Customer:
Sara Billey (billey)
Box 354350
Math dept
Seattle, WA 98105

Email address: billey@math.washington.edu
Phone Number: 2066163107
Fax:
UW Status: Seattle

Billing Account: 1060155
Customer Reference:

Needed By: 04/25/2014

U.S. Libraries Only

Loansome Doc:

Journal Title: **Studies in applied mathematics**

Volume: 61 Issue: 2
Month/Year: 1979
Pages: 93-139

Article Author: Joni and Rota

Article Title: Coalgebras and bialgebras in combinatorics

ISSN: 0022-2526

English only!

Notes/Alternate Delivery:

☒ Paged to SZ 3/28/14 ☐ Paged to HS ☐ Emailed Loc

Coalgebras and Bialgebras in Combinatorics

By S.A. Joni and G.-C. Rota†*

The following material is discussed in this paper: Incidence Coalgebras for PO sets; Reduced Boolean Coalgebras; Divided Powers Coalgebra; Dirichlet Coalgebra; Eulerian Coalgebra; Faà di Bruno Bialgebra; Incidence Coalgebras for Categories; The Umbral Calculus; Infinitesimal Coalgebras; Creation and Annihilation Operators; Point Lattice Coalgebras; Restricted Placements; Cleavages; and Hereditary Bialgebras.

Dedicated to William T. Tutte on his 60th birthday.

Forse altri canterà con miglior plettro

—*Ariosto*

I. Introduction

A great many problems in combinatorics are concerned with assembling, or disassembling, large objects out of pieces of prescribed shape, as in the familiar board puzzles. Even in the seemingly simple case of finite sets, very little is known on, say, the structure of families of sets subject to restrictions. The oldest result in this direction is Sperner's theorem, which gives the structure of all maximum size families of subsets of a finite set, subject to the restriction that no set in the family may be contained in another. On the blueprint of Sperner's theorem, a host of similar results have been developed, largely in the last fifteen years, but the proofs rely more on ingenuity than on general techniques.

In more complicated cases, our understanding is even more limited; rarely, except perhaps in number theory, has a branch of mathematics been so rich in relevant problems and so poor in general ideas as to how such problems may be attacked.

This paper grew out of an attempt to make some of the combinatorial problems of assemblage available to a public of algebraists. It originated from

---

Address for correspondence: Professor G.-C. Rota, Room 2-351, M.I.T., Cambridge, MA 02139.
*Research partially supported by NSF Contract No. MCS 7820264.
†Research partially supported by NSF Contract No. MCS 7701947

Published by Elsevier North Holland, Inc. 0022-2526/79/050093+47$01.75

the realization that the notions of coalgebra, bialgebra, and Hopf algebra, recently introduced into mathematics, may give in a variety of cases a valuable formal framework for the study of combinatorial problems. Armed with this realization, we have assembled in this paper a variety of coalgebras and bialgebras which arise in combinatorics, in the hope of interesting both the combinatorist in search of a theoretical horizon, and the algebraist in search of examples which may point to new and general theorems.

The modesty of our undertaking cannot be overemphasized. We have simply given a list of coalgebras and bialgebras as possible objects of investigation, and proved only a few elementary results whenever the proofs were indispensable to the understanding of the examples.

Several of the coalgebras described below are presented here for the first time, notably puzzles, closure coalgebras, infinitesimal coalgebras, hereditary bialgebras, rook coalgebras, and cleavages. Others are drawn from previous work on the subject by P. Doubilet, M. Henle, R. W. Lawvere, S. Roman, R. Stanley, and ourselves.

It must be stressed that the coalgebras of combinatorics come endowed with a distinguished basis, and many an interesting combinatorial problem can be formulated algebraically as that of transforming this basis into another basis with more desirable properties. Thus, a mere structure theory of coalgebras—or Hopf algebras—will hardly be sufficient for combinatorial purposes.

Most of the content of this paper was developed from the Hopf Algebras and Combinatorics lectures presented by G.-C. Rota during the Umbral Calculus Conference at the University of Oklahoma on May 15–19, 1978. The authors take this opportunity to thank Professor M. Marx and Professor Robert Morris of the Mathematics Department of the University of Oklahoma for giving them an opportunity to present these ideas to a responsive audience of coalgebraists, as well as for their gracious hospitality.

II. Notation and terminology

Very little knowledge is required to read this work. Most of the concepts basic enough to be left undefined in the succeeding sections will be introduced here.

A *partial ordering relation* (denoted by $\leqslant$) on a set P is one which is reflexive, transitive, and antisymmetric (that is, $a \leqslant b$ and $b \leqslant a$ imply $a = b$). A set P together with a partial ordering relation is a *partially ordered set*, or PO set for short. For $x \leqslant y$ in P, the *segment* (or *interval*) $[x,y]$ is the collection of all elements z in P such that $x \leqslant z \leqslant y$. A PO set is said to be *locally finite* if every segment is finite. All the PO sets we shall consider will be locally finite.

A PO set P is said to have a 0 or a 1 if it has a unique minimal or maximal element. An element y is said to cover x if the segment $[x,y]$ has two elements. An *atom* of P is an element which covers a minimal element.

An *ordered ideal* in a PO set P is a subset J which has the property that if $y \in J$ and $x \leqslant y$, then $x \in J$.

The *product* $P \times Q$ of two PO sets P and Q is the set of all ordered pairs (p,q) where $p \in P$ and $q \in Q$, with $(p,q) \geqslant (r,s)$ if and only if $p \geqslant r$ and $q \geqslant s$. The product of any number of PO sets is defined similarly.

A *lattice* is a PO set where the max and min of two elements (we call them join and meet, and write them $\vee$ and $\wedge$) are defined. A *sublattice* L' of a lattice

L is a subset of L which is a lattice under the induced partial ordering such that the join and meet of any two elements in L' are the same as those in L. A *distributive lattice* is one in which for all p,q,r in L, $p\wedge(q\vee r)=(p\wedge r)\vee(p\wedge q)$ and $p\vee(q\wedge r)=(p\vee r)\wedge(p\vee q)$.

A *partition* π of a set S is a collection of pairwise disjoint nonempty subsets of S, called the *blocks* of π, whose union is S. The *lattice of partitions* $\Pi(S)$ is the set of all partitions of S ordered by *refinement*: a partition π is less than or equal to a partition σ (or π is a refinement of σ) if each block of π is contained in a block of σ. The 0 of $\Pi(S)$ is the partition having all blocks of size one, and the 1 is the partition with one block. For further study of lattices, the reader is referred to Birkhoff.

We come now to the definition of the *incidence algebra* $\mathcal{I}(P)$ of a locally finite PO set P over a field K. We shall assume throughout that K has characteristic zero. The members of $\mathcal{I}(P)$ are functions of two variables $f: P\times P\rightarrow K$ such that $f(x,y)=0$ unless $x\leqslant y$. The sum of two functions, as well as multiplication by scalars, is defined as usual. The product (or convolution) $f*g=h$ is defined by

$$h(x,y)=\sum_{z\in P} f(x,z)g(z,y).$$

Since P is locally finite, the variable z in the above sum ranges over the finite segment $[x,y]$. It is immediate that this product is associative, and the unit element δ is

$$\delta(x,y)=\begin{cases}1 & \text{if } x=y,\\ 0 & \text{otherwise}\end{cases}.$$

No further knowledge of the incidence algebra is required in the present paper; the reader is referred to [4] and [12] for studies of this algebra.

A *coalgebra* is a triple (C,Δ,ε) with C a K-vector space, $\Delta: C\rightarrow C\otimes C$ a map called *diagonalization* or *comultiplication*, and $\varepsilon: C\rightarrow K$ a map called the *counit* or *augmentation*, where Δ and ε satisfy the following commutative diagrams:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C\otimes C \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta\otimes I} \\ C\otimes C & \xrightarrow{I\otimes\Delta} & C\otimes C\otimes C \end{array} \qquad \text{(coassociativity)}, \qquad (2.1)$$

$$\begin{array}{ccccc} & & C & & \\ & \swarrow & \downarrow{\scriptstyle\Delta} & \searrow & \\ K\otimes C & & & & C\otimes K \\ & {\scriptstyle\varepsilon\otimes I}\nwarrow & & \nearrow{\scriptstyle I\otimes\varepsilon} & \\ & & C\otimes C & & \end{array} \qquad \text{(counitary property)}. \qquad (2.2)$$

Thus, coassociativity says $(I\otimes\Delta)\circ\Delta=(\Delta\otimes I)\circ\Delta$, or in words, after diagonalizing once, we can next diagonalize in either factor and obtain the same result. When we write "a coalgebra C," we mean "a coalgebra (C,Δ,ε)."

A *subcoalgebra* of a coalgebra C is a subspace W such that $\Delta(W)\subseteq W\otimes W$. A *coideal* of C is a subspace J such that $\Delta(J)\subseteq J\otimes C+C\otimes J$ and $\varepsilon(J)=0$. If $\sim$ is an equivalence relation on a basis of C such that the subspace J spanned by

$\{f-g:f\sim g\}$ is a coideal, then the quotient space $C/\sim$ can be endowed with the coalgebra structure of the quotient coalgebra of C modulo J.

A space B which is simultaneously an algebra and a coalgebra is said to be a *bialgebra* if the diagonalization Δ and counit ε are algebra maps.

Let C be a coalgebra, A an algebra, and set for $c\in C$

$$\Delta c = \sum_i c_{1i} \otimes c_{2i}.$$

We give $\mathrm{Hom}(C,A)$ an algebra structure by defining the product or convolution $f*g=h$ as follows:

$$h(c) = f*g(c) = \sum_i f(c_{1i})g(c_{2i}).$$

The unit of this algebra is $u\varepsilon$, where u is the unit of A.

Let H be a bialgebra, and let us define I in $\mathrm{Hom}(H,H)$ to be map $I(h)=h$ for all h in H. If it exists, the unique element S in H which is inverse under $*$ to I (i.e., $S*I=I*S=u\varepsilon$) is the *antipode* of H. A bialgebra with an antipode is a *Hopf algebra*. For a further study of bi-, co-, and Hopf algebras, the reader is referred to [27].

III. Section coefficients

We begin with the abstract concept of section coefficients. This concept arises as a natural generalization of the binomial coefficients. We shall see many examples in the later sections, particularly in Sec. IV–IX. Using section coefficients, one can give an alternative definition of coalgebras (with a specified basis) that does not involve commutative diagrams. Let $\mathcal{G}$ denote a set. *Section coefficients* $(i|j,k)$ of $\mathcal{G}$ arise by specifying and counting the number of ways an element i in $\mathcal{G}$ can be "cut up" into the ordered pair of pieces j,k(with j,k in $\mathcal{G}$). The *multisection coefficients* $(i|j,p,q)$ count the number of ways we can "cut" i into the ordered triple of pieces j,p,q. To get $(i|j,p,q)$ we could cut i into pieces j,k and then cut k into pieces p,q in all possible ways, and we want to get the same number if we cut i into pieces s,q and then cut s into pieces j,p in all possible ways. More precisely, section coefficients are a mapping

$$(i,j,k)\mapsto(i|j,k)\in\mathbb{Z}$$

satisfying

Given i,
the number of ordered pairs j,k (3.1)
such that $(i|j,k)\neq 0$ is finite

and

$$\sum_k (i|j,k)(k|p,q) = \sum_s (i|s,q)(s|j,p). \tag{3.2}$$

The common value of the two sides of (3.2) is denoted $(i|j,p,q)$. Iterating (3.2) allows us to define more general multisection coefficients $(i|j,k,\ldots,p,q)$.

Often, there exists a function $\varepsilon:\mathcal{G}\to K$ such that

$$\sum_j (i|j,k)\varepsilon(j) = \delta_{i,k},$$

$$\sum_k (i|j,k)\varepsilon(k) = \delta_{i,j}. \tag{3.3}$$

If $\mathcal{G}$ is a commutative semigroup (written additively), the section coefficients are called *bisection coefficients* if they satisfy

$$(i+j|p,q) = \sum_{\substack{p_1+p_2=p\\ q_1+q_2=q}} (i|p_1,q_1)(j|p_2,q_2). \tag{3.4}$$

In words, cutting up $i+j$ is the same as cutting i and j individually and piecing back together.

Example 3.1 (Binomial coefficients): The binomial coefficients are defined by

$$(n|j,k) = \begin{cases} \dfrac{n!}{j!k!} & \text{if } \; n=j+k, \\ 0 & \text{otherwise.} \end{cases}$$

They count the number of ways a set with n element can be "cut up" into two disjoint sets of size j and $k(=n-j)$. Usually, we write $\binom{n}{j}$ for these coefficients. The condition (3.2) is easily seen to be satisfied, since

$$(n|j,p,q) = \begin{cases} \dfrac{n!}{j!p!q!} & \text{if } \; j+p+q=n, \\ 0 & \text{otherwise,} \end{cases}$$

and for $j+p+q=n$,

$$(n|j,p,q) = \frac{n!}{j!(p+q)!}\,\frac{(p+q)!}{p!q!} = \frac{n!}{(j+p)!q!}\,\frac{(j+p)!}{j!p!}.$$

The well-known Vandermonde convolution identity

$$\binom{i+j}{p} = \sum_{p_1+p_2=p} \binom{i}{p_1}\binom{j}{p_2}$$

shows that the binomial coefficients are bisection coefficients.

Each collection of section coefficients satisfying (3.1), (3.2), and (3.3) gives rise to a coalgebra C in the following way: we associate to each i in $\mathcal{G}$ the variable x_i and let C be the free vector space spanned by the x_i's. The counit ε is the function defined in (3.3), and the diagonalization Δ is defined by

$$\Delta x_i = \sum_{j,k} (i|j,k) x_j \otimes x_k. \tag{3.5}$$

In our examples it is often the case that there exists a unique "0" in $\mathcal{I}$ such that $(i|0,j)=(i|j,0)=\delta_{ij}$, and the counit ε is given by

$$\varepsilon(x_j) = \begin{cases} 1 & \text{if } j=0 \\ 0 & \text{otherwise.} \end{cases}$$

The condition (3.2) gives that C is coassociative. C is cocommutative if and only if for all i,j,k, $(i|j,k)=(i|k,j)$. In addition, if the section coefficients are bisection coefficients, and we set $x_i x_j = x_{i+j}$, then C is a bialgebra. This is so because

$$\begin{aligned} \Delta(x_i)\Delta(x_j) &= \Big(\sum_{p_1,q_1} (i|p_1,q_1) x_{p_1} \otimes x_{q_1}\Big)\Big(\sum_{p_2,q_2} (j|p_2,q_2) x_{p_2} \otimes x_{q_2}\Big) \\ &= \sum_{p,q} \sum_{\substack{p_1+p_2=p \\ q_1+q_2=q}} (i|p_1,q_1)(j|p_2,q_2) x_{p_1+p_2} \otimes x_{q_1+q_2} \\ &= \sum_{p,q} (i+j|p,q) x_p \otimes x_q \\ &= \Delta x_{i+j}. \end{aligned}$$

Good references for this section include [4], [11], [16], and [17].

IV. Incidence coalgebras for partially ordered sets

Many of the coalgebras arising from the study of combinatorial problems are incidence or reduced incidence coalgebras of locally finite PO sets. The duals of these coalgebras, namely the incidence and reduced incidence algebras for PO sets, have been objects of intensive study during the last fifteen years. In this section, we give the abstract setting, definitions, and some basic results. In Sec. V–IX we work out some of the fundamental examples.

