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The Computer Searches for Pascal Conics*

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Abstract—An approach to discovering new theorems in geometry using numerical examples is discussed. Four classes of theorems related to Pascal conics have been discovered by this approach. Two of them are new.

Keywords—Geometry theorem proving, Pseudo-division, Wu's method, Pascal's theorem, Pascal line, Conic, Pascal conic, Numerical examples.

1. INTRODUCTION

This paper reports our effort to discover and prove new theorems in elementary geometry using computers. Provers based on Wu's method [1–4] are the basic tool in this effort. Since 1983, from time to time, we have proved a few geometry statements unknown to us. They were possibly new theorems. But later on, we found (or were informed) that they were known results in geometry. It is rather frustrating. Elementary geometry, one of oldest branches of mathematics, seems to have been so thoroughly studied for more than 2000 years that any possibly "new" results might be found in someone else's work. However, some among those "unknown" geometry statements proved by our prover still survive up to now: we still do not know whether these results are known to others. Among them are the three Pascal conic theorems "discovered" by Wu and us.

To discover new theorems in geometry using provers based on Wu's method or other algebraic methods (e.g., the Gröbner basis method), the hard thing is to form plausible conjectures. Once we come up with a conjecture that is within the scope of the methods, proovers based on these methods can prove them automatically.

The statement of the first Pascal conic theorem was discovered by Wu based on an ingenious guess; then he confirmed the statement using his program [2]. The statements of the other two Pascal conic theorems were discovered by our technique "*searching for correct conjectures using numerical examples.*" Once we find a conjecture numerically correct, we can use our theorem prover to confirm the conjecture.

In Section 2, we will give a brief introduction to Wu's method. In Section 3, we will propose the Pascal conic problem and explain how we can find correct conjectures by using numerical examples. In Section 4, we will discuss some further topics related to numerical examples.

2. A BRIEF REVIEW OF WU'S METHOD

In this section, we give a brief review of Wu's method. The reader can find the detailed presentations in [2,3,5,6]. Wu's method was introduced as a mechanical method to prove those

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statements in geometry for which, in their algebraic form, the hypotheses and the conclusion can be expressed by polynomial equations.

For such a geometry statement, after adopting an appropriate coordinate system, the hypotheses can be expressed by a set of polynomial equations:

$$\begin{aligned} h_1(u_1, \ldots, u_d, x_1, \ldots, x_t) &= 0, \\ h_2(u_1, \ldots, u_d, x_1, \ldots, x_t) &= 0, \\ &\vdots \\ h_n(u_1, \ldots, u_d, x_1, \ldots, x_t) &= 0, \end{aligned} \tag{2.1}$$

and the conclusion is also a polynomial equation $g(u_1, \ldots, u_d, x_1, \ldots, x_t) = 0$, where $h_1, \ldots, h_n$ and g are polynomials in $\mathbf{Q}[u_1, \ldots, u_d, x_1, \ldots, x_t]$, where $\mathbf{Q}$ is the field of rational numbers. Variables $u_1, \ldots, u_d$ are parameters or independent variables, and variables $x_1, \ldots, x_t$ are algebraically dependent on the u's *under normal conditions*, being restricted by (2.1). Thus, the corresponding algebraic statement would be

$$\forall ux\,[(h_1 = 0 \wedge \cdots \wedge h_n = 0) \Rightarrow g = 0]. \tag{2.2}$$

However, (2.2) is not exact because it is valid only under some additional conditions connected with nondegeneracy. Those additional conditions can be produced automatically by the method. We use the following example to illustrate how the method works.

EXAMPLE 2.3. (Pascal's Theorem) Let A_0, A_1, A_2, A_3, A_4 and A_5 be six points on a circle (O). Let $P = A_0A_1 \cap A_3A_4$, $Q = A_1A_2 \cap A_4A_5$ and $S = A_2A_3 \cap A_5A_0$. Show that P, Q, and S are collinear (Figure 1).

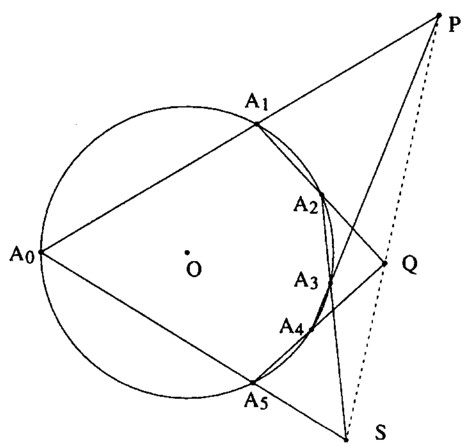

Figure 1. Pascal's theorem.

