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Bruhat Order of Coxeter Groups and Shellability

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1. INTRODUCTION

Let (W, S) be a Coxeter group and denote by W^J the Bruhat partially ordered set of minimal coset representatives modulo a parabolic subgroup W_J, $J \subseteq S$. We introduce in this paper a method of representing the maximal chains of intervals in W^J with strings of integers, which shows that intervals in W^J are lexicographically shellable. This technical information is then used to uncover some notable properties of Bruhat order of a combinatorial, topological and algebraic nature.

First of all, a number of results about the Möbius function of Bruhat order are derived. These include the formulas of D.-N. Verma [14] and V. Deodhar [5], which in this setting find a place among related results in a wider theoretical framework.

Secondly, it is shown that the simplicial complex of chains $w_0 > w_1 > \cdots > w_k$ in an open interval $(w, w')^J$ in W^J triangulates a sphere or a cell. The first case occurs exactly when $(w, w')^J$ is *full*, in the sense that all elements between w and w' in W are also in the quotient W^J.

Thirdly, consider the polynomial ring in variables corresponding to the elements of an interval $(w, w')^J$ modulo the ideal generated by all products of incomparable elements. This ring is shown to be Gorenstein if $(w, w')^J$ is full and Cohen–Macaulay in general. Furthermore, in the first case the Hilbert series $F(z)$ of the ring satisfies a functional equation $F(1/z) = (-1)^n F(z)$.

Lexicographic shellability of Bruhat order has previously been shown for the symmetric groups by P. Edelman [6], and for the classical Weyl groups and their quotients by R. Proctor [9]. Their method depends on special combinatorial representations of the group elements and does not seem extendable beyond the classical cases. The topological results reported above had been conjectured for finite Weyl groups by C. de Concini and R. Stanley. We are grateful to them for communicating their conjectures to us. We are also grateful to C. de Concini and V. Lakshmibai for informing us

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about their work [3], which shows that the Cohen–Macaulayness of homogeneous coordinate rings of certain generalized Schubert varieties depends on the Cohen–Macaulayness of certain of the rings considered here. Finally, we want to express our deep gratitude and affection to A. Garsia, who with inimitable enthusiasm encouraged this work.

2. Basic Properties of Bruhat Order

A *Coxeter group* is a pair (W, S) where W is a group and S is a distinguished set of generators of W such that

(i) $s^2 = e$, for all $s \in S$,

(ii) $(s_i s_j)^{p_{ij}} = e$, $p_{ij} \geqslant 2$, for all $s_i \neq s_j$ in S such that $s_i s_j$ is of finite order, and

(iii) all other relations among the generators are implied by (i) and (ii).

In other words, (i) and (ii) give a presentation of W. Important examples of Coxeter groups are the Weyl groups of root systems and the symmetry groups of regular polytopes and tessellations. The purpose for this section is to review some fundamental properties of the partial ordering of a Coxeter group which is known as Bruhat order. The facts which we state are per se well-known, however we find it desirable to in this manner make the foundations for the paper explicit. Proofs and further details can be found in Bourbaki [2], Deodhar [5] and Verma [14, 15].

For the remainder of this section, let (W, S) be a fixed Coxeter group. If $w = s_1 s_2 \cdots s_q$, $w \in W$, $s_i \in S$, we call the word $s_1 s_2 \cdots s_q$ in the alphabet S an *expression* for w. The length $l(w)$ of $w \in W$ is the least integer q for which an expression $w = s_1 s_2 \cdots s_q$ exists. Such an expression $w = s_1 s_2 \cdots s_q$ of minimal length $q = l(w)$ is said to be *reduced.*

Let T be the set of conjugates of S, i.e., $T = \{wsw^{-1} \mid w \in W, s \in S\}$. The elements of T are commonly called *reflections.*

2.1. Definition. For $w, w' \in W$, it is said that w precedes w' in *Bruhat order*, written $w \leqslant w'$, if there exist reflections $t_1, t_2,..., t_m \in T$ such that $w' = wt_1 t_2 \cdots t_m$ and $l(wt_1 t_2 \cdots t_i) > l(wt_1 t_2 \cdots t_{i-1})$ for $i = 1, 2,..., m$.

Our basic tool for working with Bruhat order is the following result, which is due to Verma [15].

2.2. Strong Exchange Property. For $w \in W$, $w = s_1 s_2 \cdots s_q$ a reduced expression, and $t \in T$ the following conditions are equivalent:

(i) $wt < w$;

(ii) $t = s_q s_{q-1} \cdots s_i s_{i+1} \cdots s_q$ for some i, $1 \leqslant i \leqslant q$;

(iii) $wt = s_1 s_2 \cdots \hat{s}_i \cdots s_q$ for some i, $1 \leqslant i \leqslant q$ (s_i deleted).

When they hold the integer i of the last two conditions is uniquely determined.

The significant part of this statement is that (i) implies (ii) and (iii)—the other parts are quite easy to establish. The strong exchange property can be used to prove the important fact that Bruhat order can be characterized in terms of the word–subword relation.

2.3. Subword Property. Let $w' = s_1 s_2 \cdots s_q$ be a reduced expression. Then $w \leqslant w'$ if and only if there is a reduced expression

$$w = s_{i_1} s_{i_2} \cdots s_{i_k} \qquad \text{with} \quad 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant q.$$

The subword property reveals the *left–right symmetry* of Bruhat order. Whereas Definition 2.1 and the strong exchange property 2.2 are formulated in terms of action by reflections $t \in T$ on the right, the characterization of Bruhat order by subwords is impartial in this respect. This shows that we could equally well have started out defining Bruhat order by action of reflections $t \in T$ on the left and ended up with the same ordering of W. By this left–right symmetry any computation "done on the right" can be mirrored into a corresponding computation "done on the left."

