Combinatorial Hopf algebras in quantum field theory I

Héctor Figueroa†

†Departamento de Matemáticas, Universidad de Costa Rica,
San Pedro 2060, Costa Rica

José M. Gracia-Bondía‡

‡Departamento de Física Teórica I, Universidad Complutense,
Madrid 28040, Spain

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Abstract

This manuscript stands at the interface between combinatorial Hopf algebra theory and renormalization theory. Its plan is as follows: Section 1 is the introduction, and contains as well an elementary invitation to the subject. The rest of part I, comprising Sections 2–6, is devoted to the basics of Hopf algebra theory and examples, in ascending level of complexity. Part II turns around the all-important Faà di Bruno Hopf algebra. Section 7 contains a first, direct approach to it. Section 8 gives applications of the Faà di Bruno algebra to quantum field theory and Lagrange reversion. Section 9 rederives the related Connes–Moscovici algebras. In Part III we turn to the Connes–Kreimer Hopf algebras of Feynman graphs and, more generally, to incidence bialgebras. In Section 10 we describe the first. Then in Section 11 we give a simple derivation of (the properly combinatorial part of) Zimmermann's cancellation-free method, in its original diagrammatic form. In Section 12 general incidence algebras are introduced, and the Faà di Bruno bialgebras are described as incidence bialgebras. In Section 13, deeper lore on Rota's incidence algebras allows us to reinterpret Connes–Kreimer algebras in terms of distributive lattices. Next, the general algebraic-combinatorial proof of the cancellation-free formula for antipodes is ascertained; this is the heart of the paper. The structure results for commutative Hopf algebras are found in Sections 14 and 15. An outlook section very briefly reviews the coalgebraic aspects of quantization and the Rota–Baxter map in renormalization.

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Part I: Basic Combinatorial Hopf Algebra Theory

1 Why Hopf algebras?

Quantum field theory (QFT) aims to describe the fundamental phenomena of physics at the shortest scales, that is, higher energies. In spite of many practical successes, QFT is mathematically a problematic construction. Many of its difficulties are related to the need for *renormalization*. This complicated process is at present required to make sense of quantities very naturally defined, that we are however unable to calculate without incurring infinities. The complications are of both analytical and combinatorial nature.

Since the work by Joni and Rota [1] on incidence coalgebras, the framework of Hopf algebras (a dual concept to groups in the spirit of noncommutative geometry) has been recognized as a very sophisticated one at our disposal, for formalizing the art of combinatorics. Now, recent developments (from 1998 on) have placed Hopf algebras at the heart of a noncommutative geometry approach to physics. Rather unexpectedly, but quite naturally, akin Hopf algebras appeared in two previously unrelated contexts: perturbative renormalization in quantum field theories [2–4] and index formulae in noncommutative geometry [5].

Even more recently, we have become aware of the neglected coalgebraic side of the quantization procedure [6, 7]. Thus, even leaving aside the role of "quantum symmetry groups" in conformal field theory, Hopf algebra is invading QFT from both ends, both at the foundational and the computational level. The *whole* development of QFT from principles to applications might conceivably be subtended by Hopf algebra.

The approach from quantum theoretical first principles is still in its first infancy. This is one reason why we have focused here on the understanding of the contributions by Kreimer, Connes, and Moscovici from the viewpoint of algebraic combinatorics —in particular in respect of incidence bialgebra theory. In other words (in contrast with [8] for instance), we examine Rota's and Connes and Kreimer's lines of thought in parallel. Time permitting, we will return in another article to the perspectives broached in [6, 7], and try to point out ways from the outposts into still unconquered territory.

In reference [9] Broadhurst and Kreimer declare: "[In renormalization theory] combinations of diagrams. . . can provide cancellations of poles, and may hence eliminate pole terms". The practical interest of this is manifest. In fact, the ultimate goal of tackling the Schwinger–Dyson equations in QFT is linked to general characterization problems for commutative Hopf algebras [10]. This is one of the reasons why we have provided a leisurely introduction to the subject, splashing pertinent examples to chase dreariness away, and, we hope, leading the audience almost imperceptibly from the outset towards the deeper structure questions undertaken in Sections 14 and 15 of Part IV.

The study of the more classical Faà di Bruno Hopf algebras, effected mostly in Part II, serves as a guiding thread of this survey. As well as their applications, in particular to the Lagrange reversion formula —a subject we find fascinating. The Faà di Bruno algebras, denoted $\mathcal{F}(n)$, are of the same general type as the Kreimer–Connes–Moscovici Hopf algebras; they are in fact Hopf subalgebras of the Connes–Moscovici Hopf algebras $\mathcal{H}_{\mathrm{CM}}(n)$.

As hinted at above, the latter appeared in connection with the index formula for transversally elliptic operators on a foliation. These canonical Hopf algebras depend only on the codimension n of the foliation, and their action on the transverse frame bundle simplifies decisively the associated computation of the index, that takes place on the cyclic cohomology of $\mathcal{H}_{\rm CM}(n)$. One of our main results is a theorem describing $\mathcal{H}_{\rm CM}(n)$ as a kind of bicrossedproduct Hopf algebra of $\mathcal{F}(n)$ by the (action of/coaction on) the Lie algebra of the affine transformation group. This is implicit in [5], but there the construction, in the words of G. Skandalis [11], is performed "by hand", using the action on the frame bundle. As the $\mathcal{H}_{\rm CM}(n)$ reappear in other contexts, such as modular forms [12], a more abstract characterization was on the order of the day.

Another focus of interest is the comprehensive investigation of the range of validity of Zimmermann's combinatorial formula of QFT [13], in the algebraic-combinatorial context. It so happens that the 'natural' formulae for computing the antipodes of the algebras we are dealing with are in some sense inefficient. This inefficiency lies in the huge sets of cancellations that arise in the final result.

A case in point is the alternating sum over chains characteristic of incidence Hopf algebras. The Faà di Bruno bialgebras are incidence bialgebras, corresponding to the family of partially ordered sets (posets) that are partition lattices of finite sets. In relation with them at the end of his authoritative review [14] on antipodes of incidence bialgebras, Schmitt in 1987 wrote: "[The Lagrange reversion formula] can be viewed as a description of what remains of the alternating sum [over chains] after all the cancellations have been taken into account. We believe that understanding exactly how these cancellations take place will not only provide a direct combinatorial proof of the Lagrange inversion formula, but may well yield analogous formulas for the antipodes of Hopf algebras arising from classes of geometric lattices other than partition lattices".

Unbeknown, Zimmermann's formula had been performing this trick in renormalization theory by then for almost twenty years. And already in [15], the Dyson–Salam method for renormalization was thoroughly reexamined and corrected, and its equivalence (in the context of the so-called BPHZ renormalization scheme) to Bogoliubov's and Zimmermann's methods was studied. Inspired by the book [15] and the work by Kreimer and Connes, the present authors a couple of years ago investigated the Hopf algebra theory underpinnings of these equivalences in [16, 17].

This is the punchline: on the one hand, the Connes–Kreimer algebras of rooted trees and Feynman diagrams can be subsumed under the theory of incidence algebras of distributive lattices (see [18] for the latter); on the other, Zimmermann's formula can be incorporated into the mainstream of combinatorial Hopf algebra theory as the cancellation-free formula for the antipode of any such incidence algebra. We show all this in Sections 11 and 13 of Part III. Haiman and Schmitt [19] eventually found the equivalent of Zimmermann's formula for Faà di Bruno algebras. This is also subsumed here. Thus the trade between QFT and combinatorial Hopf algebra theory is not one-way.