We can let $O = (u_1, 0)$, $A_0 = (0,0)$, $A_1 = (x_1, u_2)$, $A_2 = (x_2, u_3)$, $A_3 = (x_3, u_4)$, $A_4 = (x_4, u_5)$, $A_5 = (x_5, u_6)$, $P = (x_7, x_6)$, $Q = (x_9, x_8)$, and $S = (x_{11}, x_{10})$. Then the hypothesis can be expressed by the following set of 11 equations:

$h_1 = x_1^2 - 2u_1x_1 + u_2^2 = 0,$	$OA_0 = OA_1,$
$h_2 = x_2^2 - 2u_1x_2 + u_3^2 = 0,$	$OA_0 \equiv OA_2,$
$h_3 = x_3^2 - 2u_1x_3 + u_4^2 = 0,$	$OA_0 \equiv OA_3,$
$h_4 = x_4^2 - 2u_1x_4 + u_5^2 = 0,$	$OA_0 \equiv OA_4,$
$h_5 = x_5^2 - 2u_1x_5 + u_6^2 = 0,$	$OA_0 \equiv OA_5,$
$h_6 = (u_5 - u_4)x_7 + (-x_4 + x_3)x_6 + u_4x_4 - u_5x_3 = 0,$	Points P, A_3 and A_4 are collinear,

$h_7 = u_2x_7 - x_1x_6 = 0,$	Points P, A_0 and A_1 are collinear,
$h_8 = (u_6 - u_5)x_9 + (-x_5 + x_4)x_8 + u_5x_5 - u_6x_4 = 0,$	Points Q, A_4 and A_5 are collinear,
$h_9 = (u_3 - u_2)x_9 + (-x_2 + x_1)x_8 + u_2x_2 - u_3x_1 = 0,$	Points Q, A_1 and A_2 are collinear,
$h_{10} = u_6x_{11} - x_5x_{10} = 0,$	Points S, A_5 and A_0 are collinear,
$h_{11} = (u_4 - u_3)x_{11} + (-x_3 + x_2)x_{10} + u_3x_3 - u_4x_2 = 0,$	Points S, A_2 and A_3 are collinear.

Also, the conclusion that points P, Q, and S are collinear can be expressed by the equation $g=(x_8-x_6)x_{11}-(x_9-x_7)x_{10}+x_6x_9-x_7x_8=0$. As we know, the formula $(h_1=0 \wedge \cdots \wedge h_{11}=0) \Rightarrow g=0$ is usually not valid because of the missing nondegenerate conditions. The method can produce sufficiently many nondegenerate conditions to make the above formula valid.

The next step is to transform $h_1, \ldots, h_{11}$ into triangular form $f_1, \ldots, f_{11}$; that is, each f_i introduces only one new dependent variable x_i. The triangulation procedure is very similar to the Gauss elimination. The reader can invent his own easily, using pseudo-divisions (see, e.g., [5]). A complete triangulation algorithm was implicit in Ritt's work [7] and rewritten in detail and referred to as Ritt's principle in [2] by Wu. For the configurations of most geometry theorems, new points are constructed one by one; that is, new dependent variables are introduced at most two by two. For such kinds of geometry theorems, triangulation can be done according to *the geometric constructions*, as our theorem prover actually does for such kinds of *constructive geometric statements* (see [4,8]). For example, in this problem, we can let $f_{10} = \text{prem}(h_{11}, h_{10}, x_{11})$, $f_8 = \text{prem}(h_9, h_8, x_9)$, $f_6 = \text{prem}(h_7, h_6, x_7)$,[1] and $f_i = h_i$ for $i = 1,2,3,4,5,7,9,11$. Then $f_1, \ldots, f_{11}$ are in triangular form:

