Abstract. Let $g$ be a complex simple Lie algebra of rank $\ell$, $h$ the Coxeter number, $m_1$, $m_2$, \ldots, $m_\ell$ the exponents of $g$, and $C$ the Cartan matrix. Then

$$2^{2\ell} \prod_{i=1}^{\ell} \sin^2 \frac{m_i \pi}{2h} = \det C.$$ 

1. INTRODUCTION. Here, we attempt to explain in simple terms the meaning of the symbols appearing in the formula

$$2^{2\ell} \prod_{i=1}^{\ell} \sin^2 \frac{m_i \pi}{2h} = \det C.$$ 

In particular, we address the Cartan matrix, the rank, the exponents, and the Coxeter number of a complex simple Lie algebra. For more details, see [8] and [9]. In this article, we consider only finite-dimensional Lie algebras.

Cartan matrices appear in the classification of simple Lie algebras over the complex numbers. A Cartan matrix is associated to each such Lie algebra. It is a $\ell \times \ell$ square matrix, where $\ell$ is the rank of the Lie algebra. The Cartan matrix encodes all the properties of the simple Lie algebra it represents. Let $g$ be a complex simple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, and $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ a basis of simple roots for the root system $\Delta$ of $\mathfrak{h}$ in $g$. The elements of the Cartan matrix $C$ are given by

$$c_{ij} := 2 \frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle},$$

where the inner product is induced by the Killing form. The $\ell \times \ell$-matrix $C$ is invertible and called the Cartan matrix of $g$. The detailed machinery for constructing the Cartan matrix from the root system can be found, e.g., in [8, p. 55] or [10, p. 111]. In the following example, we give full details for the case of $sl(4, \mathbb{C})$, which is of type $A_3$.

Example 1. Let $E$ be the hyperplane of $\mathbb{R}^4$ for which the coordinates sum to 0 (i.e., vectors orthogonal to $(1, 1, 1, 1)$). Let $\Delta$ be the set of vectors in $E$ of length $\sqrt{2}$ with integer coordinates. There are 12 such vectors in all. We use the standard inner product in $\mathbb{R}^4$ and the standard orthonormal basis $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$. Then, it is easy to see that $\Delta = \{\epsilon_i - \epsilon_j | i \neq j\}$. The vectors

$$\alpha_1 = \epsilon_1 - \epsilon_2,$$

$$\alpha_2 = \epsilon_2 - \epsilon_3,$$

and

$$\alpha_3 = \epsilon_3 - \epsilon_4.$$
form a basis of the root system in the sense that each vector in $\Delta$ is a linear combination of these three vectors with integer coefficients, either all nonnegative or all nonpositive. For example, $\epsilon_1 - \epsilon_3 = \alpha_1 + \alpha_2$, $\epsilon_2 - \epsilon_4 = \alpha_2 + \alpha_3$, and $\epsilon_1 - \epsilon_4 = \alpha_1 + \alpha_2 + \alpha_3$. Therefore, $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$, and the set of positive roots $\Delta^+$ is given by

$$\Delta^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}.$$  

Define the matrix $C$ using (1). It is clear that $c_{ii} = 2$ and

$$c_{i,i+1} = 2 \frac{(\alpha_i, \alpha_{i+1})}{(\alpha_{i+1}, \alpha_{i+1})} = -1 \text{ for } i = 1, 2.$$  

Similar calculations lead to the following form of the Cartan matrix

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$  

The complex simple Lie algebras are classified as:

$$A_l, B_l, C_l, D_l, E_6, E_7, E_8, F_4, G_2.$$  

Traditionally, $A_l, B_l, C_l, D_l$ are called the classical Lie algebras, while $E_6, E_7, E_8, F_4, G_2$ are called the exceptional Lie algebras. Moreover, for any Cartan matrix there exists just one complex simple Lie algebra up to isomorphism that gives rise to it. The classification is due to Killing and Cartan. According to A. J. Coleman [4], the classification paper of Killing is the greatest mathematical writing of all time (after Euclid’s Elements and Newton’s Principia). In Table 1, we list the determinants for the Cartan matrices of complex simple Lie algebras.