▶ We did not want to assume that the readership is familiar with all the notions of Hopf algebra theory; and some find a direct passage to the starchy algebraist's diet too abrupt. This is why we start, in the footsteps of [20], with a motivational discussion —that experts should probably skip. In the spirit of noncommutative geometry, suppose we choose to study a set S via a commutative algebra $\mathcal{F}(S)$ of complex functions on it (we work with complex numbers for definiteness only: nearly everything we have to say applies when $\mathbb{C}$ is replaced by any unital commutative $\mathbb{Q}$-algebra; also, in some parts of this paper it will be clear from the context when we work with real numbers instead of $\mathbb{C}$). The tensor product of algebras $\mathcal{F}(S) \otimes \mathcal{F}(S)$ has the algebra structure given by

$$(f \otimes g)(f' \otimes g') = ff' \otimes gg'.$$

Also, there is a one-to-one map $\sigma : \mathcal{F}(S) \otimes \mathcal{F}(S) \to \mathcal{F}(S \times S)$ given by $f \otimes g(s, s') \mapsto f(s)g(s')$. The image of σ consists of just those functions h of two variables for which the vector space spanned by the partial maps $h_{s'}(s) := h(s, s')$ is of finite dimension. Suppose now the set $S \equiv G$ is a group. Then there is much more to $\mathcal{F}(G)$ than its algebra structure. The multiplication in G induces an algebra map $\rho : \mathcal{F}(G) \to \mathcal{F}(G \times G)$ given by

$$\rho[f](x, y) = f(xy) =: y \triangleright f(x) =: f \triangleleft x(y),$$

where the functions $y \triangleright f$, $f \triangleleft x$ are respectively the right translate of f by y and the left translate of f by x. Then $\rho[f] \in \operatorname{im} \sigma$ iff f is such that its translates span a finite-dimensional space. An example is given by the space of polynomials over the additive group $\mathbb{C}$ acting on itself by translation. A function with this property is called a *representative* function; representative functions clearly form a subalgebra $\mathcal{R}(G)$ of $\mathcal{F}(G)$. In summary, there is an algebra map $\Delta = \sigma^{-1} \circ \rho$ from $\mathcal{R}(G)$ to $\mathcal{R}(G) \otimes \mathcal{R}(G)$ that we express by

$$f(xy) = \sum_j f_{j(1)}(x) \otimes f_{j(2)}(y) := \Delta f(x, y).$$

This gives a linearized form of the group operation. Now, the associative identity $f((xy)z) = f(x(yz))$ imposes

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta,$$

where we denote id the identity map in the algebra of representative functions. Moreover, $f \mapsto f(1_G)$ defines a map $\eta : \mathcal{R}(G) \to \mathbb{C}$ (called the *augmentation* map) that, in view of $f(x1_G) = f(1_G x) = f(x)$ verifies

$$(\mathrm{id} \otimes \eta) \circ \Delta = (\eta \otimes \mathrm{id}) \circ \Delta = \mathrm{id}.$$

The basic concept of *coalgebra* in Hopf algebra theory abstracts the properties of the triple $(\mathcal{R}(G), \Delta, \eta)$.

Let us go a bit further and consider, for a complex vector space V, representative maps $G \to V$ whose translates span a finite-dimensional space of maps, say $\mathcal{R}_V(G)$. Let $(f_1, \dots, f_n)$ be a basis for the space of translates of $f \in \mathcal{R}_V(G)$, and express

$$y \triangleright f(x) = \sum_{i=1}^n c_i(y) f_i(x).$$

In particular,

$$y \triangleright f_j(x) = \sum_{i=1}^n c_{ij}(y) f_i(x)$$

defines the c_{ij}; also $f = \sum_{i=1}^n \eta(f_i)c_i$. Now, the c_i are representative, since

$$\sum_{i=1}^{n} c_i(zy)f_i(x) = z \triangleright (y \triangleright f)(x) = \sum_{i=1}^{n} c_i(y) z \triangleright f_i(x) = \sum_{i,j=1}^{n} c_j(y)c_{ij}(z)f_i(x),$$

implying $y \triangleright c_i = \sum_{j=1}^n c_j(y)c_{ij}$. In consequence, the map

$$v \otimes f \mapsto f(.)v$$

from $V \otimes \mathcal{R}(G)$ to $\mathcal{R}_V(G)$ is bijective, and we identify these two spaces. All this points to the importance of $\mathcal{R}(G)$. But in what wilderness are its elements found? A moment's reflection shows that linear forms on any locally finite G-module provide representative functions. Suppose V is such a module, and let T denote the representation of G on it. A map $\gamma_T : V \to V \otimes \mathcal{R}(G)$ is given by

$$\gamma_T(v)(x) = T(x)v.$$

The fact $T(x)T(y) = T(xy)$ means that γ_T 'interacts' with Δ:

$$(\gamma_T \otimes \mathrm{id}) \circ \gamma_T = (\mathrm{id}_V \otimes \Delta) \circ \gamma_T;$$

and the fact $T(1_G) = \mathrm{id}_V$ forces

$$(\mathrm{id}_V \otimes \eta) = \mathrm{id}_V.$$

One says that (V, γ_T) is a *comodule* for $(\mathcal{R}(G), \Delta, \eta)$. A Martian versed in the 'dual' way of thinking could find this method a more congenial one to analyze group representations.

Let us add that the coalgebraic aspect is decisive in the applications of Hopf algebra theory to renormalization; those interested mainly in this aspect might consult now our invitation at the beginning of Section 10. Also, after extracting so much mileage from the coalgebraic side of the group properties, one could ask, what is the original algebra structure of $\mathcal{R}(G)$ good for? This question we shall answer in due course.

2 Précis of bialgebra theory

We assume familiarity with (associative, unital) algebras; but it is convenient here to rephrase the requirements for an algebra A in terms of its two defining maps, to wit $m : A \otimes A \to A$ and $u : \mathbb{C} \to A$, respectively given by $m(a, b) := ab$; $u(1) = 1_A$. They must satisfy:

- Associativity: $m(m \otimes \mathrm{id}) = m(\mathrm{id} \otimes m) : A \otimes A \to A$;
- Unity: $m(u \otimes \mathrm{id}) = m(\mathrm{id} \otimes u) = \mathrm{id} : \mathbb{C} \otimes A = A \otimes \mathbb{C} = A \to A$.

In the following we omit the sign $\circ$ for composition of linear maps. These two properties correspond, respectively, to the commutativity of the diagrams

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "m \otimes \mathrm{id}_A"] \arrow[d, "\mathrm{id}_A \otimes m"'] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A, \end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{C} \otimes A \arrow[r, "u\otimes \mathrm{id}_A"] \arrow[d, leftrightarrow] & A \otimes A \arrow[d, "m"] \\
A \arrow[r, "\mathrm{id}_A"] & A,
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes \mathbb{C} \arrow[r, "\mathrm{id}_A\otimes u"] \arrow[d, leftrightarrow] & A \otimes A \arrow[d, "m"] \\
A \arrow[r, "\mathrm{id}_A"] & A.
\end{tikzcd}$$

The unnamed arrows denote the natural identifications $\mathbb{C} \otimes A = A = A \otimes \mathbb{C}$ respectively given by $\lambda \otimes a \mapsto \lambda a$ and $a \otimes \lambda \mapsto \lambda a$.