$$
\begin{aligned}
f_1 &= x_1^2 - 2u_1x_1 + u_2^2 = 0,\\
f_2 &= x_2^2 - 2u_1x_2 + u_3^2 = 0,\\
f_3 &= x_3^2 - 2u_1x_3 + u_4^2 = 0,\\
f_4 &= x_4^2 - 2u_1x_4 + u_5^2 = 0,\\
f_5 &= x_5^2 - 2u_1x_5 + u_6^2 = 0,\\
f_6 &= (u_2x_4 - u_2x_3 + (-u_5 + u_4)x_1)\, x_6 - u_2u_4x_4 + u_2u_5x_3 = 0,\\
f_7 &= u_2x_7 - x_1x_6 = 0,\\
f_8 &= ((u_3 - u_2)x_5 + (-u_3 + u_2)x_4 + (-u_6 + u_5)x_2 + (u_6 - u_5)x_1)\, x_8 + (-u_3 + u_2)u_5x_5,\\
&\quad + (u_3 - u_2)u_6x_4 + (u_2u_6 - u_2u_5)x_2 + (-u_3u_6 + u_3u_5)x_1 = 0,\\
f_9 &= (u_3 - u_2)x_9 + (-x_2 + x_1)x_8 + u_2x_2 - u_3x_1 = 0,\\
f_{10} &= ((u_4 - u_3)x_5 - u_6x_3 + u_6x_2)\, x_{10} + u_3u_6x_3 - u_4u_6x_2 = 0,\\
f_{11} &= u_6x_{11} - x_5x_{10} = 0.
\end{aligned}
$$

The above three pseudo-divisions correspond to the three constructions: taking intersections $P = A_0A_1 \cap A_3A_4$, $Q = A_1A_2 \cap A_4A_5$, and $S = A_2A_3 \cap A_5A_0$.

Then the next step is to do the following successive pseudo-divisions.

$R_{11} = \text{prem}(g, f_{11}, x_{11}),$	R_{11} is a polynomial with 6 terms.
$R_{10} = \text{prem}(R_{11}, f_{10}, x_{10}),$	R_{10} is a polynomial with 16 terms.
$R_9 = \text{prem}(R_{10}, f_9, x_9),$	R_9 is a polynomial with 42 terms.
$R_8 = \text{prem}(R_9, f_8, x_8),$	R_8 is a polynomial with 196 terms.
$R_7 = \text{prem}(R_8, f_7, x_7),$	R_7 is a polynomial with 160 terms.

[1] We denote $\text{prem}(h_7, h_6, x_7)$ the pseudo-remainder of h_7 by h_6 in the variable x_7. For pseudo-division algorithm see, e.g., [3].

$R_6 = \mathrm{prem}(R_7, f_6, x_6)$,	R_6 is a polynomial with 228 terms.
$R_5 = \mathrm{prem}(R_6, f_5, x_5)$,	R_5 is a polynomial with 272 terms.
$R_4 = \mathrm{prem}(R_5, f_4, x_4)$,	R_4 is a polynomial with 272 terms.
$R_3 = \mathrm{prem}(R_4, f_3, x_3)$,	R_3 is a polynomial with 228 terms.
$R_2 = \mathrm{prem}(R_3, f_2, x_2)$,	R_2 is a polynomial with 144 terms.
$R_1 = \mathrm{prem}(R_2, f_1, x_1)$,	$R_1 = 0$.

The last expression $R_1 = 0$ means that we have proved the theorem. To see this, let us recall the simple and important *remainder formula* for successive pseudo-divisions of g with respect to a triangular form $f_1, \ldots, f_r$ (see, for example, [3,5]):

$$I_1^{s_1} \cdots I_r^{s_r} g = Q_1 f_1 + \cdots + Q_r f_r + R_1,$$

where the I_k are the leading coefficients of the f_k in the variables x_k; s_i are nonnegative integers, and Q_j are polynomials. Since $R_1 = 0$ and all $f_i = 0$, we can infer $g = 0$ if we assume $I_k \neq 0$ ($k = 1, \ldots, r$). The subsidiary conditions $I_k \neq 0$ are usually connected with nondegeneracy. In our case, they are

$$\begin{aligned}
I_6 &= u_2x_4 - u_2x_3 - (u_5 - u_4)x_1 \neq 0,\\
I_7 &= u_2 \neq 0,\\
I_8 &= (u_3 - u_2)x_5 + (-u_3 + u_2)x_4 + (-u_6 + u_5)x_2 + (u_6 - u_5)x_1 \neq 0,\\
I_9 &= u_3 - u_2 \neq 0,\\
I_{10} &= (u_4 - u_3)x_5 - u_6x_3 + u_6x_2 \neq 0,\\
I_{11} &= u_6 \neq 0.
\end{aligned}$$

By a more careful analysis (this can be done mechanically; see [4,8]), $I_7 \neq 0$, $I_9 \neq 0$, and $I_{11} \neq 0$ are redundant, while $I_6 \neq 0$, $I_8 \neq 0$, and $I_{10} \neq 0$ have the geometric meanings: each of the three pairs of lines, A_0A_1 and A_3A_4, A_1A_2 and A_4A_5, and A_2A_3 and A_5A_0, has a normal intersection (only one common point). Note that I_6, I_8, and I_{10} are produced from the three pseudo-divisions in the triangular procedure.