<table>
<thead>
<tr>
<th>Lie algebra</th>
<th>$A_n$</th>
<th>$B_n$</th>
<th>$C_n$</th>
<th>$D_n$</th>
<th>$E_6$</th>
<th>$E_7$</th>
<th>$E_8$</th>
<th>$F_4$</th>
<th>$G_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>det</td>
<td>$n + 1$</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Simple Lie algebras over $\mathbb{C}$ are classified by using the associated Dynkin diagram, a graph whose vertices correspond to the elements of $\Pi$. Each pair of vertices $\alpha_i, \alpha_j$ are connected by

$$m_{ij} = 4(\alpha_i, \alpha_j)^2 \frac{1}{(\alpha_i, \alpha_i)(\alpha_j, \alpha_j)}$$  

edges, where

$$m_{ij} \in \{0, 1, 2, 3\}.$$  

To a given Dynkin diagram $\Gamma$ with $n$ nodes, we associate the Coxeter adjacency matrix, which is the $n \times n$ matrix $A = 2I - C$, where $C$ is the Cartan matrix.

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characteristic polynomial $a_n(x)$ of the adjacency matrix is simply related to the characteristic polynomial of the Cartan matrix $p_n(x)$. In fact,

$$a_n(x) = p_n(x + 2).$$

It follows that, if $\lambda$ is an eigenvalue of the adjacency matrix, then $2 + \lambda$ is an eigenvalue of the corresponding Cartan matrix.

Let us recall the definition of exponents for a simple complex Lie group $G$; see [3], [5], [12]. We form the de Rham cohomology groups $H^i(G, \mathbb{C})$ and the corresponding Poincaré polynomial of $G$:

$$p_G(t) = \sum_{i=1}^{\ell} b_i t^i,$$

where $b_i = \dim H^i(G, \mathbb{C})$ are the Betti numbers of $G$. The De Rham groups encode topological information about $G$. Following work of Cartan, Ponrjagin, and Brauer, Hopf proved that the cohomology algebra is isomorphic to that of a finite product of $\ell$ spheres of odd dimension, where $\ell$ is the rank of $G$. This result implies that

$$p_G(t) = \prod_{i=1}^{\ell} \left(1 + t^{2m_i+1}\right).$$

The positive integers $\{m_1, m_2, \ldots, m_\ell\}$ are called the exponents of $G$. They are also the exponents of the Lie algebra $\mathfrak{g}$ of $G$. We can also extract the exponents from the root space decomposition of $\mathfrak{g}$ following methods developed by R. Bott, A. Shapiro, R. Steinberg, A. J. Coleman, and B. Kostant. We describe a method that was observed empirically by A. Shapiro and R. Steinberg, and proved in a uniform way by B. Kostant. If $\alpha \in \Delta^+$, then we can express $\alpha$ uniquely as a sum of simple roots, $\alpha = \sum_{i=1}^{\ell} n_i \alpha_i$, where $n_i$ are nonnegative integers. The height of $\alpha$ is defined to be the number

$$\text{ht}(\alpha) = \sum_{i=1}^{\ell} n_i.$$

The procedure follows.

**Proposition 1.** Let $b_j$ be the number of $\alpha \in \Delta^+$ such that $\text{ht}(\alpha) = j$. Then $b_j - b_{j+1}$ is the number of times $j$ appears as an exponent of $\mathfrak{g}$. Furthermore,

$$\ell = b_1 \geq b_2 \geq \cdots \geq b_{h-1},$$

where $h$ is the Coxeter number. Moreover, the partition of $r = |\Delta^+|$ as defined by the above sequence is conjugate to the partition

$$h - 1 = m_\ell \geq m_{\ell-1} \geq \cdots \geq m_1 = 1.$$

The Coxeter number is determined by the relation $h\ell = 2r$. 

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Example 2. Consider the previous example of $A_3$. Recall that $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ and the set of positive roots $\Delta^+$ is given by

$$\Delta^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}.$$ 

Therefore, $r = 6$ and the Coxeter number is determined by $h \ell = 2r$, i.e., $3h = 12$, or $h = 4$. The number of roots of height 1 is 3; therefore, $b_1 = 3$. Similarly, $b_2 = 2$ and $b_3 = 1$. Therefore, the exponents are $m_3 = 3$, $m_2 = 2$, and $m_1 = 1$.

The exponents of a complex simple Lie algebra are given in Table 2.

**Table 2.** Exponents and Coxeter number for root systems

<table>
<thead>
<tr>
<th>Root system</th>
<th>Exponents</th>
<th>Coxeter number</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>$1, 2, 3, \ldots, n$</td>
<td>$n + 1$</td>
</tr>
<tr>
<td>$B_n$</td>
<td>$1, 3, 5, \ldots, 2n - 1$</td>
<td>$2n$</td>
</tr>
<tr>
<td>$C_n$</td>
<td>$1, 3, 5, \ldots, 2n - 1$</td>
<td>$2n$</td>
</tr>
<tr>
<td>$D_n$</td>
<td>$1, 3, 5, \ldots, 2n - 3, n - 1$</td>
<td>$2n - 2$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$1, 4, 5, 7, 8, 11$</td>
<td>$12$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$1, 5, 7, 9, 11, 13, 17$</td>
<td>$18$</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$1, 7, 11, 13, 17, 19, 23, 29$</td>
<td>$30$</td>
</tr>
<tr>
<td>$F_4$</td>
<td>$1, 5, 7, 11$</td>
<td>$12$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$1, 5$</td>
<td>$6$</td>
</tr>
</tbody>
</table>

Note in the table, the well-known duality of the set of exponents

$$m_i + m_{\ell + 1 - i} = h,$$  \hspace{1cm} (3)

where $h$ is the Coxeter number.

