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On the structure of Brauer's centralizer algebras

By Hans Wenzl[†]

In his paper [Br], R. Brauer defined algebras motivated by the following basic problem of classical invariant theory: Let G be a group of linear transformations on a vector space V and let $\pi^{\otimes f}$ be the representation of G on $V^f = V \otimes \cdots \otimes V$, the f-th tensor power of V. Then the question is how does $\pi^{\otimes f}$ decompose into irreducible representations of G. One way of studying this problem is to consider the algebra $B_f(G)$ of centralizers, i.e. the algebra of linear maps y on V^f such that $y\pi^{\otimes f}(g) = \pi^{\otimes f}(g)y$ for all $g \in G$. This approach was successful for $G = \mathrm{Gl}(n)$, where $B_f(\mathrm{Gl}(n))$ is a quotient of kS_f, the group algebra of the symmetric group. So the decomposition of $\pi^{\otimes f}$ was obtained from knowledge about that algebra.

In this paper, we will study a sequence of algebras, denoted by $D_f(n)$ which play a similar role for other classical Lie groups. More precisely, if G is the orthogonal group $\mathrm{O}(n)$ or the symplectic group $\mathrm{Sp}(2m)$, the corresponding $B_f(G)$'s are quotients of Brauer's $D_f(n)$ and $D_f(-2m)$ respectively (see [HW1], Theorem (2.10)). If $n > f$, $D_f(n)$ is semisimple (which in this paper means that it is a direct sum of full matrix algebras over k) and its structure does not depend on n. In the other case, however, $D_f(n)$ is no longer semisimple, which makes it difficult to determine the relevant semisimple quotient. In fact, H. Weyl was unable to use this "somewhat enigmatic algebra" directly in the determination of the decomposition of $\pi^{\otimes f}$ and therefore was obliged to resort to different methods still considered "mysterious" by other authors (see [Wy], V.5 and p. 159 and [ABP]).

The algebras $D_f(n)$ have been studied by various authors mainly using combinatorial methods (see [Be], [Bw], [EK], [HW1-3] and [S]). Based on their results and extensive computations, P. Hanlon and D. Wales conjectured that

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$D_f(n)$ is semisimple for all $f \in \mathbf{N}$ if n is not an integer. We will prove their conjecture in this paper, thereby also giving a simple inductive procedure to determine the decomposition of $D_f(n)$ into full matrix rings. Moreover, if n is an integer, we completely determine the structure of the semisimple quotient of $D_f(n)$ in which Brauer and Weyl were interested.

The main tools for proving these results come from the study of subfactors of type II_1 von Neumann factors and its recent applications in knot theory (see [J1,2], [Wn] and [BW]). Similar methods were already used to determine the structure of an algebra derived from recently discovered link invariants (see [BW]). These algebras contain as limiting cases Brauer's algebras $D_f(n)$. The connection with link invariants, however, will not be explored in this paper. Therefore we will proceed directly from Brauer's original definitions.

Let us briefly explain the main technical devices which we shall employ. We single out a special element of $D_{f+1}(n)$, denoted by e_f (which comes from the contraction on the f-th and $(f+1)$-th factor of V^{f+1}), to define a map (which is usually called a conditional expectation) from $D_f(n)$ onto $D_{f-1}(n)$. Via this conditional expectation, we obtain a trace τ_n with the following remarkable property: If its restriction to $D_f(n)$ is nondegenerate, then $D_{f+1}(n)$ is semisimple. In this case the structure of the ideal generated by e_f can be determined by a simple inductive procedure from the structure of $D_{f-1}(n)$ and $D_f(n)$. Such constructions were first used in V. Jones' paper [J1] on subfactors for finite von Neumann algebras and were extended in [Wn], [BW] and [GHJ].

Using the representation theory of the odd dimensional orthogonal groups, we provide an easy criterion to determine whether τ_n is nondegenerate or not. We only have to evaluate special polynomials derived from Weyl's dimension formulas at $x = n$ (see [EK]). The conjecture of Hanlon and Wales follows then from the fact that all these polynomials have integer roots.

After factoring over the annihilator of τ_n, our methods can also be extended to the case when n is an integer. Using essentially the same inductive procedure as in the semisimple case, we determine the structure of a particularly interesting quotient of $D_f(n)$. If n is a positive or an even negative integer, it coincides with $B_f(\mathrm{O}(n))$ in the first case and with $B_f(\mathrm{Sp}(-n))$ in the second case. Similar characterizations can also be found for the other cases.

I would like to thank Vaughan Jones for bringing the preprints of Hanlon and Wales to my attention.

1. Adjoining idempotents to semisimple algebras

We consider extensions for a pair of (finite dimensional) semisimple algebras $A \subset B$, obtained by adjoining an idempotent e which is closely related to a conditional expectation from B onto A. For convenience the term *semisimple* will be used in this paper as a synonym for *a direct sum of full matrix algebras*. The letter k will always denote a field of characteristic 0 and $k(x)$ will denote the field of rational functions over k.

Let $M_n(k)$ (or just M_n) denote the algebra of all $n \times n$ matrices over k. So if A and B are semisimple algebras (in our restricted sense), we can write them as $A = \oplus A_i$ and $B = \oplus B_j$ with $A_i \cong M_{a_i}(k)$ and $B_j \cong M_{b_j}(k)$ for appropriate natural numbers a_i and b_j. If A is a subalgebra of B, any simple B_j module is also an A module. Let g_{ij} be the number of simple A_i modules in its decomposition into simple A modules. The matrix $G = (g_{ij})$ is called the inclusion matrix for $A \subset B$.

The inclusion of A in B is conveniently described by a so-called Bratteli diagram. This is a graph with vertices arranged in 2 lines. In one line, the vertices are in one-to-one correspondence with the minimal direct summands A_i of A, in the other one with the summands B_j of B. Then a vertex corresponding to A_i is joined with a vertex corresponding to B_j by g_{ij} edges. If A and B have the same identity, there is an easy way of computing the square root b_j of the dimension of B_j. We just add up all the square roots of the dimensions of the A_i's to which B_j is joined by edges (with multiplicities).

We can also interpret the numbers g_{ij} in the following way: Let p_i be a minimal idempotent of A_i and let $p_i = \Sigma q_m$, where the q_m's are mutually orthogonal minimal idempotents of B. This decomposition is not unique in general. But for any such decomposition there will be exactly g_{ij} idempotents in B_j. As an easy consequence we obtain that

$$p_i B p_i \cong \bigoplus_j M_{g_{ij}}. \tag{1}$$

We will describe, as an example, the inclusion of kS_{f-1} in kS_f, where kS_{f-1} and kS_f are the group algebras of the corresponding symmetric groups. Let, for $f \geq 0$, Λ_f be the set of all Young diagrams with f nodes. We will write a specific Young diagram λ as an m-tuple $[\lambda_1, \ldots, \lambda_m]$ where λ_i is the number of nodes in the i-th row. The empty Young diagram in Λ_0 is denoted by [0]. It is well-known that the simple components $kS_{f,\lambda}$ of S_f are labeled by Young diagrams λ with f nodes. So the Bratteli diagram for $kS_{f-1} \subset kS_f$ is obtained in the following way:

We draw for each $\mu \in \Lambda_{f-1}$ and each $\lambda \in \Lambda_f$ a vertex and connect two vertices by an edge if and only if the corresponding μ is obtained from the corresponding λ by taking away one node from λ. The inclusion diagram for $kS_2 \subset kS_3$ is shown in the upper half of Figure 1 (with $[1^m] = [1, \ldots, 1]$ (m times)).

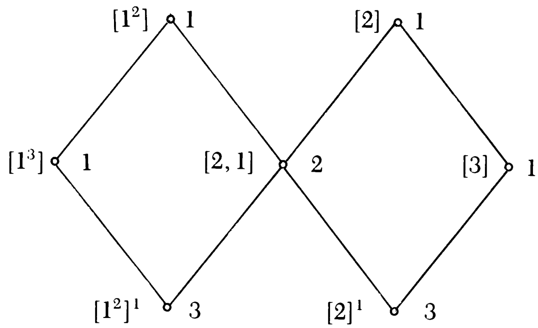


FIGURE 1

An important role will be played by traces, i.e. functionals tr: $B \to k$ such that $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ for all $a, b \in B$. As there is up to scalar multiples only one trace on $M_n(k)$, any trace tr on $B = \oplus B_j$ is completely determined by a vector (t_j), where $t_j = \mathrm{tr}(p_j)$ and p_j is a minimal idempotent of B_j. A trace tr on B is called *nondegenerate* if for any $b \in B$ there is a $b' \in B$ such that $\mathrm{tr}(bb') \neq 0$. It is easy to check that tr is nondegenerate if and only if $t_j \neq 0$ for each j.

