,$$

every reflection is conjugate either to R_1 or to R_m. R_1 reverses each of the opposite edges a_1a_2, c_1c_2 of β_m; R_m transposes the opposite vertices a_m, c_m. Since β_m has $2m(m - 1)$ edges and $2m$ vertices, the total number of reflections is

$$m(m - 1) + m = m^2.$$

(iii)[48] $\begin{bmatrix} 3^n \\ 3 \\ 3 \end{bmatrix}$. $\quad m = n + 3, \quad h = 2(m - 1).$

By Lemma 16.1, every reflection is conjugate to that which reverses the opposite edges a_1a_2, c_1c_2 of β_m. Hence the number of reflections is $m(m - 1)$.

---

[46] For the case when $n = 3$, cf. Goursat **1**, 90.
[47] For the case when $n = 2$, *ibid.*, 85.
[48] For the case when $n = 1$, *ibid.*, 82.

(These same reflections occur in $[3^{m-2}, 4]$, whose remaining m reflections generate $[\ ]^m$.)[49]

(iv) $[k]$. $m = 2$, $h = k$.

If k is odd, every reflection reverses one side of $\{k\}$ and leaves the opposite vertex invariant; so the number is k.

If k is even, R_1 reverses each of two opposite sides of $\{k\}$, while R_2 leaves two opposite vertices invariant; so the number is $\frac{1}{2}k + \frac{1}{2}k$.[50] (Although R_1 and R_2 are not conjugate in $[k]$, these same operations are conjugate in the larger group $[2k]$.)

(v) $[3, 5]$. $m = 3$, $h = 10$.

Every plane of symmetry of $\{3, 5\}$ passes through one pair of opposite edges, and bisects another pair; so the number of planes is 15.

(vi)[51] $[3, 4, 3]$. $m = 4$, $h = 12$.

Since

$$(R_1R_2)^3 = (R_3R_4)^3 = 1,$$

there are only two sets of mutually conjugate reflections. But there is a reflection that transposes two opposite vertices of $\{3, 4, 3\}$, and another that transposes two opposite octahedra. Hence the total number is $12 + 12$.[52] (*All* these reflections are conjugate in the group of symmetries of $t_{1,2}\{3, 4, 3\}$.[53])

(vii)[54] $[3, 3, 5]$. $m = 4$, $h = 30$.

There is a reflection that transposes two opposite vertices of $\{3, 3, 5\}$. By Lemma 16.1, every reflection is of this kind. Hence the number is 60.

(viii) $\begin{bmatrix} 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}$. $m = 6$, $h = 12$.

---

[49] $[\,]^m$ means $[\,] \times [\,] \times \cdots$ (to m terms), which might also be written $[2^{m-1}]$. (See *Preface* for the cases $m = 2, 3$.)

The relation between these groups is epitomized in the symbolic equation

$$[3^{m-2}, 4] = \begin{bmatrix} 3^{m-3} \\ 3 \\ 3 \end{bmatrix} + [\,]^m \ (m \geqq 3).$$

[50] $[k] = [\frac{1}{2}k] + [\frac{1}{2}k]$ (k even).

[51] Goursat **1**, 86.

[52] $[3, 4, 3] = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$.

[53] This is an example of the "augmented group" defined in the corollary to Theorem 14. It is derived from [3, 4, 3] by adjoining the operation considered in Robinson **1**.

[54] Goursat **1**, 88.

P_1 transposes the two opposite α_5's[55]

$$a_1b_1c_{23}c_{24}c_{25}c_{26}, \qquad a_2b_2c_{13}c_{14}c_{15}c_{16}$$

of 2_{21}. Since there are 72 α_5's, the number of reflections is 36. (These correspond, of course, to the double-sixes on the cubic surface.)