An alternative way to view the exponents is by considering the corresponding Coxeter group. A Coxeter graph is a simple graph $\Gamma$ with $n$ vertices and edge weights $m_{ij} \in \{3, 4, \ldots, \infty\}$. We define $m_{ij} = 1$ and $m_{ij} = 2$ if node $i$ is not connected with node $j$. By convention, if $m_{ij} = 3$, then the edge is often not labeled. If $\Gamma$ is a Coxeter graph with $n$ vertices, then we define a bilinear form $B$ on $\mathbb{R}^n$ by choosing a basis $e_1, e_2, \ldots, e_n$ and setting

$$B(e_i, e_j) = -2 \cos \frac{\pi}{m_{ij}}.$$ 

If $m_{ij} = \infty$, then we define $B(e_i, e_j) = -2$. We also define, for $i = 1, 2, \ldots, n$, the reflections

$$\sigma_i(e_j) = e_j - B(e_i, e_j)e_i.$$ 

Let $S = \{\sigma_i \mid i = 1, \ldots, n\}$. The Coxeter group $W(\Gamma)$ is the group generated by the reflections in $S$. The group $W$ has the presentation

$$W = (\sigma_1, \sigma_2, \ldots, \sigma_n \mid \sigma_i^2 = 1, (\sigma_i \sigma_j)^{m_{ij}} = 1).$$

It is well known that $W(\Gamma)$ is finite if and only if $B$ is positive definite. A Coxeter element (or transformation) is a product of the form

$$\sigma_{\alpha(1)}\sigma_{\alpha(2)} \cdots \sigma_{\alpha(n)} \in S_n.$$
If the Coxeter graph is a tree, then the Coxeter elements are in a single conjugacy class in \(W\). A Coxeter polynomial for the Coxeter system \((W, S)\) is the characteristic polynomial of the matrix representation of a Coxeter element. For Coxeter systems whose graphs are trees, the Coxeter polynomial is uniquely determined.

**Example 3.** Consider a Coxeter system with graph \(A_3\). The bilinear form is defined by the Cartan matrix

\[
C = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2
\end{pmatrix}.
\]

![Figure 1.](image)

The reflection \(\sigma_1\) is determined by the action

\[
\sigma_1(e_1) = -e_1, \quad \sigma_1(e_2) = e_1 + e_2, \quad \sigma_1(e_3) = e_3.
\]

It has the matrix representation

\[
\sigma_1 = \begin{pmatrix}
-1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Similarly,

\[
\sigma_2 = \begin{pmatrix}
1 & 0 & 0 \\
1 & -1 & 1 \\
0 & 0 & 1
\end{pmatrix}, \quad \text{and} \quad \sigma_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & -1
\end{pmatrix}.
\]

A Coxeter element is defined by

\[
R = \sigma_1\sigma_2\sigma_3 = \begin{pmatrix}
0 & 0 & -1 \\
1 & 0 & -1 \\
0 & 1 & -1
\end{pmatrix}.
\]

The Coxeter polynomial is the characteristic polynomial of \(R\), which is

\[
x^3 + x^2 + x + 1.
\]

Note that \(R\) has order 4, i.e., \(R^4 = I\). The order of the Coxeter element is called the Coxeter number. The roots of the Coxeter polynomial are \(-1, -i, i\). We can write \(i = i^1, -1 = i^2, -i = i^3\). The integers 1, 2, 3 are the exponents of \(W\). In general, the order of the Coxeter (or Weyl or Killing) group is

\[
(m_1 + 1)(m_2 + 1) \cdots (m_l + 1),
\]

where \(m_i\) are the exponents. In the present example, \(W = S_4\).

To prove the sine formula, we use the following two facts.
Lemma 1. The eigenvalues of $C$ occur in pairs $\{\xi, 4 - \xi\}.$

See [2, p. 345] for a general proof. A proof of this lemma is given in section 8 using the duality of the exponents and the following lemma.