Let us recall that if tr is a nondegenerate trace on B, the map $b \in B \mapsto \mathrm{tr}(b\,\cdot\,) \in B^*$ is an isomorphism between B and its dual B^* (where as usual $\mathrm{tr}(b\,\cdot\,)$ denotes the map $x \mapsto \mathrm{tr}(bx)$). Let tr be nondegenerate on both A and B. Using the isomorphism above for A and A^*, we obtain for every $b \in B$ a necessarily unique $\varepsilon_A(b) \in A$ such that $\mathrm{tr}(b\,\cdot\,)_{|A} = \mathrm{tr}(\varepsilon_A(b)\,\cdot\,)_{|A}$. The linear map ε_A: $B \to A$, $b \mapsto \varepsilon_A(b)$ is called the trace-preserving conditional expectation from B onto A, where the element $\varepsilon_A(b) \in A$ is uniquely determined by the equation

$$\mathrm{tr}(\varepsilon_A(b)a) = \mathrm{tr}(ba) \quad \text{for all } a \in A. \tag{2}$$

We obtain from this equation and the faithfulness of tr the following properties of ε_A:

(a) $\varepsilon_A(a_1 b a_2) = a_1 \varepsilon_A(b) a_2$ for all $a_1, a_2 \in A$ and $b \in B$ and in particular $\varepsilon_A(a) = a$ for all $a \in A$.

(b) ε_A is nondegenerate; i.e. for all $0 \neq b \in B$ there are $b_1, b_2 \in B$ such that $\varepsilon_A(b_1 b) \neq 0$ and $\varepsilon_A(b b_2) \neq 0$.

We will moreover assume that B is contained in an algebra C and that there is an element $e \in C$ such that

(3)
(a) $e^2 = e$,
(b) $ebe = e\varepsilon_A(b) = \varepsilon_A(b)$ for all $b \in B$,
(c) The map $a \in A \mapsto ae$ is an injective homomorphism with $1e = e$.

An important example for such a situation can be obtained in the following way (see [J1], §3.1): Let B be represented via left regular representation on itself. For convenience, the isomorphic image of B in this representation will also be denoted by B. If B is regarded as representation space, it will be denoted by B_ξ and its elements by b_ξ with $b \in B$. We take as C the set $L(B_\xi)$ of all linear maps on B_ξ. As in [J1] we define an idempotent e_A on B_ξ by $e_A b_\xi = \varepsilon_A(b)_\xi$. It follows from this definition that

$$(e_A b e_A)b'_\xi = (e_A b)\varepsilon_A(b')_\xi = (\varepsilon_A(b)e_A)b'_\xi = (e_A\varepsilon_A(b))b'_\xi \text{ for all } b' \in B.$$

Using again (2) (a) we show that e_A is an idempotent and that $(ae_A)b_\xi = (e_A a)b_\xi$ for all $a \in A$ and $b \in B$. Finally, the equation $(ae_A)1_\xi = a_\xi$ shows that the map $a \in A \mapsto ae_A$ is injective.

LEMMA (1.1). (a) *Any element $x \in \langle B, e\rangle$ can be written as a linear combination of elements of B and elements of the form $b_1 e b_2$ with $b_1, b_2 \in B$. In particular, any element of the ideal generated by e can be written as a linear combination of elements of the form $b_1 e b_2$.*

(b) *ε_A can be extended to $\langle B, e\rangle$ such that $exe = \varepsilon_A(x)e$ for every $x \in \langle B, e\rangle$. This extension is unique in the following sense: If e' is another idempotent in some extension C' of B with properties* (3)(a)–(c) *and if x' is the element in $\langle B, e'\rangle$ obtained by replacing every occurrence of e in x by e', then $\varepsilon_A(x) = \varepsilon_A(x')$.*

(c) *If $x \in \langle B, e\rangle$, there exist unique elements b and b' in B such that $xe = be$ and $ex = eb'$.*

(d) *Let $C = L(B_\xi)$ and let $x \in \langle B, e_A\rangle$. Then $(e_A x)b_\xi = (\varepsilon_A(xb))_\xi$ and $x = 0$ if and only if $\varepsilon_A(b_1 x b_2) = 0$ for all $b_1, b_2 \in B$.*

Proof. (a) This is an immediate consequence of (3)(a).

(b) By (a) it is enough to show the statement for $x = b_1 e b_2$ with $b_1, b_2 \in B$. But then $exe = eb_1eb_2e = \varepsilon_A(b_1)\varepsilon_A(b_2)e$. Hence $\varepsilon_A(x)$ is uniquely determined by (3)(c). By the same computations, we obtain $\varepsilon_A(b_1e'b_2) = \varepsilon_A(b_1)\varepsilon_A(b_2)$ for any e' with (3)(a)–(c).

(c) Again we can assume that $x \in B$ or $x = b_1eb_2$ with $b_1, b_2 \in B$. In the second case we have $xe = b_1\varepsilon_A(b_2)e$. If $\tilde{b} \in B$ such that $\tilde{b}e = xe$, we have for all elements $c \in B$, $0 = ec(\tilde{b} - b)e = \varepsilon_A(c(\tilde{b} - b))e$. Hence $\tilde{b} = b$ as ε_A is nondegenerate.

(d) The proof of the first statement is an easy computation. If $x \neq 0$, there exists $b_2 \in B$ such that $xb_{2\xi} = b'_\xi \neq 0$. As ε_A is nondegenerate, there is $b_1 \in B$ such that $\varepsilon_A(b_1b') \neq 0$. Hence $\varepsilon_A(b_1xb_2)_\xi = e_Ab_1(xb_2)_\xi = \varepsilon_A(b_1b')_\xi \neq 0$. The other direction is trivial.

Recall that we can identify B with a direct sum of full matrix algebras. Using the transposition of matrices, we obtain an involution $*: B \to B$, $b \mapsto b^*$

such that $(ab)^* = b^*a^*$ for all $a, b \in B$ (where $*$ depends on the matrix representation of B). The involution $*$ defines a ("right regular") representation ρ of B on B_ξ by $\rho(b)b'_\xi = (b'b^*)_\xi$. If J: $B_\xi \to B_\xi$ is defined by $Jb_\xi = b^*_\xi$, it is easy to check that $J\rho(b)J$ is equal to left multiplication by b. So ρ is equivalent to the left regular representation of B.

We will moreover assume in the following that $*$ also leaves A invariant, i.e. $a^* \in A$ for all $a \in A$. This is possible as A is isomorphic to a direct sum of full matrix algebras. Using J as above, one shows that $\rho_{|A}$ is equivalent to the representation of A on B_ξ given by left multiplication.

We can now completely determine the structure of the extension $\langle B, e_A \rangle$ of B. If A and B are C^* algebras, this result follows from [J1, §3.2]. A comprehensive treatment of this and related questions can be found in [GHJ].

Proposition (1.2). (a) *Let A, B,* tr *and ε_A be as above. Then the algebra $\langle B, e_A \rangle$ is isomorphic to the centralizer of A on B_ξ. In particular, it is semisimple.*

(b) *There is a one-to-one correspondence between the simple components of A and $\langle B, e_A \rangle$ such that if $p \in A_i$ is a minimal idempotent, pe_A is a minimal idempotent of $\langle B, e_A \rangle_i$. Under this correspondence, the inclusion matrix for $B \subset \langle B, e_A \rangle$ is the transposed G^t of the inclusion matrix for $A \subset B$.*

(c) $\langle B, e_A \rangle = Be_AB$.

Proof. (a) By the remarks above it is enough to show that $\langle B, e_A \rangle$ is isomorphic to the centralizer $\rho(A)'$ of $\rho(A)$. By (2)(a), we have

$$e_A\rho(a)b_\xi = (\varepsilon_A(b)a^*)_\xi = \rho(a)e_Ab_\xi \qquad \text{for all } a \in A \text{ and } b \in B.$$

So $\langle B, e_A \rangle$ and its double centralizer $\langle B, e_A \rangle''$ are contained in $\rho(A)'$. On the other hand, it is well-known that if a linear operator x on B_ξ commutes with B, it is right multiplication by the element $b \in B$, uniquely determined by $x1_\xi = b_\xi$. Note that $(e_Ab)1_\xi = \varepsilon_A(b)_\xi$ is equal to $(be_A)1_\xi = b_\xi$ only if $\varepsilon_A(b) = b$, i.e. if $b \in A$. So x is in the centralizer of $\langle B, e_A \rangle$ only if $x = \rho(b^*) \in \rho(A)$. Taking centralizers, we obtain from this $\langle B, e_A \rangle'' \supset \rho(A)'$. By Jacobson's density theorem, (a) follows as soon as we have shown that $\langle B, e_A \rangle$ is semisimple.