Select a subset $J \subseteq S$, and let W_J be the subgroup generated by J in W. Subgroups of the form W_J are called *parabolic*.

2.4. Definition. $W^J = \{w \in W \mid ws > w \text{ for all } s \in J\}$.

The significance of the set W^J stems from the fact that every element $w \in W$ can be factored $w = u \cdot v$, with $u \in W^J$ and $v \in W_J$, in one and only one way, and then $l(w) = l(u) + l(v)$. This shows that each element $u \in W^J$ is the unique member of its coset uW_J having minimal length. Since the parabolic subgroups W_J as a rule are not normal in W there is no induced group structure on the quotient $W^J \cong W/W_J$. However, the Bruhat partial ordering of W^J, obtained by restricting the partial order on W, is significant.

From now on we will consider Bruhat order on the quotient posets W^J. The case of the full group can be obtained by setting $J = \varnothing$.

2.5. Lemma. *Assume that $w \in W^J$, $w' \in W$, $w' > w$ and $l(w') = l(w) + 1$. Then either $w' \in W^J$ or $w' = ws$ for some $s \in J$.*

2.6. Chain Property. If $w, w' \in W^J$, $w < w'$, then all maximal chains $w' = u_0 > u_1 > \cdots > u_r = w$ in W^J have the same length $r = l(w') - l(w)$.

Suppose now that (W, S) is a *finite* Coxeter group and $J \subseteq S$. Then the Bruhat order of W^J has a greatest element w_0^J, so $w_0^J \geqslant w \geqslant e$ for all $w \in W^J$. Hence, the results of later sections, which are formulated for intervals, apply in particular to the entire posets W^J for finite Coxeter groups.

3. Lexicographically Shellable Posets

Let P be a finite poset (i.e., partially ordered set). We say that P is *bounded* if there exist a top element $\hat{1} \in P$ and a bottom element $\hat{0} \in P$ so that $\hat{1} \geqslant x \geqslant \hat{0}$ for all $x \in P$. We say that P is *graded* if it is bounded and satisfies the chain property: all maximal chains $\hat{1} = x_0 > x_1 > \cdots > x_r = \hat{0}$ in P have the same *length* r. In a graded poset P there is a natural corank-function, *viz*., for $x \in P$ the *corank* $\rho(x)$ is the common length of all maximal chains from $\hat{1}$ to x. By the *length* of P we mean $\rho(\hat{0})$. We say that y *covers* x in P, and write $y \to x$, if $y > x$ and $y > z > x$ for no $z \in P$. The corank-function can then be characterized by (i) $\rho(\hat{1}) = 0$ and (ii) $y \to x$ implies $\rho(x) = \rho(y) + 1$.

Let P be a graded poset of length r. We are going to describe a type of labeling of the maximal chains of P which will be a fundamental tool throughout this paper. It will be the convention to always read chains in P from top to bottom. Given a maximal chain $\mathbf{m}$: $\hat{1} = x_0 \to x_1 \to \cdots \to x_r = \hat{0}$ we associate a label $\lambda(\mathbf{m}) = (\lambda_1(\mathbf{m}), \lambda_2(\mathbf{m}), \ldots, \lambda_r(\mathbf{m})) \in \mathbb{Z}^r$, where we think of the integer $\lambda_i(\mathbf{m})$ as being associated with the *edge* (or covering relation) $x_{i-1} \to x_i$. The idea is perhaps best conveyed by saying that we label $\mathbf{m}$ edge-wise from top to bottom with integers.

3.1. Example. Figure 1 shows a graded poset and a labeling of its four maximal chains.

The first requirement of a useful labeling is the following.

(L1) If two maximal chains $\mathbf{m}$ and $\mathbf{m}'$ coincide along their first d edges, $1 \leqslant d \leqslant r$, then $\lambda_i(\mathbf{m}) = \lambda_i(\mathbf{m}')$ for $i = 1, 2, \ldots, d$.