(ix) $$\begin{bmatrix} 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}. \qquad m = 7, \qquad h = 18.$$

P_1 transposes the two opposite β_6's

$$c_{13} \cdots c_{18}C_{23} \cdots C_{28}, \qquad C_{13} \cdots C_{18}c_{23} \cdots c_{28}$$

of 3_{21}. Since there are 126 β_6's, the number of reflections is 63. (These correspond[56] to the 63 *period characteristics* of genus 3.)

(x) $$\begin{bmatrix} 3, 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}. \qquad m = 8, \qquad h = 30.$$

P_1 transposes the two opposite vertices c_{12}, c_{21} of 4_{21}. Since there are 240 vertices, the number of reflections is 120.[57]

THEOREM 17. *Every irreducible finite g.g.r. can be generated by two operations.*

For all finite groups of the form $[k_1, k_2, \cdots, k_{m-1}]$, this was proved by Todd,[58] the two generating operations being, except in one case, R_1 and R.

In the case of $[3^n]$, putting $R = R_1R_2 \cdots R_{n+1}$, we have

$$R_{i+1} = R^iR_1R^{-i} \qquad (i \leqq n).$$

In the case of $[3^n, k]$, putting $R = R_1R_2 \cdots R_{n+2}$, we have

$$R_{i+1} = R^iR_1R^{-i} \qquad (i \leqq n),$$
$$R_{n+2} = R^nR_1(R^{-1}R_1)^nR.$$

This result includes $[3^n, 4]$, $[k]$, $[3, 5]$, $[3, 3, 5]$, and incidentally also the infinite groups $[\infty]$, $[3, 6]$. Todd[59] generates the exceptional group $[3, 4, 3]$, by means of $R_1R_2R_4$ and $R_1R_3R_4$.

In the case of $\begin{bmatrix} 3^n \\ 3 \\ 3 \end{bmatrix}$, we define

$$U \equiv N_{n-1}N_{n-2} \cdots NO, \qquad V \equiv POQ,$$

[55] Coxeter **2,** 165.
[56] Du Val **1,** 58.
[57] *Ibid.*, 47, 49.
[58] Todd **1,** 226.
[59] *Ibid.*, 228.

and observe that

$$\begin{aligned} U^{-1}OU &= ONONO \qquad \text{(since } O \text{ is permutable with } N_1, \cdots, N_{n-1}) \\ &= N, \\ U^{-1}N_iU &= ON \cdots N_{i-1}N_iN_{i+1}N_iN_{i+1}N_iN_{i-1} \cdots NO \\ &= ON \cdots N_{i-1}N_{i+1}N_{i-1} \cdots NO \\ &= N_{i+1}, \end{aligned}$$

$$V^{-1}OV = QPQ = P, \qquad VOV^{-1} = PQP = Q,$$

and

$$U^{-1}VU = ONPOQNO = OPNONQO = OPONOQO = POPNQOQ$$
$$= POQNPOQ = VNV.$$

Since $N = N^{-1}$, we now have

$$\begin{aligned} N &= VU^{-1}V^{-1}UV, \\ N_i &= U^{-i}NU^i, \\ O &= UNU^{-1}, \\ P &= V^{-1}UNU^{-1}V, \\ Q &= VUNU^{-1}V^{-1}. \end{aligned}$$

When n is even, we can alternatively generate the group by means of O and

$$R' \equiv N_{n-1}N_{n-2} \cdots NPOQ.^{60}$$

In fact,

$$\begin{aligned} N &= OR'^{n+1}OR'^{-n+1}O, \\ N_i &= R'^{-i}NR'^i, \\ P &= R'^{-(n+1)}OR'^{n+1}, \\ Q &= R'OR'^{-1} \qquad (n \text{ even}^{61}). \end{aligned}$$

The remaining groups are the symmetry groups of the polytopes 2_{21}, 3_{21}, 4_{21}, which were considered in Coxeter **1**.[62] However, we shall reconsider them here, in order to show that they can all be generated by the special operations O, R.

---

[60] By Theorem 11, R' is conjugate to R (which has OP in place of PO). But the permutable operations P, Q could never be distinguished in terms of O and R.