Let $x \in \operatorname{rad}(\langle B, e_A \rangle)$ be in the radical of $\langle B, e_A \rangle$. If $x \neq 0$, there are $b_1, b_2 \in B$ such that $\varepsilon_A(b_1xb_2) \neq 0$ by Lemma (1.1)(d). But then $0 \neq e_Ab_1xb_2e_A = \varepsilon_A(b_1xb_2)e_A \in \operatorname{rad}(\langle B, e_A \rangle)$ and therefore also $\varepsilon_A(b_1xb_2)e_A \in \operatorname{rad}(Ae_A)$. As $Ae_A \cong A$, it follows that $0 \neq \varepsilon_A(b_1xb_2) \in \operatorname{rad}(A)$, a contradiction to A being semisimple.

(b) If p is a minimal idempotent of A, it follows from (3)(b) that pe_Axpe_A is a multiple of pe_A for any $x \in \langle B, e_A \rangle$. Hence pe_A is a minimal idempotent in $\langle B, e_A \rangle$. Obviously the centers of $\rho(A)$ and $\langle B, e_A \rangle = \rho(A)'$ are the same. So, by the remarks above, $z \mapsto JzJ$ provides a one-to-one correspondence between the centers of A, represented on B_ξ by left multiplication, and of $\langle B, e_A\rangle$. Moreover, if z is the minimal central idempotent of A with $zp \neq 0$, also $(e_A pJzJ)z_\xi = \varepsilon_A(p)_\xi = p_\xi \neq 0$. The statement about the inclusion matrix for $B \subset \langle B, e_A\rangle''$ follows from [Bou, §3, ex. 17] (see also [J1, (3.2.3)].

For (c), we note that $ze_A \neq 0$ for any minimal central idempotent of $\langle B, e_A\rangle$ by (b). Hence the two-sided ideal generated by e_A has to be the whole algebra. The rest follows from Lemma (1.1)(a).

The structure of $\langle B, e_A\rangle$ can now be easily determined by just reflecting the Bratteli diagram for $A \subset B$ about the line of B and then adding up the dimensions. Figure 1 shows the structure of $\langle B, e_A\rangle$ for our example $kS_2 \subset kS_3$.

The next theorem shows that the special case treated in Proposition (1.2) is the "smallest" algebra generated by B and a projection e with properties (a)–(c) (see also [Wn, Prop. (1.2)] for A and B finite dimensional C^* algebras). A similar proof appears in [BW, Th. 3.5].

THEOREM (1.3). *Let $A \subset B$ be (finite dimensional) semisimple algebras and let* tr *be a nondegenerate trace on B such that also its restriction to A is nondegenerate. Let ε_A be the trace-preserving conditional expectation onto A and let e be as in* (3). *Then $\langle B, e\rangle$ is a direct sum of full matrix algebras, which decomposes as*

$$\langle B, e\rangle \cong \langle B, e_A\rangle \oplus \tilde{B},$$

where $\langle B, e_A\rangle$ is as in Proposition (1.2) *and $\tilde{B}$ is isomorphic to a subalgebra of B. In particular, the ideal generated by e is isomorphic to the semisimple algebra $\langle B, e_A\rangle$.*

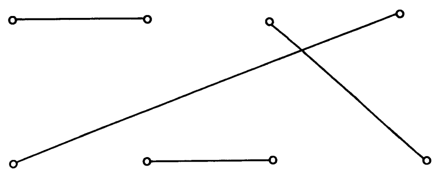
Proof. As already mentioned we will not distinguish in notation between $B \subset C$ and the image of B acting on B_ξ by left regular representation. Let $\Phi\colon B \cup \{e\} \to L(B_\xi)$ be defined by $\Phi(b) = b$ for $b \in B$ and $\Phi(e) = e_A$. Obviously, Φ extends to a homomorphism from $\langle B, e\rangle$ onto $\langle B, e_A\rangle$, provided it is well-defined. Let $x \in \langle B, e\rangle$ with $x = 0$. Then $eb_1xb_2e = \varepsilon_A(b_1xb_2)e = 0$ for $b_1, b_2 \in B$. By Lemma (1.1)(b), we also have $\varepsilon_A(b_1\Phi(x)b_2) = 0$ for all $b_1, b_2 \in B$; hence $\Phi(x) = 0$ by Lemma (1.1)(d).

On the other hand, if $x \in \ker\Phi$ and $b_2 \in B$, there exists a unique $b \in B$ such that $(xb_2)e = be$ by Lemma (1.1)(c). By Lemma (1.1)(b) and (d), we have for any $b_1 \in B$ that $\varepsilon_A(b_1xb_2) = \varepsilon_A(b_1\Phi(x)b_2) = 0$. As ε_A is nondegenerate, $b = 0$ and $xb_2e = 0$. Similarly, we also show that $eb_1x = 0$ for any $b_1 \in B$.

So if $x \in \ker\Phi \cap BeB$, x is annihilated by both $\ker\Phi$ and BeB. Note that by (3)(b), $BeB = \langle e\rangle$, the ideal generated by e and that $\Phi(BeB) = \langle B, e_A\rangle$ by Lemma (1.1)(a) and Proposition (1.2). Hence BeB and $\ker\Phi$ generate $\langle B, e\rangle$ and therefore $x = 0$. In particular, BeB is isomorphic to $\langle B, e_A\rangle$. Obviously, the quotient $\langle B, e\rangle / BeB$ is generated by the image of B; hence it is also semisimple. From this follows the general statement.

2. Brauer's algebras

We will first define Brauer's algebras D_f over $k(x)$. For $f = 0$, $D_0 = k(x)$. For $f > 0$, a linear basis of the $k(x)$ algebra D_f is given by graphs with f edges and $2f$ vertices, arranged in 2 lines of f vertices each. In these graphs each edge belongs to exactly 2 vertices and each vertex belongs to exactly one edge. So an example for a graph in D_4 would be



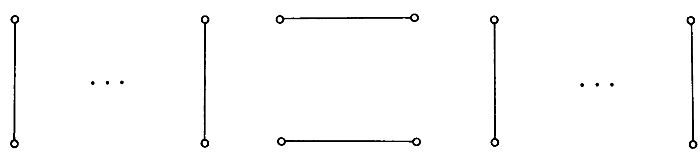
It is easy to see that we have $2f - 1$ possibilities to join the first vertex with another one, then $2f - 3$ possibilities for the next one and so on. So the dimension of D_f is $1 \cdot 3 \cdot 5 \ldots (2f - 1)$. To define the multiplication in D_f, it is enough to define the product ab for 2 graphs a and b. This is done similarly as with braids by the following rule.

(a) Draw b below a.

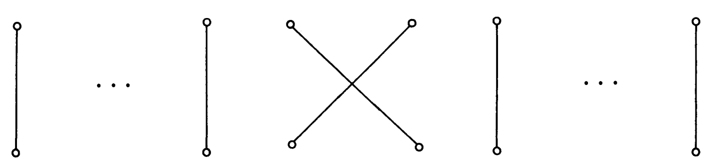
(b) Connect the i-th upper vertex of b with the lower i-th vertex of a.

(c) Let d be the number of cycles in the new graph obtained in (b) and let c be this graph without the cycles. Then $ab = x^d c$.