Lemma 2. The roots of $a_n(x)$ are

$$2 \cos \frac{m_j \pi}{h},$$

where $m_j$ are the exponents of $g$ and $h$ is the Coxeter number of $g.$

In this article, we demonstrate this lemma in an elementary fashion using only basic properties of Chebyshev polynomials. Nevertheless, for the interested reader we give some references on how to prove this lemma without a case by case verification; there is an empirical procedure due to Coxeter, which can be used to find the roots of the Coxeter polynomial. If $\zeta$ is a primitive $h$ root of unity (where $h$ is the Coxeter number), then the roots of the Coxeter polynomial are $\zeta^m$, where $m$ runs over the exponents of the corresponding root system [3], [6]. This observation allows the calculation of the Coxeter polynomial for each root system. It also explains the duality (3) since non-real eigenvalues of $R$ appear in conjugate pairs. Coxeter also observed that

$$h\ell = 2r,$$

where $r$ is the number of positive roots. Using (4) as the only empirical fact, Coleman proved in [3] the procedure of Coxeter. The proof of (4) is in the classic 1959 paper of Kostant [12]. Knowing the roots of the Coxeter polynomial, it is straightforward to determine the spectrum of the Cartan matrix $C$; see, e.g., [2, Theorem 2]. This in turn determines the roots of $a_n(x)$ via the relation (2).

Alternatively, we can compute the Coxeter polynomial as follows. Define the associated polynomial

$$Q_n(x) = x^n a_n\left(x + \frac{1}{x}\right).$$

$Q_n(x)$ turns out to be an even, reciprocal polynomial of the form

$$Q_n(x) = f_n(x^2).$$

The polynomial $f_n$ is the Coxeter polynomial of the underlying graph. For more details, see [7]. The procedure in [7] allows us to connect the roots of the Coxeter polynomial with the roots of $a_n(x).$ Let $\theta_j = \frac{m_j \pi}{h}.$ Then the roots of the Coxeter polynomial are $e^{2i\theta_j},$ while the roots of $a_n(x)$ are $2 \cos \theta_j = 2 \cos \frac{m_j \pi}{h}.$

2. CHEBYSHEV POLYNOMIALS. To compute $p_n(x)$ explicitly, we use the following result (see [6] and [11]).

Proposition 2. Let $C$ be the $n \times n$ Cartan matrix of a simple Lie algebra over $\mathbb{C}.$ Let $p_n(x)$ be its characteristic polynomial and define $q_n(x) = \det(2xI + A).$ Then

$$p_n(x) = q_n\left(\frac{x}{2} - 1\right) \quad \text{and} \quad a_n(x) = q_n\left(\frac{x}{2}\right).$$
The polynomial $q_n$ is related to Chebyshev polynomials as follows:

$$A_n : q_n = U_n,$$

$$B_n, C_n : q_n = 2T_n, \text{ and}$$

$$D_n : q_n = 4xT_{n-1}$$

where $T_n$ and $U_n$ are the Chebyshev polynomials of first and second kind, respectively.

Proof. We give an outline of the proof. Note that

$$q_n \left( \frac{x}{2} - 1 \right) = \det \left( 2 \left( \frac{x-2}{2} \right) I_n + A \right)$$

$$= \det \left( xI_n - 2I_n + A \right)$$

$$= \det \left( xI_n - 2I_n + 2I_n - C \right)$$

$$= \det \left( xI_n - C \right) = p_n(x).$$

Furthermore, the matrix $A$ for classical Lie algebras has the form

$$A = \begin{pmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & & \ddots & \ddots & 1 \\
1 & 0 & 1 \\
1 & D
\end{pmatrix},$$

where $D$ is

$$(0), \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

for the cases $A_n, B_n, C_n, D_n$, respectively. The proof is by induction on $n$. Suppose that the result is proved for $n$ and $n-1$. To get $q_{n+1}(x)$, expand the determinant of $2xI_{n+1} + A$ by the first two rows to obtain

$$q_{n+1}(x) = 2xq_n(x) - q_{n-1}.$$ 

This is the recurrence relation satisfied by the Chebyshev polynomials $T_n$ and $U_n$. Sections 5 and 6 cover in detail the cases $A_n$ and $B_n$, respectively. In section 7, we outline the case of $D_n$. Finally, note that

$$a_n(x) = p_n(x + 2) = q_n \left( \frac{x+2}{2} - 1 \right) = q_n \left( \frac{x}{2} \right).$$