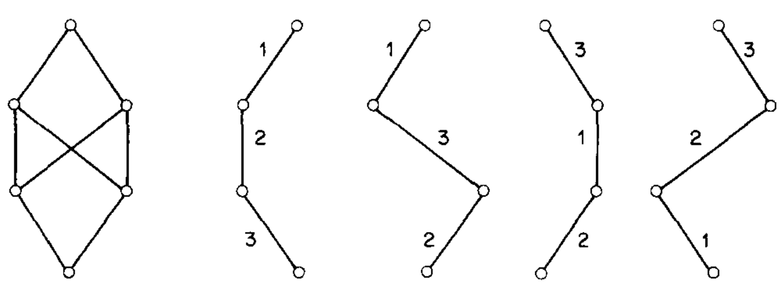

Figure 1

By a *rooted interval* $([x, y], \mathbf{c})$ in P we shall mean a pair where $[x, y]$ is an interval, i.e., $x \leqslant y$ and $[x, y] = \{z \in P \mid x \leqslant z \leqslant y\}$, and $\mathbf{c}$ is a saturated chain from $\hat{1}$ to y, say $\mathbf{c}$: $\hat{1} = c_0 \to c_1 \to \cdots \to c_e = y$. It is important to notice that if the maximal chains of P have a labeling which obeys axiom (L1) and $([x, y], \mathbf{c})$ is a rooted interval, then the maximal chains of $[x, y]$ receive an induced labeling which also obeys (L1). Specifically, if $\mathbf{n}$ is a maximal chain in $[x, y]$ we get the induced label $\lambda'(\mathbf{n}) = (\lambda'_1(\mathbf{n}), \lambda'_2(\mathbf{n}),..., \lambda'_f(\mathbf{n})) \in \mathbb{Z}^f$, where $f = \rho(x) - \rho(y)$, by concatenating $\mathbf{c}$ followed by $\mathbf{n}$ with an arbitrary saturated chain $\mathbf{c}'$ from x to $\hat{0}$ to get a maximal chain $\mathbf{m} = \mathbf{c} * \mathbf{n} * \mathbf{c}'$ in P and then setting $\lambda'_i(\mathbf{n}) = \lambda_{e+i}(\mathbf{m})$, $i = 1, 2,..., f$. By abuse of notation and language we now drop the "prime" and the word "induced" when we consider the labeling of maximal chains of a rooted interval. Recall that the *lexicographic order* of $\mathbb{Z}^f$ is a linear ordering defined as follows: $\mathbf{a} = (a_1, a_2,..., a_f) \in \mathbb{Z}^f$ precedes $\mathbf{b} = (b_1, b_2,..., b_f) \in \mathbb{Z}^f$, which we write $\mathbf{a} <_L \mathbf{b}$, if and only if $a_i < b_i$ in the first coordinate where they differ. Our second requirement of a labeling is the following.

(L2) For every rooted interval $([x, y], \mathbf{c})$ in P there is a unique maximal chain $\mathbf{m}_0$ in $[x, y]$ whose label $\lambda(\mathbf{m}_0)$ is increasing, $\lambda_1(\mathbf{m}_0) \leqslant \lambda_2(\mathbf{m}_0) \leqslant \cdots \leqslant \lambda_f(\mathbf{m}_0)$, and if $\mathbf{m}$ is any other maximal chain in $[x, y]$ then $\lambda(\mathbf{m}_0) <_L \lambda(\mathbf{m})$.

It is simple to verify that the labeling of maximal chains which was suggested in Example 3.1 satisfies both (L1) and (L2).

3.2. Definition. A labeling of the maximal chains of a graded poset P which obeys conditions (L1) and (L2) will be called an *L-labeling*. When an L-labeling is possible P is said to be *lexicographically shellable*, or *L-shellable* for short.

The notion of lexicographic shellability was introduced in [1], where, however, a more restrictive definition of the concept was used. Nevertheless, the proofs from [1] can be carried over almost verbatim if only care is taken to replace "interval" by "rooted interval" at the appropriate places. We are going to illustrate this for a key theorem from [1] which finds important applications later in this paper.

A finite simplicial complex Δ is said to be *pure d-dimensional* if all maximal faces are of dimension d. A pure d-dimensional complex Δ is said to be *shellable* if its maximal faces can be arranged in sequence $\sigma_1, \sigma_2,..., \sigma_t$ in such a way that $\bar{\sigma}_j \cap (\bigcup_{i=1}^{j-1} \bar{\sigma}_i)$ is a pure $(d-1)$-dimensional complex for $j = 2, 3,..., t$ (here $\bar{\sigma}_i = \{\tau \mid \tau \subseteq \sigma_i\}$). Such an ordering of the maximal faces is called a *shelling*.

To a finite partially ordered set P one can associate the simplicial complex $\Delta(P)$ of all chains $x_0 > x_1 > \cdots > x_k$, often called the *order complex* of P.

Clearly, the maximal faces of $\Delta(P)$ are the maximal chains of P. Also, if P is a graded poset of length r then $\Delta(P)$ is pure r-dimensional. Notice that if P is graded and $\bar{P} = P - \{\hat{0}, \hat{1}\}$, then $\Delta(P)$ is shellable if and only if $\Delta(\bar{P})$ is shellable.

3.3. THEOREM. *If P is lexicographically shellable then the order complex $\Delta(P)$ is shellable.*

Proof. We propose to show that any linear ordering of the set $\mathscr{M}$ of maximal chains which extends the lexicographic ordering of the labels is a shelling order. So assign a linear order, denoted "$\prec$" to $\mathscr{M}$ such that $\lambda(\mathbf{m}) <_L \lambda(\mathbf{m}')$ implies $\mathbf{m} \prec \mathbf{m}'$. We have to prove that if $\mathbf{k} \prec \mathbf{m}$ for $\mathbf{k}, \mathbf{m} \in \mathscr{M}$, then there exists an $\mathbf{h} \in \mathscr{M}$ such that $\mathbf{h} \prec \mathbf{m}$, $(\mathbf{k} \cap \mathbf{m}) \subseteq (\mathbf{h} \cap \mathbf{m})$, and $|\mathbf{h} \cap \mathbf{m}| = |\mathbf{m}| - 1$.

Consider two maximal chains in P, $\mathbf{k}\colon \hat{1} = k_0 \to k_1 \to \cdots \to k_r = \hat{0}$ and $\mathbf{m}\colon \hat{1} = m_0 \to m_1 \to \cdots \to m_r = \hat{0}$, and suppose that $\mathbf{k} \prec \mathbf{m}$. Let d be the greatest integer such that $k_i = m_i$ for $i = 0, 1, \ldots, d$, and let g be the least integer such that $d < g$ and $k_g = m_g$. Then $g - d \geqslant 2$ and $d < i < g$ implies that $k_i \neq m_i$. Now consider the rooted interval $([m_g, m_d], \hat{1} \to m_1 \to \cdots \to m_d)$. The chain $m_d \to m_{d+1} \to \cdots \to m_g$ cannot be the unique maximal chain of this interval with increasing label because then axiom (L2) would force $\lambda(\mathbf{m}) <_L \lambda(\mathbf{k})$ contrary to the assumption that $\mathbf{k} \prec \mathbf{m}$. Consequently, the label $\lambda(\mathbf{m})$ must have a descent $\lambda_e(\mathbf{m}) > \lambda_{e+1}(\mathbf{m})$ for some e with $d < e < g$. Then in the rooted interval $([m_{e+1}, m_{e-1}], \hat{1} \to m_1 \to \cdots \to m_{e-1})$ the chain $m_{e-1} \to m_e \to m_{e+1}$ has a decreasing label so by axiom (L2) there is a chain $m_{e-1} \to x \to m_{e+1}$ whose label comes earlier in the lexicographic order. If we let $\mathbf{h}\colon \hat{1} \to m_1 \to \cdots \to m_{e-1} \to x \to m_{e+1} \to m_{e+2} \to \cdots \to \hat{0}$ it follows that $\lambda(\mathbf{h}) <_L \lambda(\mathbf{m})$, hence $\mathbf{h} \prec \mathbf{m}$, and the construction shows that $\mathbf{h} \cap \mathbf{m} = \mathbf{m} - \{m_e\} \supseteq \mathbf{k} \cap \mathbf{m}$. ∎