We will later need the following examples: Let e_i and g_i denote the graphs

$e_i =$ 

and $\qquad i \qquad i+1$

$g_i =$ 

Then it is easy to check by pictures (see Figure 2) that

(4) $$e_i^2 = xe_i$$

and

(5) $$e_i e_{i-1} e_i = e_i \quad \text{and} \quad e_{i-1} e_i e_{i-1} = e_{i-1}.$$

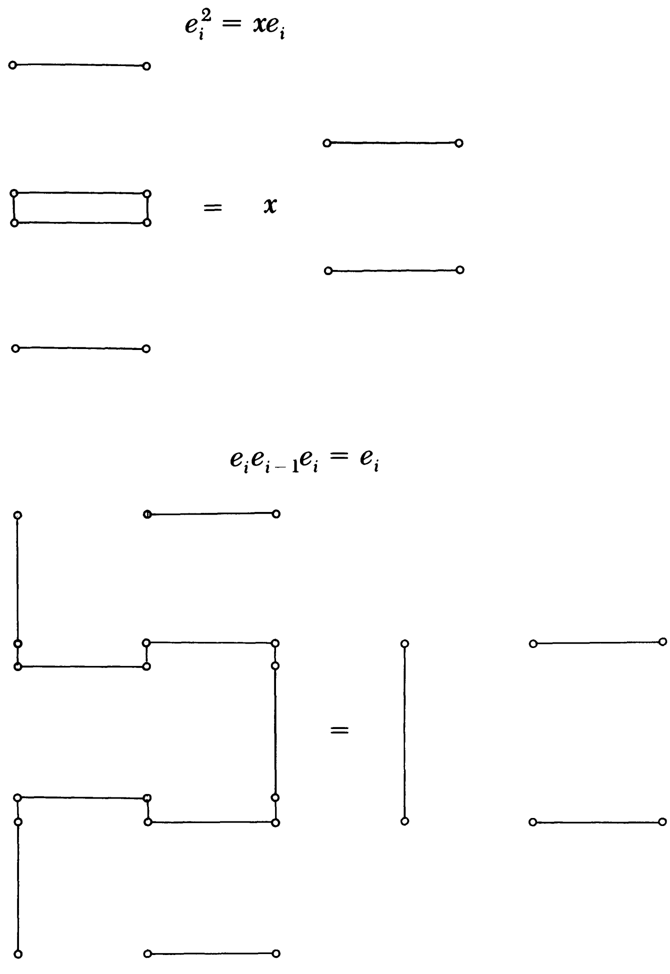


FIGURE 2

We will call an edge horizontal if it joins 2 vertices in the same row. Note that there are as many horizontal edges in the upper row as there are in the lower one. Whenever a graph p has no horizontal edges, it can be regarded as a permutation π connecting the i-th lower vertex to the $\pi(i)$-th upper vertex. It is easy to check that the multiplication of graphs is compatible with the composition of permutations under this identification. We will therefore refer to graphs without horizontal lines as permutation graphs or just as permutations. So, obviously, D_f contains $k(x)S_f$ as a subalgebra. We also remark that for any graph $b \in D_f$ and a permutation graph p the graph bp is obtained by permuting the vertices of the lower row of b by π^{-1} and pb is the graph obtained by permuting the vertices of the upper row of b by π.

We finally remark that D_f can be identified with the subalgebra of D_{f+1} spanned linearly by all graphs with a vertical edge on their right hand sides.

The k algebra $D_f(n)$ has a linear basis labeled by the same graphs. The multiplication is defined as in D_f except that every occurrence of x is replaced by n. The relationship between D_f and $D_f(n)$ will be studied in Lemma (2.3). As we will have to divide by our parameter n later, we will always assume $n \neq 0$ even though $D_f(0)$ is well-defined.

Most of the following results can also be obtained from [BW], Lemma (3.1) and the proof of Theorem (3.7) in connection with Section 5. We prove them here directly without using generators and relations or link invariants.

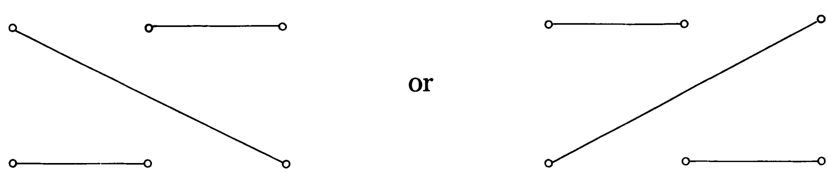
PROPOSITION (2.1). (a) *Any graph $d \in D_f$ is either already in D_{f-1} or it can be written in the form $a\chi b$ with $a, b \in D_{f-1}$ and $\chi \in \{g_{f-1}, e_{f-1}\}$. In particular, $\{e_1, g_1, \ldots, e_{f-1}, g_{f-1}\}$ generate D_f as an algebra.*

(b) *The ideal generated by e_{f-1} coincides with the linear span I_f of graphs containing horizontal edges. Every graph $b \in I_f$ can be written as $b = b_1 e_{f-1} b_2$ with $b_1, b_2 \in D_{f-1}$. So $I_f = \langle e_{f-1} \rangle$ is contained in the algebra generated by D_{f-1} and e_{f-1}.*

(c) $D_f/I_f \cong k(x)S_f$.

(d) *The same statements hold for $D_f(n)$ and the ideal $I_f(n) \subset D_f(n)$ generated by graphs containing horizontal edges (with x replaced by n).*

Proof. (a) Let b be a graph in D_f which is not in D_{f-1}. We have to consider 3 cases, depending on whether the last 2 vertices belong to two, one or no horizontal edge. We will consider the case with one horizontal edge in detail. It is easy to see that for appropriate permutations p_1 and p_2 in D_{f-1} the last six vertices of $p_1 b p_2$ are connected by edges in one of the following ways.



Let us assume the first case. Then $p_1bp_2 = b'e_{f-1}e_{f-2}$ with $b' \in D_{f-3}$. But then

$$b = (p_1^{-1}b')e_{f-1}(e_{f-2}p_2^{-1})$$

has the desired form.

In the other 2 cases, it is easy to check that there are permutations $p_1, p_2 \in D_{f-1}$ and an element $b' \in D_{f-2}$ such that $p_1bp_2 = b'g_{f-1}$ if the edges belonging to the last two vertices are not horizontal and $p_1bp_2 = b'e_{f-1}$ if the last two vertices belong to two horizontal edges. Again, we only need to solve for b to show the claim.

(b) It is easy to show (see also the remark after (5)) that the product of two graphs contains at least as many horizontal lines as either of the graphs. So I_f is an ideal. We prove the second statement of (b) by induction on f. Let $b \in D_f$ be a graph containing at least one horizontal line. If $b \in D_{f-1}$, then, by the induction assumption and (5),

$$b = b_1e_{f-2}b_2 = (b_1e_{f-2})e_{f-1}(e_{f-2}b_2)$$

with $b_1, b_2 \in D_{f-2}$. Otherwise, we can assume by (a) that $b = b_1\chi b_2$ with $b_1, b_2 \in D_{f-1}$ and $\chi \in \{g_{f-1}, e_{f-1}\}$. If $\chi = g_{f-1}$, either b_1 or b_2 has to be in I_{f-1}. Let us assume $b_1 \in I_{f-1}$. Then $b_1 = b_{1,1}e_{f-2}b_{1,2}$ with $b_{1,1}, b_{1,2} \in D_{f-2}$. Hence $b = b_{1,1}e_{f-2}g_{f-1}b_{1,2}b_2$ as g_{f-1} commutes with every element of D_{f-2}. But then it can be checked easily by drawing the graphs that

$$e_{f-2}g_{f-1} = e_{f-2}e_{f-1}g_{f-2}.$$

The case $b_2 \in I_{f-1}$ is similar.

(c) Just note that D_f decomposes as a vector space into the direct sum of the span of the permutation graphs and I_f.

(d) We can use the same proofs for $D_f(n)$ as in (a)–(c).

In the next proposition we will construct conditional expectations and traces, which will relate Brauer's algebras to the concepts of Section 1.

PROPOSITION (2.2). *Let the notation be as in Proposition* (2.1).

(a) *For each* $b \in D_f$ *there exists a unique* $\varepsilon_{f-1}(b) \in D_{f-1}$ *such that* $e_fbe_f = x\varepsilon_{f-1}(b)e_f$ *and* $\varepsilon_{f-1}(b) = b$ *for* $b \in D_{f-1}$.

(b) *There exists a linear functional* τ *on* D_f *defined inductively by* $\tau(1) = 1$ *and* $\tau(b) = \tau(\varepsilon_{f-1}(b))$ *for* $b \in D_f$.

(c) τ *is uniquely determined, inductively, by* $\tau(b_1\chi b_2) = (1/x)\tau(b_1b_2)$ *for* $\chi \in \{e_{f-1}, g_{f-1}\}$ *and* $b_1, b_2 \in D_{f-1}$.

(d) $\tau(b'\varepsilon_{f-1}(b)) = \tau(b'b)$ *for* $b \in D_f$ *and* $b' \in D_{f-1}$.

(e) *There exist a map from* $D_f(n)$ *onto* $D_{f-1}(n)$, *also denoted by* ε_{f-1}, *and a trace* τ_n *on* $D_f(n)$ *such that* $\tau_n(b'\varepsilon_{f-1}(b)) = \tau_n(b'b)$ *for* $b \in D_f(n)$ *and* $b' \in D_{f-1}(n)$.