Definition 2.1. A *coalgebra* C is a vector space with the structure obtained by reversing the arrows in the diagrams above characterizing an algebra. A coalgebra is therefore also described by two linear maps: the coproduct $\Delta : C \to C \otimes C$ (also called diagonalization, or "sharing"), and the counit (or augmentation) $\eta : C \to \mathbb{C}$, with requirements:

- Coassociativity: $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta : C \to C \otimes C \otimes C$;
- Counity: $(\eta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \eta)\Delta = \mathrm{id} : C \to C$.

These conditions, therefore, simply describe the commutativity of the diagrams "dual" to the above, namely

$$\begin{tikzcd}
C \otimes C \otimes C & C \otimes C \arrow[l, "\Delta\otimes \mathrm{id}_C"'] \\
C \otimes C \arrow[u, "\mathrm{id}_C\otimes\Delta"] & C \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{C} \otimes C \arrow[d, leftrightarrow] & C \otimes C \arrow[l, "\eta\otimes \mathrm{id}_C"'] \\
C & C, \arrow[l, "\mathrm{id}_C"] \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
C \otimes \mathbb{C} \arrow[d, leftrightarrow] & C \otimes C \arrow[l, "\mathrm{id}_C\otimes\eta"'] \\
C & C. \arrow[l, "\mathrm{id}_C"] \arrow[u, "\Delta"']
\end{tikzcd}$$

In general, the prefix 'co' makes reference to the process of reversing arrows in diagrams. With $\Delta a := \sum_j a_{j(1)} \otimes a_{j(2)}$, the second requirement is explicitly given by:

$$\sum_j \eta(a_{j(1)})\, a_{j(2)} = \sum_j a_{j(1)}\, \eta(a_{j(2)}) = a. \tag{2.1}$$

Since it has a left inverse, Δ is always injective.

If $\Delta a_{j(1)} = \sum_k a_{jk(1)(1)} \otimes a_{jk(1)(2)}$ and $\Delta a_{j(2)} = \sum_l a_{jl(2)(1)} \otimes a_{jl(2)(2)}$, the first condition is

$$\sum_{jk} a_{jk(1)(1)} \otimes a_{jk(1)(2)} \otimes a_{j(2)} = \sum_{jl} a_{j(1)} \otimes a_{jl(2)(1)} \otimes a_{jl(2)(2)}.$$

Thus coassociativity corresponds to the idea that, in decomposing the 'object' a in sets of three pieces, the order in which the breakups take place does not matter.

In what follows we alleviate the notation by writing simply $\Delta a = \sum a_{(1)} \otimes a_{(2)}$. Sometimes even the $\sum$ sign will be omitted. Since $\sum a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)} = \sum a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)}$, we can write

$$\Delta^2 a = a_{(1)} \otimes a_{(2)} \otimes a_{(3)}, \quad \Delta^3 a = a_{(1)} \otimes a_{(2)} \otimes a_{(3)} \otimes a_{(4)},$$

and so on, for the n-fold coproducts. Notice that $(\eta \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id})\Delta^n = \Delta^{n-1}$ (n id factors understood). And similarly when η is in any other position. We analogously consider the n-fold products $m^n : A \otimes \cdots \otimes A \to A$, with $n+1$ factors A understood.

Like algebras, coalgebras have a tensor product. The coalgebra $C \otimes D$ is the vector space $C \otimes D$ endowed with the maps

$$\Delta_\otimes(c \otimes d) = \sum c_{(1)} \otimes d_{(1)} \otimes c_{(2)} \otimes d_{(2)}; \qquad \eta_\otimes(c \otimes d) = \eta(c)\eta(d). \tag{2.2}$$

That is $\Delta_\otimes = (\mathrm{id} \otimes \tau \otimes \mathrm{id})(\Delta_C \otimes \Delta_D)$, in parallel with $m_\otimes = (m_A \otimes m_B)(\mathrm{id} \otimes \tau \otimes \mathrm{id})$ for algebras.

A counital coalgebra (or comultiplicative) map $\ell : C \to D$ between two coalgebras is a linear map that preserves the coalgebra structure,

$$\Delta_D \ell = (\ell \otimes \ell)\Delta_C : C \to D \otimes D; \qquad \eta_D \ell = \eta_C : C \to \mathbb{C}.$$

Once more these properties correspond to the commutativity of the diagrams

$$\begin{tikzcd} C \otimes C \arrow[d, "\ell\otimes\ell"'] & C \arrow[l, "\Delta_C"'] \arrow[d, "\ell"] \\ D \otimes D & D, \arrow[l, "\Delta_D"'] \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "\ell"] \arrow[dr, "\eta_C"'] & & D \arrow[dl, "\eta_D"] \\ & \mathbb{C}, & \end{tikzcd}$$

obtained by reversing arrows in the diagrams that express the homomorphism properties of linear maps between unital algebras.

(Non)commutativity of the algebra and coalgebra operations is formulated with the help of the "flip map" $\tau(a \otimes b) := b \otimes a$. The algebra A is *commutative* if $m\tau = m : A \otimes A \to A$; likewise, the coalgebra C is called *cocommutative* if $\tau\Delta = \Delta : C \to C \otimes C$. These properties correspond, respectively, to the commutativity of the diagrams

$$\begin{tikzcd} A \otimes A \arrow[rr, "\tau"] \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\ & A, & \end{tikzcd} \qquad \begin{tikzcd} C \otimes C & & C \otimes C \arrow[ll, "\tau"'] \\ & C. \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] & \end{tikzcd}$$

For commutative algebras, the map m is a homomorphism, and similarly Δ is a coalgebra map for cocommutative coalgebras. The same space C with the coalgebra structure given by $\tau\Delta$ is called the *coopposite* coalgebra C^{cop}.

A *subcoalgebra* of C is a subspace Z such that

$$\Delta Z \subseteq Z \otimes Z.$$

A coalgebra without nontrivial subcoalgebras is *simple*. The direct sum of all simple subcoalgebras of C is called its coradical $R(C)$.

A very important concept is that of *coideal*. A subspace J is a coideal of C if

$$\Delta J \subseteq J \otimes C + C \otimes J \quad \text{and} \quad \eta(J) = 0.$$

The kernel of any coalgebra map ℓ is a coideal. In effect,

$$\Delta_D \ell(\ker \ell) = 0 = (\ell \otimes \ell)\Delta_C \ker \ell,$$

forces $\Delta_C \ker \ell \subseteq \ker \ell \otimes C + C \otimes \ker \ell$, and moreover

$$\eta_C(\ker \ell) = \eta_D \ell(\ker \ell) = 0.$$

If J is a coideal, then C/J has a (unique) coalgebra structure such that the canonical projection $C \xrightarrow{q} C/J$ is a coalgebra map —see the example at the end of the section.