At this point let us establish some notational conventions. The cardinality of a finite set A will be denoted by $|A|$. For a positive integer n let $[n] = \{1, 2, \ldots, n\}$. Suppose that P is a lexicographically shellable poset of length $n+1$ with a given L-labeling λ of the set $\mathscr{M}$ of maximal chains. For a maximal chain $\mathbf{m} \in \mathscr{M}$ define the *descent set* $D_\lambda(\mathbf{m}) = \{i \in [n] \mid \lambda_i(\mathbf{m}) > \lambda_{i+1}(\mathbf{m})\}$. For any subset $E \subseteq [n]$ define the *rank-selected subposet*

$$P_E = \{x \in P \mid \rho(x) \in E \cup \{0, n+1\}\}.$$

Thus P_E is a graded poset of length $|E| + 1$. In fact, by [1, Theorem 4.1] the order complex $\Delta(P_E)$ is shellable for all E. Let us write $\mu_E(x, y)$ for the Möbius function $\mu(x, y)$ computed on the rank-selected subposet P_E (see Rota [10] for information about Möbius functions).

3.4. THEOREM. $(-1)^{|E|+1}\mu_E(\hat{0},\hat{1})=|\{\mathbf{m}\in\mathcal{M}\mid D_\lambda(\mathbf{m})=E\}|$.

This result is due to Stanley [11]. The proof (cf. [1, p. 164]) goes through without significant modifications, so we omit repeating it here.

4. Lexicographic Shellability of Bruhat Order

Let (W,S) be a Coxeter group and $J\subseteq S$. Let $w<w'$ in Bruhat order, $w,w'\in W^J$. The following special notation will remain in force throughout the rest of this paper: $[w,w']=\{u\in W\mid w\leqslant u\leqslant w'\}$, $[w,w']^J=\{u\in W^J\mid w\leqslant u\leqslant w'\}$, $(w,w')=\{u\in W\mid w<u<w'\}$ and $(w,w')^J=\{u\in W^J\mid w<u<w'\}$. The closed interval $[w,w']^J$ in the poset W^J is said to be *full* if $[w,w']^J=[w,w']$, and similarly for open intervals. If $l(w')=q$ then one can see from the subword property 2.3 that $|[e,w']|\leqslant 2^q$. It follows that $[w,w']^J$ is finite. In view of the chain property 2.6 it is then clear that the interval $[w,w']^J$ is a *graded* poset with corank-function $\rho(u)=l(w')-l(u)$.

We will now describe a labeling of the maximal chains of $[w,w']^J$. Fix once and for all a reduced expression $w'=s_1s_2\cdots s_q$. Suppose that $l(w')-l(w)=r$ and let $\mathbf{m}\colon w'=w_0\to w_1\to\cdots\to w_r=w$ be a maximal chain in $[w,w']^J$. To $\mathbf{m}$ we assign a label $\lambda(\mathbf{m})$ in the following manner. By the strong exchange property $w_1=w_0t_1=s_1s_2\cdots\hat{s}_i\cdots s_q$, where the deleted generator s_i is uniquely determined. Let $\lambda_1(\mathbf{m})=i$. Now repeat the process. After k steps we have reached w_k and after k deletions obtained a uniquely determined reduced subword expression $w_k=s_{j_1}s_{j_2}\cdots s_{j_{q-k}}$, $1\leqslant j_1<j_2<\cdots<j_{q-k}\leqslant q$. Again, $w_{k+1}=w_kt_{k+1}=s_{j_1}s_{j_2}\cdots\hat{s}_{j_l}\cdots s_{j_{q-k}}$ where the deleted generator s_{j_l} is uniquely determined. Let $\lambda_{k+1}(\mathbf{m})=j_l$. Hence, the idea is to label by the positions of the generators which are successively deleted from the chosen reduced expression for w' as we go down the maximal chain from w' to w.

4.1. EXAMPLE. Let W be the dihedral group of order six on two generators $S=\{a,b\}$ (or equivalently, the symmetric group S_3). The Bruhat ordering of this group is depicted in Fig. 1. Choosing "*aba*" as reduced expression for the top element the labels of the four maximal chains are

$$\lambda(aba\to ba\to a\to\varnothing)=(1,2,3),$$

$$\lambda(aba\to ba\to b\to\varnothing)=(1,3,2),$$

$$\lambda(aba\to ab\to b\to\varnothing)=(3,1,2),$$

$$\lambda(aba\to ab\to a\to\varnothing)=(3,2,1).$$

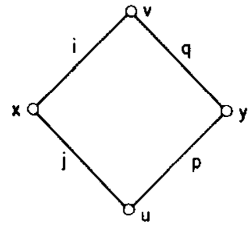

FIGURE 2

4.2. THEOREM. *Let (W, S) be a Coxeter group, $J \subseteq S$, $w, w' \in W^J$ and $w < w'$. Then every labeling of the maximal chains of $[w, w']^J$, which is induced by a reduced expression $w' = s_1 s_2 \cdots s_q$ as described, is an L-labeling. In particular, $[w, w']^J$ is lexicographically shellable.*

It will be convenient for later arguments to have an explicit description of the length 2 case (cf. Fig. 2).