[61] When n is odd, $R'^{n+2} = S$, so that

$$R'^{n+1}OR'^{-(n+1)} = R'^{-1}OR' = QPNONPQ$$

and

$$R'^{-(n+1)}OR'^{n+1} = R'OR'^{-1} = Q.$$

[62] 408 (11.63), 406 (11.52), 409 (11.71).

We can express the generators of $\begin{bmatrix} 3^n \\ 3^2 \\ 3 \end{bmatrix}$ in terms of O and

$$R \equiv N_{n-1} N_{n-2} \cdots NOPP_1 Q,$$

by observing that

$$\begin{aligned}
R^{-1} OR &= QP_1 PONONOPP_1 Q = QP_1 PNPP_1 Q \\
&= N, \\
R^{-1} N_i R &= QP_1 PONN_1 \cdots N_{i-1} N_i N_{i+1} N_i N_{i+1} N_i N_{i-1} \cdots N_1 NOPP_1 Q \\
&= QP_1 PONN_1 \cdots N_{i-1} N_{i+1} N_{i-1} \cdots N_1 NOPP_1 Q \\
&= N_{i+1}, \\
R^{-2} N_{n-1} R^2 &= QP_1 PONN_1 \cdots N_{n-2} N_{n-1} QP_1 PONN_1 \cdots N_{n-2} \\
&\qquad N_{n-1} N_{n-2} \cdots N_1 NOPP_1 QN_{n-1} N_{n-2} \cdots N_1 NOPP_1 Q \\
&= QP_1 PONN_1 \cdots N_{n-2} QP_1 PONN_1 \cdots N_{n-3} N_{n-1} N_{n-2} \\
&\qquad N_{n-1} N_{n-2} N_{n-1} N_{n-3} \cdots N_1 NOPP_1 QN_{n-2} \cdots N_1 NOPP_1 Q \\
&= QP_1 PONN_1 \cdots N_{n-2} QP_1 PONN_1 \cdots N_{n-3} \\
&\qquad N_{n-2} N_{n-3} \cdots N_1 NOPP_1 QN_{n-2} \cdots N_1 NOPP_1 Q \\
&= \cdots \\
&= QP_1 PONQP_1 PONOPP_1 QNOPP_1 Q \\
&= P_1 PP_1 QOQPNONONPQOQP_1 PP_1 \\
&= PP_1 POQOPOPOQOPP_1 P = PP_1 POPOPP_1 P = POP \\
&= OPO, \\
RPR^{-1} &= N_{n-1} N_{n-2} \cdots NOPP_1 PP_1 PON \cdots N_{n-2} N_{n-1} \\
&= P_1, \\
R^{-1}PR &= QP_1 POPOPP_1 Q = QOQ \\
&= OQO,
\end{aligned}$$

whence

$$(17.1) \qquad \begin{cases} N_{i-1} = R^{-i} OR^i & (N = N_0), \\ P = OR^{-(n+2)} OR^{n+2} O, \\ P_1 = ROR^{-(n+2)} OR^{n+2} OR^{-1}, \\ Q = OR^{-1} OR^{-(n+2)} OR^{n+2} ORO. \end{cases}$$

These relations can be simplified considerably when $n = 2$, and slightly when $n = 3$.

For $\begin{bmatrix} 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}$ $(n = 2, R = N_1 NOPP_1 Q)$, we find

$$R^3 OR^{-3} = OQO,$$

so that

$$\begin{aligned} N_1 &= R^{-2} OR^2, \\ N &= R^{-1} OR, \\ P &= R^4 OR^{-4}, \\ P_1 &= R^5 OR^{-5}, \\ Q &= OR^3 OR^{-3} O. \end{aligned}$$