(f) *Let* $b \in D_f$. *If* $b(n)$ *is defined*, $\tau_n(b(n)) = \tau(b)(n)$.

Proof. (a) It can be checked easily by pictures that for any graph $b \in D_f$, both the f-th and $(f+1)$-th upper and lower vertices of $e_f b e_f$ are connected by horizontal edges. If we interpret the rest of this graph as an element $b' \in D_{f-1}$, we obtain immediately $e_f b e_f = b' e_f$. Then just define $\varepsilon_{f-1}(b) = (1/x)b'$. Obviously this map can be extended to D_f by linearity. If $b \in D_{f-1}$, then b commutes with e_f. Hence $e_f b e_f = b e_f e_f = x b e_f$, which shows the second statement.

(b) The functional τ is well-defined because $\varepsilon_{f-1}(b) = b$ for all $b \in D_{f-1}$.

(c) It follows from (5) and a similar picture for g_{f-1} that $e_f \chi e_f = e_f$ for $\chi \in \{g_{f-1}, e_{f-1}\}$. Hence $\varepsilon_{f-1}(b_1 \chi b_2) = (1/x) b_1 b_2$ as e_f commutes with all elements of D_{f-1}.

(d) As e_f commutes with b', $\varepsilon_{f-1}(b'b) = b'\varepsilon_{f-1}(b)$. Then the claim follows from (b).

(e) This is clear.

(f) This follows by induction on f.

Property (c) is called the Markov property, which plays a central role for link invariants (see [J2] and (for the Kauffman invariant) [BW]). Note that by the just proven proposition we can use Theorem (1.3) for $e = (1/x)e_f$, $A = D_{f-1}$ and $B = D_f$, provided that τ is nondegenerate on both D_{f-1} and D_f. In this case the ideal I_{f+1} generated by e_f is isomorphic to $\langle D_f, e_{D_{f-1}} \rangle$. A similar statement holds for the $D_f(n)$'s. The question about when these traces are nondegenerate will be settled in the next section.

In the remainder of this section we will prove a lemma which will relate $D_f(n)$ to the more easily manageable D_f. Slightly more generally, let A be a finite dimensional, not necessarily semisimple $k(x)$ algebra given by the $k(x)$ basis $\{b_1, b_2, \ldots, b_m\}$. Assume that the multiplication of basis elements is given by formulas $b_s b_r = \sum_{i=1}^m \alpha_{s,r,i} b_i$ with $\alpha_{s,r,i} \in k(x)$ for $i, r, s = 1, 2, \ldots, m$. If for a given $n \in k$ all these rational functions are well-defined, we can define the k algebra $A(n)$ by "evaluating A at $x = n$". This means that we have a linear basis $\{b_1(n), \ldots, b_m(n)\}$ of $A(n)$ such that $b_s(n) b_r(n) = \sum_{i=1}^m \alpha_{s,r,i}(n) b_i(n)$.

Similarly, we define for $a \in A$ with $a = \sum_{i=1}^m \alpha_i b_i$ the element $a(n) = \sum_{i=1}^m \alpha_i(n) b_i(n) \in A(n)$, provided $\alpha_i(n)$ is well-defined for $i = 1, 2, \ldots, m$. It is easy to see that the map $a \in A \mapsto a(n) \in A(n)$ is a partially defined, surjective ring homomorphism.

A set $S = \{p_i,\ i = 1, 2, \dots, r\}$ of idempotents is called a partition of unity if $p_i p_j = p_j p_i = 0$ for $i \neq j$ and if $\sum_{i=1}^{r} p_i = 1$. We will need the following presumably well-known results.

LEMMA (2.3). *Let $\{p_i,\ i = 1, 2, \dots, r\}$ and $\{q_j,\ j = 1, 2, \dots, s\}$ be partitions of unity in A such that $p_i(n)$ and $q_j(n)$ are defined for all possible indices and let q be an idempotent in A such that $q(n)$ is defined. Then*

(a) $\dim_k p_i(n)A(n)q_j(n) = \dim_{k(x)} p_i A q_j$ *for* $i = 1, 2, \dots, r$ *and* $j = 1, 2, \dots, s$.

(b) *If $A \cong M_d(k(x))$ and if $A(n)$ is semisimple, then $A(n) \cong M_d(k)$.*

(c) *Let both qAq and $q(n)A(n)q(n)$ be semisimple. Then $qAq \cong k(x) \otimes q(n)A(n)q(n)$.*

Proof. (a) Let $a_1(n), \dots, a_t(n)$ be a basis of $p_i(n)A(n)q_j(n)$. Let, for $l = 1, 2, \dots, t$, a_l be the corresponding linear combination of $b_1, \dots, b_m$. We define $a_l' = p_i a_l q_j \in p_i A q_j$. Obviously $a_l'(n) = p_i(n) a_l q_j(n) = a_l(n)$ and the a_l''s are linearly independent because they are already for $x = n$. Hence $\dim_k p_i(n)A(n)q_j(n) \leq \dim_{k(x)} p_i A q_j$. But $A = \oplus_{i,j} p_i A q_j$ as a vector space. Hence the dimensions have to be equal as $\dim_{k(x)} A = \dim_k A(n)$.

(b) This is obviously true for $d = 1$ by (a). Let $p \in A$ be such that Ap is a minimal left ideal of A. By the semisimplicity of $A(n)$ we can find $a \in A$ such that $ap(n)$ is well-defined and $(ap)^2(n) \neq 0$. As Ap is minimal, $(ap)^2 = \alpha ap$ for some $\alpha \in k(x)$ with $\alpha(n) \neq 0$. Hence $p_1 = (1/\alpha)ap$ is an idempotent such that $p_1(n)$ is also a well-defined idempotent in $A(n)$. By the induction assumption for $d - 1$, we have a partition of unity $\{p_2, p_3, \dots, p_d\}$ in $(1 - p_1)A(1 - p_1)$ which is also well-defined for $x = n$. It follows from (a) that $\dim_k p_i(n)A(n)p_j(n) = \dim_{k(x)} p_i A p_j = 1$ for $i, j = 1, 2, \dots, d$. We thus obtain sufficiently many matrix units in $A(n)$.

(c) Let $z_1, \dots, z_s$ be the minimal central idempotents of A. If $z_i(n)$ is defined, it obviously must be a central idempotent of $q(n)A(n)q(n)$. If it were not defined, we could find a $d \in N$ such that if $z_i' = (x - n)^d z_i$, $z_i'(n)$ is defined and not equal to 0. But then $(z_i'(n))^2 = [(x - n)^d z_i'](n) = 0$. This would mean that $z_i'(n)$ is a nonzero central nilpotent in the semisimple algebra $q(n)A(n)q(n)$ which is not possible. The $z_i(n)$'s are nonzero by (a) and $Az_i \cong k(x) \otimes A(n)z_i(n)$ by (b).

3. Representations of $D_f(n)$

As pointed out in the introduction, the definition of the algebras $D_f(n)$ was motivated by representations of Lie groups. We will go back to this definition and define representations of $D_f(n)$ onto $B_f(\mathrm{O}(n))$ for n odd. Using results of the representation theory of these groups (mainly the formulas for the dimensions of their irreducible representations) we will obtain information about the traces τ_n not only for these values of n but also about τ and τ_n for arbitrary $n \in k$. A comprehensive study of traces on centralizer algebras for infinite tensor products can be found in [Wa].

Let n be a positive integer and let e_{ij} be matrix units for $i, j = 1, 2, \ldots, n$. Then it is easy to check that for

$$G = \sum e_{ij} \otimes e_{ji}$$

and

$$E = \sum e_{ij} \otimes e_{ij}$$

G and E are in $B_2(\mathrm{O}(n))$ for $n \geq 3$. Note also that G is the "flip"; i.e. $G(\xi \otimes \eta) = \eta \otimes \xi$, and that E corresponds to Weyl's trace operation (or contraction); i.e. it maps $\xi \otimes \eta$ onto a multiple of the vector $\delta_{i_1 i_2} = \Sigma v_i \otimes v_i$, where $v_1, v_2, \ldots, v_n$ is an orthonormal basis of V (see [Wy], V.6).