Given a locally finite PO set $(P, \leqslant)$, the *incidence coalgebra* $\mathcal{C}(P)$ (over K, a field of characteristic zero) is the free vector space spanned by the indeterminates $[x,y]$, for all intervals (or segments) $[x,y]$ in P. The diagonalization Δ and counit ε are given by

$$\Delta[x,y] = \sum_{x \leqslant z \leqslant y} [x,z] \otimes [z,y] \tag{4.1}$$

and

$$\varepsilon([x,y]) = \begin{cases} 1 & \text{if } x=y, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Here, the section coefficients are

$$([x_1,x_2]|[y_1,y_2],[z_1,z_2]) = \begin{cases} 1 & \text{if } x_1=y_1,\ x_2=z_2 \text{ and } y_2=z_1, \\ 0 & \text{otherwise.} \end{cases}$$

It is immediate that $\mathcal{C}(P)$ is coassociative. Moreover, it is cocommutative if and only if the order relation is trivial, i.e., no two elements of P are comparable.

Note that $\mathcal{C}^*(P)=\mathrm{Hom}(\mathcal{C}(P),K)$ is isomorphic to $\mathcal{I}(P)$, the incidence algebra of P, since if $f,g\in\mathcal{C}^*(P)$, then

$$f\circ g[x,y] = \sum_{x\leq z\leq y} f[x,z]\,g[z,y]$$

which is precisely the definition of $f*g$ in $\mathcal{I}(P)$.

It is frequently the case in enumeration problems that the full incidence coalgebra is not required; rather, we want to work with a smaller quotient coalgebra of $\mathcal{C}(P)$. These quotient coalgebras, called *reduced incidence coalgebras*, are obtained by taking suitable equivalence relations on P.

DEFINITION 4.1. An equivalence relation $\sim$ on the segments of P is said to be *order compatible* if the subspace spanned by the collection $\{[x,y]-[u,v]\,|\,[x,y]\sim[u,v]\}$ is a coideal.

Whenever $\sim$ is order compatible, the quotient space $\mathcal{C}(P)/\sim$ is isomorphic to a quotient coalgebra of $\mathcal{C}(P)$ (see [27, p.22]). In general, there is no simple criteria expressible in terms of the partial ordering to decide when an equivalence relation on P is order-compatible. A useful sufficient condition due to D.A. Smith [4, p. 276] is the following.

PROPOSITION 4.1. *An equivalence relation $\sim$ on the segments of P is order compatible if whenever $[x,y]\sim[u,v]$ there exists a bijection ϕ, depending in general on $[x,y]$, of $[x,y]$ onto $[u,v]$ such that $[x_1,y_1]\sim[\phi(x_1),\phi(y_1)]$ for all $x\leq x_1\leq y_1\leq y$.*

Note that the linear dual $(\mathcal{C}(P)/\sim)^*$ is isomorphic to the reduced incidence algebra $\mathcal{I}(P)/\sim$.

If $\sim$ is an order compatible equivalence relation on P, we call the nonempty equivalence classes of $\mathcal{C}(P)/\sim$ *types*, and we think of $\mathcal{C}(P)/\sim$ as the vector space spanned by the variables x_α associated to each type α. Each such reduced incidence coalgebra gives rise to a collection of section coefficients $(\alpha|\beta,\gamma)$, where $(\alpha|\beta,\gamma)$ counts the number of distinct z in any interval $[x,y]$ of type α such that $[x,z]$ is of type β and $[z,y]$ is of type γ, and the diagonalization in $\mathcal{C}(P)/\sim$ is given by

$$\Delta x_\alpha = \sum (\alpha|\beta,\gamma)\,x_\beta\otimes x_\gamma,$$

where the sum ranges over all ordered pairs of types β,γ.

The *standard reduced incidence coalgebra* is obtained from the equivalence relation

$[x,y]\sim[u,v]$ if and only if $[x,y]$ is isomorphic to $[u,v]$.

One way of obtaining bialgebras of combinatorial interest is to form reduced incidence coalgebras. We shall return several times to the question of when a reduced incidence coalgebra is a bialgebra.

The following definition is motivated by the fact that the lattice of closed ideals of an incidence algebra is distributive [4].

DEFINITION 4.2. A *combinatorial coalgebra* is a coalgebra whose lattice of subcoalgebras is distributive.

The characterization of all combinatorial coalgebras is an opening problem. At present, we can prove

THEOREM 4.1. *Every (full) incidence coalgebra is a combinatorial coalgebra.*

Proof: Let W be a subcoalgebra of $\mathcal{C}(P)$. If $[x,y]$ is in W, then for all $x \leqslant w \leqslant z \leqslant y$, $[w,z]$ is in W. This is seen as follows: If $x \leqslant z \leqslant y$, then the term $[x,z] \otimes [z,y]$ occurs in $\Delta[x,y]$. The occurrence of the segment $[x,z]$ (and $[z,y]$) is unique, and all segments are linearly independent. Thus, we must have $[x,z]$ and $[z,y]$ in W. Since $[x,z] \in W$, the same argument applies and gives that for all $x \leqslant w \leqslant z \leqslant y$, we must have $[w,z]$ in W. Thus the collection of segments of W forms an order ideal in the PO set of all segments of P, Seg(P), ordered by inclusion. Conversely, if J is an order ideal in Seg(P),, then $\Delta J \subseteq J \otimes J$, so that the linear span of J forms a subcoalgebra of $\mathcal{C}(P)$. Therefore, the lattice of subcoalgebras of $\mathcal{C}(P)$ is isomorphic to the lattice of order ideals of Seg(P). A well-known theorem of Birkhoff states that the lattice of order ideals of any PO set is distributive, and our proof is complete.

V. Reduced Boolean coalgebras

The Boolean PO set (lattice) $\mathfrak{B}$ consists of all finite sets of positive integers ordered by inclusion. The minimum element of this lattice is the empty set. The Boolean incidence coalgebra $\mathcal{C}(\mathfrak{B})$ is spanned by all segments $[A,B]$ with

$$\Delta[A,B] = \sum_{A \subseteq C \subseteq B} [A,C] \otimes [C,B].$$

1. Boolean coalgebras

The *Boolean coalgebra* $\mathfrak{B}$ is the coalgebra spanned by all sets of (positive) integers, with (for $A \in \mathfrak{B}$)

$$\Delta A = \sum_{\substack{A_1 \cap A_2 = \varnothing \\ A_1 \cup A_2 = A}} A_1 \otimes A_2 \tag{5.1}$$

and

$$\varepsilon(A) = \begin{cases} 1 & \text{if } A = \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

Note that in (5.1), A_1, A_2 is an ordered pair. This coalgebra is isomorphic to the reduced Boolean incidence coalgebra obtained by setting $[A,B] \sim [C,D]$ if and only if $B-A=D-C$. Thus, each set A represents the equivalence class of all segments $[B,C]$ such that $C-B=A$.

2. Binomial coalgebras

For each integer $s>0$, we define the binomial coalgebra B_s to be vector space $K[x_1,x_2,\ldots,x_s]$ with

$$\Delta x_1^{n_1}\cdots x_s^{n_s} = \sum_{(m_1,\ldots,m_s)<(n_1,\ldots,n_s)} \binom{n_1}{m_1}\cdots\binom{n_s}{m_s} x_1^{m_1}\cdots x_s^{m_s} \otimes x_1^{n_1-m_1}\cdots x_s^{n_s-m_s} \tag{5.2}$$

and

$$\varepsilon(x_1^{n_1}\cdots x_s^{n_s}) = \begin{cases} 1 & \text{if } n_1=\cdots=n_s=0, \\ 0 & \text{otherwise.} \end{cases}$$

Each binomial coalgebra is seen to be the Boolean incidence coalgebra modulo the coideal generated by a compatible $\sim$ as follows: For $s=1$, the (univariate) binomial coalgebra $B_1=K[x]$ is obtained by setting $[A,B]\sim[C,D]$ if and only if $|B-A|=|D-C|$. This is the standard reduced incidence coalgebra. Here the section coefficients are the binomial coefficients $\binom{n}{k}$.

For $s=2$, we set $[A,B]\sim[C,D]$ if and only if the numbers of even and odd integers in $B-A$ and $D-C$ are equal. For general s, we set $[A,B]\sim[C,D]$ if and only if for all $k=1,2,\ldots,s$,

$$|\{i\in B-A \mid i\equiv k \bmod s\}| = |\{j\in D-C \mid j\equiv k \bmod s\}|.$$

It is easy to verify that the binomial coalgebras are cocommutative bialgebras, and in fact, Hopf algebras with the antipode S given by $S(x_i)=-x_i$. In addition, the dual B_s^* is isomorphic to the algebra of formal exponential power series in s variables. A final heuristic remark: "$B_\infty \cong \mathfrak{B}$."

3. Polynomial sequences of Boolean and binomial type

A polynomial sequence $p_n(x)$ is said to be of *binomial type* if

$$\deg p_n(x) = n \qquad \text{for all } n, \tag{5.3}$$

and

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y). \tag{5.4}$$

Let us rephrase (5.4) in the landuage of bialgebras. The polynomial ring $K[x,y]$ is seen to be isomorphic to $K[x]\otimes K[x]$ under the mapping $x\mapsto x\otimes 1$ and $y\mapsto 1\otimes x$. By linearity, for any polynomials $q(x)$ and $r(y)$, $q(x)\mapsto q(x)\otimes 1$ and $r(y)\mapsto 1\otimes r(x)$. Thus, (5.4) can be restated

$$p_n(x\otimes 1+1\otimes x) = \sum_{k=0}^{n}\binom{n}{k}p_k(x)\otimes p_{n-k}(x). \tag{5.5}$$

A map p mapping the binomial coalgebra $K[x]$ to itself is a *coalgebra map* if $\Delta\circ p=(p\otimes p)\circ\Delta$. Thus, a polynomial sequence is of binomial type if and only if it is the image of $\{x^n\}$ under an invertible coalgebra map p. This is seen as follows. Let $p_n(x)$ denote the image of x^n under p. Since $K[x]$ is a bialgebra, we have

$$(\Delta\circ p)x^n = \Delta p_n(x) = p_n(\Delta x) = p_n(x\otimes 1+1\otimes x), \tag{5.6}$$

and clearly

$$((p\otimes p)\circ\Delta)x^n = \sum_{k=0}^{n}\binom{n}{k}p_k(x)\otimes p_{n-k}(x). \tag{5.7}$$

Therefore, if p is an invertible coalgebra map, $\deg p_n(x)=n$ and (5.5) holds, and conversely.

Multivariate polynomial sequences of binomial type, $\{p_{n_1\ldots n_s}(x_1,\ldots,x_s)\}$, are similarly seen to correspond to invertible coalgebra maps of B_s to itself.

Examples of sequences of polynomials of binomial type include x^n, $(x)_n= x(x-1)\cdots(x-n+1)$, $x(x-na)^{n-1}$, and the Laguerre, Gould, and exponential polynomial sequences. The reader is referred to [3] and [5] for further examples, and to [18] for their multivariate analogs.

A polynomial sequence indexed by the finite subsets of a set $\{p_A(x)\}$ is said to be of *Boolean type* if

$$p_A(x+y) = \sum_{A_1+A_2=A} p_{A_1}(x)p_{A_2}(y), \tag{5.8}$$

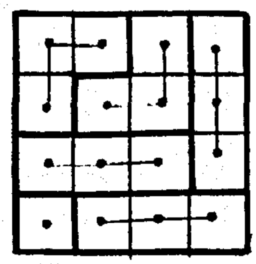
or equivalently, if $p_A(x)$ is the image of A under a coalgebra map from $\mathfrak{B}$ to $K[x]$. [Usually, we require that $\deg p_A(x)=|A|$.] Chromatic polynomials of graphs provide combinatorially interesting examples of polynomials of Boolean type. Given a graph G, the chromatic polynomial of G, $\mathcal{X}_G(x)$, counts the number of proper colorings (i.e. assignments of colors to the vertices of G so that no edge connects two vertices of the same color) of G with x colors. Given a subset H of the vertex set of G, we think of H as the full subgraph of G obtained by restricting the vertex set of G to H. Similarly, we denote by $G\setminus H$ the graph obtained by restricting the vertex set of G to $G-H$. Tutte, in [28], states

$$\mathcal{X}_G(x+y) = \sum_{H}\mathcal{X}_H(x)\mathcal{X}_{G\setminus H}(y). \tag{5.9}$$

This is not difficult to verify, since every proper coloring of G in $x+y$ different colors decomposes uniquely into the proper colorings of the subgraph H colored with the x colors and $G \setminus H$ colored with the y colors, and conversely. Polynomials of Boolean type were first studied by J. P. S. Kung and T. Zaslavsky.

4. Puzzles

Everyone is familiar with solitaire games where several flat pieces of wood or cardboard are to be assembled into a required shape, for example, a square, as in the following figure:



Little is known at present of the underlying mathematical theory that might lead, for example, to an algorithm for verifying that an assigned shape can be assembled out of a given set of pieces. We shall develop here the very first step in such a program, namely, the precise definition of a puzzle as a very special type of coalgebra. The definition of comultiplication is in fact a natural rendering of the combinatorial operation of cutting up an object into a set of pieces.

Before introducing the general definition, we shall describe the coalgebra associated with the puzzles in the above picture. We shall develop the construction in two steps. In the first step we define the placement coalgebra; in the second step we decribe a quotient coalgebra of the placement coalgebra, modulo a certain coideal. The quotient coalgebra will be called the puzzle or the piece coalgebra, and we shall see that the difficulty of the puzzle is carried in the structure of this coideal.

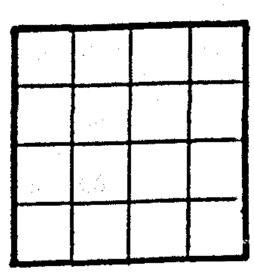
The *pieces* of the puzzle are

a  2 pieces

b  3 pieces (5.10)

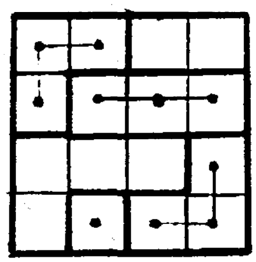
c  1 piece.

The *board* is the four-by-four square



on which pieces are to be placed. The squares are labeled by Cartesian coordinates.

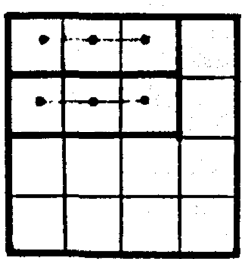
A *placement* of some of the pieces on the board is a subset of the board obtained by placing some of the pieces on the board without overlapping. For example the placement



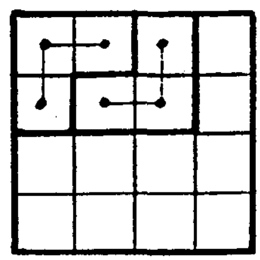
(5.11)

is obtained by placing two pieces of shape a, a piece of shape b, and a piece of shape c, as indicated. In a placement, no more than the alloted number of pieces is allowed.