4.3. LEMMA. *Let $[u, v]$ be an interval of length 2 in W, and let $v = s_1 s_2 \cdots s_k$ be a reduced expression. Then*

(α) *there is a unique chain $v \to x \to u$ with increasing label (i, j), $i < j$;*

(β) *there is a unique chain $v \to y \to u$ with decreasing label (q, p), $q > p$;*

(γ) *$i < q$; and*

(δ) *if $u, v \in W^J$ then $x \in W^J$.*

Proof of lemma. Among all reduced expressions for u which are subwords of $s_1 s_2 \cdots s_k$, choose $u = s_1 s_2 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$ so that $i < j$ and j is *minimal*. Let $t_j = s_k s_{k-1} \cdots s_j s_{j+1} \cdots s_k$.

Suppose that $u t_j < u$. Then by the strong exchange property applied to the reduced expression $u = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$, either (i) $t_j = t_g = s_k s_{k-1} \cdots s_g s_{g+1} \cdots s_k$, $j < g \leqslant k$, or (ii) $t_j = t_e = s_k \cdots \hat{s}_j \cdots s_e \cdots \hat{s}_j \cdots s_k$, $i < e < j$, or (iii) $t_j = t_d = s_k \cdots \hat{s}_j \cdots \hat{s}_i \cdots s_d \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$, $1 \leqslant d < i$. The first case (i) yields $v = v t_j t_g = s_1 \cdots \hat{s}_j \cdots \hat{s}_g \cdots s_k$ which is impossible since the expression $v = s_1 s_2 \cdots s_k$ is reduced. The two other cases yield (ii) $u = u t_e t_j = s_1 \cdots \hat{s}_i \cdots \hat{s}_e \cdots s_j \cdots s_k$ and (iii) $u = u t_d t_j = s_1 \cdots \hat{s}_d \cdots \hat{s}_i \cdots s_j \cdots s_k$, so both violate the choice of reduced expression $u = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$ with j minimal.

Let $x = u t_j = s_1 s_2 \cdots \hat{s}_i \cdots s_j \cdots s_k$. We have shown in the preceding paragraph that $x = u t_j > u$, and the subword property shows that $x = s_1 \cdots \hat{s}_i \cdots s_k < v$. The chain $v \to x \to u$ has increasing label (i, j), and it will be shown below that there can be at most one chain in $[u, v]$ with increasing label. Thus, part (α) is done.

For part (β) we simply mirror the computations of part (α) by the left–right symmetry. Thus, choose a reduced subword expression $u = s_1 s_2 \cdots \hat{s}_p \cdots$

$\hat{s}_q \cdots s_k$ such that $p < q$ and p is *maximal*, and so on. By choice of j, $i < j \leqslant q$, hence (γ).

Finally, if $u \in W^J$ and $x \notin W^J$ then, by Lemma 2.5, $x = us$ for some $s \in J$. Hence $t_j = s$, and so $vs = vt_j = s_1 \cdots \hat{s}_j \cdots s_k < v$, which shows that $v \notin W^J$. ∎

Proof of theorem. Since the deleted generator is at each step uniquely determined, it is evident that if two maximal chains of $[w, w']^J$ coincide along their d first edges then also the first d entries in their labels coincide. Thus, axiom (L1) is immediately verified.

We must verify axiom (L2) for each rooted interval ($[u, v]^J$, $w' = w_0 \to w_1 \to \cdots \to w_e = v$) in $[w, w']^J$. But starting with the given reduced expression $w' = s_1 s_2 \cdots s_q$ and moving down the saturated chain $w' \to w_1 \to \cdots \to w_e = v$ we produce a uniquely determined reduced expression $v = s_{i_1} s_{i_2} \cdots s_{i_{q-e}}$, $1 \leqslant i_1 < i_2 < \cdots < i_{q-e} \leqslant q$, and the induced labeling of the maximal chains of $[u, v]^J$ as a rooted interval in $[w, w']^J$ is equivalent to the labeling of $[u, v]^J$ directly obtained starting from the reduced expression $v = s'_1 s'_2 \cdots s'_{q-e} = s_{i_1} s_{i_2} \cdots s_{i_{q-e}}$. Hence, no generality is lost if we verify (L2) only for the entire interval $[w, w']^J$.