A coalgebra filtered as a vector space is called a *filtered coalgebra* when the filtering is compatible with the coalgebra structure; that is, there exists a nested sequence of subspaces C_n such that $C_0 \subsetneq C_1 \subsetneq \dots$ and $\bigcup_{n \geq 0} C_n = C$, and moreover

$$\Delta C_n \subseteq \sum_{k=0}^{n} C_{n-k} \otimes C_k.$$

▶ Given an algebra A and a coalgebra C over $\mathbb{C}$, one can define the *convolution* of two elements f, g of the vector space of $\mathbb{C}$-linear maps $\operatorname{Hom}(C, A)$, as the map $f * g \in \operatorname{Hom}(C, A)$ given by the composition

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{m} A.$$

In other words,

$$f * g(a) = \sum f(a_{(1)})\, g(a_{(2)}).$$

The triple $(\operatorname{Hom}(C, A), *, u_A \eta_C)$ is then a unital algebra. Indeed, convolution is associative because of associativity of m and coassociativity of Δ:

$$\begin{aligned}(f * g) * h &= m((f * g) \otimes h)\Delta = m(m \otimes \mathrm{id})(f \otimes g \otimes h)(\Delta \otimes \mathrm{id})\Delta \\ &= m(\mathrm{id} \otimes m)(f \otimes g \otimes h)(\mathrm{id} \otimes \Delta)\Delta = m(f \otimes (g * h))\Delta = f * (g * h).\end{aligned}$$

The unit is in effect $u_A \eta_C$:

$$\begin{aligned} f * u_A\eta_C &= m(f \otimes u_A\eta_C)\Delta = m(\mathrm{id}_A \otimes u_A)(f \otimes \mathrm{id}_{\mathbb{C}})(\mathrm{id}_C \otimes \eta_C)\Delta = \mathrm{id}_A\, f\, \mathrm{id}_C = f, \\ u_A\eta_C * f &= m(u_A\eta_C \otimes f)\Delta = m(u_A \otimes \mathrm{id}_A)(\mathrm{id}_{\mathbb{C}} \otimes f)(\eta_C \otimes \mathrm{id}_C)\Delta = \mathrm{id}_A\, f\, \mathrm{id}_C = f. \end{aligned}$$

Algebra morphisms $l : A \to B$ and coalgebra morphisms $\ell : D \to C$ respect convolution, in the following respective ways: if $f, g \in \operatorname{Hom}(C, A)$ for some coalgebra C and some algebra A, then

$$l(f * g) = l m_A(f \otimes g)\Delta = m_B(l \otimes l)(f \otimes g)\Delta = m_B(lf \otimes lg)\Delta = lf * lg, \tag{2.3}$$

$$(f * g)\ell = m(f \otimes g)\Delta_C \ell = m(f \otimes g)(\ell \otimes \ell)\Delta_D = m_A(f\ell \otimes g\ell)\Delta_D = f\ell * g\ell. \tag{2.4}$$

Definition 2.2. To obtain a *bialgebra*, say H, one considers on the vector space H both an algebra and a coalgebra structure, and further stipulates the *compatibility* condition that the algebra structure maps m and u are counital coalgebra morphisms, when $H \otimes H$ is seen as the tensor product coalgebra.

This leads (omitting the subscript from the unit in the algebra) to:

$$\Delta a \, \Delta b = \Delta(ab), \quad \Delta 1 = 1 \otimes 1, \quad \eta(a)\eta(b) = \eta(ab), \quad \eta(1) = 1; \tag{2.5}$$

in particular, η is a one-dimensional representation of the algebra H. For instance:

$$\begin{aligned} \Delta(ab) &= \Delta m(a \otimes b) = (m \otimes m)\Delta(a \otimes b) = (m \otimes m)[a_{(1)} \otimes b_{(1)} \otimes a_{(2)} \otimes b_{(2)}] \\ &= a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)} = (a_{(1)} \otimes a_{(2)})(b_{(1)} \otimes b_{(2)}) = \Delta a \, \Delta b. \end{aligned}$$

Of course, it would be totally equivalent and (to our earthlings' mind) looks simpler to postulate instead the conditions (2.5): they just state that the coalgebra structure maps Δ and η are unital algebra morphisms. But it is imperative that we familiarize ourselves with the coalgebra operations. Note that the last equation in (2.5) is redundant.

The map $u\eta : H \to H$ is an idempotent, as $u\eta u\eta(a) = \eta(a)u\eta(1_H) = \eta(a)1_H = u\eta(a)$. Therefore $H = \operatorname{im} u\eta \oplus \ker u\eta = \operatorname{im} u \oplus \ker \eta = \mathbb{C}1_H \oplus \ker \eta$. A halfway house between algebras or coalgebras and bialgebras is provided by the notions of augmented algebra, which is a quadruple (A, m, u, η), or augmented coalgebra, which is a quadruple (C, Δ, u, η), with the known properties in both cases.

A *bialgebra morphism* is a linear map between two bialgebras, which is both a unital algebra homomorphism and a counital coalgebra map. A *subbialgebra* of H is a vector subspace E that is both a subalgebra and a subcoalgebra; in other words, E, together with the restrictions of the product, coproduct and so on, is also a bialgebra and the inclusion $E \hookrightarrow H$ is a bialgebra morphism.

A *biideal* J of H is a linear subspace that is both an ideal of the algebra H and a coideal of the coalgebra H. The quotient H/J inherits a bialgebra structure.

Associated to any bialgebra H there are the three bialgebras H^{opp}, H^{cop}, H^{copp} obtained by taking opposite either of the algebra structure or the coalgebra structure or both.

Linear maps of a bialgebra H into an algebra A can in particular be convolved; but if they are multiplicative, that is, algebra homomorphisms, their convolution in general will be multiplicative only if A is commutative. In such a case if $f, g \in \mathrm{Hom}_{\mathrm{alg}}(H, A)$, then

$$f * g(ab) = f(a_{(1)})f(b_{(1)})g(a_{(2)})g(b_{(2)}) = f(a_{(1)})g(a_{(2)})f(b_{(1)})g(b_{(2)}) = f * g(a) f * g(b).$$

Similarly, the convolution of coalgebra maps of a coalgebra C into a bialgebra H is comultiplicative when H is cocommutative.

Definition 2.3. A bialgebra $\mathbb{N}$-filtered as a vector space is called a *filtered bialgebra* when the filtering is compatible with both the algebra and the coalgebra structures; that is, there exits a nested sequence of subspaces $H_0 \subsetneq H_1 \subsetneq \dots$ such that $\bigcup_{n \geq 0} H_n = H$, and moreover

$$\Delta H_n \subseteq \sum_{k=0}^{n} H_{n-k} \otimes H_k; \quad H_n H_m \subseteq H_{n+m}.$$

Connected bialgebras are those filtered bialgebras for which the first piece consists just of scalars: $H_0 = u(\mathbb{C})$; in that case $R(H) = H_0$, and the augmentations η, u are unique.

▶ Any coalgebra C is filtered via a filtering, dubbed the *coradical filtering*, whose starting piece is the coradical. When H is a bialgebra, its coradical filtering is not necessarily compatible with the algebra structure. Nevertheless, it is compatible when $R := R(H)$ is a subbialgebra of H, in particular when $H_0 = u(\mathbb{C})$. In short, the coradical filtering goes as follows: consider the sums of tensor products

$$H_R^1 := R; \quad H_R^2 := R \otimes H + H \otimes R; \quad H_R^3 := R \otimes H \otimes H + H \otimes R \otimes H + H \otimes H \otimes R;$$

and so on; then the subcoalgebras H_n are defined by

$$H_n = [\Delta^n]^{-1}(H_R^{n+1}).$$

We refer the reader to [21,22] for more details on the coradical filtering. Naturally, a bialgebra may have several different filterings.