Orthogonal polynomials are associated with three-term recurrences, which in turn correspond to tridiagonal matrices. A sequence of orthogonal polynomials $P_n(x)$ is
characterized by a three-term recurrence of the form

$$x P_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x).$$

We associate to each such recurrence, a tridiagonal matrix:

$$J_{n+1} = \begin{pmatrix}
\beta_0 & \alpha_0 & & \\
\gamma_1 & \beta_1 & \alpha_1 & \\
& \ddots & \ddots & \ddots \\
& & \gamma_{n-1} & \beta_{n-1} & \alpha_{n-1} \\
& & & \gamma_n & \beta_n
\end{pmatrix}.$$  

Each zero of the polynomial $P_n(x)$ is an eigenvalue of $J_n$, and the characteristic polynomial of $J_n$ is precisely $P_n(x)$. In our case, the Cartan matrices are either tridiagonal or close to tridiagonal. The corresponding family of orthogonal polynomials are the Chebyshev polynomials. It is remarkable that the Chebyshev polynomials give the universal solution of any second-order recurrence relation with constant coefficients; see [1, Theorem 3.1, p. 617]. Moreover, it is clear from what follows that Chebyshev polynomials were designed in order to fit nicely the theory of complex simple Lie algebras. The precise definition and some basic properties of $U_n$ and $T_n$ will be given in sections 5 and 6, respectively.

In the case of exceptional Lie algebras, we can directly compute (preferably using a symbolic manipulation package) the characteristic polynomial $a_n(x)$ for each exceptional type:

- $G_2 : a_2(x) = x^2 - 3$,
- $F_4 : a_4(x) = x^4 - 4x^2 + 1$,
- $E_6 : a_6(x) = x^6 - 5x^4 + 5x^2 - 1$,
- $E_7 : a_7(x) = x^7 - 6x^5 + 9x^3 - 3x$, and
- $E_8 : a_8(x) = x^8 - 7x^6 + 14x^4 - 8x^2 + 1$.

Lemma 2 is then easily verified.

3. THE $A_n$ SINE FORMULA. Toeplitz matrices have constant entries on each diagonal parallel to the main diagonal. Tridiagonal Toeplitz matrices are commonly the result of discretizing differential equations.

The eigenvalues of the Toeplitz matrix

$$\begin{pmatrix}
b & a & & \\
c & b & a & \\
& \ddots & \ddots & \ddots \\
& & c & b & a \\
& & & c & b
\end{pmatrix}$$

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are given by
\[ \lambda_j = b + 2a \sqrt{\frac{c}{a}} \cos \frac{j\pi}{n + 1} \quad \text{for} \quad j = 1, 2, \ldots, n. \] (6)

See, e.g., [14, p. 59].

The Cartan matrix of type $A_n$ is a tridiagonal Toeplitz matrix of the form
\[
C_{A_n} = \begin{pmatrix}
2 & -1 & & & \\
-1 & 2 & -1 & & \\
& \ddots & \ddots & \ddots & \\
& & -1 & 2 & -1 \\
& & & -1 & 2
\end{pmatrix}.
\] (7)

Taking $a = c = -1$, $b = 2$ in (6) immediately yields the following.

**Proposition 3.** The eigenvalues of $C_{A_n}$ are given by
\[ \lambda_j = 2 - 2 \cos \frac{j\pi}{n + 1} = 4 \sin^2 \frac{j\pi}{2(n + 1)} \quad \text{for} \quad j = 1, 2, \ldots, n. \]

Let $d_n$ be the determinant of $C_{A_n}$. We can compute $d_n$ by using expansion on the first row and induction. The result is $d_n = 2d_{n-1} - d_{n-2}$, $d_1 = 2$, $d_2 = 3$. This is a simple linear recurrence with solution $d_n = n + 1$.

We conclude that
\[
\prod_{j=1}^{n} 4 \sin^2 \frac{j\pi}{2(n + 1)} = n + 1.
\]

Equivalently,
\[
2^n \prod_{j=1}^{n} \sin^2 \frac{j\pi}{2(n + 1)} = n + 1 \quad \text{(the $A_n$ sine formula)}. \] (8)

We refer to this relation as the $A_n$ sine formula. This formula will be generalized for each complex simple Lie algebra.

4. PRODUCT OF DISTANCES FROM 1 TO THE OTHER ROOTS OF UNITY.
The following problem is well known. Consider a regular polygon inscribed in the unit circle with one vertex at the point $(1, 0)$. Find the product of the distances from the point $(1, 0)$ to the other $n - 1$ vertices. The answer is $n$.