The above proof took about 1.5 seconds on a Symbolics 3600. This is the simplest and most elementary part of Wu's method. Hundreds of theorems can be proved by the above simple technique, including the Pascal conic theorems discovered by Wu and us (see the next section). For a description of the complete method of Wu and examples, see [6, Chapter 4].

3. THE PASCAL CONIC PROBLEM

First we note that since Pascal's theorem has been proved to be valid for the case when points A_0, A_1, A_2, A_3, A_4, and A_5 are on a circle, it is also valid when they are on a conic. This is because under a projective mapping, a circle maps to a conic and the relation of incidence is preserved. Such six points form a *Pascal configuration.* Second, we note that the hexagon $A_0A_1A_2A_3A_4A_5$ is not necessarily convex. Let us call the intersection $A_iA_j \cap A_kA_l$ (with distinct i, j, k, and l) a *Pascal point.* There are 45 Pascal points for a given Pascal configuration. Let $[i_0\ i_1\ i_2\ i_3\ i_4\ i_5]$ be a permutation of [0 1 2 3 4 5]. Then the Pascal points $P' = A_{i_0}A_{i_1} \cap A_{i_3}A_{i_4}$, $Q' = A_{i_1}A_{i_2} \cap A_{i_4}A_{i_5}$ and $S' = A_{i_2}A_{i_3} \cap A_{i_5}A_{i_0}$ are also on the same line which is denoted by $[i_0\ i_1\ i_2\ i_3\ i_4\ i_5]$. For example, the Pascal line PQS in Figure 1 is denoted by [0 1 2 3 4 5]. Thus, we have actually proved that there are 60 3-tuples of Pascal points that are collinear. We call these 60 lines the *Pascal lines.*

The Pascal configuration was a subject studied extensively by many great geometers of the last century, including Steiner, Staudt, Cayley, Kirkman, and Salmon. For example, we have the following two theorems [9]:

STEINER'S THEOREM. *The three Pascal lines* [0 1 4 3 2 5], [2 3 0 5 4 1], *and* [4 5 2 1 0 3] *pass through a point which is called a Steiner point. There are 20 Steiner points for a given Pascal configuration.*

KIRKMAN'S THEOREM. *The three Pascal lines* [1 0 4 2 3 5], [2 3 1 5 4 0], *and* [5 4 2 0 1 3] *pass through a point which is called a Kirkman point. There exist 60 Kirkman points.*

There are further properties on the Pascal configuration. The 20 Steiner points, in turn, lie four by four on 15 lines, called the Plück lines (Plück's theorem). Corresponding to each Steiner point, there are three Kirkman points such that all four points lie upon a line, called a Cayley line (Cayley's theorem). There are 20 Cayley lines, and they pass four by four through 15 points, called the Salmon points (Salmon's theorem).

All these theorems were discovered in an *ad hoc* way over a 50-year period in the last century and, except for the last one (Salmon's theorem), all were proved by our prover. Salmon's theorem involves too many points (thus, too many polynomials in our prover), and is beyond the space limit of our computer.

In geometry, these theorems were proved by repeated use of the following theorems and their reciprocals [9]: "If two triangles be such that the lines joining corresponding vertices meet in a point (the center of homology of the two triangles), the intersections of the corresponding sides will lie in one line." "If the intersections of the opposite sides of three triangles be for each pair *the same* three points in a line, the centers of homology of the first and second, second and third, third and first, will lie in a line."