A bialgebra $H = \bigoplus_{n=0}^{\infty} H^{(n)}$ graded as a vector space is called a *graded bialgebra* when the grading # is compatible with both the algebra and the coalgebra structures:

$$H^{(n)} H^{(m)} \subseteq H^{(n+m)} \quad \text{and} \quad \Delta H^{(n)} \subseteq \bigoplus_{k=0}^{n} H^{(n-k)} \otimes H^{(k)}.$$

That is, grading $H \otimes H$ in the obvious way, m and Δ are homogeneous maps of degree zero. A graded bialgebra is filtered in the obvious way. Most often we work with graded bialgebras *of finite type*, for which the $H^{(n)}$ are finite dimensional; and, at any rate, all graded bialgebras are direct limits of subbialgebras of finite type [23, Proposition 4.13].

Two main "classical" *examples* of bialgebras, respectively commutative and cocommutative, are the space of representative functions on a compact group and the enveloping algebra of a Lie algebra.

Example 2.1. Let G be a compact topological group (most often, a Lie group). The Peter–Weyl theorem shows that any unitary irreducible representation π of G is finite-dimensional, any matrix element $f(x) := \langle u, \pi(x)v \rangle$ is a representative function on G, and the vector space $\mathcal{R}(G)$ generated by these matrix elements is a dense $*$-subalgebra of $C(G)$. Elements of this space can be characterized as those continuous functions $f : G \to \mathbb{C}$ whose translates $t \triangleright f : x \mapsto f(xt)$, for all $t \in G$, generate a finite-dimensional subspace of $C(G)$. The theorem says, in other words, that nothing of consequence is lost in compact group theory by studying just $\mathcal{R}(G)$. As we know already, the commutative algebra $\mathcal{R}(G)$ is a coalgebra, with operations

$$\Delta f(x, y) := f(xy); \quad \eta(f) := f(1_G). \tag{2.6}$$

Example 2.2. The *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the quotient of the tensor algebra $T(\mathfrak{g})$ —with $T^0(\mathfrak{g}) \simeq \mathbb{C}$— by the two sided ideal I generated by the elements $XY - YX - [X, Y]$, for all $X, Y \in \mathfrak{g}$. The word "universal" is appropriate because any Lie algebra homomorphism $\psi : \mathfrak{g} \to A$, where A is a unital associative algebra, extends uniquely to a unital algebra homomorphism $\mathcal{U}\psi : \mathcal{U}(\mathfrak{g}) \to A$.

A coproduct and counit are defined first on elements of $\mathfrak{g}$ by

$$\Delta X := X \otimes 1 + 1 \otimes X, \tag{2.7}$$

and $\eta(X) := 0$. These linear maps on $\mathfrak{g}$ extend to homomorphisms of $T(\mathfrak{g})$; for instance,

$$\Delta(XY) = \Delta X \Delta Y = XY \otimes 1 + X \otimes Y + Y \otimes X + 1 \otimes XY.$$

It follows that the tensor algebra on any vector space is a (graded, connected) bialgebra. Now

$$\Delta(XY - YX - [X,Y]) = (XY - YX - [X,Y]) \otimes 1 + 1 \otimes (XY - YX - [X,Y]),$$

Thus I is also a coideal (clearly $\eta(I) = 0$, too) and if $q : T(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ is the quotient map, then $I \subseteq \ker(q \otimes q)\Delta$; thus $(q \otimes q)\Delta$ induces a coproduct on the quotient $\mathcal{U}(\mathfrak{g})$, that becomes an irreducible bialgebra. From (2.7) and the definition of I, it is seen that $\mathcal{U}(\mathfrak{g})$ is *cocommutative*. Note also that it is a graded coalgebra. The coradical filtering of $\mathcal{U}(\mathfrak{g})$ is just the obvious filtering by degree [21].

When $\mathfrak{g}$ is the Lie algebra of G, both previous constructions are mutually dual in a sense that will be studied soon.

3 Primitive and indecomposable elements

Definition 3.1. An element a in a bialgebra H is said to be (1-)primitive when

$$\Delta a = a \otimes 1 + 1 \otimes a.$$

Primitive elements of H form a vector subspace $P(H)$, which is seen at once to be a Lie subalgebra of H with the ordinary bracket

$$[a,b] := ab - ba.$$

For instance, elements of $\mathfrak{g}$ inside $\mathcal{U}(\mathfrak{g})$ are primitive by definition; and there are no others.

Denote by $H_+ := \ker \eta$ the augmentation ideal of H. If for some $a \in H$, we can find $a_1, a_2 \in H_+$ with $\Delta a = a_1 \otimes 1 + 1 \otimes a_2$, then by the counit property $a = (\mathrm{id} \otimes \eta)\Delta a = a_1$, and similarly $a = a_2$; so a is primitive. In other words,

$$P(H) = \Delta^{-1}(H_+ \otimes 1 + 1 \otimes H_+). \tag{3.1}$$

By the counit property as well, $a \in \ker \eta$ is automatic for primitive elements. If $\ell : H \to K$ is a bialgebra morphism, and $a \in P(H)$ then $\Delta_K \ell(a) = (\ell \otimes \ell)\Delta_H a = \ell(a) \otimes 1 + 1 \otimes \ell(a)$; thus the restriction of ℓ to $P(H)$ defines a Lie algebra map $P(\ell) : P(H) \to P(K)$. If ℓ is injective, obviously so is $P(\ell)$.

Proposition 3.1. *Let H be a connected bialgebra. Write $H_{+n} := H_+ \cap H_n$ for all n. If $a \in H_{+n}$, then*

$$\Delta a = a \otimes 1 + 1 \otimes a + y, \quad \textit{where} \quad y \in H_{+n-1} \otimes H_{+n-1}. \tag{3.2}$$

Moreover, $H_1 \subseteq \mathbb{C}1 \oplus P(H)$.

Proof. Write $\Delta a = a \otimes 1 + 1 \otimes a + y$. If $a \in H_+$, then $(\mathrm{id} \otimes \eta)(y) = (\mathrm{id} \otimes \eta)\Delta a - a = 0$ and similarly $(\eta \otimes \mathrm{id})(y) = 0$ by the counity properties (2.1). Therefore $y \in H_+ \otimes H_+$, and thus

$$y \in \sum_{i=0}^{n} H_{+i} \otimes H_{+n-i}.$$

As $H_{+0} = (0)$, the first conclusion follows.

Thus, when H is a connected bialgebra, $H = H_0 \oplus \ker \eta$. If $c \in H_1$, we can write $c = \mu 1 + a$, with $\mu \in \mathbb{C}$ and $a \in H_{+1}$. Now, $\Delta a = a \otimes 1 + 1 \otimes a + \lambda(1 \otimes 1)$ for some $\lambda \in \mathbb{C}$; and $a + \lambda 1$ is primitive, thus $c = (\mu - \lambda)1 + (a + \lambda 1)$ lies in $\mathbb{C}\,1 \oplus P(H)$. ◻

In a connected bialgebra, the primitive elements together with the scalars constitute the "second step" of the coradical filtering. If H is graded, some $H^{(k)}$ might be zero; but if k is the smallest nonzero integer such that $H^{(k)} \neq 0$, compatibility of the coproduct with the grading ensures $H^{(k)} \subseteq P(H)$.