For $\begin{bmatrix} 3, 3, 3 \\ 3, 3, \\ 3, \end{bmatrix}$ $(n = 3, R = N_2 N_1 NOPP_1 Q)$,

$$R^9 = S,$$

so that

$$R^{-5} OR^5 = R^4 OR^{-4}$$

and

$$\begin{aligned} N_2 &= R^{-3} OR^3, \\ N_1 &= R^{-2} OR^2, \\ N &= R^{-1} OR, \\ P &= OR^4 OR^{-4} O, \\ P_1 &= ROR^4 OR^{-4} OR^{-1}, \\ Q &= OR^{-1} OR^4 OR^{-4} ORO. \end{aligned}$$

For $\begin{bmatrix} 3, 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}$, we cannot do better than put $n = 4$ in (17.1). Incidentally, by putting $n = 5$ we obtain a generation of the *infinite* group $\begin{bmatrix} 3, 3, 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}$ by two[63] operations. For $n > 5$ we have the (infinite) symmetry groups of Du Val's Minkowskian polytopes[64] n_{21}.

[63] Three were used in Coxeter **1**, 409.

[64] Du Val **1**, 71.

DEFINITION. If Γ is a group which contains the central inversion S,

$$\tfrac{1}{2}\Gamma$$

will denote the factor group Γ/Σ, where Σ is the subgroup of order 2 generated by S. From an abstract definition for Γ, we derive an abstract definition for $\frac{1}{2}\Gamma$ by inserting the new relation $S = 1$.

Examples.

(i) $\frac{1}{2}[2k] \sim [k]$, both having the abstract definition

$$R_1^2 = R_2^2 = (R_1 R_2)^k = 1.$$

(ii) $\frac{1}{2}[3, 4] \sim [3, 3]$.

$\frac{1}{2}[3, 4]$[65] has the abstract definition

$$R_1^2 = R_2^2 = R_3^2 = (R_1 R_2)^3 = (R_2 R_3)^4 = (R_3 R_1)^2 = (R_2 R_3 R_1)^3 = 1.$$

On putting $R_3 = R_3' R_1$, this becomes

$$R_1^2 = R_2^2 = R_3'^2 = (R_1 R_2)^3 = (R_2 R_3')^3 = (R_3' R_1)^2 = (R_2 R_3' R_1)^4 = 1,$$

which is the abstract definition for $[3, 3]$ (with the superfluous relation $R^4 = 1$).

(iii) In the notation of Coxeter **2**, 145–151, (since the central inversion is a negative operation when the number of Euclidean dimensions is odd)

$$\tfrac{1}{2}[3^n, 4] \sim [3^n, 4]' \qquad (n \text{ odd}),$$

$$\tfrac{1}{2}[3, 5] \sim [3, 5]' \qquad (\text{the icosahedral group}^{66}),$$

$$\tfrac{1}{2}\begin{bmatrix} 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix} \sim \begin{bmatrix} 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}'.$$

Du Val **1,** 58 shows that the last of these groups is the θ-characteristic group of genus 3. We now see that an abstract definition for it can be obtained by adjoining the relation

$$R^9 = 1$$

to the abstract definition[67] for $\begin{bmatrix} 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}$. This new definition is more elegant than (18.51).[68] In terms of the digits $1, 2, \cdots, 8$, whose pairs denote the twenty-eight odd θ-characteristics (or the twenty-eight bitangents of the plane quartic curve), Q (or Q_0) is simply a bifid substitution, while the other six generators are transpositions.[69] By 15 (ix) above, R permutes twenty-seven of the

[65] Fricke-Klein **1,** 71.

[66] *Ibid.*, 72.

[67] Coxeter **2,** 149 (16.74).

[68] *Ibid.*, 166.

[69] *Ibid.*, 168.

odd characteristics (or bitangents) in three cycles, namely

(17 67 34 23 78 68 58 48 38)

(57 35 24 13 18 12 28 27 16)

(36 25 14 37 26 15 47 45 56).