**Proof.** Consider the unit circle $|z| = 1$ in the complex plane. Let $\omega = e^{2\pi i/n}$. Then, the other vertices of the polygon are the roots of unity $\omega, \omega^2, \ldots, \omega^{n-1}$. These numbers, together with 1, are roots of the polynomial $z^n - 1$. Therefore,
\[ z^n - 1 = (z - 1)(z - \omega)(z - \omega^2) \cdots (z - \omega^{n-1}). \]
Divide both sides by $z - 1$ to obtain

$$z^{n-1} + z^{n-2} + \cdots + z + 1 = (z - \omega)(z - \omega^2) \cdots (z - \omega^{n-1}).$$

Substitute $z = 1$ and take absolute values to obtain the result.

Let $\theta = \frac{2\pi}{n}$. Then, $\omega^k = e^{ik\theta}$ and the distance from 1 to $\omega^k$ is

$$|1 - \omega^k| = |1 - (\cos k\theta + i \sin k\theta)| = \sqrt{2 - 2 \cos k\theta} = 2 \sin \frac{k\theta}{2} = 2 \sin \frac{\pi k}{n}.$$

As a result, we obtain the formula

$$\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}}. \quad (9)$$

This formula was standard in textbooks in the 19th century. For example, it is derived in S. L. Loney’s book *Plane Trigonometry*, one of the books that Ramanujan used to learn mathematics.

We are now in a position to give a different proof of the $A_n$ sine formula (8). Consider the formula (9) with $n$ replaced by $2(n + 1)$. It becomes

$$\prod_{k=1}^{2n+1} \sin \frac{k\pi}{2(n + 1)} = \frac{n + 1}{2^{2n}} = \frac{n + 1}{4^n}.$$

We split the product on the left-hand side in the following way:

$$\prod_{k=1}^{n} \sin \frac{k\pi}{2n + 2} \prod_{k=n+2}^{2n+1} \sin \frac{k\pi}{2n + 2}.$$

Note also that

$$\prod_{k=n+2}^{2n+1} \sin \frac{k\pi}{2n + 2} = \prod_{k=n+2}^{2n+1} \sin \left( \frac{\pi}{2n + 2} \right)$$

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\[
\begin{align*}
&= \prod_{k=n+2}^{2n+1} \sin \frac{2n+2-k}{2n+2} \pi \\
&= \prod_{j=n}^{2n-1} \sin \frac{2n-j}{2n+2} \pi.
\end{align*}
\]

This last product is the same as

\[
\prod_{k=1}^{n} \sin \frac{k\pi}{2n+2},
\]

in reversed order. Consequently,

\[
\prod_{k=1}^{n} \sin^2 \frac{k\pi}{2(n+1)} = \frac{n+1}{4^n},
\]

which yields formula (8) exactly.

5. CARTAN MATRIX OF TYPE \(A_n\). The Chebyshev polynomials form an infinite sequence of orthogonal polynomials. The Chebyshev polynomial of the second kind of degree \(n\) is usually denoted by \(U_n\). We list some properties of Chebyshev polynomials following [13], [15].

A fancy way to define the \(n\)th Chebyshev polynomial of the second kind is

\[
U_n(x) = \det \begin{pmatrix} 2x & 1 & & & \\ 1 & 2x & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 2x \\ & & & & 1 \end{pmatrix},
\]

(10)

where \(n\) is the size of the matrix. By expanding the determinant with respect to the first row, we get the recurrence

\[
U_0(x) = 1, \quad U_1(x) = 2x, \quad U_{n+1}(x) = 2x U_n(x) - U_{n-1}(x).
\]

(11)

It is easy then to compute recursively the first few polynomials:

\[
\begin{align*}
U_0(x) &= 1, \\
U_1(x) &= 2x, \\
U_2(x) &= 4x^2 - 1, \\
U_3(x) &= 8x^3 - 4x, \\
U_4(x) &= 16x^4 - 12x^2 + 1, \\
U_5(x) &= 32x^5 - 32x^3 + 6x, \text{ and} \\
U_6(x) &= 64x^6 - 80x^4 + 24x^2 - 1.
\end{align*}
\]
Letting $x = \cos \theta$, we obtain

$$U_n(x) = \frac{\sin((n + 1)\theta)}{\sin \theta}.$$ 

Knowledge of the roots of $U_n$ implies that

$$U_n(x) = 2^n \prod_{j=1}^{n} \left(x - \cos \left(\frac{j\pi}{n+1}\right)\right).$$ (12)

Setting $x = 1$ in this equation, we obtain again the $A_n$ sine formula (8).