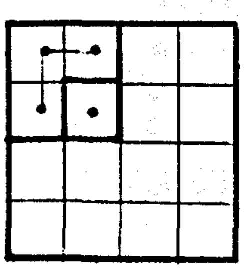
Two placements covering the same squares by distinct sets of pieces, or by pieces placed in different positions are considered to be different, for example,



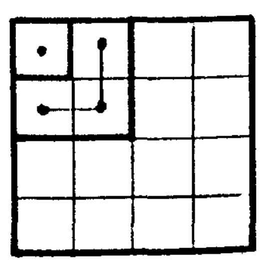
and



are distinct, as are

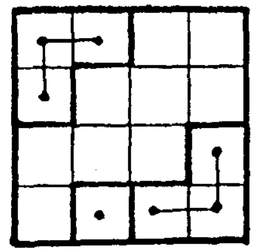


and

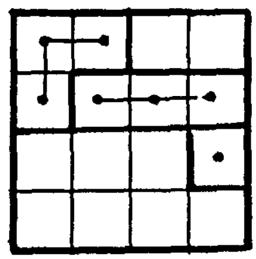


The pieces in a placement need not be adjacent. To every placement p, specified by the occupied squares and the position of the pieces, we associate a variable $x(p)$, and we denote by V the free module over the integers spanned by the variables $x(p)$ and the variable 1, which denotes the trivial placement of no pieces.

We now define a comultiplication on the module V, as follows. If p and q are placements, it is clear what is meant by saying that q is a subplacement of p. The pieces used in q are a submultiset of the pieces in p, and they are placed in the same positions. For example,



is a subplacement of the placement given in (5.11), whereas



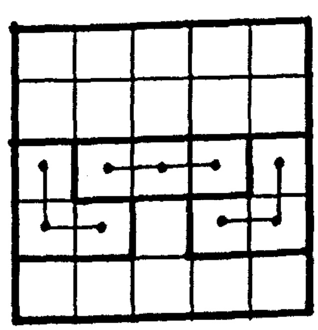
is *not* a subplacement. Thus, there is a partial ordering of placements, and we denote this PO set by P. P has a unique minimal element, the empty placement, but in general, it has no maximal element. Furthermore, for any placement p, the segment $\{q|q \leqslant p\}$ is a *Boolean algebra*; therefore, the PO set P is a *simplicial complex*. We are now ready to define the *placement coalgebra*. For any placement p, list all *ordered* pairs (q,r) such that

q and r are subplacements of p, (5.12a)
q and r do not overlap, (5.12b)
the union of q and r is the placement p. (5.12c)

Now set

$$\Delta x(p) = \sum x(q) \otimes x(r) \tag{5.13}$$

where the sum ranges over all such pairs. For example, if p is the placement



, (5.14)

and $x_1, x_2, \ldots, x_6$ are the placements shown in Fig. 1, then

$$\Delta x(p) = 1 \otimes x(p) + x_1 \otimes x_6 + x_2 \otimes x_4 + x_3 \otimes x_5 + x_4 \otimes x_2 + x_5 \otimes x_3$$
$$+ x_6 \otimes x_1 + x(p) \otimes 1.$$

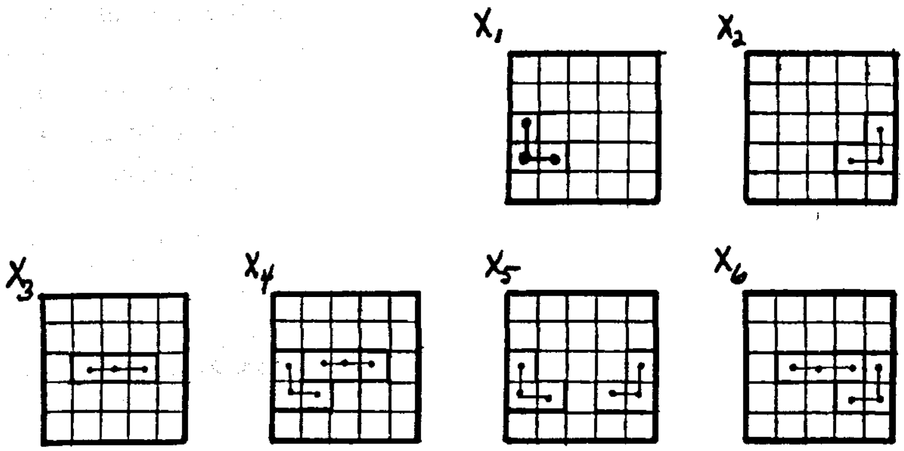


Figure 1.

It is intuitively clear that the comultiplication just defined is coassociative, in fact, it follows from the coassociativity of the Boolean coalgebra. The counit ε is defined by

$$\varepsilon(1) = 1 \text{ and } \varepsilon(x(p)) = 0 \qquad \text{for all} \quad x(p) \neq 1. \tag{5.15}$$

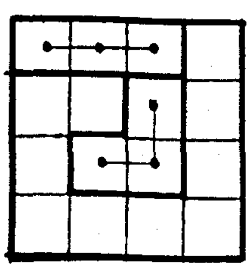
We now come to the definition of a puzzle, at least in the special case we are considering. To this end, we begin by defining an equivalence relation on placements. We shall say that $p \sim q$ when:

p and q are obtained by placing,
possibly in different positions, the same pieces
with the same multiplicity, (5.16a)

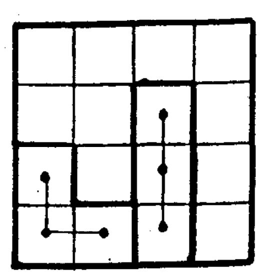
and

the placement q can be obtained from the placement p
by rigidly sliding and rotating (and possibly turning over,
depending on the rules of the game) placement p. (5.16b)

For example, any two placements of single pieces of the same shape are equivalent. As another example,



and



are equivalent.

It is immediate that the relation $\sim$ is an equivalence. An equivalence class will be called a *shape*. The equivalence classes corresponding to placements of a single piece will be called, appropriately enough, pieces.

The most important remark is that the submodule C of V generated by all elements

$$x(p)-x(q),$$

where $p\sim q$, is a coideal. Again, this is intuitively clear, but we shall verify it in detail. We have

$$\Delta x(p)=\sum_i x(p_{1i})\otimes x(p_{2i})$$

and

$$\Delta x(q)=\sum_i x(q_{1i})\otimes x(q_{2i}),$$

and it follows from the definition of equivalence that the families $\{(p_{1i},p_{2i})\}$ and $\{(q_{1i},q_{2i})\}$ of ordered pairs can be put into one-to-one correspondence in such a way that the entries are respectively equivalent. We can therefore write

$$\begin{aligned}x(p_{1i})&\otimes x(p_{2i})-x(q_{1i})\otimes x(q_{2i})\\&=[x(p_{1i})-x(q_{1i})]\otimes x(p_{2i})+x(q_{1i})\otimes[x(p_{2i})-x(q_{2i})].\end{aligned}$$

Thus, if $p\sim q$, then

$$\begin{aligned}\Delta(x(p)-x(q))&=\sum_i [x(p_{1i})-x(q_{1i})]\otimes x(p_{2i})\\&\quad+x(q_{1i})\otimes[x(p_{2i})-x(q_{2i})].\end{aligned}$$

In other words, this shows that $\Delta C\subseteq C\otimes V+V\otimes C$, and thus proves that C is a coideal (see [27, p. 18]). We can therefore take the quotient coalgebra V/C. This coalgebra generated by shapes is called a *puzzle*. If p is the placement given in (5.14), then in the puzzle (or quotient coalgebra) we have $x_1\sim x_2$ and $x_4\sim x_6$. Thus (if we represent each equivalence class by its placement of smallest index) in the puzzle

$$\Delta x(p)=1\otimes x(p)+2(x_1\otimes x_4)+x_3\otimes x_5+x_5\otimes x_3+2(x_4\otimes x_1)+x(p)\otimes 1.$$

From the preceding example it is now easy to extract the general definition of a puzzle. One begins with a finite simplicial complex P, and one associates to P a placement coalgebra in the same way as we have done above: to every p in P, one associates the set of ordered pairs (q,r) such that $q\vee r=p$ and $q\wedge r=0$.

From this, one obtains the definition of the placement coalgebra in exactly the same way. A puzzle is now generally defined as the quotient of the placement coalgebra by a coideal defined by an equivalence relation among the elements of P.

The basic problem about puzzles is to determine how many distinct shapes cover the entire board. At present, too little is known about the structures of puzzles to even hazard a conjecture on how one might approach the problem.

VI. Divided powers coalgebra

Let N denote the lattice of nonnegative integers under natural ordering. The incidence coalgebra $\mathcal{C}(N)$ is spanned by all segments $[i,j]$ with

$$\Delta[i,j] = \sum_{i<k<j} [i,k] \otimes [k,j].$$

The *divided powers coalgebra* $\mathfrak{D}$ is the vector space $K[x]$ with

$$\Delta x^n = \sum_{k=0}^{n} x^k \otimes x^{n-k}$$

and

$$\varepsilon(x^n) = \delta_{0,n}.$$

It is the standard reduced incidence coalgebra of $\mathcal{C}(N)$, and its dual $\mathfrak{D}^*$ is isomorphic to the algebra of formal power series $k[[x]]$ (with the usual multiplication). Multivariate divided powers coalgebras are similarly defined to be the standard reduced incidence coalgebra of

$$\mathcal{C}(N^s) = \mathcal{C}(\underbrace{N \times \cdots \times N}_{s \text{ times}}).$$

VII. Dirichlet coalgebra

Let Z^+ denote the lattice of positive integers ordered by divisibility, i.e., $m \leqslant n$ if and only if m divides n. The 0 of this lattice is 1. The equivalence relation on the segments of $\mathcal{C}(Z^+)$ which gives the Dirichlet coalgebra is $[i,j] \sim [k,l]$ if and only if $j/i = l/k$. Alternatively, the Dirichlet coalgebra D is the vector space spanned by the variables $\{n^x : n = 0, 1, 2, \ldots\}$, with

$$\Delta(n^x) = \sum_{pq=n} p^x \otimes q^x$$

and

$$\varepsilon(n^x) = \delta_{0,n}.$$

D has a natural algebra structure given by $n^x m^x = (nm)^x$. While D is not a bialgebra, the comultiplication is an algebra map when n and m are coprime, that is,

$$\Delta(n^x m^x) = \Delta(n^x)\Delta(m^x)$$

whenever the gcd of n and m equals 1.

The linear dual D^* is isomorphic to the algebra of formal Dirichlet series, the isomorphism being given by

$$f \mapsto \varphi(s) = \sum_n \frac{f(n^x)}{n^s}.$$

Multivariate Dirichlet coalgebras are obtained from the same equivalence relation on the incidence coalgebra $\mathcal{C}(Z^+ \times \cdots \times Z^+)$.

The standard reduced incidence coalgebra is a subcoalgebra of the Dirichlet coalgebra. Let $[i,j]$ and $[k,l]$ be two segments, and let

$$j/i = p_1^{\alpha_1} \cdots p_s^{\alpha_s} \quad \text{and} \quad l/k = q_1^{\beta_1} \cdots q_r^{\beta_r}$$

be their respective prime factorizations. The segments $[i,j]$ and $[k,l]$ are isomorphic if and only if $s=r$, and as multisets, the collections $\{\alpha_i\}$ and $\{\beta_i\}$ are the same. In other words, given n, let shape$(n)=(\lambda_1,\lambda_2,\ldots)$ where λ_k is the number of distinct primes in the factorization of n which occur precisely k times. Then $[i,j]\sim[k,l]$ if and only if shape$(j/i)=$shape(l/k).

VIII. Eulerian coalgebra

Let V denote the lattice of all finite-dimensional subspaces of a vector space of countable dimension over GF (q), ordered by inclusion. The minimal element of V is the trivial subspace. The standard reduced incidence coalgebra of $\mathcal{C}(V)$ is obtained by setting

$$[X,Y] \sim [S,T] \quad \text{if and only if} \quad \dim Y - \dim X = \dim T - \dim S.$$

The section coefficients count the number of subspaces of dimension k contained in a subspace of dimension n, which is given by the Gaussian coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)\cdots(1-q^k)(1-q)\cdots(1-q^{n-k})}.$$

If we set $[n]_q! = (1-q)(1-q^2)\cdots(1-q^n)$, then

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

The *Eulerian coalgebra* E is the vector space $K[x]$ with

$$\Delta x^n = \sum_{k=0}^{n} \left[\begin{matrix} n \\ k \end{matrix} \right]_q x^k \otimes x^{n-k}$$

and

$$\varepsilon(x^n) = \delta_{0,n}.$$

It is cocommutative, and E^* is isomorphic to the algebra of formal Eulerian power series, the isomorphism being given by

$$f \mapsto \varphi(u) = \sum \frac{f(x^n)}{[n]_q!} u^n.$$

IX. The Faà di Bruno bialgebra

The Faà di Bruno coalgebra $\mathfrak{F}$ is the standard reduced incidence coalgebra for the full lattice of partitions, Π. As such, it bears the same relationship to the lattice of partitions as does the binomial coalgebra to the lattice of subsets and the Eulerian coalgebra to the lattice of subspaces of a vector space over a finite field. In this section we shall show that this coalgebra is a bialgebra. (The proof is due to Doubilet [11].) Moreover, this bialgebra serves as a blueprint for the formulation and understanding of the general class of hereditary bialgebras presented in Sec. XVII.

The full lattice of partitions, Π, is the lattice of all set partitions of Z^+ (positive integers) having exactly one infinite block and finitely many finite blocks, ordered by refinement (see Sec. II).

Every segment $[\sigma,\tau]$ of Π is isomorphic to $\Pi_1^{\lambda_1} \times \Pi_2^{\lambda_2} \times \cdots \times \Pi_k^{\lambda_k} \cdots$, where Π_n is the lattice of partitions of an n-set, and λ_k equals the number of blocks of τ which consist of k blocks of σ. (This isomorphism can be seen by thinking of the ith block of σ as the "element" B_i, and $[\sigma,\tau]$ as a partition on the collection of B_i's, with σ as the finest partition.) To each segment $[\sigma,\tau]$ of Π, we associate the sequence $\lambda = (1,1,\ldots,1,2,\ldots,2,\ldots)$ of λ_1 ones, λ_2 twos,..., sometimes written $\lambda = (1^{\lambda_1} 2^{\lambda_2} \cdots)$ or equivalently, $x_1^{\lambda_1} x_2^{\lambda_2} \ldots = \mathbf{x}^\lambda$; λ, or $\mathbf{x}^\lambda$, is the *type* of $[\sigma,\tau]$, and clearly $[\sigma_1,\tau_1]$ is isomorphic to $[\sigma_2,\tau_2]$ if and only if they have the same type. The type $\lambda = (1^0 2^0 \cdots n^1 \ldots) = x_n$ is often written as n. We shall use the symbols $\alpha,\beta,\lambda,\nu,\mu$ to denote types.