(iv) $\frac{1}{2}\begin{bmatrix} 3, 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}$ is the group considered in Du Val **1,** 49. The "one extra relation" that was required in Coxeter **2,** 174 (at the bottom of the page) is now seen to be

$$R^{15} = 1,$$

where $$R = N_3 N_2 N_1 NOPP_1 Q.$$

(v) Of Goursat's groups,[70]

XLII is $\frac{1}{2}\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$,

XLIV is $\frac{1}{2}([\,] \times [3, 4])$,

XLV is $\frac{1}{2}[3, 4, 3]$,

XLVII is $\frac{1}{2}[3, 3, 4]$,

XLIX is $\frac{1}{2}([\,] \times [3, 5])$,

L is $\frac{1}{2}[3, 3, 5]$;

But LI $\sim [3, 3, 3]$ itself, the central inversion being absent.

Remark. If Γ (containing S) is a g.g.r. in spherical space, then $\frac{1}{2}\Gamma$ is a g.g.r. in *elliptic* space (of the same number of dimensions).

THEOREM 18. *For every finite g.g.r. in Euclidean (or spherical) space, there is a simply isomorphic group in elliptic space. Every other discrete g.g.r. in elliptic space is of the form*

$$\tfrac{1}{2}(\Gamma^{(1)} \times \Gamma^{(2)} \times \cdots),$$

where all the Γ*'s*[71] *occur among*

$[3^n, 4]$, $\begin{bmatrix} 3^n \\ 3 \\ 3 \end{bmatrix}$ (n odd), $[k]$ (k *even*), $[3, 5]$, $[3, 4, 3]$, $[3, 3, 5]$,

$$\begin{bmatrix} 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 3, 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}.$$

[70] Goursat **1,** 78, 79.

[71] There may, of course, be only one.

LEMMA 18.1. *If* Γ, *a finite g.g.r., contains* S, *then*

$$\tfrac{1}{2}([\ ] \times \Gamma) \sim \Gamma.$$

LEMMA 18.2. *If a reducible group contains the central inversion, then each component must contain its own central inversion.*

(The proofs of these lemmas are omitted, as they involve no ingenuity.)

It is obvious that any group in elliptic space can be derived from a group in spherical space by identifying antipodal points. If the spherical group does not contain the central inversion, the elliptic group is simply isomorphic (but the fundamental region of the latter is one half of that of the former). If the spherical group is of the form [ ] $\times$ Γ and contains the central inversion, the derived elliptic group is simply isomorphic with Γ, by Lemma 18.1. The remaining case is when the spherical group contains the central inversion but no component [ ] (*i.e.*, no single reflection which is permutable with all the others). By Lemma 18.2, every component of the spherical group must then contain its own central inversion.