In view of (3.2), it will be very convenient to consider the reduced coproduct Δ' defined on H_+ by

$$\Delta' a := \Delta a - a \otimes 1 - 1 \otimes a =: \sum a'_{(1)} \otimes a'_{(2)}. \tag{3.3}$$

In other words, Δ' is the restriction of Δ as a map $\ker \eta \to \ker \eta \otimes \ker \eta$; and a is primitive if and only if it lies in the kernel of Δ'. Coassociativity of Δ' (and cocommutativity, when it holds) is easily obtained from the coassociativity of Δ; and we write

$$\Delta'^n a = \sum a'_{(1)} \otimes \cdots \otimes a'_{(n)}.$$

The reduced coproduct is not an algebra homomorphism:

$$\Delta'(ab) = \Delta' a\, \Delta' b + (a \otimes 1 + 1 \otimes a)\Delta' b + \Delta' a\,(b \otimes 1 + 1 \otimes b) + a \otimes b + b \otimes a. \tag{3.4}$$

Let $p : H \to H_+$ be the projection defined by $p(a) := (\mathrm{id} - u\eta)a = a - \eta(a)\,1$; then the previous result (3.2) is reformulated as $(p \otimes p)\Delta = \Delta' p$; more generally, it follows that

$$U_n := (p \otimes p \otimes \cdots \otimes p)\Delta^n = \Delta'^n p, \tag{3.5}$$

with $n+1$ factors in $p \otimes p \otimes \cdots \otimes p$. This humble equality plays a decisive role in this work.

Theorem 3.2. *Consider now the tensor product $H \otimes K$ of two connected graded bialgebras H and K, which is also a connected graded bialgebra, with the coproduct (2.2). Then*

$$P(H \otimes K) = P(H) \otimes 1 + 1 \otimes P(K) \cong P(H) \oplus P(K) \tag{3.6}$$

in $H \otimes K$.

Proof. The last identification comes from the obvious $P(H) \otimes 1 \cap 1 \otimes P(K) = (0)$ in $H \otimes K$. Let $a \in P(H)$ and $b \in P(K)$, then

$$\begin{aligned} \Delta_\otimes(a \otimes 1 + 1 \otimes b) &= (\mathrm{id} \otimes \tau \otimes \mathrm{id})\big((a \otimes 1 + 1 \otimes a) \otimes (1 \otimes 1) + (1 \otimes 1) \otimes (b \otimes 1 + 1 \otimes b)\big) \\ &= (a \otimes 1 + 1 \otimes b) \otimes (1 \otimes 1) + (1 \otimes 1)(a \otimes 1 + 1 \otimes b), \end{aligned}$$

so $a\otimes 1+1\otimes b\in P(H\otimes K)$.

On the other hand, letting $x\in P(H\otimes K)$:

$$\Delta_{\otimes}x = x\otimes 1\otimes 1 + 1\otimes 1\otimes x. \tag{3.7}$$

Since $H\otimes K$ is a graded bialgebra we can write

$$x = a^0\otimes 1 + 1\otimes b^0 + \sum_{p,q\geq 1} a^p\otimes b^q,$$

where $a^0\in H_+$, $b^0\in K_+$, $a^p\in H^{(p)}$, $b^q\in K^{(q)}$. By (3.3) $\Delta_H a^i = a^i\otimes 1+1\otimes a^i+\sum {a^i}'_{(1)}\otimes {a^i}'_{(2)}$ for $i\geq 0$ and $\Delta_K b^j = b^j\otimes 1+1\otimes b^j+\sum {b^j}'_{(1)}\otimes {b^j}'_{(2)}$ for $j\geq 0$. Thus, after a careful book-keeping, it follows that

$$\begin{aligned}\Delta_{\otimes}x = x\otimes 1\otimes 1 + 1\otimes 1\otimes x + \sum {a^0}'_{(1)}\otimes 1\otimes {a^0}'_{(2)}\otimes 1 + \sum 1\otimes {b^0}'_{(1)}\otimes 1\otimes {b^0}'_{(2)}\\ + \sum 1\otimes b^q\otimes a^p\otimes 1 + \sum a^p\otimes 1\otimes 1\otimes b^q + R,\end{aligned}$$

where R is a sum of terms of the form $c\otimes d\otimes e\otimes f$ where at least three of the following conditions hold: $c\in H_+$, $d\in K_+$, $e\in H_+$, or $f\in K_+$. A comparison with (3.7) then gives $R=0$ and

$$\sum {a^0}'_{(1)}\otimes 1\otimes {a^0}'_{(2)}\otimes 1 = \sum 1\otimes {b^0}'_{(1)}\otimes 1\otimes {b^0}'_{(2)} = \sum a^p\otimes 1\otimes 1\otimes b^q = 0.$$

The vanishing of the third sum gives that $\sum a^p\otimes b^q = 0$, so $x = a^0\otimes 1+1\otimes b^0$, whereas the vanishing of the first and second sums gives $\sum {a^0}'_{(1)}\otimes {a^0}'_{(2)} = 0$ and $\sum {b^0}'_{(1)}\otimes {b^0}'_{(2)} = 0$, that is $a^0\in P(H)$ and $b^0\in P(K)$. □

Definition 3.2. In a connected Hopf algebra H_+ is the unique maximal ideal, and the H_+^m for $m\geq 1$ form a descending sequence of ideals. The graded algebra $Q(H) := \mathbb{C}1\oplus H_+/H_+^2$ is called the set of *indecomposables* of H.

Algebraically, the H-module $Q(H) := H_+/H_+^2$ is the tensor product of H_+ and $\mathbb{C}$ by means of $\eta: H\to\mathbb{C}$ [24, Section 2.4]. We spell this out. Given M and N, respectively a right H-module by an action ϕ_M and a left H-module by an action ϕ_N, the vector space whose elements are finite sums $\sum_j m_j\otimes n_j$ with $m_j\in M$ and $n_j\in N$, subject to the relations

$$m\phi_M(a)\otimes n = m\otimes\phi_N(a)n, \quad \text{for each} \quad a\in H;$$

is denoted $M\otimes_H N$. Note now that $\mathbb{C}$ is a (left or right) H-module by $a\triangleright\lambda = \lambda\triangleleft a := \eta(a)\lambda$. Also H_+ is a (right or left) H-module. Thus, the tensor product $H_+\otimes_H\mathbb{C}$ of H_+ and $\mathbb{C}$ over H by means of η is the graded vector space whose elements are finite sums $\sum_j s_j\otimes\beta_j$ with $s_j\in H_+$ and $\beta_j\in\mathbb{C}$, subject to the relations

$$sa\otimes\beta = s\otimes\eta(a)\beta, \quad \text{for each} \quad a\in A;$$

if $s\in H_+^2$, then $s=0$ in $H_+\otimes_H\mathbb{C}$. Similarly one defines $\mathbb{C}\otimes_H H_+\simeq H_+\otimes_H\mathbb{C}$. Notice $\mathbb{C}\otimes_H N\simeq N/H_+N$.

The quotient algebra morphism $H\to\mathbb{C}1\oplus H_+/H_+^2$ restricts to a graded linear map $q_H: P(H)\to H_+/H_+^2$; clearly this map will be one-to-one iff $P(H)\cap H_+^2 = 0$, and onto iff $P(H)+H_+^2 = H_+$.

Proposition 3.3. *If the relation*

$$P(H) \cap H_+^2 = (0),$$

implying that primitive elements are all indecomposable, holds, then H is commutative.

Proof. The commutator $[a, b]$ for a, b primitive belongs both to $P(H)$ and H_+^2; therefore it must vanish. Proceeding by induction on the degree, one sees that the bracket vanishes for general elements of H: indeed, let $[a, b] = 0$ for all $a \in H_p, b \in H_q$ and consider $[a, b]$ for, say, $a \in H_p, b \in H_{q+1}$. A straightforward computation, using (3.4), shows that $\Delta'[a, b] = 0$; so $[a, b]$ is primitive; and hence zero. □

If $\ell : H \to K$ is an onto bialgebra map, then the induced map $Q(\ell) : Q(H) \to Q(K)$ is onto.