The section coefficients $\left[\begin{matrix} n \\ \alpha,\beta \end{matrix} \right]$ count the number of partitions π contained in $[0,(1,2,\ldots,n)] \cong \Pi_n$ such that $[0,\pi]$ is of type α and $[\pi,(1,2,\ldots,n)]$ is of type β. Note that if $\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, we must have $\alpha_1 + 2\alpha_2 + \ldots + n\alpha_n = n$ and $\beta = x_{\alpha_1+\alpha_2+\cdots+\alpha_n}$. These section coefficients, known as the *Faà di Bruno coefficients*, are given explicitly by

$$\left[\begin{matrix} n \\ \alpha,\beta \end{matrix} \right] = \frac{n!}{\alpha_1!\alpha_2!\ldots\alpha_n!(1!)^{\alpha_1}(2!)^{\alpha_2}\ldots(n!)^{\alpha_n}}. \tag{9.1}$$

The explicit coalgebra structure of $\mathcal{F}$ is as follows. As a vector space, $\mathcal{F}$ is isomorphic to $K[x_1,x_2,\ldots]=K[\mathbf{x}]$. The diagonalization Δ and counit ε are given by

$$\Delta \mathbf{x}^\lambda = \sum_{\alpha,\beta} \begin{bmatrix} \lambda \\ \alpha,\beta \end{bmatrix} \mathbf{x}^\alpha \otimes \mathbf{x}^\beta \tag{9.2}$$

and

$$\varepsilon(\mathbf{x}^\lambda) = \begin{cases} 1 & \text{if } \lambda=(0,0,0,\ldots) \text{ or } (1,0,0,\ldots) \\ 0 & \text{otherwise.} \end{cases} \tag{9.3}$$

If $\sigma \leqslant \pi$ and B is a block of π, then by $\sigma \cap B$ we mean the partition of B consisting of the blocks of σ contained in B. Let $[\sigma,\tau]$ be of type $x_m x_n$, i.e., τ has two blocks B and B', where B contains m blocks of σ, and B' contains n. Suppose π is such that $\sigma \leqslant \pi \leqslant \tau$, $[\sigma\cap B, \pi\cap B]$ is of type $x_1^{\alpha_1}x_2^{\alpha_2},\ldots$, and $[\sigma\cap B', \pi\cap B']$ is of type $x_1^{\alpha'_1}x_2^{\alpha'_2}\cdots$. Then clearly $[\sigma,\pi]$ is of type $x_1^{\alpha_1+\alpha'_1}x_2^{\alpha_2+\alpha'_2}\cdots = \mathbf{x}^\alpha\mathbf{x}^{\alpha'}$. Similarly, if $[\pi\cap B,\tau\cap B]$ is of type $\mathbf{x}^\beta$ and $[\pi\cap B',\tau\cap B']$ is of type $\mathbf{x}^{\beta'}$, then $[\pi,\tau]$ is of type $\mathbf{x}^\beta\mathbf{x}^{\beta'}$. Thus, for all m,n,ν,μ,

$$\begin{bmatrix} x_m x_n \\ \nu,\mu \end{bmatrix} = \sum_{\substack{\alpha,\alpha',\beta,\beta' \\ \alpha+\alpha'=\nu,\beta+\beta'=\mu}} \begin{bmatrix} x_m \\ \alpha,\beta \end{bmatrix}\begin{bmatrix} x_n \\ \alpha',\beta' \end{bmatrix}, \tag{9.4}$$

where addition of sequences is defined by $(1^{\alpha_1}2^{\alpha_2}\cdots)+(1^{\beta_1}2^{\beta_2}\ldots)=(1^{\alpha_1+\beta_1}2^{\alpha_2+\beta_2}\ldots)$, i.e. $\mathbf{x}^\alpha\mathbf{x}^\beta=\mathbf{x}^{\alpha+\beta}$. It follows that $\begin{bmatrix} x_m x_n \\ \nu,\mu \end{bmatrix}$ is the coefficient of $\mathbf{x}^\nu\otimes\mathbf{x}^\mu$ in $\Delta(x_n)\Delta(x_m)$. This is equivalent to

$$\sum_{\nu,\mu} \begin{bmatrix} x_m x_n \\ \nu,\mu \end{bmatrix} \mathbf{x}^\nu\otimes\mathbf{x}^\mu = \left(\sum_{\alpha,\beta}\begin{bmatrix} x_m \\ \alpha,\beta \end{bmatrix}\mathbf{x}^\alpha\otimes x^\beta\right)\left(\sum_{\alpha,\beta}\begin{bmatrix} x_n \\ \alpha,\beta \end{bmatrix}\mathbf{x}^\alpha\otimes\mathbf{x}^\beta\right). \tag{9.5}$$

More generally, $\begin{bmatrix} x_1^{\lambda_1}x_2^{\lambda_2}\cdots \\ \nu,\mu \end{bmatrix}$ is the coefficient of $\mathbf{x}^\nu\otimes\mathbf{x}^\mu$ in $\Delta(x_1)^{\lambda_1}\Delta(x_2)^{\lambda_2}\cdots$. But this is just

$$\Delta\left(x_1^{\lambda_1}x_2^{\lambda_2}\cdots\right) = \Delta(x_1)^{\lambda_1}\Delta(x_2)^{\lambda_2}\cdots. \tag{9.6}$$

In addition, it is clear from (9.3) that

$$\varepsilon\left(x_1^{\lambda_1}x_2^{\lambda_2}\ldots\right) = \varepsilon(x_1)^{\lambda_1}\varepsilon(x_2)^{\lambda_2}\cdots.$$

Hence, we have shown

THEOREM 9.1. *$\mathcal{F}$ is a bialgebra under ordinary multiplication and the coalgebra structure obtained from the standard reduced incidence coalgebra of $\mathcal{C}(\Pi)$.*

Note that F is non-cocommutative. By Theorem 9.1, the space of all K-liner maps from $\mathcal{F}$ to itself, $\mathrm{Hom}(\mathcal{F},\mathcal{F})$, is an algebra with multiplication (or convolution) $*$ defined by

$$f*g(\mathbf{x}^{\lambda}) = \sum_{\alpha,\beta} \begin{bmatrix} \lambda \\ \alpha,\beta \end{bmatrix} f(\mathbf{x}^{\alpha})g(\mathbf{x}^{\beta}). \tag{9.7}$$

A function f in $\mathrm{Hom}(\mathcal{F},\mathcal{F})$ is said to be *multiplicative* if and only if for all λ, $f(x_1^{\lambda_1}\cdots x_n^{\lambda_n}\ldots)=f(x_1)^{\lambda_1}\ldots f(x_n)^{\lambda_n}\ldots$. Any such function is determined by the values it takes on the segments Π_n. Let $\mathfrak{M}(\mathcal{F})$ denote the class of multiplicative functions. The following elementary result is fundamental [4].

PROPOSITION 9.1. *The convolution of two multiplicative functions is multiplicative.*

Thus, $\mathfrak{M}(\mathcal{F})$ is a subsemigroup of the multiplicative semigroup $\mathrm{Hom}(\mathcal{F},\mathcal{F})$. If $f\in\mathfrak{M}(\mathcal{F})$, let $f(n)$ denote $f(\Pi_n)$, that is, $f([\sigma,\tau])$ for all $[\sigma,\tau]$ of type n. For $f,g\in\mathfrak{M}(\mathcal{F})$, we get from (9.1) and (9.7) that

$$f*g(n) = \sum_{\alpha_1+2\alpha_2+\ldots+n\alpha_n=n} \frac{n!f(1)^{\alpha_1}\cdots f(n)^{\alpha_n}g(\alpha_1+\ldots+\alpha_n)}{\alpha_1!\ldots\alpha_n!(1!)^{\alpha_1}\ldots(n!)^{\alpha_n}}. \tag{9.8}$$

THEOREM 9.2 (Doubilet, Rota, Stanley). *The semigroup $\mathfrak{M}(\mathcal{F})$ is anti-isomorphic to the algebra of all formal power series with zero constant term over $K[\mathbf{x}]$ in the variable u under the operation of functional composition. The anti-isomorphism is given by $f\mapsto f(u)$, where*

$$f(u) = \sum_{n=1}^{\infty} \frac{f(n)}{n!}u^n. \tag{9.9}$$

Thus $f*g(u)=g(f(u))$.

Proof: Clearly the map defined by (9.9) is a bijection, so we need only check that multiplication is preserved. Now

$$g(f(u)) = \sum_{k=1}^{\infty} \frac{g(k)}{k!}\left(\sum_{\nu=1}^{\infty}\frac{f(\nu)}{\nu!}u^{\nu}\right)^k. \tag{9.10}$$

The coefficient of u^n in the expansion of

$$\left(\sum_{\nu=1}^{\infty}\frac{f(\nu)}{\nu!}u^{\nu}\right)^k$$

is

$$\sum_{\substack{\nu_1+\ldots+\nu_k=n \\ \nu_i\geq 1}} \frac{f(\nu_1)\cdots f(\nu_k)}{\nu_1!\ldots\nu_k!} = \sum \frac{k!}{\alpha_1!\ldots\alpha_n!}\,\frac{f(1)^{\alpha_1}\ldots f(n)^{\alpha_n}}{(1!)^{\alpha_1}\ldots(n!)^{\alpha_n}},$$

where the summation is taken over $\alpha_1+2\alpha_2+\ldots+n\alpha_n=n$ and $\alpha_1+\ldots+\alpha_n=k$, since there are $k!/\alpha_1!\ldots\alpha_n!$ ways of ordering the partition $\alpha_1+2\alpha_2+\ldots+n\alpha_n=n$. When we multiply (9.11) by $g(k)/k!$ and sum over all k, we obtain (9.8), and the proof follows.

$\mathcal{F}$ is not a Hopf algebra, since $\Delta(x_1)=x_1\otimes x_1$. It can be realized as a Hopf algebra in $K[\mathbf{x}]$ localized at x_1, with

$$\Delta\left(\frac{1}{x_1}\right)=\frac{1}{x_1}\otimes\frac{1}{x_1}.$$

A proof of the existence of the antipode is given by demonstrating that a certain recursion can be carried out. An explicit formula can be obtained using the Lagrange inversion theorem [19].

In $K[\mathbf{x}]$ localized at x_1, $\mathfrak{M}(\mathcal{F})$ is anti-isomorphic to the group of all invertible (under functional composition) formal power series. The inverse of any function can be obtained by composition of this function with the antipode S. For a more detailed discussion of the Hopf-algebra aspects of $\mathcal{F}$, we refer the reader to [11], [19].

X. Incidence coalgebras for categories

Certain enumeration problems (see [4, p. 283]) lead to counting over structures more complicated than a single PO set. The concept of a *Mobius category* [21], gives one such structure. To extend the notion of incidence and reduced incidence coalgebras for PO sets to these situations, we are led to define incidence coalgebras for categories.

A *locally finite* category is a category in which for each morphism f, the collection of pairs of morphisms $\{(f_1,f_2):f_1\circ f_2=f\}$ is finite.

Given a locally finite category M, the incidence coalgebra $\mathcal{C}(M)$ is the free vector space over K spanned by the indeterminates f, where f is a morphism of M with coalgebra structure given by

$$\Delta f=\sum_{f_1\circ f_2=f} f_1\otimes f_2. \tag{10.1}$$

and

$$\varepsilon(f)=\begin{cases}1 & \text{if } f=id_p \text{ for some object } p \text{ in } M,\\ 0 & \text{otherwise.}\end{cases} \tag{10.2}$$

Let $\sim$ denote an equivalence relation on the morphisms of M. The *subspace generated by* $\sim$ is the subspace of $\mathcal{C}(M)$ spanned by the collection $\{f-g: f\sim g\}$. We say $\sim$ is *compatible* if the subspace generated by $\sim$ is a coideal. A *reduced incidence coalgebra* $\mathcal{C}(M)/\sim$ is the quotient coalgebra of $\mathcal{C}(M)$ modulo the coideal generated by a compatible $\sim$ relation.

Given two morphisms f,g in M, we say that g *divides* f if there exists morphisms h,k in M such that $f=h\circ g\circ k$. Let $[f]$ denote the subcategory generated by $\{g|g \text{ divides } f\}$. The *standard reduced* incidence coalgebra is obtained via the following equivalence on the morphisms: We set $f\sim g$ if and only if $[f]$ is isomorphic (as a subcategory) to $[g]$. Clearly the subspace generated by this equivalence relation is a coideal.

The range of meaningful reduced incidence coalgebras for categories is much larger than those of PO sets. For example, the *inner reduced* incidence coalgebra arises by setting $f\sim g$ if and only if there exists an invertible morphism h in M such that $[f]\simeq h\circ[g]\circ h^{-1}$, and the *strongly reduced* incidence coalgebra arises by setting $f\sim g$ if and only if there exists a category isomorphism φ: $[f]\to[g]$ such that $\varphi(f)=g$.

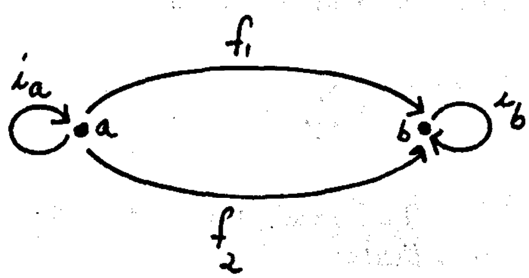
Example 10.1: Every locally finite PO set P can be viewed as a locally finite category as follows: the objects of M are the elements (or vertices) of P, and there is a unique morphism $f_{x,y}x\to y$ if and only if $x\leqslant y$. Clearly, if $x\leqslant z\leqslant y$, then $f_{x,z}\circ f_{z,y}=f_{x,y}$ and $[f_{x,y}]$ corresponds to the interval $[x,y]$. There are no invertible morphisms (other than the trivial ones, i.e. $f_{x,x}$), so that in this case, the inner reduced incidence coalgebra is the full incidence coalgebra. Moreover, in this case the standard reduced and strongly reduced incidence coalgebra are isomorphic.

Example 10.2: Every finite group G can be viewed as a locally finite category. The category has only one object, and each morphism f_g corresponds to an element of G. Composition of morphisms is given by $f_{g_1}\circ f_{g_2}=f_{g_1g_2}$, and if e is the identity element of G, f_e is the identity morphism. In contrast to the case of PO sets, the standard reduced incidence coalgebra for this category is isomorphic to the trivial category (consisting of one object and one morphism), whereas the inner reduced incidence coalgebra is isomorphic to the category of conjugacy classes of G. Indeed, in the inner reduced coalgebra, we have $f\sim g$ if and only if there exists an h such that $f=hgh^{-1}$, that is, f is conjugate to g. Let J denote the subspace generated by $\sim$. If $f\sim g$, then to each pair (f_1,f_2) such that $f_1f_2=f$ there correspond a unique pair (g_1,g_2) such that $g_1g_2=g$ and $f_i\sim g_i$. The correspondence is given explicitly by $g_i=h^{-1}f_ih$. Therefore

$$\begin{aligned}\Delta(f-g) &= \sum_i f_{1i}\otimes f_{2i}-\sum_i h^{-1}f_{1i}h\otimes h^{-1}f_{2i}h\\ &= \sum_i\left[(f_{1i}-h^{-1}f_{1i}h)\otimes f_{2i}+h^{-1}f_{1i}h\otimes(f_{2i}-h^{-1}f_{2i}h)\right]\\ &\subseteq J\otimes\mathcal{C}(M)+\mathcal{C}(M)\otimes J.\end{aligned}$$

Hence J is a coideal and $\mathcal{C}(M)/\sim$ is isomorphic to the category of conjugacy classes of G, as asserted. In the strongly reduced incidence category we have $f\sim g$ if and only if there exists a group automorphism φ such that $\varphi(f)=g$.