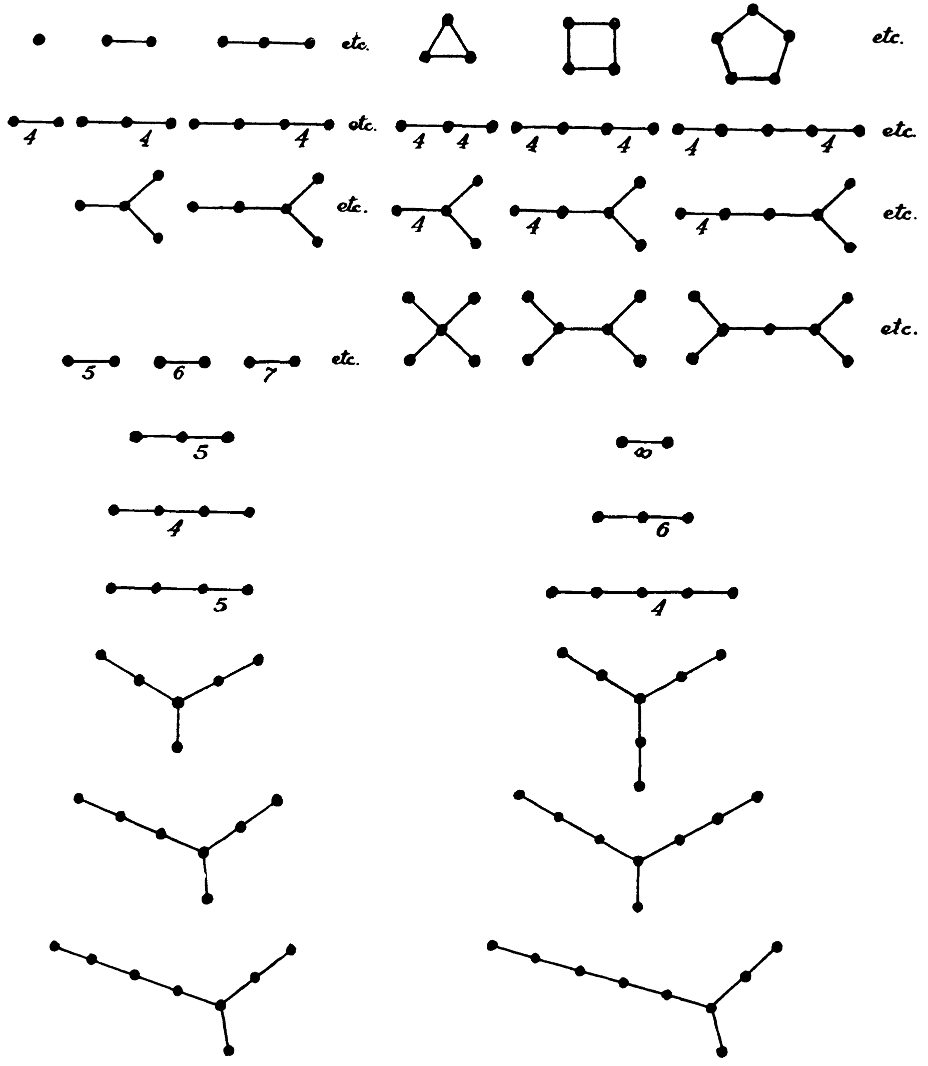
TABLE OF IRREDUCIBLE FINITE GROUPS GENERATED BY REFLECTIONS

Group	Order[72]	m	h	Central inversion?
$[3^n]$	$(n+2)!$	$n+1$	$n+2$	Only when $n = 0$
$[3^n, 4]$	$2^{n+2}(n+2)!$	$n+2$	$2(n+2)$	Yes
$\begin{bmatrix} 3^n \\ 3 \\ 3 \end{bmatrix}$	$2^{n+2}(n+3)!$	$n+3$	$2(n+2)$	Only when n is odd
$[k]$	$2k$	2	k	Only when k is even
$[3, 5]$	120	3	10	Yes
$[3, 4, 3]$	1152	4	12	Yes
$[3, 3, 5]$	14400	4	30	Yes
$\begin{bmatrix} 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}$	51840	6	12	No
$\begin{bmatrix} 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}$	2903040	7	18	Yes
$\begin{bmatrix} 3, 3, 3, 3 \\ 3, 3 \\ 3 \end{bmatrix}$	696729600	8	30	Yes

[72] Coxeter **2**, 159.

GRAPHICAL REPRESENTATION OF THE IRREDUCIBLE GROUPS GENERATED BY REFLECTIONS

(*Finite*) (*Infinite*)



Here each dot represents a *mirror*, i.e. the prime of a generating reflection. The links indicate the relative positions of the various mirrors. When two dots are joined by a link marked k, the corresponding mirrors are inclined at angle π/k. When a link is unmarked, we understand $k = 3$. When two dots are not (directly) linked, the mirrors are perpendicular.[73]

[73] *Cf.* Coxeter **4.**

By Theorem 9, the above graphs represent *all* irreducible discrete groups generated by reflections. By Theorem 6, the graph for any *reducible* discrete g.g.r. is obtained by juxtaposing several of these, repetitions being allowed.

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