We wondered whether there were further linear properties (collinearity or concurrency) of the Pascal configuration. Equipped with a modern computer, we took a completely different approach to search for new (linear) properties of a given Pascal configuration. First, we assign numerical coordinates (floating-point numbers) to each point of the starting configuration. Then we calculate new lines and new points, searching for concurrent lines or collinear points. If we find one, there might be a theorem. Then we can use our prover to check whether the conjecture is really a theorem. We call this approach "*search for correct conjecture using numerical examples.*" This approach was first used in 1983 [5]. Based on this approach, it seems that the great geometers of the last century were so clever that they found all linear properties, from Steiner's theorem to Salmon's theorem. It was Wu who first raised the following question:

THE PASCAL CONIC PROBLEM. *Are there six points among the 45 Pascal points that lie on the same conic?*

We will call such a conic a *Pascal conic.*[2] There are many trivial degenerate Pascal conics. Among the 45 points, there are many 6-tuples of points that are on one or two of the 15 lines formed by the original 6 points. There are other kinds of degenerate conics due to Pascal's theorem: 6 points are on two (different) Pascal lines.

Besides the above degenerate Pascal conics, are there other nondegenerate Pascal conics? Based on his strong geometric intuition, Wu came up with a conjecture. Using his program, he confirmed the conjecture [2]. By permutations of the six points $A_0, \ldots, A_5$, Wu actually found a class of 60 Pascal conics. However, Wu did not solve the problem completely: are there other Pascal conics besides the 60 conics he found? That is, are there other 6-tuples of the Pascal points that are on the same conic? Using our approach of "search for correct conjecture using numerical examples," we searched all 8,145,060 ($= \binom{45}{6}$) 6-tuples of points from the 45 Pascal points and found the following 4 classes of Pascal conics:

[2]As we know, five points generally determine a conic. Six points are generally not on the same conic.

Class	Number of Elements
1	60
2	45
3	90
4 (found by Wu in 1980)	60

The representative elements of each class are

$A_0A_1 \cap A_2A_3,\ A_0A_1 \cap A_2A_4,\ A_0A_2 \cap A_1A_3,\ A_0A_2 \cap A_1A_4,\ A_0A_3 \cap A_1A_2,\ A_0A_4 \cap A_1A_2.$

$A_0A_1 \cap A_2A_3,\ A_0A_1 \cap A_4A_5,\ A_0A_2 \cap A_1A_3,\ A_0A_3 \cap A_1A_2,\ A_0A_4 \cap A_1A_5,\ A_0A_5 \cap A_1A_4.$

$A_0A_1 \cap A_2A_3,\ A_0A_1 \cap A_4A_5,\ A_0A_2 \cap A_1A_4,\ A_0A_3 \cap A_1A_5,\ A_0A_4 \cap A_1A_2,\ A_0A_5 \cap A_1A_3.$

$A_0A_1 \cap A_2A_3,\ A_0A_1 \cap A_4A_5,\ A_0A_2 \cap A_3A_4,\ A_0A_3 \cap A_2A_5,\ A_1A_4 \cap A_2A_5,\ A_1A_5 \cap A_3A_4.$

Corresponding to each class, we have a conjecture that is numerically correct. Thus, we have the following four conjectures:

CONJECTURE 1. *Given six points A_0, A_1, A_2, A_3, A_4, and A_5 on a conic, the six Pascal points $A_0A_1 \cap A_2A_3$, $A_0A_1 \cap A_2A_4$, $A_0A_2 \cap A_1A_3$, $A_0A_2 \cap A_1A_4$, $A_0A_3 \cap A_1A_2$, and $A_0A_4 \cap A_1A_2$ are on the same conic.*

CONJECTURE 2. *Given six points A_0, A_1, A_2, A_3, A_4, and A_5 on a conic, the six Pascal points $A_0A_1 \cap A_2A_3$, $A_0A_1 \cap A_4A_5$, $A_0A_2 \cap A_1A_3$, $A_0A_3 \cap A_1A_2$, $A_0A_4 \cap A_1A_5$, and $A_0A_5 \cap A_1A_4$ are on the same conic.*

CONJECTURE 3. *Given six points A_0, A_1, A_2, A_3, A_4, and A_5 on a conic, the six Pascal points $A_0A_1 \cap A_2A_3$, $A_0A_1 \cap A_4A_5$, $A_0A_2 \cap A_1A_4$, $A_0A_3 \cap A_1A_5$, $A_0A_4 \cap A_1A_2$, and $A_0A_5 \cap A_1A_3$ are on the same conic.*

CONJECTURE 4. *Given six points A_0, A_1, A_2, A_3, A_4, and A_5 on a conic, the six Pascal points $A_0A_1 \cap A_2A_3$, $A_0A_1 \cap A_4A_5$, $A_0A_2 \cap A_3A_4$, $A_0A_3 \cap A_2A_5$, $A_1A_4 \cap A_2A_5$, and $A_1A_5 \cap A_3A_4$ are on the same conic.*

We have used our theorem prover to confirm that the above four conjectures are true (under certain nondegenerate conditions). In the case when the six points A_0, A_1, A_2, A_3, A_4, and A_5 are on a circle, see [6, Examples 9–12], where the exact nondegenerate conditions are also listed. We also confirmed the above four conjectures in projective geometry and affine geometry.