As we have seen in Sec. IV (Theorem 4.1), the lattice of subcoalgebras of the incidence coalgebra for PO sets is distributive. This is also trivially true for the lattice of subcoalgebras of the incidence coalgebra for a group G, because there are no proper subcoalgebras. It is, however, in general false. For example, let M be the category



where $i_a \circ f_j = f_j$ and $f_j \circ i_b = f_j, j=1,2$. The lattice of subcoalgebras of this category is not a distributive lattice. This is easily seen as follows: Let $L(A)$ denote the linear span of A, and set $M_1 = L(i_a, i_b)$; $M_2 = L(f_1, i_a, i_b)$; $M_3 = L(f_2, i_a, i_b)$; $M_4 = L(f_1 + f_2, i_a, i_b)$. Then each M_i is a subcoalgebra of $\mathcal{C}(M)$, and

$$M_2 \wedge (M_3 \vee M_4) = M_2,$$

whereas

$$(M_2 \wedge M_3) \vee (M_2 \wedge M_4) = M_1$$

In fact, the segment $[M_1, \mathcal{C}(M)]$ is isomorphic to the lattice of subspaces of a two-dimensional vector space over K, and it is well known that this lattice is not distributive.

XI. The umbral calculus

The binomial bialgebra has been studied in great detail, in particular with regard to applications to combinatorics, in a series of papers beginning with Mullin and Rota [2], followed by Kahaner, Odlyzko, and Rota [3] and finally Roman and Rota [5]. Elegant expositions of the results of Mullin and Rota were given by Aigner [6], Garsia [14], Liu [22], and several others. We shall summarize the main lines of this theory, keeping in mind that these results should act as blueprints for yet to be carried out generalizations to the more complex bialgebras and coalgebras arising in combinatorics, some of which are described in the rest of the present paper.

The comultiplication

$$\Delta x^n = \sum_k \binom{n}{k} x^k \otimes x^{n-k}, \tag{11.1}$$

on the algebra of polynomials $p(x)$ of one variable, defines a bialgebra structure. The dual algebra on linear functions L—where we denote by $\langle L | p(x) \rangle$ the action of the linear functional L on the polynomial $p(x)$—is seen to be

$$\langle L_1 L_2 | x^n \rangle = \sum_k \binom{n}{k} \langle L_1 | x^k \rangle \langle L_2 | x^{n-k} \rangle. \tag{11.2}$$

The dual algebra, with the augmentation ε acting as the identity, has been called the *umbral algebra* by Roman and Rota. The umbral algebra is isomorphic to the algebra of formal power series under the map

$$L \to \sum_n \langle L|x^n\rangle \frac{t^n}{n!}, \tag{11.3}$$

even in a topological sense. The formal power series thus associated to a linear functional is said to be its *indicator*.

The algebra of shift-invariant operators on polynomials is the algebra of all linear operators T mapping polynomials into polynomials, such that $TE^a = E^aT$, where E^a is the shift operator mapping $p(x)\to p(x+a)$, for all a. It turns out that the umbral algebra is also isomorphic to the algebra of shift invariant operators under the map sending the linear functional L to the operator Q given by

$$Qx^n = \sum_k \binom{n}{k}\langle L|x^k\rangle x^{n-k}.$$

A coalgebra isomorphism U, that is, a one-to-one onto linear operator on polynomials such that

$$\Delta Ux^n = \sum_k \binom{n}{k} Ux^k \otimes Ux^{n-k} \tag{11.4}$$

has been called an *umbral operator* by Mullin and Rota. The adjoint of an umbral operator is an isomorphism of the umbral algebra, and conversely, with due respect to topology. The sequence $p_n(x) = Ux^n$, where U is an umbral operator, is said to be of *binomial type*, and is characterized by the identity

$$p_n(x+a) = \sum_k \binom{n}{k} p_k(a)p_{n-k}(x). \tag{11.5}$$

Sequences of binomial type are of frequent occurrence in combinatorics, and have motivated much of the work on the umbral calculus. For example the sequences $(x)_n = x(x-1)\cdots(x-n+1)$, $x(x-na)^{n-1}$, and the Laguerre polynomials are of binomial type.

A *delta functional* L is a linear functional such that $\langle L|1\rangle = 0$ and $\langle L|x\rangle \neq 0$. To every delta functional one can associate two polynomial sequences of binomial type: the *associated sequence* $p_n(x)$ uniquely defined by the biorthogonality requirements

$$\langle L^k|p_n(x)\rangle = n!\,\delta_{k,n}, \tag{11.6}$$

and the *conjugate sequence* $q_n(x)$, defined by

$$q_n(x) = \sum_k \langle L^k|x^n\rangle \frac{x^k}{k!}. \tag{11.7}$$

Conversely, every sequence of binomial type, $p_n(x)$, is the associated sequence and the conjugate sequence of unique delta functionals, say L and $\tilde{L}$, which are said to be *reciprocal*.

A shift-invariant operator Q associated to a delta functional L is said to be a *delta operator*. If $p_n(x)$ is the associated sequence of the linear functional L, then the identity $Qp_n(x) = np_{n-1}(x)$ shows that the sequence $p_n(x)$ is related to the delta operator Q in a manner analogous to D and x^n. This leads to the generalization to delta operators of several classical formulas of the calculus; as the simplest example, Taylor's formula generalizes to

$$p(x+a) = \sum_n \frac{p_n(a)}{n!} Q^n p(x). \tag{11.8}$$

For example, for the sequence $p_n(x) = (x)_n$, the delta operator Q is the difference operator Δ defined by $\Delta p(x) = p(x+1) - p(x)$. Every delta operator Q equals the product DP, where $Dp(x) = p'(x)$ is the ordinary derivative, and the inverse operator P^{-1} exists. The operator P is called the *transfer operator* of the sequence $p_n(x)$. We come now to the first basic fact of the umbral calculus, which is the transfer formula:

$$p_n(x) = xP^{-n}x^{n-1}, \tag{11.9}$$

where P is the transfer operator of the sequence of binomial type $p_n(x)$. This formula is closely related to the Lagrange inversion formula for formal power series [15].

To introduce the next basic fact, we consider the operator x mapping $p(x)$ to $xp(x)$. The operator $Q' = Q\mathrm{x} - \mathrm{x}Q$ is called the *Pincherlé derivative* of the operator Q, and is also shift-invariant if Q is. Now, if Q is the delta operator of the sequence $p_n(x)$, then the *recurrence formula*

$$p_n(x) = x(Q')^{-1} p_{n-1}(x) \tag{11.10}$$

gives another way of explicitly computing a sequence of binomial type.

We now come to the fundamental fact of the umbral calculus. If $p_n(x)$ is a sequence of binomial type, then its generating function is of the form

$$\sum_n \frac{p_n(x)}{n!} t^n = \exp\left[x\left(a_1 t + \frac{a_2}{2!} t^2 + \quad\right)\right] = e^{xf(t)} \tag{11.11}$$

for some formal power series $f(t)$ such that $a_0 = 0$ and $a_1 \neq 0$, (a delta series, for short) and conversely. If $p_n(x)$ is the associated sequence for the delta functional L with indicator $g(t)$, then the series $f(t)$ and $g(t)$ are inverse in the sense of functional composition, that is, $f(g(t)) = g(f(t)) = t$. Furthermore, if $p_n(x)$ is the conjugate sequence of the delta functional $\tilde{L}$, then $f(t)$ is the indicator of $\tilde{L}$.

Functional composition is also related to umbral operators. It turns out that every umbral operator U is uniquely related to a delta series $u(t)$, and if L has indicator $f(t)$, then the linear functional $U^*(L)$ has the indicator $f(u(t))$; the converse is also true.

The coalgebraic statement of this fact leads to the interpretation and rigorization of the classical technique of treating indices as exponents, from which the umbral calculus derives its name. If $p_n(x)=\sum_k a_{n,k}x^k$ and $q_k(x)$ are sequences of polynomials of binomial type, then the polynomial sequence

$$r_n(x) = \sum_k a_{n,k}q_k(x) = p_n(\mathbf{q}(x)) \tag{11.12}$$

is called the *umbral composition* of the sequences $p_n(x)$ and $q_n(x)$. It turns out that the sequence $r_n(x)$ is also of binomial type; furthermore, if the indicators of the delta functionals L and M with respect to which $p_n(x)$ and $q_n(x)$ are the associated sequences are, respectively, $f(t)$ and $g(t)$, then the corresponding indicator of the sequence $r_n(x)$ is the functional composition $f(g(t))$.

Among many other facts of the umbral calculus which cannot be mentioned here—but some of which will be found in the memoir of Roman and Rota—we mention the extension of the preceding results to other module actions of the umbral algebra; in fact, it would be of the utmost interest to classify all such module actions. For example, a natural action is defined on the ring of inverse formal power series

$$f(t) = \sum_{n>1} \frac{a_n}{t^n}$$

by sending t^{-n} to $(t+a)^{-n}$, thus defining the operator E^a, and then taking a suitable closure. In this way, one can define "inverse" analogs of all sequences of binomial type; for example,

$$(x)_{-n} = \frac{1}{(x+1)(x+2)\cdots(x+n)},$$

leading to a generalization of the classical theory of factorial series.

XII. Infinitesimal coalgebras; the Newtonian coalgebra

Recall that a bialgebra A is a vector space which is simultaneously an algebra and a coalgebra such that the comultiplication Δ is an "endomorphism" of A (as an algebra). The analogy between endomorphisms and derivations leads us to define an *infinitesimal coalgebra A* to be a vector space which is simultaneously an algebra and a coalgebra (possibly without a counit) such that the comultiplication Δ is a derivation of A in the sense that for p,q in A,

$$\Delta(pq) = (\Delta p)(q\otimes 1) + (1\otimes p)(\Delta q). \tag{12.1}$$

In this section we shall present only one infinitesimal coalgebra, the Newtonian coalgebra. The study of this coalgebra should provide a prototype for the general study of infinitesimal coalgebras.

Let us recall the definition of the Newton divided differences. The 0th divided difference is

$$[f:x_0] = f(x_0).$$

The first divided difference is

$$[f:x_0,x_1] = \frac{f(x_0)-f(x_1)}{x_0-x_1},$$

the second

$$[f:x_0,x_1,x_2] = \frac{[f:x_0,x_1]-[f:x_1,x_2]}{x_0-x_2},$$

and the kth divided difference $[f:x_0,\ldots,x_k]$ is obtained by iteration.

A polynomial sequence $\{p_n(x)\}$ [with $p_0(x)\equiv 1$] is said to be of *Newtonian type* if

$$\frac{p_n(x)-p_n(y)}{x-y} = \sum_{k=0}^{n-1} p_k(x)p_{n-k-1}(y). \tag{12.2}$$

Two examples of such sequences are $\{x^n\}$, and $\{(x+a)^n\}$ for any a. There are two coalgebras within which we can study these polynomial sequences. As vector spaces and as algebras, both are isomorphic to $K[x]$. The first coalgebra we shall consider is the *Newtonian coalgebra*, denoted N. The comultiplication in N is

$$\Delta p(x) = \frac{p(x)-p(y)}{x-y} \tag{12.3}$$

and is easily checked to be coassociative. There is no counit in N. Moreover, it is immediate to verify that

$$\Delta(pq) = \Delta p\,(q\otimes 1) + (1\otimes p)\Delta q,$$

so that N is an infinitesimal coalgebra. The kth divided difference $[p:x_0,\ldots,x_k] = \Delta^k p(x)$. This coalgebra setting gives an elegant proof of Newton's formula, namely

$$f(x) = \sum_{k=0}^{\infty} (x-x_0)\cdots(x-x_k)[f:x_0,\ldots,x_k].$$

In the dual algebra, the following striking relationship between divided differences and ordinary differentiation is seen. Let us set, for all p,

$$\langle \varepsilon_p | f(x) \rangle = f(p).$$

Then

$$\langle \varepsilon_p \varepsilon_q | f(x) \rangle = \frac{f(p)-f(q)}{p-q},$$

and

$$\langle \varepsilon_p^2 | f(x) \rangle = f'(p).$$

An extensive study of the theory within this setting has been pursued by S. Roman [25].

A different approach was taken by Garsia and Joni. Using the umbral machinery with the "differentiation" operator A defined by

$$Ax^n = x^{n-1}$$

and the "multiplication" operator B defined by

$$Bx^n = \frac{x^{n+1}}{n+1},$$

they define a polynomial sequence $\{q_n(x)\}$ $[q_0(x) \equiv 1]$ to be of *Newjonian type* if

$$\frac{xq_n(x)-yq_n(y)}{x-y} = \sum_{k=0}^{n} q_k(x) q_{n-k}(y). \tag{12.4}$$

Note that $q_n(0)=0$ for all $n \geq 1$. Examples of such sequences are $\{x^n\}$, $\{x(x+a)^{n-1}\}$.

Here, the underlying coalgebra structure is given by

$$\Delta p(x) = \frac{xp(x)-yp(y)}{x-y}$$

and

$$\varepsilon(x^n) = \begin{cases} 1, & n=0, \\ 0 & \text{otherwise} \end{cases}$$

(that is, ε is evaluation at zero).

In this setting, all of the results within the umbral calculus, appropriately modified (see [13]) for the "differentiation" A and "multiplication" B, apply. For example it is not difficult to show that $\{q_n(x)\}$ is of Newjonian type if and only if

$$\sum_{n=0}^{\infty} q_n(x)u^n = \frac{1}{1-xf(u)},$$

where $f(u)$ is an invertible (under functional composition) formal power series.

It turns out that polynomial sequences of Newtonian and Newjonian type are essentially the same class of polynomial sequences. Indeed, we have

THEOREM 12.1. *A polynomial sequence $\{p_n(x)\}$ is of Newtonian type if and only if the polynomial sequence $\{q_n(x)\}$ defined by*

$$q_n(x) = \begin{cases} 1, & n=0, \\ xp_{n-1}(x), & n \geq 1 \end{cases}$$

is of Newjonian type.

The Newjonian coalgebra setting provided the machinery for the explicit computation of the "Newtonian analogs" of many of the classical polynomial sequences (e.g. Laguerre, Abel, exponential, Gould, etc.).

XIII. Creation and annihilation operators

The creation and annihilation operators we present here generalize those of quantum field theory.

Let $\{(i|j,k)\}$ be a collection of section coefficients satisfying the extra condition that for each ordered pair (j,k), the set $\{i:(i|j,k)\neq 0\}$ is finite, and let C be the coalgebra defined by these section coefficients and a given counit ε (see Sec. III). Creation and annihilation operators are linear maps from C to itself defined as follows:
for each $j\in\mathcal{I}$, *the creation operator* K_j is

$$K_j x_k = \sum_i (i|j,k)x_i \tag{13.1}$$

and the *annihilation operator* A_j is

$$A_j x_k = \sum_i (k|j,i)x_i. \tag{13.2}$$

If the section coefficients are all equal to zero or one, and if, in addition, for each j,k there is at most one i such that $(i|j,k)=1$, then the creation operator K_j acting on x_k gives the "piece" i obtained by piecing k and j together if it exists, and zero otherwise. Similarly, the annihilation operator A_j acting on x_k gives the piece i which when added to the piece j is the piece k, if such a piece exists, and zero otherwise.