Let us first prove that two distinct maximal chains in $[w, w']^J$ cannot both have increasing labels. This is clear for length 1 so we may inductively suppose that it has been shown for length $r-1$. Suppose that in $[w, w']^J$ there are two maximal chains $\mathbf{m}$: $w' = w_0 \to w_1 \to \cdots \to w_r = w$ and $\mathbf{m}'$: $w' = w'_0 \to w'_1 \to \cdots \to w'_r = w$ with increasing labels $\lambda(\mathbf{m}) = (i_1, i_2, ..., i_r)$ and $\lambda(\mathbf{m}') = (j_1, j_2, ..., j_r)$. Then $w = s_1 \cdots \hat{s}_{i_1} \cdots \hat{s}_{i_2} \cdots \hat{s}_{i_r} \cdots s_q = s_1 \cdots \hat{s}_{j_1} \cdots \hat{s}_{j_2} \cdots \hat{s}_{j_r} \cdots s_q$. Assume that $i_r < j_r$, and let $t_{j_r} = s_q s_{q-1} \cdots s_{j_r} s_{j_r+1} \cdots s_q$. Then $w'_{r-1} = wt_{j_r} = s_1 \cdots \hat{s}_{i_1} \cdots \hat{s}_{i_2} \cdots \hat{s}_{i_r} \cdots \hat{s}_{j_r} \cdots s_q$, so $l(w'_{r-1}) \leqslant l(w) - 1$ which contradicts $w'_{r-1} \to w$. Hence, $j_r \leqslant i_r$, and by symmetry, $i_r \leqslant j_r$. The equality $i_r = j_r$ implies that $w_{r-1} = w'_{r-1}$. Since the interval $[w_{r-1}, w']^J$ by the induction assumption is known not to admit two distinct maximal chains with increasing labels, we must conclude that $\mathbf{m} = \mathbf{m}'$. In particular, for $r = 2$ this also completes the proof of Lemma 4.3.

Let $\mathbf{m}_0$: $w' = w_0 \to w_1 \to \cdots \to w_r = w$ be that maximal chain in $[w, w']^J$ whose label $\lambda(\mathbf{m}_0)$ comes first in lexicographic order. Suppose that $\lambda(\mathbf{m}_0) = (\lambda_1(\mathbf{m}_0), \lambda_2(\mathbf{m}_0), ..., \lambda_r(\mathbf{m}_0))$ has a descent $\lambda_i(\mathbf{m}_0) > \lambda_{i+1}(\mathbf{m}_0)$, $1 \leqslant i \leqslant r-1$. Then in the rooted interval ($[w_{i+1}, w_{i-1}]$, $w' \to w_1 \to \cdots \to w_{i-1}$), $w_{i-1} \to w_i \to w_{i+1}$ is the chain with decreasing label. By part (α) of Lemma 4.3 we can replace this chain by one, say $w_{i-1} \to x_0 \to w_{i+1}$, with increasing label. This replacement produces a new maximal chain $\mathbf{m}_1$: $w' = w_0 \to w_1 \to \cdots \to w_{i-1} \to x_0 \to w_{i+1} \to \cdots \to w_r = w$. By part ($\delta$) $\mathbf{m}_1$ is in W^J and by part (γ) $\lambda(\mathbf{m}_1) <_L \lambda(\mathbf{m}_0)$. This contradicts the choice of $\mathbf{m}_0$, so we must conclude that $\mathbf{m}_0$ has increasing label $\lambda_1(\mathbf{m}_0) < \lambda_2(\mathbf{m}_0) < \cdots < \lambda_r(\mathbf{m}_0)$. ∎

5. Consequences of Shellability

(A) *Combinatorial Consequences*

Let (W, S) be a Coxeter group and consider the Möbius function μ_E computed on a rank-selected subposet P_E of an interval $P = [w, w']$.

5.1. Theorem. *Suppose $[w, w']$ is an interval in W and $l(w') - l(w) = n + 1$. Then for each $E \subseteq [n]$:*

(i) $(-1)^{|E|+1} \mu_E(w, w') \geqslant 1$, *and*

(ii) $\mu_E(w, w') = (-1)^n \mu_{[n]-E}(w, w')$.

Proof. Let $w' = s_1 s_2 \cdots s_q$ be a reduced expression and let λ be the induced labeling of the set $\mathscr{M}$ of maximal chains of $[w, w']$. Statement (i) is by Theorem 3.4 equivalent to the existence of a maximal chain $\mathbf{m}$ with descent set $D_\lambda(\mathbf{m}) = E$.

Assume that in every interval shorter than $[w, w']$ there exist maximal chains with all possible descent sets. For intervals of length 2 this was verified in Lemma 4.3. Suppose first that $1 \notin E$. Let $\mathbf{m}_0$: $w' = w_0 \to w_1 \to \cdots \to w_{n+1} = w$ be the unique maximal chain with increasing label $\lambda(\mathbf{m}_0)$. In the shorter interval $[w, w_1]$ with the induced labeling it is by assumption possible to find a maximal chain $w_1 = v_1 \to v_2 \to \cdots \to v_{n+1} = w$ with prescribed descent set corresponding to E. The maximal chain $\mathbf{m}$: $w' \to v_1 \to v_2 \to \cdots \to v_{n+1} = w$ satisfies $\lambda_1(\mathbf{m}) = \lambda_1(\mathbf{m}_0) < \lambda_2(\mathbf{m}_0) \leqslant \lambda_2(\mathbf{m})$, since $\lambda(\mathbf{m}_0) \leqslant_L \lambda(\mathbf{m})$, so we can conclude that $D_\lambda(\mathbf{m}) = E$. Thus, (i) is proved for the case $1 \notin E$. The case $1 \in E$ then follows via (ii).