(The terminology "indecomposable elements" used for instance in this section, is somewhat sloppy, as in fact the indecomposables are defined only modulo H_+^2. However, to avoid circumlocutions, we shall still use it often, trusting the reader not be confused.)

4 Dualities

Consider the space C^* of all linear functionals on a coalgebra C. One identifies $C^* \otimes C^*$ with a subspace of $(C \otimes C)^*$ by defining

$$f \otimes g(a \otimes b) = f(a)g(b), \tag{4.1}$$

where $a, b \in C; f, g \in C^*$. Then C^* becomes an algebra with product the restriction of Δ^t to $C^* \otimes C^*$; with ^{t} denoting transposed maps. We have already seen this, as this product is just the convolution product:

$$fg(a) = \sum f(a_{(1)})g(a_{(2)});$$

and $u_{C^*}1 = \eta$ means the unit is η^t.

It is a bit harder to obtain a coalgebra by dualization of an algebra A —and thus a bialgebra by a dualization of another. The reason is that m^t takes A^* to $(A \otimes A)^*$ and there is no natural mapping from $(A \otimes A)^*$ to $A^* \otimes A^*$; if A is not finite dimensional, the inverse of the identification in (4.1) does not exist, as the first of these spaces is larger than the second.

In view of these difficulties, the pragmatic approach is to focus on the (strict) pairing of two bialgebras H and K, where each may be regarded as included in the dual of the other. That is to say, we write down a bilinear form $\langle a, f\rangle := f(a)$ for $a \in H$ and $f \in K$ with implicit inclusions $K \hookrightarrow H^*, H \hookrightarrow K^*$. The transposing of operations between the two bialgebras boils down to the following four relations, for $a, b \in H$ and $f, g \in K$:

$$\begin{gathered}\langle ab, f\rangle = \langle a \otimes b, \Delta_K f\rangle, \qquad \langle a, fg\rangle = \langle \Delta_H a, f \otimes g\rangle, \\ \eta_H(a) = \langle a, 1\rangle, \quad \text{and} \quad \eta_K(f) = \langle 1, f\rangle.\end{gathered} \tag{4.2}$$

The nondegeneracy conditions allowing us to assume that $H \hookrightarrow K^*$ and $K \hookrightarrow H^*$ are: (i) $\langle a, f\rangle = 0$ for all $f \in K$ implies $a = 0$, and (ii) $\langle a, f\rangle = 0$ for all $a \in H$ implies $f = 0$. It is plain that H is a commutative bialgebra iff K is cocommutative.

The two examples at the end of Section 3 are tied up by duality as follows. Let G be a compact connected Lie group whose Lie algebra is $\mathfrak{g}$. The function algebra $\mathcal{R}(G)$ is a commutative bialgebra, whereas $\mathcal{U}(\mathfrak{g})$ is a cocommutative bialgebra. Moreover, representative functions are smooth [25]. On identifying $\mathfrak{g}$ with the space of left-invariant vector fields on the group manifold G, we can realize $\mathcal{U}(\mathfrak{g})$ as the algebra of left-invariant differential operators on G. If $X \in \mathfrak{g}$, and $f \in \mathcal{R}(G)$, we define

$$\langle X, f\rangle := Xf(1) = \frac{d}{dt}\bigg|_{t=0} f(\exp tX),$$

and more generally, $\langle X_1 \dots X_n, f\rangle := X_1(\cdots(X_n f)\cdots)(1)$; we also set $\langle 1, f\rangle := f(1)$. This yields a duality between $\mathcal{R}(G)$ and $\mathcal{U}(\mathfrak{g})$. Indeed, the Leibniz rule for vector fields, namely $X(fh) = (Xf)h + f(Xh)$, gives

$$\begin{aligned} \langle X, fh\rangle &= Xf(1)h(1) + f(1)Xh(1) = (X \otimes 1 + 1 \otimes X)(f \otimes h)(1 \otimes 1) \\ &= \Delta X(f \otimes h)(1 \otimes 1) = \langle \Delta X, f \otimes h\rangle; \end{aligned} \tag{4.3}$$

while

$$\begin{aligned} \langle X \otimes Y, \Delta f\rangle &= \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} (\Delta f)(\exp tX \otimes \exp sY) = \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} f(\exp tX \exp sY) \\ &= \frac{d}{dt}\bigg|_{t=0} (Yf)(\exp tX) = X(Yf)(1) = \langle XY, f\rangle. \end{aligned}$$

The necessary properties are easily checked. Relation (4.3) shows that $\Delta X = X \otimes 1 + 1 \otimes X$ encodes the Leibniz rule for vector fields.

► A more normative approach to duality is to consider instead the subspace A° of A^* made of functionals whose kernels contain an ideal of finite codimension in A. Alternatively, A° can be defined as the set of all functionals $f \in A^*$ for which there are functionals $g_1, \dots, g_r; h_1, \dots h_r$ in A^* such that $f(ab) = \sum_{j=1}^r g_j(a)h_j(b)$; that is to say, A° is the set of functions on the monoid of A that are both linear and representative. It can be checked that m^t maps A° to $A^\circ \otimes A^\circ$, and so $(A^\circ, m^t|_{A^\circ}, u^t)$ defines a coalgebra structure on A°, with $\eta_{A^\circ}(f) = f(1_A)$.

Given a bialgebra (H, m, u, Δ, η), one then sees that $(H^\circ, \Delta^t, \eta^t, m^t, u^t)$ is again a bialgebra, called the *finite dual* or Sweedler dual of H; the contravariant functor $H \mapsto H^\circ$ defines a duality of the category of bialgebras into itself. In the previous case of a dual pair (H, K), we actually have $K \hookrightarrow H^\circ$ and $H \hookrightarrow K^\circ$.

If G is a group, $\mathbb{C}G$ denotes the group algebra of G, that is, the complex vector space freely generated by G as a basis, with product defined by extending linearly the group multiplication of G, so 1_G is the unit in $\mathbb{C}G$. Endowed with the coalgebra structure given by (the linear extensions of) $x \to x \otimes x$ and $\eta(x) := 1$, it is a cocommutative bialgebra. In view of the discussion of Section 1, $\mathcal{R}(G)$ is the Sweedler dual of $\mathbb{C}G$.

In a general bialgebra H, a nonzero element g is called *grouplike* if $\Delta g := g \otimes g$; for it $\eta(g) = 1$. The product of grouplike elements is grouplike.

The *characters* of a bialgebra H (further discussed in the following section) are by definition the multiplicative elements of H^*. They belong to H°, as for them $m^t f = f \otimes f$. Then, the set $G(H^\circ)$ of grouplike elements of H° coincides with the set of characters of H. If $H = \mathcal{R}(G)$, the map $x \to x^0$ given by $x^0(f) = f(x)$ gives a map $G \to G(\mathcal{R}^0(G))$; it will become clear soon that it is a homomorphism of groups.

Among the interesting elements of the dual of a bialgebra, there are also the derivations or *infinitesimal characters*: these are linear functionals δ satisfying

$$\delta(ab) = \delta(a)\eta(b) + \eta(a)\delta(b) \quad \text{for all} \quad a, b \in H.$$

This entails $\delta(1) = 0$. The previous relation can also be written as $m^t(\delta) = \delta \otimes \eta + \eta \otimes \delta$, which shows that infinitesimal characters belong to H° as well, and are primitive there. Thus the Lie algebra of primitive elements of H° coincides with the Lie algebra $\mathrm{Der}_\eta H$ of infinitesimal characters.