We note that the matrix $2xI + A$, where $A$ is the adjacency matrix of the $A_n$ graph, coincides with the matrix in (10). Consequently, we conclude that

$$q_n(x) = U_n(x).$$

Hence, $a_n(x) = U_n \left(\frac{1}{2}\right)$. It is well known that the roots of $U_n$ are given by

$$x_k = \cos \frac{k\pi}{n+1} \text{ for } k = 1, 2, \ldots, n,$$

as we already observed in (12). We can write them in the form $x_k = \cos k\theta$, where $\theta = \frac{\pi}{n+1}$. The roots of $a_n(x)$ are then

$$\lambda_k = 2 \cos \frac{k\pi}{n+1} = 2 \cos k\theta \text{ for } k = 1, 2, \ldots, n,$$

i.e., the roots of $a_n(x)$ are

$$2 \cos \frac{m_i \pi}{h}$$

where $m_i$ are the exponents of $A_n$ and $h$ is the Coxeter number.

6. CARTAN MATRIX OF TYPE $B_n$ AND $C_n$. The Chebyshev polynomials of the first kind are denoted by $T_n(x)$. They are defined by the recurrence:

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

Using this recursion, we may compute the first few polynomials as follows:

$$T_0(x) = 1,$$
$$T_1(x) = x,$$
$$T_2(x) = 2x^2 - 1,$$
$$T_3(x) = 4x^3 - 3x,$$
$$T_4(x) = 8x^4 - 8x^2 + 1,$$
$$T_5(x) = 16x^5 - 20x^3 + 5x,$$ and
$$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1.$$
It is well known that \( \cos(n \theta) \) can be expressed as a polynomial in \( \cos(\theta) \). For example,

\[
\begin{align*}
\cos(0 \theta) &= 1, \\
\cos(1 \theta) &= \cos \theta, \\
\cos(2 \theta) &= 2(\cos \theta)^2 - 1, \text{ and} \\
\cos(3 \theta) &= 4(\cos \theta)^3 - 3(\cos \theta).
\end{align*}
\]

More generally, we have that

\[
\cos(n \theta) = T_n(\cos \theta).
\]

The roots of \( T_n(x) \) are well known:

\[
T_n(x) = 2^n \prod_{j=1}^{n} \left[ x - \cos \left( \frac{(2j - 1)\pi}{2n} \right) \right].
\] (13)

Setting \( x = 1 \) in this equation quickly leads to the formula

\[
2^n \prod_{j=1}^{n} \sin^2 \left( \frac{(2j - 1)\pi}{4n} \right) = 2 \text{ (the } B_n \text{ sine formula).}
\] (14)

We refer to this relation as the \( B_n \) sine formula.

The Cartan matrix is a tridiagonal matrix of the form

\[
C_{B_n} = \begin{pmatrix}
2 & -1 & & \\
-1 & 2 & -1 & \\
& \ddots & \ddots & \ddots \\
& & -1 & 2 & -1 \\
& & & -1 & 2 & -2 \\
& & & & -1 & 2
\end{pmatrix}.
\] (15)

Since the Cartan matrix of type \( C_n \) is the transpose of this matrix, we consider only the Cartan matrix of type \( B_n \). Using expansion on the first row, it is easy to prove that \( \det(C_{B_n}) = 2 \).

Likewise, by expanding the determinant of the matrix \( 2x I + A \) with respect to the first row, we obtain the recurrence

\[
q_1(x) = 2x, \quad q_2(x) = 4x^2 - 2, \text{ and } q_{n+1}(x) = 2xq_n(x) - q_{n-1}(x).
\]

Define \( q_0(x) = 2 \). The recurrence implies that \( q_n(x) = 2T_n(x) \), where \( T_n \) is the \( n \)th Chebyshev polynomial of the first kind. The roots of \( T_n(x) \) are given by

\[
x_k = \cos \left( \frac{(2k - 1)\pi}{2n} \right) \quad \text{for} \quad k = 1, 2, \ldots, n.
\]

That is,

\[
x_k = \cos(2k - 1)\theta \quad \text{for} \quad k = 1, 2, \ldots, n,
\]
where \( \theta = \frac{\pi}{2n} \). We conclude that the roots of \( a_n(x) \) are

\[
\lambda_k = 2 \cos \left( \frac{(2k-1)\pi}{2n} \right) = 2 \cos (2k-1)\theta \quad \text{for} \quad k = 1, 2, \ldots, n.
\]

We have verified that Lemma 2 holds in this case.

7. CARTAN MATRIX OF TYPE \( D_n \). The Cartan matrix of type \( D_n \) is a matrix of the form

\[
C_{D_n} = \begin{pmatrix}
2 & -1 & 0 & & \\
-1 & 2 & -1 & & \\
& -1 & 2 & -1 & \\
& & -1 & 2 & -1 \\
& & & -1 & 2
\end{pmatrix}.
\]

(16)

Note that the matrix is no longer tridiagonal. Using expansion on the first row and induction, we find that \( \det(C_{D_n}) = 4 \).