By [Br, (19) and §5], there is an isomorphism between $D_2(n)$ and $B_2(\mathrm{O}(n))$ for $n \geq 3$ mapping G to g_1 and E to e_1. More generally, let us define elements G_i and E_i of $(M_n)^f = M_n \otimes \cdots \otimes M_n$ (f times) by

$$G_i = 1 \otimes \cdots \otimes 1 \otimes G \otimes 1 \otimes \cdots \otimes 1$$

and

$$E_i = 1 \otimes \cdots \otimes 1 \otimes E \otimes 1 \otimes \cdots \otimes 1$$

where we plug in G and E for the i-th and $(i+1)$-th factor. Then it follows again by [Br, (19) and §5], that Φ: $e_i \mapsto E_i$ and $g_i \mapsto G_i$ defines a homomorphism from $D_f(n)$ onto $B_f(\mathrm{O}(n))$. If n is large enough (say $n > 2f$), this representation is faithful and semisimple (see [Wy], V.5). In particular, the quotient $D_f(n)/I_f(n) \cong kS_f$ splits as a direct summand. Note that if $z(n)$ is the central idempotent corresponding to $I_f(n)$ and if p_λ is a Young idempotent for $kS_f \subset D_f(n)$, then $q_\lambda(n) = (1 - z(n))p_\lambda$ is a minimal idempotent of $D_f(n)/I_f(n)$. Hence, as Φ is injective,

$$g \in \mathrm{O}(n) \mapsto \Phi(q_\lambda(n))\pi^{\otimes f}(g)\Phi(q_\lambda(n))$$

is an irreducible representation of $\mathrm{O}(n)$. By [EK], there exists a polynomial P_λ, derived from Weyl's dimension formulas such that the dimension $\dim_{\lambda, n}$ of this representation is given by

$$\dim_{\lambda, n} = P_\lambda(n).$$

The polynomial P_λ can be written in the following form: Let $\tilde{\lambda}_j$ be the number of nodes in the j-th column and let

$$c(\lambda) = \prod_{(i,j)}^{\lambda} \left(\lambda_i + \tilde{\lambda}_j + 1 - i - j\right),$$

with the product taken over all pairs (i, j) specifying row and column of a node of λ. Then P_λ can be written as

$$P_\lambda(x) = 1/c(\lambda) \prod_{i \geq j}^{\lambda} \left(x + \lambda_i + \lambda_j - i - j\right) \times \prod_{i<j}^{\lambda} \left(x - \tilde{\lambda}_i - \tilde{\lambda}_j + i + j - 2\right). \tag{6}$$

We see immediately that

(a) All roots of P_λ are integers.

(b) The smallest and the second smallest roots are $2 - 2\lambda_1$ and $3 - \lambda_1 - \lambda_2$.

(c) The largest root is $\tilde{\lambda}_1 + \tilde{\lambda}_2 - 1$.

Let tr denote the usual normalized trace on $(M_n)^f \cong M_{n^f}$. If tr′ is the normalized trace on M_n, we have

$$\mathrm{tr}\left(a_1 \otimes \cdots \otimes a_f\right) = \prod \mathrm{tr}'(a_i).$$

It follows from the considerations above that

$$\mathrm{tr}\left(\Phi(q_\lambda(n))\right) = P_\lambda(n)/n^f. \tag{7}$$

As any minimal idempotent p of $B_f(G)$ corresponds to an irreducible representation of G, it follows $\mathrm{tr}(p) \neq 0$ in general. Hence tr is nondegenerate as $B_f(\mathrm{O}(n))$ is semisimple.

We note here that the corresponding matrices G_i and E_i for an even dimensional vector space also generate $B_f(\mathrm{O}(n))$ but not $B_f(\mathrm{SO}(n))$.

LEMMA (3.1). *Let G_i, E_i and* tr *be defined for $n = 2m + 1$ as above and let τ and τ_n be as in Section 2.*

(a) $\mathrm{tr}(E_i) = \mathrm{tr}(G_i) = 1/n$.

(b) $\mathrm{tr}(aE_{f-1}b) = \mathrm{tr}(aG_{f-1}b) = (1/n)\mathrm{tr}(ab)$ *for* $a, b \in B_{f-1}(\mathrm{O}(n))$.

(c) *τ and τ_n are traces on D_f and $D_f(n)$ for arbitrary $n \in k$. For n a positive odd integer we have* $\tau_n = \mathrm{tr} \circ \Phi$.

(d) *Let $J_f(n) = \{a \in D_f(n),\ \tau_n(ab) = 0$ for all $b \in D_f(n)\}$ be the annihilator of τ_n in $D_f(n)$. Then $J_f(n)$ is a two-sided ideal and $J_f(n) \subset J_{f+1}(n)$.*

Proof. (a) is a straightforward computation.

(b) $(M_n)^{f-1} \otimes 1$ is the span of elements $b \in (M_n)^f$ of the form $b = b_1 \otimes e_{ij} \otimes 1$ with $b_1 \in (M_n)^{f-2}$ and $i, j = 1, 2, \ldots, n$. For these elements it is easily verified by a direct matrix computation that $\operatorname{tr}(E_{f-1}b) = \operatorname{tr}(G_{f-1}b) = (1/n)\operatorname{tr}(b)$. The general case follows from the trace property.

(c) It follows from (a), (b) and Proposition (2.2)(c) that $\tau_n = \operatorname{tr} \circ \Phi$. Hence it is a trace on $D_f(n)$ for $n = 2m + 1$, $m = 1, 2, \ldots$. So for any two graphs $b_1, b_2 \in D_f$ we have

$$\tau(b_1b_2)(n) = \tau_n(b_1(n)b_2(n)) = \tau_n(b_2(n)b_1(n)) = \tau(b_2b_1)(n) \qquad \text{for } n = 3, 5, 7, \ldots .$$

Hence $\tau(b_1b_2)$ and $\tau(b_2b_1)$ have to be the same rational functions. The case for arbitrary $n \in k$ follows from this.

(d) As τ_n is a trace, $J_f(n)$ is a two-sided ideal. Note that if $a \in J_f(n)$, $\tau_n(ab_1\chi b_2) = (1/n)\tau_n(ab_1b_2) = 0$ for $b_1, b_2 \in D_f(n)$ and $\chi \in \{g_f, e_f\}$. Hence, by Propositions (2.1)(a) and (2.2)(c) $J_f(n) \subset J_{f+1}(n)$.

Let Γ_f be the set of all Young diagrams with $k \leq f$ nodes such that $k \geq 0$ and $f - k$ is even.

THEOREM (3.2). (a) *The $k(x)$ algebra D_f is semisimple. It decomposes into a direct sum of full matrix algebras $D_{f,\lambda}$, where $\lambda \in \Gamma_f$. A simple $D_{f,\lambda}$ module $V_{f,\lambda}$ decomposes as a D_{f-1} module into a direct sum*

$$V_{f,\lambda} = \bigoplus V_{f-1,\mu},$$

where $V_{f-1,\mu}$ is a simple $D_{f-1,\mu}$ module and μ ranges over all Young diagrams obtained from λ by removing or (if λ contains fewer than f nodes) adding one node. The weight vector of τ is given by $(P_\lambda(x)/x^f)_{\lambda \in \Gamma_f}$.

(b) *Let $D_{f-1}(n)$ and $D_f(n)$ be semisimple. Then the following statements are equivalent*

(i) *τ_n is nondegenerate on $D_f(n)$.*

(ii) *$D_{f+1}(n)$ is semisimple and $D_{f+1}(n) \otimes k(x) \cong D_{f+1}$.*

In this case, the weight vector of τ_n is given by $(P_\lambda(n)/n^f)_{\lambda \in \Gamma_f}$.

Proof. We shall prove (a) and (b), (i) $\Rightarrow$ (ii) by induction on f. The statements are trivially true for $f = 0$ and $f = 1$. Assume that D_{f-1} and D_f are semisimple and τ is a nondegenerate trace on them. It follows from Propositions (2.1)(b) and (2.2) that the conditions of Theorem (1.3) are satisfied for $A = D_{f-1}$, $B = D_f$, $e = (1/x)e_f$ and $\operatorname{tr} = \tau$. So the ideal $\langle e_f \rangle \subset D_{f+1}$ is isomorphic to Jones' extension for $D_{f-1} \subset D_f$ by that theorem and Proposition (2.1)(b). By Proposition (1.2)(b) and the induction assumption, the simple components of

$\langle e_f \rangle$ are labeled by the elements of Γ_{f-1}. In particular, $\langle e_f \rangle$ is semisimple. By Proposition (2.1)(b) and (c) the quotient $D_{f+1}/\langle e_f \rangle \cong k(x)S_{f+1}$ is also semisimple. Hence D_{f+1} is semisimple.