For example, let us look at the situation when C is an incidence coalgebra for a PO set. An easy computation gives

$$A_{[x,y]}[u,v] = \begin{cases} [y,v] & \text{if } \quad x=u \text{ and } x \leqslant y \leqslant v, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly

$$K_{[x,y]}[u,v] = \begin{cases} [x,v] & \text{if } \quad y=u, \\ 0 & \text{otherwise.} \end{cases}$$

In a puzzle, $K_j x_k$ gives a list (with multiplicities) of all possible x_i obtainable by piecing together x_j and x_k. Similarly, $A_j x_k$ gives a list (with multiplicities) of all possible pieces x_i such that x_i and x_j can be pieced together to form x_k.

Straightforward computations give

$$K_p A_j x_k = \sum_{i,q} (k|j,i)(q|p,i) x_q,$$

and

$$A_j K_p x_k = \sum_{i,q} (i|p,k)(i|j,q) x_q.$$

If the section coefficients are bisection coefficients, then C is a bialgebra, and in addition,

$$A_k x_{i+j} = \sum_{p_1+p_2=k} (A_{p_1} x_i)(A_{p_2} x_j).$$

PROPOSITION 13.1. *If $\mathcal{G}$ is a commutative semigroup (written additively), then*

$$K_j K_l = K_l K_j = K_{j+l} \quad \textit{and} \quad A_j A_l = A_l A_j = A_{j+l}$$

if and only if for all j,k,l,q

$$\sum_p (k|j,p)(p|l,q) = \sum_p (k|l,p)(p|j,q)$$
$$= (k|j+l,q). \tag{13.3}$$

Proof:

$$K_jK_lx_q = K_j\Big(\sum_p (p|l,q)x_p\Big)$$

$$= \sum_{p,k} (p|l,q)(k|j,p)x_k$$

and

$$K_{j+l}x_q = \sum_k (k|j+l,q)x_k.$$

Therefore, $K_jK_l = K_{j+l}$ if and only if

$$(k|j+l,q) = \sum_p (p|l,q)(k|j,p).$$

The same argument using K_lK_j completes the proof for creation operators, and the analogous argument holds for annihilation operators.

The coalgebra C, considered as a vector space, has a natural inner product:

THEOREM 13.1. *The bilinear form*

$$\langle x_i|x_j\rangle_C = \sum_q (i|j,q)\varepsilon(x_q)$$

on C is symmetric and nondegenerate.

Proof: Since ε is the counit of C, we have

$$x_i = \sum_{j,q} (i|j,q)\varepsilon(x_q)x_j. \tag{13.4}$$

Equating coefficients of the x's on both sides gives

$$\langle x_i|x_j\rangle_C = \sum_q (i|j,q)\varepsilon(x_q) = \delta_{i,j}.$$

THEOREM 13.2. *Relative to the symmetric form $\langle\ ,\ \rangle_C$, A_j and K_j are adjoint operators.*

Proof: We show that

$$\langle A_ix_j|x_k\rangle_C = \langle x_j|K_ix_k\rangle_C$$

for all i,j,k. Expanding the left side gives

$$\langle A_ix_j|x_k\rangle_C = \sum_p (j|i,p)\langle x_p,x_k\rangle_C = (j|i,k)$$

since $\langle x_p, x_k \rangle_C = \delta_{p,k}$. Similarly, the right-hand side gives

$$\langle x_j | K_i x_k \rangle_C = \sum_p (p|i,k) \langle x_j, x_p \rangle = (j|i,k),$$

as desired.

In the following examples we see how creation and annihilation operators cut up and piece together sets and partitions.

Example 13.1: For the binomial coalgebra, the creation and annihilation operators are easily seen to be

$$K_j x^k = \binom{j+k}{j} x^{k+j}$$

and

$$A_j x^k = \begin{cases} \binom{k}{j} x^{k-j} & \text{if } j < k, \\ 0 & \text{otherwise.} \end{cases}$$

Example 13.2: Let X and Y be subsets. The creation and annihilation operators for the Boolean coalgebra are

$$K_X Y = \begin{cases} X \cup Y & \text{if } X \cap Y = \varnothing, \\ 0 & \text{otherwise} \end{cases}$$

and

$$A_X Y = \begin{cases} Y - X & \text{if } X \subseteq Y, \\ 0 & \text{otherwise.} \end{cases}$$

Example 13.3: The creation and annihilation operators for the Faá di Bruno coalgebra are a bit more complicated than those of the previous two examples. Let α, β, and λ denote types of partitions. The creation operators K_α are, by definition,

$$K_\alpha \mathbf{x}^\beta = \sum_\lambda (\lambda | \alpha, \beta) \mathbf{x}^\lambda.$$

Since $(\lambda|\alpha,\beta) \neq 0$ only if $\beta_1 + 2\beta_2 + \ldots + n\beta_n = \alpha_1 + \alpha_2 + \ldots + \alpha_n$, $\lambda_1 + 2\lambda_2 + \ldots + n\lambda_n = \alpha_1 + 2\alpha_2 + \ldots + n\alpha_n$, and $\lambda_1 + \lambda_2 + \ldots + \lambda_n = \beta_1 + \beta_2 + \ldots + \beta_n$, the types $\mathbf{x}^\lambda$ occurring in $K_\alpha \mathbf{x}^\beta$ [with multiplicites $(\lambda|\alpha,\beta)$] are seen to be the types obtainable by merging, in all possible ways, the blocks of a partition of type α so that the resulting partition has the same number of blocks as β. The multiplicities count the number of ways in which a given type can occur.

The annihilation operators A_α are

$$A_\alpha \mathbf{x}^\beta = \sum_\lambda (\beta|\alpha,\lambda)\mathbf{x}^\lambda,$$

so here we must have $\alpha_1+2\alpha_2+\ldots+n\alpha_n=\beta_1+2\beta_2+\ldots+n\beta_n$, $\lambda_1+2\lambda_2+\ldots+n\lambda_n=\alpha_1+\ldots+\alpha_n$, and $\lambda_1+\lambda_2+\ldots+\lambda_n=\beta_1+\beta_2+\ldots+\beta_n$ for the type $\mathbf{x}^\lambda$ to occur in $A_\alpha \mathbf{x}^\beta$. Thus we obtain a list of all the types of partitions (with appropriate multiplicities) of a set of size $\alpha_1+\ldots+\alpha_n$ with the same number of blocks as β.

XIV. Point-lattice coalgebras

Let $\mathcal{L}$ be a finite point lattice, that is, a lattice in which every element is the supremum of a set of atoms. It is well known and easily proved that $\mathcal{L}$ is isomorphic to the lattice of closed sets relative to the closure operation defined on subsets of the set $\mathcal{A}$ of atoms by

$$\bar{A}=\{p\in\mathcal{A}|p\leqslant \sup A\} \qquad \text{for} \quad A\subseteq\mathcal{A}.$$

The closure operation enjoys the properties

$$A\subseteq\bar{A}, \tag{14.1a}$$

$$\bar{\bar{A}}=\bar{A}, \tag{14.1b}$$

$$\text{if} \quad A\subseteq B, \quad \text{then} \quad \bar{A}\subseteq\bar{B} \tag{14.1c}$$

(but *not*, in general $\overline{A\cup B}=\bar{A}\cup\bar{B}$). The complements of closed sets, called *open sets*, can be characterized even more simply by

(1) the union of any family of open sets is an open set,
(2) every open set is the union of the minimal nonempty open sets it contains.

Thus, every point lattice can be represented as the family of all open sets in a closure relation where the join in the lattice is set-theoretic union. In the following we shall assume that $\mathcal{L}$ is so represented by a fixed set $\mathcal{A}$. We shall further assume that $\mathcal{L}$ has a unique minimal element, which is represented by the empty set. This representation of $\mathcal{L}$ allows us to define a very interesting coalgebra structure on $\mathcal{A}$. As a vector space, this coalgebra $\mathcal{C}(\mathcal{L})$ is isomorphic to the free vector space over K with basis consisting of all open sets of $\mathcal{A}$. For each open set $A\subseteq\mathcal{A}$, the diagonalization is

$$\Delta A=\sum_{\substack{A_1,A_2 \text{ open}\\ A_1\cap A_2=\varnothing\\ A_1\cup A_2=A}} A_1\otimes A_2, \tag{14.3}$$

and the counit is

$$\varepsilon(A) = \begin{cases} 1 & \text{if } A = \varnothing \\ 0 & \text{otherwise.} \end{cases} \tag{14.4}$$

Since the union of open sets is again an open set, it follows immediately that the above diagonalization (14.3) is coassociative. Since point lattices occur in many combinatorial investigations, the study of this class of coalgebras should prove very interesting. We give three examples of point-lattice coalgebras.

Example 14.1 *(The Boolean coalgebra):* Finite-dimensional Boolean coalgebras arise from the point lattice of subsets of $\{1,2,\ldots,n\}$. This lattice can be represented as follows: the minimal nonempty open sets are the sets consisting of one element, i.e., the sets $\{j\}$, for $1 \leqslant j \leqslant n$. Thus, every subset is an open set. Hence, for each A,

$$\Delta A = \sum_{\substack{A_1 \cap A_2 = \varnothing \\ A_1 \cup A_2 = A}} A_1 \otimes A_2,$$

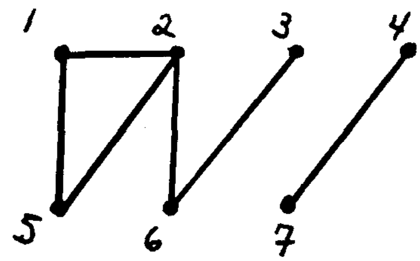
so that these coalgebras are isomorphic to subcoalgebras of the Boolean coalgebra defined in Sect. V.

Example 14.2 *(The $n \times n$ board):* Let $\mathscr{Q}$ denote the collection of the n^2 squares $\{a_{ij}\}_{i,j=1}^{n}$ on an $n \times n$ square board. Our point lattice $\mathfrak{L}$ is represented by the following family of open subsets of $\mathscr{Q}$: the minimal nonempty open sets are the *lines* of the board, where a line, by definition, is either a row or column. The open sets consist of all possible unions of lines, so each open set A is uniquely determined by the two subsets of $\{1,2,\ldots,n\}$

$$R(A) = \{i \mid \text{row } i \text{ is in } A\} \quad \text{and} \quad C(A) = \{j \mid \text{column } j \text{ is in } A\}.$$

Two open sets A_1 and A_2 can have $A_1 \cap A_2 = \varnothing$ if and only if either $|R(A_1)| = |R(A_2)| = 0$ or $|C(A_1)| = |C(A_2)| = 0$. Thus, our comultiplication Δ breaks up open sets which are unions of rows or unions of columns, and leaves intact any open set which is a combination of both rows and columns.

Example 14.3 *(Graphs):* Let $\mathcal{G} = (V,E)$ be an undirected graph with vertex set V, $|V| < \infty$, and edge set E. Here, our point lattice is the family of open subsets of V defined as follows: the minimal nonempty open sets are (unordered) pairs of vertices p and q such that there is an edge in E connecting p and q. We shall sometimes write (p,q) to denote such an edge. An open set A is a subset of V such that for each $p \in A$, there exists a $q \in A$ such that (p,q) is an edge in E. (Note that q need not be unique.) Our comultiplication gives all ways of dividing an open vertex set A into two disjoint sets A_1 and A_2 such that each vertex in A_i, $i = 1,2$, remains connected to some other vertex in A_i. For example, let $\mathcal{G}$ be the graph given by the following figure:



$$V = \{1,2,\ldots,7\}.$$

Then $\{1,5,2\}\otimes\{3,6,4,7\}$ occurs in ΔV, whereas $\{1,2,3,5\}\otimes\{4,6,7\}$ does not.

An element p in any lattice $\mathcal{L}$ is said to be a *join-irreducible* element if it cannot be expressed as the join of two incomparable elements of $\mathcal{L}$. Every element in $\mathcal{L}$ is the supremum of a set of join irreducibles, and $\mathcal{L}$ is isomorphic to the lattice of closed sets relative to the closure operation defined on subsets of the set of join irreducibles J by

$$\bar{A} = \{p\in J \mid p \leqslant \sup A\} \qquad \text{for} \quad A \subseteq J.$$

Thus, the construction given for the point-lattice coalgebras extends in the obvious way to a construction for general lattices.

XV. Restricted placements

A fundamental concept in the study of permutations with restricted positions is that of a *non-taking subset* of a board. A non-taking subset of a board is a collection of squares $\{a_{ij}\}$ such that no two squares have the same row or column index. They are best visualized as follows: if we place a rook on each square in a given set A, then A is non-taking if and only if no rook can "take" any other rook, that is, no two rooks are in the same row or column.

In this section we shall give a very general setting for the construction of non-taking sets; non-taking sets of boards arise as one special case. Another special case gives totally unconnected collections of vertices in graphs, which are closely related to the problem of colorings of graphs. Within this context we are lead to a very natural interpretation of Möbius inversion for a large class of lattices, and a coalgebra closely associated with enumerations of non-taking sets.

In order that this paper may be reasonably self-contained, we give a brief sketch of Möbius inversion for an arbitrary locally finite PO set P. The reader is referred to [1] for a more complete discussion. In enumeration, we often wish to calculate $f(y)$, a function on P, and it turns out to be much easier to calculate

$$g(x) = \sum_{y \geqslant x} f(y).$$

As an example, if P is the lattice of subsets of $\{1,2,\ldots,n\}$, and $f(A)$ is the number of permutations of $\{1,2,\ldots,n\}$ whose set of fixed points is precisely A, then it is easy to see that

$$g(A) = \sum_{B \supseteq A} f(B) = \text{the number of permutations whose set of fixed points contains } A$$
$$= (n-|A|)!.$$

We obtain the values of f (in terms of the values of g) via Möbius inversion. The *zeta function* is the function in the incidence algebra $\mathcal{I}\ (P)$ defined by

$$\zeta(x,y) = \begin{cases} 1 & \text{if} \quad x \leqslant y, \\ 0 & \text{otherwise.} \end{cases}$$

The inverse of ζ (under $*$), μ, is the *Möbius function*. That is, μ satisfies, for all $x \leqslant y$,

$$\sum_{x \leqslant z \leqslant y} \mu(x,z)\zeta(z,y) = \sum_{x \leqslant z \leqslant y} \zeta(x,z)\mu(z,y)$$
$$= \delta_{x,y} = \begin{cases} 1 & \text{if} \quad x=y, \\ 0 & \text{otherwise.} \end{cases} \tag{15.1}$$

THEOREM 15.1 (Möbius inversion). *Let f and g be functions on a given PO set P such that*

$$g(x) = \sum_{y \geqslant x} f(y). \tag{15.2}$$

Then

$$f(x) = \sum_{y} \mu(x,y)g(y). \tag{15.3}$$

Proof: Equation (15.2) states that

$$g(y) = \sum_{z \geqslant y} f(z) = \sum_{z} \zeta(y,z)f(z).$$

Thus, multiplying both sides by $\mu(x,y)$ and summing over y gives

$$\sum_{y} \mu(x,y)g(y) = \sum_{z}\sum_{y} \mu(x,y)\zeta(y,z)f(z)$$
$$= \sum_{z} \delta_{x,z}f(z) = f(x).$$

As in Sec. XIV, we shall assume that we are given a point lattice $\mathcal{L}$, represented as a family of open subsets of a set $\mathcal{C}$. We shall call nonempty minimal open sets *forbidden sets*. Given $\mathcal{L}$, we construct a new lattice $\mathrm{St}(\mathcal{L})$, the *lattice of stars of* $\mathcal{L}$, as follows: for each $p \in \mathcal{C}$, the *star of* p, $\mathrm{st}(p)$, is the union of all forbidden (minimal nonempty open) sets containing p. If A is any subset of $\mathcal{C}$, we set

$$\mathrm{st}(A) = \bigcup_{p \in A} \mathrm{st}(p).$$

We say that an element S in $\mathcal{L}$ is a *star* if and only if $S = \mathrm{st}(A)$ for some $A \subseteq \mathcal{C}$. (Note that A is, in general, not unique.) If $A \subseteq B$, then $\mathrm{st}(A) \subseteq \mathrm{st}(B)$. We say that A *generates* S if $\mathrm{st}(A) = S$ and for all $A' \subsetneq A$, $\mathrm{st}(A') \subsetneq \mathrm{st}(A)$. The lattice $\mathrm{St}(\mathcal{L})$ consists of all stars of $\mathcal{L}$, ordered by inclusion, where the join is set-theoretic union. $\mathrm{St}(\mathcal{L})$ is, in general, not a sublattice of $\mathcal{L}$. Indeed, the meets in the two lattices are not necessarily the same, since if S and T are stars, then in $\mathrm{St}(\mathcal{L})$, their meet will be the maximal star contained in $S \cap T$, whereas in $\mathcal{L}$, their meet is the maximal open set contained in $S \cap T$. Moreover, $\mathrm{St}(\mathcal{L})$ need not be a point lattice. We shall give an example of this later in this section.