By expanding the determinant of the matrix \( 2x I + A \) with respect to the first row, we deduce the recurrence

\[
q_2(x) = 4x^2, \quad q_3(x) = 8x^3 - 4x, \quad \text{and} \quad q_{n+1}(x) = 2xq_n(x) - q_{n-1}.
\]

Define \( q_1(x) = 4x \). It is clear that \( q_n(x) = 4xT_{n-1}(x) \), where \( T_n \) is the \( n \)th Chebyshev polynomial of the first kind.

We now proceed to verify Lemma 2 in the case of \( D_n \). We have that

\[
a_n(x) = 2xT_{n-1}\left( \frac{x}{2} \right).
\]

The equation \( a_n(x_0) = 0 \) is therefore equivalent to \( 2x_0T_{n-1}(x_0) = 0 \). As a result, either \( x_0 = 0 \) or \( x_0 = 2 \cos \left( \frac{(2k-1)\pi}{2(n-1)} \right) \) for \( k = 1, 2, \ldots, n-1 \).

8. THE SINE FORMULA. We have seen that the roots of \( a_n(x) \) are

\[
2 \cos \frac{m_i \pi}{h},
\]

where \( m_i \) are the exponents of \( g \) and \( h \) is the Coxeter number of \( g \). Let \( \theta_i = \frac{m_i \pi}{h} \). Then the roots of \( a_n(x) \) are \( \lambda_i = 2 \cos \theta_i \), for \( i = 1, 2, \ldots, \ell \).

Recall the duality property of the exponents (3):

\[
m_i + m_{\ell+1-i} = h.
\]

It follows that

\[
m_i \frac{\pi}{h} + m_{\ell+1-i} \frac{\pi}{h} = \pi.
\]

This leads to

\[
\theta_i + \theta_{\ell+1-i} = \pi.
\]
Using this formula, we can infer a relationship satisfied by the roots of $p_n(x)$, which are

$$\xi_i = 4\cos^2 \frac{\theta_i}{2}.$$ 

Namely,

$$\xi_i + \xi_{\ell+1-i} = 4\cos^2 \frac{\theta_i}{2} + 4\cos^2 \frac{\theta_{\ell+1-i}}{2} = 4 \left( \cos^2 \frac{\theta_i}{2} + \cos^2 \frac{\theta_{\ell+1-i}}{2} \right) = 4 \left( \cos^2 \frac{\theta_i}{2} + \sin^2 \frac{\theta_i}{2} \right) = 4. $$

In other words, we have proved Lemma 1: The eigenvalues of the Cartan matrix occur in pairs $\xi, 4 - \xi$.

**Theorem 1.** Let $\mathfrak{g}$ be a complex simple Lie algebra of rank $\ell$, $h$ the Coxeter number, $m_1, m_2, \ldots, m_\ell$ the exponents of $\mathfrak{g}$, and $C$ the Cartan matrix. Then

$$2^{2\ell} \prod_{i=1}^{\ell} \sin^2 \frac{m_i \pi}{2h} = \det C.$$ 

**Proof.** The roots of $a_n(x)$ are

$$2\cos \frac{m_i \pi}{h},$$

where $m_i$ are the exponents of $\mathfrak{g}$ and $h$ is the Coxeter number. Hence,

$$a_n(x) = \prod_{i=1}^{\ell} \left( x - \left( 2 \cos \frac{m_i \pi}{h} \right) \right).$$

Setting $x = 2$, we obtain

$$a_n(2) = \prod_{i=1}^{\ell} \left( 2 - \left( 2 \cos \frac{m_i \pi}{h} \right) \right) = 2^{\ell} \prod_{i=1}^{\ell} \left( 1 - \cos \frac{m_i \pi}{h} \right) = 2^{\ell} \prod_{i=1}^{\ell} 2 \sin^2 \frac{m_i \pi}{2h} = 2^{2\ell} \prod_{i=1}^{\ell} \sin^2 \frac{m_i \pi}{2h}. $$
To establish the formula, we calculate $a_n(2)$. We have

$$p_n(x) = (x - \xi_1)(x - \xi_2) \cdots (x - \xi_\ell) = (x - (4 - \xi_1))(x - (4 - \xi_2)) \cdots (x - (4 - \xi_\ell)).$$

This implies that $p_n(4) = \xi_1\xi_2\cdots\xi_\ell = \det C$. Since $a_n(x) = p_n(x + 2)$, we conclude that

$$a_n(2) = p_n(4) = \det C.$$

ACKNOWLEDGMENTS. The author wishes to thank the reviewers of this article for their helpful comments, which led to a considerable improvement of the presentation.

REFERENCES


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