Let $V_{f+1,\lambda}$ be a simple $\langle e_f \rangle$ module. As $\langle e_f \rangle \cong \langle D_f, e_{D_{f-1}} \rangle$, $V_{f+1,\lambda}$ can be considered as a direct sum of exactly those $V_{f,\mu}$ which contain a simple $D_{f-1,\lambda}$ module by Proposition (1.2)(b). So the claim follows by the induction assumption from the inclusion $D_{f-1} \subset D_f$. If $V_{f+1,\lambda}$ is annihilated by e_f, it can be regarded as an S_{f+1} module, for which the decomposition into simple S_f modules is well-known (see §1).

Let $p_\lambda \in D_{f-1,\lambda}$ be a minimal idempotent. By Proposition (1.2)(b) and (4), $(1/x)e_f p_\lambda$ is a minimal idempotent of $D_{f+1,\lambda}$ and

$$\tau\big((1/x)e_f p_\lambda\big) = \big(1/x^2\big)\tau(p_\lambda) = P_\lambda(x)/x^{f+1}$$

by the induction assumption.

Let z be the central idempotent for $\langle e_f \rangle$ and let $q_\lambda = (1-z)p_\lambda$ with p_λ a Young idempotent for $\lambda \in \Lambda_{f+1}$. It follows from the remark before Lemma (3.1) that $q_\lambda(n)$ is defined for odd n with $n > 2(f+1)$. Using Lemma (3.1)(c) and (7), we obtain

$$\tau(q_\lambda)(n) = \tau_n\big(q_\lambda(n)\big) = \mathrm{tr} \circ \Phi\big(q_\lambda(n)\big) = \dim_{\lambda,n}/n^{f+1} = P_\lambda(n)/n^{f+1}.$$

Hence, as the rational functions $\tau(q_\lambda)(x)$ and $P_\lambda(x)/x^{f+1}$ coincide at infinitely many points, they have to be identical. In particular, they are nonzero. This shows the statement about the weight vector of τ.

Let us assume (b), (i). By Lemma (3.1)(d), we have $J_{f-1}(n) \subset J_f(n)$, so that τ_n is also nondegenerate on $D_{f-1}(n)$. It follows from the same arguments as in the proof for (a) that $D_{f+1}(n)$ is semisimple and that the weight vector of τ_n on $D_{f+1}(n)$ is equal to $(P_\lambda(n)/n^{f+1})_{\lambda \in \Gamma_{f+1}}$. The proof of the converse implication will be given after Corollary (3.5).

The following corollary proves the conjecture of Hanlon and Wales which was mentioned above.

Corollary (3.3). *If n is not an integer, $D_f(n)$ is semisimple and the trace τ_n is nondegenerate on $D_f(n)$ for all $f \in \mathbf{N}$.*

Proof. The statement is obviously true for $D_0(n)$ and $D_1(n)$. Let us assume the claim for $D_{f-1}(n)$ and $D_f(n)$. Then also $D_{f+1}(n)$ is semisimple by the just shown part of Theorem (3.3)(b). By (6)(a) all the entries of the vector $(P_\lambda(n)/n^{f+1})_{\lambda \in \Gamma_{f+1}}$ are nonzero. Hence τ_n is nondegenerate on $D_{f+1}(n)$.

Theorem (3.2) gives an easy inductive procedure to compute the dimensions of the simple D_f modules (similar procedures can also be found in [Be], [K] or [S] for special values of x). For small f, this can be conveniently done using Bratteli diagrams (see §1). One only has to reflect the inclusion pattern for $D_{f-1} \subset D_f$ about the line of D_f to obtain the structure of $\langle e_f \rangle$. Examples are given at the end of [BW].

The techniques of the proof of Theorem (3.2) can also be used if τ_n is degenerate on $D_f(n)$ by factoring over its annihilator. So let $B_f(n) = D_f(n)/J_f(n)$ and let $\rho^{(f)}$ be the quotient map from $D_f(n)$ onto $B_f(n)$. It follows from Lemma (3.1)(d) that $\rho^{(f+1)}(D_f(n))$ is isomorphic to $\rho^{(f)}(D_f(n)) = B_f(n)$, where we regard $D_f(n)$ as a subalgebra of $D_{f+1}(n)$ as usual. We will therefore omit the index f of $\rho^{(f)}$.

For $n = 2m + 1 > 0$, we can show easily that $B_f(n) \cong B_f(\mathrm{O}(n))$. Indeed, the kernel of Φ is contained in the kernel of ρ by Lemma (3.1)(d). On the other hand, tr is a nondegenerate trace on $B_f(\mathrm{O}(n))$ by the remark before Lemma (3.1). Hence the other inclusion also holds.

For even $n > 0$ the same proof can be applied to show that $B_f(n)$ is isomorphic to $B_f(\mathrm{O}(n))$.

To determine the structure of the $B_f(n)$'s, we need to define a special class of Young diagrams. As usual, we say that a Young diagram μ is a subdiagram of the Young diagram λ, denoted by $\mu < \lambda$, if μ can be obtained from λ by taking away appropriate nodes.

A Young diagram λ is said to be *n-permissible* if $P_\mu(n) \neq 0$ for all subdiagrams $\mu \leq \lambda$. The n-permissible Young diagrams of Γ_f and Λ_f are denoted by $\Gamma_f^{(n)}$ and $\Lambda_f^{(n)}$ respectively.

THEOREM (3.4). (a) *Let $\lambda \in \Lambda_f$. If all subdiagrams of λ are n-permissible, then there exists a minimal idempotent $q_\lambda \in D_{f,\lambda}$ such that $q_\lambda(n)$ is well-defined.*

(b) *$B_f(n)$ is the direct sum of full matrix algebras $B_{f,\lambda}$ where $\lambda \in \Gamma_f^{(n)}$. A simple $B_{f,\lambda}(n)$ module $\overline{V}_{f,\lambda}$ decomposes as a $B_{f-1}(n)$ module into a direct sum*

$$\overline{V}_{f,\lambda} = \bigoplus \overline{V}_{f-1,\mu}$$

with μ ranging over all n-permissible Young diagrams obtained from λ by removing or (if λ contains fewer than f nodes) by adding one node.

Proof. As usual, the proof goes by induction on f with $f = 0$ and $f = 1$ being trivial. The trace τ_n is nondegenerate on $B_{f-1}(n)$ and $B_f(n)$ by definition of these algebras. So we can show as in the proof of Theorem (3.2) that $\langle \rho(e_f) \rangle$ is semisimple and that $\overline{D}_{f+1}(n) = D_{f+1}(n)/(\langle e_f \rangle \cap J_{f+1}(n))$ is isomorphic to the semisimple algebra $\langle \rho(e_f) \rangle \oplus C$ with $C \cong kS_{f+1}$. Obviously τ_n and ρ are also well-defined on this quotient of $D_{f+1}(n)$. To find the structure of $B_{f+1}(n)$, we have to determine which simple components of $\overline{D}_{f+1}(n)$ are annihilated by τ_n.

We label the simple components of C by Young diagrams as usual. Let $\lambda \in \Lambda_{f+1}$ be such that $\rho(C_\lambda) \neq 0$, and let V be a simple C_λ module. If we regard V as a $B_f(n)$ module, it decomposes into a direct sum of simple $B_{f,\lambda'}(n)$ modules with $\lambda' < \lambda$ by the branching rule for $kS_f \subset kS_{f+1}$ (see also the proof of Theorem (3.2)). Hence all proper subdiagrams μ of λ have to be n-permissible by the induction assumption.

Let λ' be an n-permissible subdiagram of λ and let $q_{\lambda'}$ be a minimal idempotent of $D_{f,\lambda'}$. By Theorem (3.2)(a) and (1) we have $q_{\lambda'}D_{f+1}q_{\lambda'} \cong k(x)^s$, where s is the number of diagrams on $f-1$ or $f+1$ nodes which are contained in or contain λ'.

By the induction assumption for (a), we can choose $q_{\lambda'}$ such that $q_{\lambda'}(n)$ is defined. As all subdiagrams of λ' are also n-permissible, we show as for D_{f+1} that $\bar{q}_{\lambda'}(n)\overline{D}_{f+1}(n)\bar{q}_{\lambda'}(n)$ is isomorphic to k^s. By Lemma (2.3)(a) we also have $q_{\lambda'}(n)D_{f+1}(n)q_{\lambda'}(n) \cong k^s$. Hence there exists a minimal idempotent $q_\lambda \in D_{f+1,\lambda}$ such that $q_\lambda(n)$ is well-defined by Lemma (2.3), which shows (a).