A subset A of $\mathcal{C}$ will be said to be *non-taking* when for all $p \neq q$ in A, $p \notin \mathrm{st}(q)$ and $q \notin \mathrm{st}(p)$. We define two functions f and g on $\mathrm{St}(\mathcal{L})$ as follows: given a star S, let $g(S)$ be the number of non-taking sets whose star contains S, and let $f(S)$ be the number of non-taking sets whose star equals S. Clearly,

$$g(S) = \sum_{T \supseteq S} f(T), \tag{15.4}$$

where T ranges over all stars. Hence, by Möbius inversion,

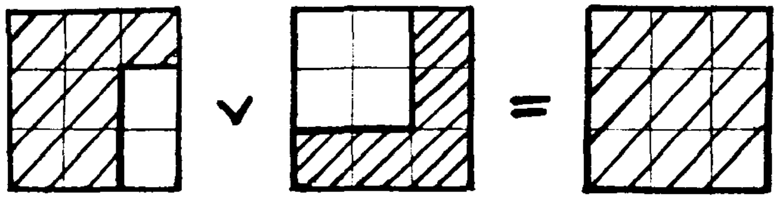
$$f(S) = \sum_{T \supseteq S} \mu(S,T) g(T), \tag{15.5}$$

and we have exhibited a combinatorial interpretation of Möbius inversion over any lattice of stars.

For our first example, let us return to the problem of rooks on an $n \times n$ board. As in Example 14.2, our point lattice $\mathcal{L}$ is represented by the family of open subsets of $\mathcal{C}$ (where $\mathcal{C}$ is the collection of squares $\{a_{ij}\}$ of the board) whose forbidden sets are lines. The minimal nonempty stars of $\mathcal{L}$ are the unions of the two lines through each square a_{ij}. Thus, the number of atoms of $\mathrm{St}(\mathcal{L})$ is n^2, and since every star is a union of these minimal stars, $\mathrm{St}(\mathcal{L})$ is a point lattice. A non-taking set $A \subseteq \mathcal{C}$ is, by definition, a set such that for each $a_{ij} \neq a_{pq}$ in A, $a_{pq} \notin \mathrm{st}(a_{ij})$ and $a_{ij} \notin \mathrm{st}(a_{pq})$. Clearly there is a bijection between these sets and all possible placements on the $n \times n$ board of non-taking rooks. Recall that for an open set A, $R(A) = \{i | \text{row } i \text{ is in } A\}$ and $C(A) = \{j | \text{column } j \text{ is in } A\}$. Let $r(A) = |R(A)|$ and $c(A) = |C(A)|$. If A generates the star S, then $|A| = \max(r(S), c(S))$. The number of sets generating S equals the number of maps from a set with $\max(r(S), c(S))$ elements *onto* a set with $\min(r(S), c(S))$ elements. Moreover, A is non-taking if and only if $r(\mathrm{st}(A)) = c(\mathrm{st}(A))$. Thus, if S is a star such that $r(S)=c(S)=m$, then S is generated by $m!$ non-taking sets of size m. Therefore, we have shown

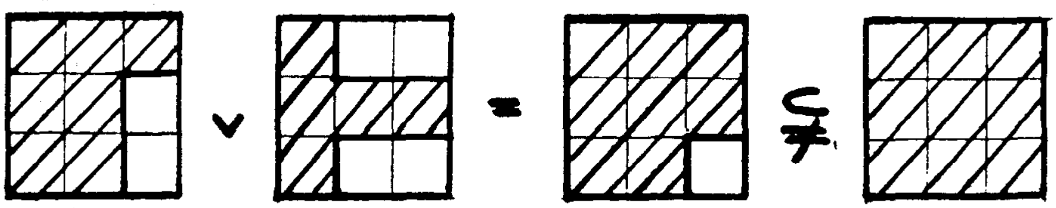
$$f(S)=\begin{cases} m! & \text{if } \; r(S)=c(S)=m, \\ 0 & \text{otherwise.}\end{cases} \tag{15.6}$$

For $n \geqslant 3$, the lattice $\mathrm{St}(\mathfrak{L})$ does not satisfy the chain condition. This is easily seen, since



(15.7)

while



(15.8)

In general, T is a successor of S in $\mathrm{St}(\mathfrak{L})$ if and only if

$$r(S) \leqslant r(T) \leqslant r(S+1),$$

and

$$c(S) \leqslant c(T) \leqslant c(S+1). \tag{15.9}$$

Thus the number of successors of S is given by

$$[n-r(S)]+[n-c(S)]+[n-r(S)][n-c(S)]$$
$$=[n+1-r(S)][n+1-c(S)]-1.$$

Moreover, if for $S \subseteq W$, we set $\mathrm{Succ}(S,W)=\{T \subseteq W \mid T \text{ is a successor of } S\}$, then $|\mathrm{Succ}(S,W)|$ is

$$\nu(S,W)=[r(W)+1-r(S)][c(W)+1-c(S)]-1. \tag{15.10}$$

Let us set, for $S \subseteq W$ and $2 \leqslant k \leqslant \nu(S,W)$,

$$c(S,W;k)=|\{Y \subseteq \mathrm{Succ}(S,W): |Y|=k \text{ and } \sup Y=W\}|. \tag{15.11}$$

The cross-cut theorem [1] for Möbius functions of lattices gives

$$\mu(S,W)=\sum_{k \geqslant 2}(-1)^k c(S,W;k). \tag{15.12}$$

While the constants in (15.11) are not tremendously difficult to calculate in any given interval, no closed formula is known at this time.

Finally, let us calculate the values of $g(S)$. If T is a star generated by a non-taking set and $T \supseteq S$, then $r(T) = c(T) = m \geqslant \max(r(S), c(S))$, and there are precisely

$$\binom{n-r(S)}{m-r(S)}\binom{n-c(S)}{m-c(S)}$$

such T. Hence, there are

$$\binom{n-r(S)}{m-r(S)}\binom{n-c(S)}{m-c(S)}m!$$

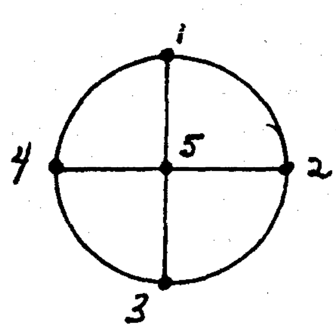
non-taking sets of size m whose star contains S. Therefore, we have

$$g(S) = \sum_{m=\max(r(S),c(S))}^{n} \binom{n-r(S)}{m-r(S)}\binom{n-c(S)}{m-c(S)}m!. \tag{15.13}$$

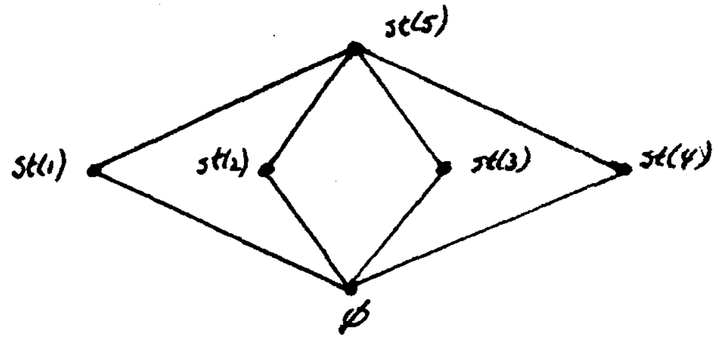
Let us now turn to the case of graphs. As in Example 14.3, given a finite graph $\mathcal{G} = (V, E)$, our lattice $\mathcal{L}$ is represented by the family of open subsets of V whose forbidden sets consist of two-point subsets $\{p,q\}$ such that (p,q) is an (undirected) edge in E. The minimal nonempty stars will be a collection of st(p)'s, but not every st(p) is necessarily minimal. A two-subset $\{p,q\}$ is nontaking if and only if $p \notin \mathrm{st}(q)$ and $q \notin \mathrm{st}(p)$, that is (p,q) is not an edge of $\mathcal{G}$. Thus, nontaking subsets correspond to collections of vertices where no two are connected by an edge of $\mathcal{G}$.

A proper coloring of a graph is a placement of colors, one on each vertex of $\mathcal{G}$, such that no edge connects two vertices of the same color. Clearly, the maximum number of vertices we can color with one color is equal to $\max_A \{|A| : A \text{ in non-taking}\}$. The minimum number of colors needed to properly color a graph is equal to the smallest k such that there exists a collection of pairwise disjoint non-taking subsets $A_1, A_2, \ldots, A_k$ whose union is V.

Since the class of all finite graphs is extremely general, one would not expect to be able to obtain general formulas for the functions f and g. However, in many specific cases, they are very simple. As an example, let $\mathcal{G}$ be the graph



There are six stars in V, and St($\mathfrak{L}$) is



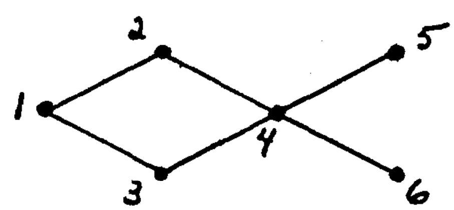
The non-taking subsets are $\varnothing$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{1,3\}$, and $\{2,4\}$. Thus,

$$f(\mathrm{st}(j)) = \begin{cases} 1, & 0 \leqslant j \leqslant 4, \\ 3, & j=5, \end{cases} \tag{15.14}$$

and

$$g(\mathrm{st}(j)) = \begin{cases} 8, & j=0, \\ 4, & 1 \leqslant j \leqslant 4, \\ 3, & j=5. \end{cases} \tag{15.15}$$

Now consider the graph



Here the lattice St($\mathfrak{L}$) is *not* a point lattice. Indeed the minimal nonempty stars are $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{4,5\}$, and $\{4,6\}$. Thus, $\mathrm{st}(4)=\{2,3,4,5,6\}$ is not a minimal star, and is not the join of the minimal nonempty stars contained in it.

Let us recall that our point lattice $\mathfrak{L}$ is represented by a family of open subsets of the set $\mathcal{Q}$. For some sets $A \subseteq \mathcal{Q}$, the enumeration of the non-taking subsets contained in A can be reduced to counting the number of non-taking subsets contained in certain subsets of A. The precise determination of when this occurs leads us to the following definition. We say that two subsets A and B *split* (or form a splitting of) a subset W if $A \cup B = W$, $A \cap \mathrm{st}(B) = \varnothing$, and $\mathrm{st}(A) \cap B = \varnothing$. Let A and B split W, and suppose S is a non-taking subset of W. Clearly, $S \cap A$ and $S \cap B$ are non-taking subsets of A and B, respectively. Conversely, if S is a non-taking subset of A, and T is a non-taking subset of B, then $S \cup T$ is a non-taking subset of W. To see this let $s \in S$, $t \in T$, and $s \in \mathrm{st}(t)$. Then $s \in A \cap \mathrm{st}(B)$, but $A \cap \mathrm{st}(B)$ is empty. Let $r(W;k)$ denote the number of non-taking subsets of W of size k. Then for any splitting A,B of W, we have shown

$$r(W,n)=\sum_{k=0}^{n}\binom{n}{k}r(A,k)r(B,n-k). \tag{15.16}$$

This identity is a generalization of those given by Henle for morphs relative to certain dissects [17]. If $\mathcal{Q}$ itself admits a nontrivial splitting A,B, then $\mathcal{L}$ is the direct product of the lattices of open subsets of A and B.

The *Henle coalgebra* $\mathcal{H}(\mathcal{L})$, associated to a point lattice $\mathcal{L}$, studies the splits of $\mathcal{Q}$. More precisely, $\mathcal{H}(\mathcal{L})$ is the vector space over K with basis consisting of all subsets of $\mathcal{Q}$. The diagonalization Δ and counit ε are given by

$$\Delta A=\sum_{\substack{(A_1,A_2)\text{ ordered}\\ \text{splits of } A}} A_1\otimes A_2 \tag{15.17}$$

and

$$\varepsilon(A)=\begin{cases}1 & \text{if } A=\varnothing,\\ 0 & \text{otherwise}\end{cases}. \tag{15.18}$$

That the comultiplication Δ is coassociative is easily verified. The full $n\times n$ rook board admits no nontrivial splittings. However, if W is the subset shown in Fig. 2, then

$$\Delta W=\varnothing\otimes W+W_1\otimes W_2+W_2\otimes W_1+W\otimes\varnothing.$$

A graph $\mathcal{G}$ with admit a splitting if and only if it has more than one connected component.

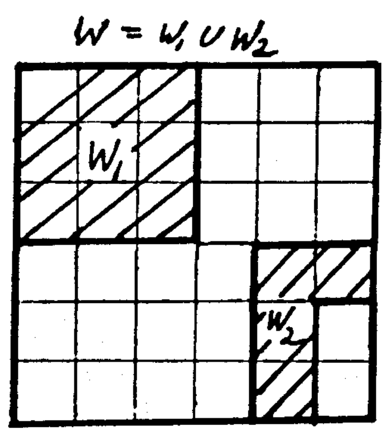


Figure 2.

XVI. Cleavages

We shall next discuss a class of coalgebras that generalizes the classical notion of the shuffle algebra to partially ordered sets. A family Σ of PO sets is called an SBC family (suitable for building cleavages) when it satisfies the following condition:

if P is a partially ordered set in Σ,
and if Q is a partially ordered subset of P in Σ,
then the partially ordered subset P-Q also belongs to Σ. (16.1)

Under this condition we define a *cleavage* of a PO set P in Σ as an ordered pair (Q,R) of subsets of P—with the inherited partial order—such that $Q \cap R = \varnothing$, $Q \cup R = P$, and Q and R belong to Σ.