Now, let $\mathbf{m}$ be a maximal chain in $[w, w']$ for which $D_\lambda(\mathbf{m}) = E$. The label $\lambda(\mathbf{m})$ has been derived from a reduced expression $w' = s_1 s_2 \cdots s_q$ where the positions of the generators are numbered from left to right. If we instead label them from right to left, $s'_i = s_{q+1-i}$, $w' = s'_q s'_{q-1} \cdots s'_1$, then the derived label $\lambda'(\mathbf{m})$, $\lambda'_i(\mathbf{m}) = q + 1 - \lambda_i(\mathbf{m})$, has a descent where $\lambda(\mathbf{m})$ has an ascent and *vice versa*. The proof of Theorem 4.2 can be mirrored by left–right symmetry into a proof that λ' is an L-labeling. For this it should be noticed that since $[w, w']$ is a full interval the requirement in that proof to remain inside W^J, which otherwise would present an obstacle, disappears. We have shown that $\{\mathbf{m} \in \mathscr{M} \mid D_\lambda(\mathbf{m}) = E\} = \{\mathbf{m} \in \mathscr{M} \mid D_{\lambda'}(\mathbf{m}) = [n] - E\}$. By Theorem 3.4 this implies $(-1)^{|E|+1} \mu_E(w, w') = (-1)^{n-|E|+1} \mu_{[n]-E}(w, w')$, which is (ii). ∎

5.2. Corollary (Verma [14]). $\mu(w, w') = (-1)^{l(w') - l(w)}$.

Proof. This is the case $E = [n]$ of part (ii). ∎

Consider next the Möbius function μ_E computed on a rank-selected subposet P_E of an interval $P = [w, w']^J$ in a quotient W^J, $J \subseteq S$. All that can be said in general here is that $(-1)^{|E|+1} \mu_E(w, w') \geqslant 0$. However, for the entire interval the Möbius function μ can be explicitly determined.

5.3. THEOREM (Verma [14]–Deodhar [5]). *Computed on an interval* $[w, w']^J$ *in* W^J:

(i) $\mu(w, w') = (-1)^{l(w')-l(w)}$, *if* $[w, w']^J$ *is full,*

(ii) $\mu(w, w') = 0$, *otherwise.*

Proof. Part (i) is merely a restatement of Corollary 5.2. Let $\mathbf{m}$ be the unique maximal chain with strictly decreasing label which we know exists in the full interval $[w, w']$. For part (ii) we must show that $\mathbf{m}$ is not present in $[w, w']^J$. In other words, we must show that $\mathbf{m} \subseteq [w, w']^J$ implies that $[w, w']^J$ is full.

Denote by $\mathscr{M}$ and $\mathscr{M}^J$ the collections of maximal chains in $[w, w']$ and $[w, w']^J$, respectively. Pick any $x \in [w, w']$, and then select $\mathbf{m}_0 \in \mathscr{M}$ such that $x \in \mathbf{m}_0$. If the label $\lambda(\mathbf{m}_0)$ has any ascent $\lambda_i(\mathbf{m}_0) < \lambda_{i+1}(\mathbf{m}_0)$ then by Lemma 4.3 we can replace one element of $\mathbf{m}_0$ to obtain a maximal chain $\mathbf{m}_1 \in \mathscr{M}$ such that $\lambda_i(\mathbf{m}_1) > \lambda_{i+1}(\mathbf{m}_1)$ and $\lambda(\mathbf{m}_0) <_L \lambda(\mathbf{m}_1)$. If again $\lambda(\mathbf{m}_1)$ has an ascent we can switch $\mathbf{m}_1$ into $\mathbf{m}_2 \in \mathscr{M}$ so that $\lambda(\mathbf{m}_1) <_L \lambda(\mathbf{m}_2)$. After a finite number of switches we must reach a chain $\mathbf{m}_k \in \mathscr{M}$, $\lambda(\mathbf{m}_{k-1}) <_L \lambda(\mathbf{m}_k)$, such that $\lambda(\mathbf{m}_k)$ has no ascent. Hence, $\mathbf{m}_k$ equals $\mathbf{m}$, the unique member of $\mathscr{M}$ with strictly decreasing label. Notice that part (δ) of Lemma 4.3 shows that if $\mathbf{m}_i \in \mathscr{M}^J$ then $\mathbf{m}_{i-1} \in \mathscr{M}^J$ for $i = 1, 2, \ldots, k$. Thus, if $\mathbf{m} \subseteq [w, w']^J$ so that $\mathbf{m}_k \in \mathscr{M}^J$, then it follows that $\mathbf{m}_0 \in \mathscr{M}^J$ and, in particular, that $x \in \mathbf{m}_0 \subseteq [w, w']^J$. ∎

In connection with the above results we want to remark that the chain property 2.6 can be deduced in a direct way from the strong exchange property 2.2 and Lemma 2.5, without (as in [5]) a previous knowledge of the Möbius function.

(B) *Topological Consequences*

Let (W, S) be a Coxeter group, $J \subseteq S$, and consider the order complex of an open interval $(w, w')^J$. For a simplicial complex Δ let $|\Delta|$ denote its geometric realization.

5.4. THEOREM. *Let Δ be the simplicial complex of chains in the open interval $(w, w')^J$, and suppose that $l(w') - l(w) = d + 2 \geqslant 2$. Then*

(i) $|\Delta|$ *is a d-sphere, if* $(w, w')^J$ *is full,*

(ii) $|\Delta|$ *is a d-cell, otherwise.*

Proof. The complex Δ is clearly pure d-dimensional. Lemma 4.3 shows that every interval $[u, v]$ in W of length 2 has exactly two maximal chains. One concludes that (i) every $(d-1)$-face of Δ is included in exactly two d-faces if $(w, w')^J$ is full, and (ii) every $(d-1)$-face of Δ is included in at most two d-faces and some $(d-1)$-face is included in only one d-face if $(w, w')^J$ is not full. By Theorems 4.2 and 3.3 the complex Δ is shellable, and a result of Danaraj and Klee [4, p. 444] shows that a shellable complex Δ triangulates a sphere under condition (i) and triangulates a cell under condition (ii). ∎

We remark that Theorem 5.4 implies Theorem 5.3, since the Möbius function $\mu(w, w')$ equals the reduced Euler characteristic of the complex Δ.