Conjecture 1 has nothing to do with Pascal configuration, because only five points $A_0, \dots, A_4$ are involved in the statement. But it is also a nontrivial theorem. Later, we found it in Coxeter's textbook *Projective Geometry* [10]. Coxeter attributed it to a contemporary mathematician S. Schuster. However, neither reference nor the proof was given in [10]. Conjecture 4 is due to Wu. Our contribution is Conjecture 2 and Conjecture 3. The following are the input (the statement) and the diagram of Conjecture 2 [6, Example 10].

((cons-seq A_0 A_1 A_2 O A_3 A_4 A_5 P_0 P_1 P_2 P_3 P_4 P_5 X Y Z)
(eqdistant O A_0 O A_1)
(eqdistant O A_0 O A_2)
(eqdistant O A_0 O A_3)
(eqdistant O A_0 O A_4)
(eqdistant O A_0 O A_5) ;Points $A_0, \dots$ A_5 are on the circle (O)
(collinear A_0 A_1 P_0)
(collinear A_2 A_3 P_0) ;$P_0 = A_0\ A_1 \cap A_2\ A_3$
(collinear A_0 A_1 P_1)
(collinear A_4 A_5 P_1) ;$P_1 = A_0\ A_1 \cap A_4\ A_5$

(collinear A_0 A_2 P_2)
(collinear A_1 A_3 P_2) ; $P_2 = A_0\ A_2 \cap A_1\ A_3$
(collinear A_0 A_3 P_3)
(collinear A_1 A_2 P_3) ; $P_3 = A_0\ A_3 \cap A_1\ A_2$
(collinear A_0 A_4 P_4)
(collinear A_1 A_5 P_4) ; $P_4 = A_0\ A_4 \cap A_1\ A_5$
(collinear A_0 A_5 P_5)
(collinear A_1 A_4 P_5) ; $P_5 = A_0\ A_5 \cap A_1\ A_4$
(collinear X P_0 P_1)
(collinear X P_3 P_4)
(collinear Y P_1 P_2)
(collinear Y P_4 P_5)
(collinear Z P_2 P_3)
(collinear Z P_5 P_0)
(collinear X Y Z)) ;Conclusion: X, Y, and Z are collinear, which
;is equivalent to $P_0, \ldots, P_5$ are on the same conic.

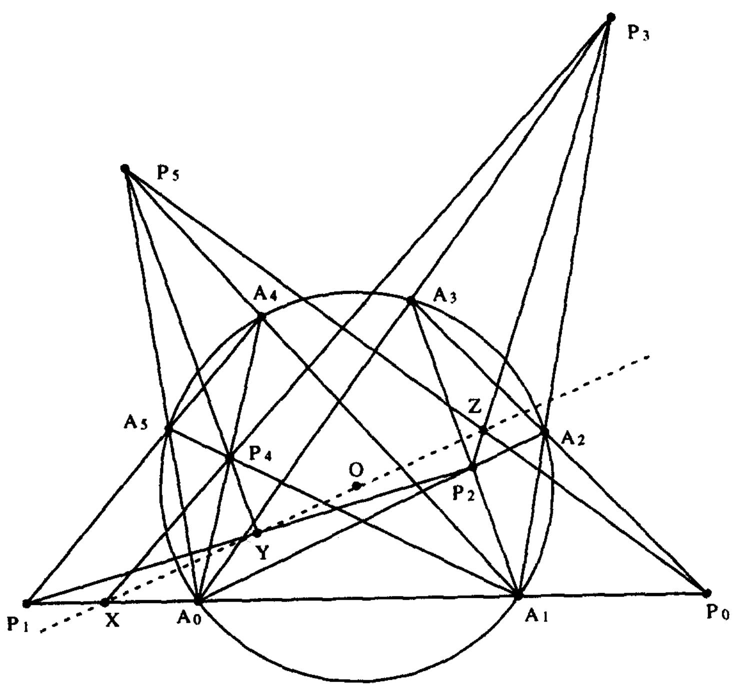

Figure 2. Conjecture 2.