Using the partially defined ring homomorphism from D_{f+1} onto $D_{f+1}(n)$, we find that $q_\lambda(n)$ is annihilated by $I_{f+1}(n)$ and by all central idempotents $z_\mu \in kS_{f+1} \subset D_{f+1}(n)$ for $\mu \neq \lambda$. Hence $\bar{q}_\lambda \in C_\lambda$. As τ_n factors over $\langle e_f \rangle \cap J_{f+1}(n)$, we have by Proposition (2.2)(f),

$$\tau_n(\bar{q}_\lambda(n)) = \tau_n(q_\lambda(n)) = \tau(q_\lambda)(n) = P_\lambda(n)/n^{f+1}.$$

So $q_\lambda(n) \in J_{f+1}(n)$ if and only if $P_\lambda(n) = 0$. By the semisimplicity of $\overline{D}_{f+1}(n)$ this is equivalent to $\rho(C_\lambda) = 0$.

By the induction assumption for $B_{f-1}(n)$ and Proposition (2.1)(b), the simple components of $\langle \rho(e_f) \rangle$ are labeled by the elements of $\Gamma_{f-1}^{(n)}$.

As for the D_f's, the branching rule gives us a way to compute the dimensions of the simple $B_f(n)$ modules. It only remains to determine the n-permissible Young diagrams. Note that part (a) of the following corollary restates Corollary (3.3). Part (c) relates the $B_f(n)$'s to the algebras in which Brauer and Weyl were primarily interested.

COROLLARY (3.5). (a) *If $n \in k$ is not an integer, all Young diagrams are n-permissible. In this case $D_f(n) \cong B_f(n)$ and its decomposition into full matrix rings is the same as for D_f.*

(b) *If n is a nonzero integer, a Young diagram λ is n-permissible if and only if:*

(i) *Its first 2 columns contain at most n nodes for n positive.*

(ii) *It contains at most m columns for $n = -2m$ a negative even integer.*

(iii) *Its first 2 rows contain at most $2 - n$ nodes for n odd and negative.*

(c) *If n is a positive integer, $B_f(n) \cong B_f(\mathrm{O}(n))$. If n is negative and odd, $B_f(n) \cong B_f(\mathrm{O}(2-n))$. For $n = -2m < 0$, $B_f(n)$ is isomorphic to $B_f(\mathrm{Sp}(2m))$.*

Proof. Part (a) follows from (6)(a). Part (c) follows for positive n from the remarks before Theorem (3.4). Part (b) is an easy consequence of (6), (b) and (c).

Let $\tilde{\lambda}$ be the Young diagram with λ_1 nodes in the first *column*, λ_2 nodes in the second, etc. Then it can be easily shown by induction using Theorem (3.4), that for odd negative n, $B_{f,\lambda}(n) \cong B_{f,\tilde{\lambda}}(2-n)$ for any n-permissible Young diagram λ and therefore $B_f(n) \cong B_f(2-n)$.

The proof of the remaining case of part (c) follows the lines of the one for the orthogonal groups.

Let us label a basis of a $2m$ dimensional vector space in pairs by $1, 1', 2, 2', \ldots m, m'$. We obtain a corresponding labeling for the matrix units of $M_{2m}(k)$. Let

$$\tilde{E} = \sum e_{ij} \otimes e_{i'j'} + e_{i'j'} \otimes e_{ij} - e_{ij'} \otimes e_{i'j} - e_{i'j} \otimes e_{ij'}.$$

The elements $\tilde{E}_i$ and G_i are defined as at the beginning of this section. Then $G_1, \ldots G_{f-1}, \tilde{E}_1, \ldots \tilde{E}_{f-1}$ generate $B_f(\mathrm{Sp}(2m))$ (see for instance [Br]). It can be checked by explicit matrix computations that $-G_i$ and $-\tilde{E}_i$ are compatible with the relations for $\hat{G}_i$ and $\hat{E}_i$ in [BW, §5] with $x = -2m$. Hence $g_i \mapsto -G_i$ and $e_i \mapsto -\tilde{E}_i$ defines a representation of $D_f(-2m)$, the image of which is $B_f(\mathrm{Sp}(2m))$. The rest of the proof goes as in Lemma (3.1) and in the remark before Theorem (3.4).

As an example, the Bratteli diagram of $B_f(3) \cong B_f(\mathrm{O}(3))$ is shown in Figure 3 for $f = 0, 1, \ldots, 4$.

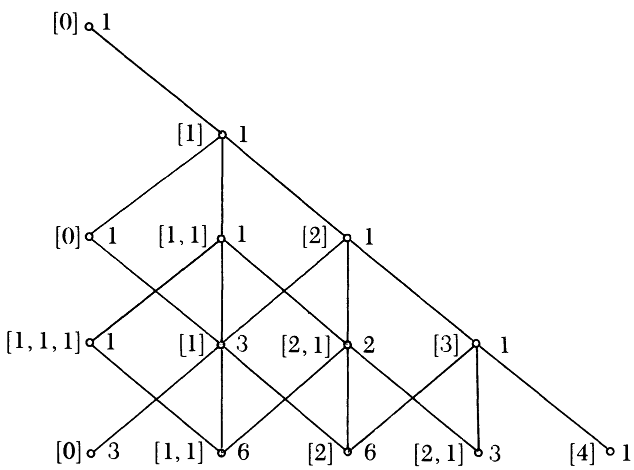


FIGURE 3

Conclusion of the proof of Theorem (3.2). Let us assume that $D_0(n), \ldots, D_f(n)$ are semisimple. By the induction assumption, τ_n has to be nondegenerate on $D_0(n), \ldots, D_{f-1}(n)$. Assume τ_n is degenerate on $D_f(n)$. Then $D_{f-1}(n) \cong B_{f-1}(n)$, while $\dim_k B_f(n) < \dim_k D_f(n)$. So the semisimple quotient $\langle \bar{e}_f \rangle \oplus kS_{f+1}$ of $D_f(n)$ (see the proof of Theorem (3.4)) cannot be a faithful representation of $D_{f+1}(n)$. On the other hand, it has as many simple components as D_{f+1}. So if $D_{f+1}(n)$ were semisimple, it would have more simple components than D_{f+1}, a contradiction to Lemma (2.3)(c).

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REFERENCES

[ABP] M. ATIYAH, R. BOTT, and V. K. PATODI, On the heat equation and the index theorem, Inv. Math. 19 (1973), 279–330.

[Be] A. BERELE, A Schensted-type correspondence for the symplectic group, J. Comb. Th. Series A (1986), 320–328.

[BW] J. BIRMAN, and H. WENZL, Braids, link polynomials and a new algebra, Trans. AMS (to appear).

[Br] R. BRAUER, On algebras which are connected with the semisimple continuous groups, Ann. of Math. 38 (1937), 854–872.

[Bou] N. BOURBAKI, *Algèbre*, Ch. 8, Hermann, 1959.

[Bw] W. BROWN, The semisimplicity of ω_f^n, Ann. of Math. 63 (1956), 324–335.

[EK] N. EL SAMRA, and R. KING, Dimensions of irreducible representations of the classical Lie groups, J. Phys. A: Math. Gen. 12 (1979), 2317–2328.

[GHJ] F. GOODMAN, P. DE LA HARPE, and V. F. R. JONES, Coxeter-Dynkin diagrams and towers of algebras (to appear).

[HW1] P. HANLON, and D. WALES, On the decomposition of Brauer's centralizer algebras, J. Alg. (to appear).

[HW2] ———, Eigenvalues connected with Brauer's centralizer algebras, J. Alg. (to appear).

[HW3] ———, Computing the discriminants of Brauer's centralizer algebras, preprint, California Institute of Technology.

[J1] V. F. R. JONES, Index for subfactors, Inv. Math. 72 (1983), 1–25.

[J2] ———, A polynomial invariant for links via von Neumann algebras, Bull. AMS 12 (1985), 103–111.

[K] R. KING, Modification rules and products of irreducible representations of the unitary, orthogonal and symplectic groups, J. Math. Phys. 12 (1971), 1588–1598.

[M] F. MURNAGHAN, *The Theory of Group Representations*, Dover, 1938.

[S] S. SUNDARAM, *On the Combinatorics of Representations of* Sp(2n, C), Memoirs of the AMS (to appear).

[Wa] A. WASSERMANN, Automorphic actions of compact groups on operator algebras, thesis, University of Pennsylvania, 1981.

[Wn] H. WENZL, Representations of Hecke algebras and subfactors, thesis, University of Pennsylvania, 1985; to appear in Inv. Math. (1988).

[Wy] H. WEYL, *The Classical Groups*, Princeton University Press, Rev. 1946.

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