The *cleavage coalgebra* of the family Σ, $C\ (\Sigma)$, is now defined as follows. Associate a variable x_P to each P in Σ, including 1 for the empty PO set, and let $C\ (\Sigma)$ be the vector space having the X_P's as a basis. Set

$$\Delta x_P = \sum_{(Q,R)} x_Q \otimes x_R, \tag{16.2}$$

where the sum ranges over all cleavages (Q,R) of P. The counit $\varepsilon(x_P)$ is zero unless $x_P = 1$, and $\varepsilon(1) = 1$; the verification of coassociativity is immediate.

We shall call a family $\tilde{\Sigma}$ of *types* (i.e. isomorphism classes) of partially ordered sets a *reduced SBC family* when it satisfies the following condition:

if α is a type in $\tilde{\Sigma}$,
if P is a partially ordered set of type α,
if Q is a partially ordered subset of P,
and if the type β of Q belongs to $\tilde{\Sigma}$, then the type γ
of the partially ordered subset $P - Q$ belongs to $\tilde{\Sigma}$. (16.3)

Clearly, if Σ is an SBC family, and if $\tilde{\Sigma}$ is the family of types (or isomorphism classes) of Σ, then $\tilde{\Sigma}$ is a reduced SBC family. The *reduced cleavage coalgebra* of the family $\tilde{\Sigma}$ is the vector space $C(\tilde{\Sigma})$ freely spanned by the variables x_α associated to each type α, including 1 for the empty PO set, with

$$\Delta x_\alpha = \sum (\alpha | \beta, \gamma) x_\beta \otimes x_\gamma, \tag{16.4}$$

where the sum ranges over all ordered pairs (β, γ) of types in $\tilde{\Sigma}$ such that a partially ordered set P of type α contains a cleavage of type (β, γ).

The section coefficients $(\alpha | \beta, \gamma)$ are integers counting the number of cleavages of type (β, γ) in a partially ordered set of type α. The counit is the obvious one, and again the verification of coassociativity is immediate.

If Σ is an SBC family and $\tilde{\Sigma}$ is the family of types of Σ, then the reduced cleavage coalgebra $C(\tilde{\Sigma})$ is isomorphic to the quotient of the cleavage coalgebra

$C(\Sigma)$ modulo the coideal generated by $x_P - x_Q$ for all isomorphic PO sets P, Q in Σ.

Examples of SBC and reduced SBC families are not abundant, and we shall only give three.

Example 16.1. Let Σ be the family of all finite linearly ordered subsets; $\tilde{\Sigma}$ is the family of all types of finite linearly ordered subsets. The cleavage coalgebra $C(\Sigma)$ is isomorphic to the Boolean coalgebra. The reduced cleavage coalgebra $C(\tilde{\Sigma})$ turns out to be isomorphic to the shuffle coalgebra. It is well known that this coalgebra is a bialgebra, where the noncommutative multiplication is simply juxtaposition.

Example 16.2. Let Σ be the family of all finite forests, and $\tilde{\Sigma}$ the reduced family consisting of all types of finite forests, considered as PO sets. Clearly Σ is an SBC family. $\tilde{\Sigma}$ defines an interesting reduced cleavage coalgebra, the tree coalgebra, which does not seem to have been studied. We do not know whether the tree coalgebra can be significantly turned into a bialgebra.

Example 16.3: Let Σ be the family of all finite PO sets; $\tilde{\Sigma}$, the reduced family of all types of PO sets. The associated reduced cleavage coalgebra, we conjecture, should have some notable universal mapping characterization, generalizing the universal properties of the shuffle coalgebra.

Several SBC subfamilies (reduced SBC subfamilies) of PO sets defined by restricting the length or width of the PO sets (types) allowed give subcoalgebras of the cleavage (reduced cleavage) coalgebra. For example, one can take all PO sets (types of PO sets) with the property that in each P, no chain exceed in length an integer n prescribed in advance.

The cleavage and reduced cleavage coalgebras can be viewed as generalizations of the incidence and reduced incidence coalgebras. Very probably, other coalgebras "in between" these two extremal cases can be defined.

XVII. Hereditary bialgebras

We come now to the description of a class of bialgebras—indeed, of Hopf algebras—which are probably the richest in structure and combinatorial applications. They are obtained from hereditary classes of matroids, a notion which we proceed to discuss briefly.

Recall that a *matroid* $M(S)$ on a (finite) set S is a closure relation defined on the subsets of S which enjoys the MacLane-Steinitz exchange property: if A is any subset of S, $\bar{A}$ its closure, and p, q elements of S such that $q \in \overline{A \cup p}$ but $q \notin \bar{A}$, then $p \in \overline{A \cup q}$. We shall need only a few elementary concepts from the theory of matroids; further details can be found in the books by Crapo and Rota [9] and by Welsh. The *direct sum* of two matroids $M(S_1)$ and $M(S_2)$ on disjoint sets S_1 and S_2 is defined as $M(S_1 + S_2)$ by setting $\overline{A_1 \cup A_2} = \bar{A}_1 \cup \bar{A}_2$, where $A_i \subseteq S_i$. A matroid is said to be *connected* when it is not isomorphic to a nontrivial direct sum of two matroids. Every matroid $M(S)$ is uniquely the direct sum of connected matroids $M(S_i)$ obtained from the blocks S_i of a suitable partition of the set S. A segment of a matroid is defined as follows. Let

A and B be closed sets of $M(S)$, and let $A \subseteq B$. The *segment* $M(A,B;S)$ is the matroid defined as the set $B-A$ with, for $C \subseteq B-A$, the closure $\tilde{C}$ of C to be $\tilde{C}=\overline{C \cup A}-A$.

In the following we denote by Greek letters isomorphism classes, or *types*, of matroids. The lattice of closed sets of a matroid is called a *geometric lattice*. Two non-isomorphic matroids may have isomorphic geometric lattices. In fact, among all non-isomorphic types of matroids having isomorphic geometric lattices, there is one which is canonically associated with the geometric lattice L as follows: the set S is the set of atoms of the lattice L (that is, elements covering the minimum element), and for $A \subseteq S$, one sets $\overline{A}=\{p \in S: p \leqslant \sup A\}$. This matroid is called the *combinatorial geometry* associated to the geometric lattice L.

The geometric lattice of the segment matroid $M(A,B;S)$ is isomorphic to the segment $[A,B]$ in the geometric lattice L of the matroid $M(S)$.

We come now to our main notion. A *hereditary class* H of matroids is a family of types of combinatorial geometries with the following properties:

(1) If α and β belong to H, then the direct sum $\alpha+\beta$ is a combinatorial geometry, and it belongs to H. The geometric lattice of $\alpha+\beta$ is the product, in the sense of partially ordered sets, of the geometric lattices of α and β.

(2) If $M(A,B;S)$ is a segment of a matroid $M(S)$ and the type of $M(S)$ is in H, then the combinatorial geometry of the type of the matroid $M(A,B;S)$ also belongs to H.

(3) If α belongs to H and α is isomorphic to the nontrivial direct sum $\alpha=\alpha_1+\alpha_2$ of combinatorial geometries, then $\alpha_i \in H$.

Let H be a hereditary class, with types α,β,γ in H. The section coefficient $(\alpha|\beta;\gamma)$ of H is defined to be the number of closed sets A in a matroid $M(S)$ of type α such that the segment $M(\varnothing,A;S)$ is of type β and the segment $M(A,S;S)$ is of type γ. It is easy to see that this number depends only on the types α,β,γ.

We have the important

PROPOSITION 17.1. *The section coefficients of a hereditary class of matroids are section coefficients.*

Proof: We have to prove the identity

$$\sum_{\gamma}(\alpha|\beta,\gamma)(\gamma|\pi,\sigma) = (\alpha|\beta,\pi,\sigma)$$
$$= \sum_{\delta}(\alpha|\delta,\sigma)(\delta|\beta,\pi).$$

Let $(\alpha|\beta,\pi,\sigma)$ be the number of pairs of closed sets $A \subseteq B$ of a matroid $M(S)$ of type α such that $M(\varnothing,A;S)$ is of type β, $M(A,B;S)$ is of type π, and $M(B,S;S)$ is of type σ. The first sum is obtained by fixing A and letting B vary, whereas the second sum is obtained by fixing B and letting A vary.

More important is the

THEOREM 17.1. *The section coefficients associated with a hereditary class of matroids H are bisection coefficients.*

Proof: We define a semigroup structure on the hereditary class H by taking direct sums as addition. For clarity, we shall prove the bilinear identity for the special case $(\alpha|\beta,\gamma)$ where $\alpha=\alpha_1+\alpha_2$ and the α_i are connected (nontrivial) types; the general case is similar. Thus, we need to show that

$$(\alpha_1+\alpha_2|\beta,\gamma) = \sum_{\substack{\beta_1+\beta_2=\beta \\ \gamma_1+\gamma_2=\gamma}} (\alpha_1|\beta_1,\gamma_1)(\alpha_2|\beta_2,\gamma_2).$$

To this end, let $A \subseteq S$ be a closed set of $M(S)$, and let $M(S)$ be the direct sum of $M(S_1)$ and $M(S_2)$. Then the matroid $M(\varnothing,A;S)$ is isomorphic to the direct sum of the matroid $M(\varnothing,A_1;S_1)$ and $M(\varnothing,A_2;S_2)$ where $A_i=A\cap S_i$. Similarly, the matroid $M(A,S;S)$ is isomorphic to the direct sum of $M(A_1,S_1;S_1)$ and $M(A_2,S_2;S_2)$. Counting, we obtain the desired identity.

Thus, associating the variable x_α to each type of the hereditary class H, we obtain a bialgebra where the underlying algebra is the polynomial algebra in the variables x_α for which α is a *connected* type, and the comultiplication is defined by

$$\Delta x_\alpha = \Sigma(\alpha|\beta,\gamma)x_\beta \otimes x_\gamma.$$

The augmentation is defined in the obvious way.

The bialgebras obtained by this construction will be called *hereditary bialgebras*. We list some of the examples previously discussed.

(1) The Boolean algebra of subsets of finite sets turns out to be a hereditary bialgebra, which is in fact the binomial bialgebra.

(2) The Faà di Bruno bialgebra is the hereditary bialgebra obtained by taking the bond closure (see [1]) on graphs which are direct sums of complete graphs.

(3) The Eulerian coalgebra is also associated—although rather trivially—with a hereditary bialgebra. One takes all direct sums of matroids whose geometric lattices are the lattices of all subspaces of a vector space over a finite field. If α is connected, then $\Delta(x_\alpha)$ agrees with the definition already given.

Other notable hereditary classes of matroids are (4) all finite sets of points in projective space over a fixed field; (5) all series-parallel networks, (6) all graphs, (7) all planar graphs; (8) all unimodular matroids.

Each hereditary bialgebra leads to a generalization of the umbral calculus, for which the umbral calculus in one variable, outlined in Sec. XI, is the blueprint. We believe the development of such "hereditary" calculi to be one of the most promising prospects of present-day combinatorics.

In the preceding theorem (Theorem 17.1), an essential role is played by the very special factorization properties of matroids. Thus, the notion of hereditary bialgebras can be extended to any family of PO sets where one can prove the factorization properties required to make the above proof work. One such class is the class of semimodular lattices. The discovery of the most general such class, if any, may well lead to a class of bialgebras sharing a simple axiomatic definition.

In closing, we remark that the detailed study of hereditary bialgebras should have as some of its goals the extension to hereditary bialgebras of the exponential formula of the binomial bialgebra, as well as generalizations of the Lagrange inversion formula.

References

Items listed in the bibliographies of the principal references, unless specifically cited in this work, will not be repeated.

Principal references

1. G.-C. Rota, On the foundations of combinatorial theory I: Theory of Möbius functions, *Z. Wahrscheinlichkeistheorie* 2: 340–368 (1964).
2. R. Mullin and G.-C. Rota, On the foundations of combinatorial theory III: Theory of binomial enumeration, *Graph Theory Appl.* 168 (1970).
3. G.-C. Rota, D. Kahaner, and A. Odlyzko, Finite operator calculus, *J. Math. Anal. Appl.* 42: 685–760 (1973).
4. P. Doubilet, G.-C. Rota, and R. Stanley, On the foundations of combinatorial theory VI: The idea of a generating function, in *Sixth Berkeley Symposium on Mathematical Statistics and Probability*, 2, Berkeley U. P., 1972, pp. 267–318.
5. S. Roman and G.-C. Rota, The umbral calculus, *Advances in Math.* 27(2): 95–188 (1978).

Additional references

6. M. Aigner, *Kombinatorik*, Vol. I and II, Springer-Verlag, 1975–76.
7. J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics*, McGraw-Hill, New York, 1964.
8. N. Bourbaki, *Algebra*, Hermann, Paris, 1970, Chapter III.
9. H. Crapo and G.-C. Rota, *Combinatorial Geometries*, M.I.T. Press, Cambridge, Mass., 1971.
10. J. A. Dieudonné, *Introduction to the Theory of Formal Groups*, M. Dekker, New York, 1973.
11. P. Doubilet, Studies in partitions and permutations, Ph.D. thesis, M.I.T., 1974.
12. R. Feinberg, Characterization of incidence algebras, Ph.D. thesis, Univ. of Wisconsin, 1974.
13. J. Fillmore and S. G. Williamson, A linear algebra setting for the Mullin-Rota theory of polynomials of binomial type, *Linear and Multilinear Algebra* 1: 67 (1973).
14. A. M. Garsia, An expose of the Mullin-Rota theory of polynomials of binomial type, *Linear and Multilinear Algebra*, 1: 43 (1973).
15. A. M. Garsia and S. A. Joni, A new expression for umbral operators and power series inversion, *Proc. Amer. Math. Soc.* 64: 179–185 (1977).
16. J. R. Goldman and G.-C. Rota, On the foundations of combinatorial theory IV: Finite vector spaces and Eulerian generating functions, *Studies in Appl. Math.* XLIX (3): 239–258 (1970).
17. M. Henle, Binomial enumeration on dissects, *Trans. Amer. Math. Soc.* 202: 1–38 (1975).
18. S. A. Joni, Polynomials of binomial type and the Lagrange inversion formula, Ph.D. thesis, Univ. of California, 1977.
19. S. A. Joni, Antipodes and inversion of formal series, *J. Algebra* (1979), to appear.

20. R. W. LAWVERE, The convolution ring of a small category, to appear.
21. P. LEROUX, Les categories de Möbius, *Cahiers Topologie Gom. Différentielle* XVI (3): 280–283 (1975).
22. C. L. LIU, *Topics in Combinatorial Mathematics*, M.A.A., Washington, D.C., 1972.
23. G. POLYA, Picture writing, *Amer. Math. Monthly* 63: 689 (1956).
24. S. ROMAN, The algebra of divided differences, in preparation.
25. G.-C. ROTA and D. A. SMITH, Enumeration under group action, *Annali Scuola Norm. Sup. Ser. IV* IV:637–646 (1977).
26. H. SCHUBERT, *Categories*, Springer, Berlin, 1972.
27. M. E. SWEEDLER, *Hopf Algebras*, Benjamin, New York, 1969.
28. W. T. TUTTE, On dichromatic polynomials, *J. Combinatorial Theory* 2: 301 (1967).

CARNEGIE-MELLON UNIVERSITY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

(Received September 15, 1978)