The program automatically assigns coordinates to the points as follows. $A_1 = (u_1, 0)$, $A_0 = (0,0)$, $A_2 = (u_2, u_3)$, $O = (x_2, x_1)$, $A_3 = (x_3, u_4)$, $A_4 = (x_4, u_5)$, $A_5 = (x_5, u_6)$, $P_0 = (x_6, 0)$, $P_1 = (x_7, 0)$, $P_2 = (x_9, x_8)$, $P_3 = (x_{11}, x_{10})$, $P_4 = (x_{13}, x_{12})$, $P_5 = (x_{15}, x_{14})$, $X = (x_{16}, 0)$, $Y = (x_{18}, x_{17})$, and $Z = (x_{20}, x_{19})$.

Once the coordinates have been assigned, it is straightforward to generate the hypothesis and conclusion polynomials (there are 20 equations $h_1, \ldots, h_{20}$ for the hypothesis) by machines. Then we can use the method in the preceding section to prove the theorem. It took about 30 seconds on a Symbolics 3600. The number of terms in the largest polynomial is 534. Recently, we repeated the proof using a fast machine (NeXT turbo station). It took about 10 seconds to complete the proof.

4. FURTHER DISCUSSION

We have given a complete solution numerically, but not *mathematically*: Classes 2–4 have been *mathematically* proved (by our prover) to be Pascal conic classes, but we have only *numerically* observed the fact that there are no Pascal conic classes among all 8,145,060 6-tuples of points from the 45 Pascal points other than those we have found. Mathematically, this fact has not been proved. To prove a 6-tuple of the Pascal points is not on the same conic, we can use a counterexample: first assign the coordinates of the six original points $A_0, \dots, A_5$ certain exact numerical values (a numerical instance); then calculate whether the 6-tuple of the Pascal points is not on the same conic. If the calculation is exact and if the numerical instance is not degenerate, then we can conclude that "the 6 Pascal points are not on the same conic."

Since in the algebraic specification of the problem, we have polynomials with the leading dependent variables of degree two, the numerical computation has to involve floating-point numbers; hence, it is not exact.

However, we can specify the condition that A_0, A_1, A_2, A_3, A_4, and A_5 are on the same conic "linearly" using the converse of Pascal's theorem.

THE CONVERSE OF PASCAL'S THEOREM. *Let A_0, A_1, A_2, A_3, A_4, and A_5 be six points. If $P = A_0A_1 \cap A_3A_4$, $Q = A_1A_2 \cap A_4A_5$, and $S = A_2A_3 \cap A_5A_0$ are collinear, then A_0, A_1, A_2, A_3, A_4, and A_5, are on a conic.*

This is a known theorem, which has also been proved by our prover [6, Example 7].

Now after transforming the hypotheses into polynomial equations and triangularizing them into a triangular form, we have a set of equations *linear* in their leading variables; that is, all leading dependent variables are of degree one in the equations. Thus, after assigning 6 exact fractional numbers to the 6 independent variables $u_1, \dots, u_6$, we can solve all dependent variables in exact fractional numbers. Then we check whether the 6 Pascal points are not on the same conic.

The calculation for each 6-tuple takes about 0.06 seconds on the NeXT turbo station with the speed of 25 MIPS. Thanks to the referee's suggestion, we have used this approach to search all 8,145,060 6-tuples; the computation took about 142 hours to complete. Our computation shows that the four classes we found are the only classes of Pascal conics. Thus, we have the following theorem.

THEOREM 4.1. *Given six points A_0, A_1, A_2, A_3, A_4, and A_5 on a conic, there exist only four classes of nondegenerate Pascal conics.*

The reader might wonder why we did not use the big fractional number calculation at the very beginning. The reason was that we completed this research during the 80's, and the computer used at that time (Symbolics 3600) was about 10 times slower than the machine we are using now. Hence, using big fractional numbers would take at least one month on a single computer at that time.

Alternatively, we might pick up one representative 6-tuple from each equivalent class and use the theorem prover to prove or disprove whether it is a Pascal conic. We picked up a few samples and used our prover to experiment with them; the time needed for each sample is, on the average, about 10 seconds on our current fast machine. However, collecting representative elements of equivalent classes is both time and memory consuming. Actually, we failed to do so because of the memory limit on our current machine.

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