(C) *Algebraic Consequences*

Let P be a partial order on the set $\{x_1, x_2, \ldots, x_f\}$. Let k be a field or $k = \mathbb{Z}$ and define $R_P = k[x_1, x_2, \ldots, x_f]/I_P$, where I_P is the ideal in the polynomial ring $k[x_1, x_2, \ldots, x_f]$ generated by all monomials $x_i x_j$ for which $x_i \nleqslant x_j$ and $x_j \nleqslant x_i$ in P. It has been known for some time that if the order complex $\Delta(P)$ is shellable then R_P is a Cohen–Macaulay ring (cf. Hochster [8] and Stanley [12]). In fact, this knowledge provided the *raison d'être* for the notion of lexicographic shellability in [1]. Also, Hochster and Stanley have shown that if $\Delta(P)$ triangulates a sphere (or, a multiple cone over a sphere) then R_P is a Gorenstein ring [8, p. 211; 12, p. 57]. Combined with this information the results of this paper show the following.

5.5. THEOREM. *Let (W, S) be a Coxeter group, $J \subseteq S$, and let $P = [w, w']^J$ or $P = (w, w')^J$, $l(w') - l(w) \geqslant 2$. Then*

(i) *R_P is Gorenstein, if $[w, w']^J$ is full,*

(ii) *R_P is Cohen–Macaulay, otherwise.*

The k-algebra R_P has a standard grading which is induced by giving all variables x_i degree one, and the Cohen–Macaulay property for R_P is equivalent to the existence of homogeneous elements $\theta_1, \theta_2, \ldots, \theta_d$ and $\eta_1, \eta_2, \ldots, \eta_t$ such that

$$R_P = \bigoplus_{i=1}^{t} \eta_i k[\theta_1, \theta_2, \ldots, \theta_d]$$

as a k-module. The ring R_P of a shellable poset P has been carefully investigated by Garsia, who in particular gives an explicit recipe for how such θ's and η's can be chosen [7, Theorem 4.2]. Garsia's result in combination with the analysis of Bruhat order in Section 4 yields the following decomposition, which formulates the Cohen–Macaulayness of Bruhat order in very explicit terms. Consider an interval $[w, w']^J$ such that $l(w') - l(w) =$

$n+1 \geqslant 2$. For $i = 1, 2, \ldots, n$; if $u_1, u_2, \ldots, u_{k_i}$ are the elements in $[w, w']^J$ of length $l(w)+i$, then let $\theta_i = u_1 + u_2 + \cdots + u_{k_i}$. Let $\mathscr{M}^J$ as usual denote the set of maximal chains in $[w, w']^J$ and consider $\mathscr{M}^J$ labeled as in Section 4. If $\mathbf{m}: w' = v_0 \to v_1 \to \cdots \to v_{n+1} = w$ is in $\mathscr{M}^J$ and $D_\lambda(\mathbf{m}) = \{m_1, m_2, \ldots, m_g\} \subseteq [n]$, then let $\eta(\mathbf{m}) = v_{m_1} v_{m_2} \cdots v_{m_g}$.

5.6. THEOREM. $R_{(w,w')^J} = \oplus_{\mathbf{m}\in\mathscr{M}^J} \eta(\mathbf{m})\, k[\theta_1, \theta_2, \ldots, \theta_n]$.

Let us finally consider the Hilbert series $F(R_P, z)$ of the graded algebra $R_P = R_0 \oplus R_1 \oplus \cdots \oplus R_i \oplus \cdots$. Recall that $F(R_P, z) = \sum_{i=0}^{\infty} (\dim_k R_i)\, z^i$, where $\dim_k$ denotes vector space dimension (or Abelian group rank if $k = \mathbb{Z}$). For the poset $P = (w, w')^J$ of length $n-1$ we get, from Theorem 5.6 or by direct combinatorial reasoning, that

$$F(R_{(w,w')^J}, z) = (1-z)^{-n} \sum_{\mathbf{m}\in\mathscr{M}^J} z^{|D_\lambda(\mathbf{m})|}. \tag{5.7}$$

For full intervals this leads to the following functional identity (cf. Stanley [13, Theorem 4.1]).

5.8. THEOREM. *Suppose that (w, w') is an interval in W, $l(w') - l(w) = n+1$, and let $F(z) = F(R_{(w,w')}, z)$. Then*

$$F(1/z) = (-1)^n F(z).$$

Proof. Let $(1-z)^n F(z) = \sum_{i=0}^{n} h_i z^i$. Formula (5.7) shows that $h_i = |\{\mathbf{m} \in \mathscr{M} \mid |D_\lambda(\mathbf{m})| = i\}|$, and part (ii) of Theorem 5.1 shows for any $E \subseteq [n]$ that $|\{\mathbf{m} \in \mathscr{M} \mid D_\lambda(\mathbf{m}) = E\}| = |\{\mathbf{m} \in \mathscr{M} \mid D_\lambda(\mathbf{m}) = [n] - E\}|$. It follows by summation over all E of cardinality i that $h_i = h_{n-i}$. Hence, $(1-z)^n F(1/z) = (-z)^n (1 - 1/z)^n F(1/z) = (-1)^n z^n \sum_{i=0}^{n} h_i z^{-i} = (-1)^n \sum_{i=0}^{n} h_{n-i} z^{n-i} = (-1)^n (1-z)^n F(z)$. ∎

The last few results have, for convenience, been formulated only for open intervals. It is easy to find the corresponding statements for closed intervals. For instance, $R_{[w,w']^J} = \oplus_{\mathbf{m}\in\mathscr{M}^J} \eta(\mathbf{m})\, k[w, \theta_1, \theta_2, \ldots, \theta_n, w']$, and if $[w, w']^J$ is full the Hilbert series of this ring satisfies $F(1/z) = (-1)^n z^2 F(z)$.

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