► When A is a graded algebra of finite type, one can consider the space

$$A' := \bigoplus_{n \geq 0} A^{(n)^*},$$

where $\langle A^{(n)^*}, A^{(m)} \rangle = 0$ for $n \neq m$, and there is certainly no obstacle to define the graded coproduct on homogeneous components of A' as the transpose of

$$m : \sum_{k=0}^{n} A^{(k)} \otimes A^{(n-k)} \to A^{(n)}.$$

If the algebra above is a bialgebra H, one obtains in this way a subbialgebra H' of H°, called the *graded dual* of H. Certainly (H, H') form a nondegenerate dual pair. Note $H'' = H$.

If I is a linear subspace of H graded of finite type, we denote by $I^\perp$ its orthogonal in H'. Naturally $I^{\perp\perp} = I$.

Proposition 4.1. *For a graded connected bialgebra of finite type H,*

$$P(H')^\perp = \mathbb{C}1 \oplus H_+^2. \tag{4.4}$$

Proof. Let $p \in P(H')$. Using (4.2) we obtain $\langle p, 1 \rangle = \eta_{H'}(p) = 0$. Also, for $a_1, a_2 \in H_+$

$$\langle p, a_1 a_2 \rangle = \langle p \otimes 1 + 1 \otimes p, a_1 \otimes a_2 \rangle = \eta_H(a_2)\langle p, a_1 \rangle + \eta_H(a_1)\langle p, a_2 \rangle = 0.$$

Thus $\mathbb{C}1 \oplus H_+^2 \subseteq P(H')^\perp$. Use of (4.2) again easily gives $(\mathbb{C}1 \oplus H_+^2)^\perp \subseteq P(H')$. Then $P(H')^\perp \subseteq \mathbb{C}1 \oplus H_+^2$, and therefore $P(H')^\perp = \mathbb{C}1 \oplus H_+^2$. □

In general $H' \subsetneq H^\circ$. As an example, let the polynomial algebra $H = \mathbb{C}[Y]$ be endowed with the coproduct associated to translation,

$$\Delta Y^n = \sum_{k=0}^{n} \binom{n}{k} Y^k \otimes Y^{n-k}, \tag{4.5}$$

which is a homomorphism in view of the Vandermonde identity; this is the so-called *binomial bialgebra*. Consider the elements $f^{(n)}$ of H^* defined by

$$\langle f^{(n)}, Y^m \rangle = \delta_{nm}.$$

Obviously any $\phi \in H^*$ can be written as

$$\phi = \sum_{n \geq 0} c_n f^{(n)},$$

where the complex numbers c_n are given by $c_n = \langle \phi, Y^n \rangle$. Now, write $f := f^{(1)}$; notice that f is primitive since

$$\langle \Delta^t f, Y^n \otimes Y^m \rangle = \langle f, Y^n\, Y^m \rangle = \langle f, Y^{n+m} \rangle = \delta_{1n}\delta_{0m} + \delta_{0n}\delta_{1m} = \langle f \otimes \eta + \eta \otimes f, Y^n \otimes Y^m \rangle.$$

On the other hand,

$$\langle f^2, Y^n \rangle = \langle f \otimes f, \Delta Y^n \rangle = \sum_{k=0}^{n} \binom{n}{k} f(Y^{n-k}) f(Y^k).$$

Since $f(Y^k) = 0$ unless $k = 1$, then $f^2 = 2! f^{(2)}$. A simple induction entails $f^n = n! f^{(n)}$. Thus, ϕ can be written as $\phi = \sum_{n \geq 0} d_n f^n$ with $d_n = \frac{c_n}{n!}$; so $H^* \simeq \mathbb{C}[[f]]$, the algebra of formal (exponential, if you wish) power series in f.

It is also rather clear what $\mathbb{C}[Y]'$ is: in terms of the $f^{(n)}$ it is the divided powers bialgebra, namely the bialgebra with basis $f^{(n)}$ for $n \geq 0$, where the product and coproduct are, respectively, given by

$$f^{(n)} f^{(m)} = \binom{n+m}{n} f^{(n+m)} \quad \text{and} \quad \Delta f^{(n)} = \sum_{k=0}^{n} f^{(n-k)} \otimes f^{(k)}.$$

We can conclude that $\mathbb{C}[Y]' = \mathbb{C}[f]$.

Consider now ϕ in the Sweedler dual H°. By definition there exists some (principal) ideal $I = (p(Y))$, with p a (monic) polynomial, such that $\phi(I) = 0$. Therefore we shall first describe all the ϕ that vanish on a given ideal I. We start with the case $p(Y) = (Y - \lambda)^r$ for some $\lambda \in \mathbb{C}$ and $r \in \mathbb{N}$. Let $\phi_\lambda := \sum_{n \geq 0} \lambda^n f^n = \exp(\lambda f)$. The set $\{\, (Y-\lambda)^m : m \geq 0 \,\}$ is also a basis of H. As before, one can consider the elements $g_\lambda^{(m)}$ of H^* defined by $\langle g_\lambda^{(m)}, (Y-\lambda)^l \rangle = \delta_{lm}$. We are going to prove that $g_\lambda^{(m)} = f^{(m)}\, \phi_\lambda$. Indeed

$$\begin{aligned}
\langle f^{(m)} \phi_\lambda, (Y-\lambda)^l \rangle &= \langle f^{(m)} \phi_\lambda, \sum_{k=0}^{l} \binom{l}{k} (-\lambda)^{l-k} Y^k \rangle \\
&= \sum_{k=0}^{l} \binom{l}{k} (-\lambda)^{l-k} \langle f^{(m)} \otimes \phi_\lambda, \Delta Y^k \rangle \\
&= \sum_{k=0}^{l} \sum_{j=0}^{k} \binom{l}{k} \binom{k}{j} (-\lambda)^{l-k} \langle f^{(m)} \otimes \phi_\lambda, Y^j \otimes Y^{k-j} \rangle \\
&= \sum_{k=0}^{l} \sum_{j=0}^{k} \binom{l}{k} \binom{k}{j} (-\lambda)^{l-k} f^{(m)}(Y^j) \phi_\lambda(Y^{k-j}).
\end{aligned}$$

Since $f^{(m)}(Y^j)$ vanish if $m > j$, it is clear that $\langle f^{(m)}\phi_\lambda, (Y-\lambda)^l\rangle = 0$ if $m > l$. If $m = l$ only one term survives and $\langle f^{(m)}\phi_\lambda, (Y-\lambda)^m\rangle = 1$. On the other hand, if $m < l$ then

$$\begin{aligned}\langle f^{(m)}\phi_\lambda, (Y-\lambda)^l\rangle &= \sum_{k=m}^{l} \binom{l}{k}\binom{k}{m}(-1)^{l-k}\lambda^{l-m} \\ &= \frac{\lambda^{l-m}}{m!}\sum_{k=m}^{l}\binom{l}{k}(-1)^{l-k}k(k-1)\cdots(k-m+1).\end{aligned}$$

Successive derivatives of the binomial identity give

$$l(l-1)\cdots(l-m+1)(x-1)^{l-m} = \sum_{k=m}^{l}\binom{l}{k}(-1)^{l-k}k(k-1)\cdots(k-m+1)x^{k-m},$$

therefore

$$0 = \sum_{k=m}^{l}\binom{l}{k}(-1)^{l-k}k(k-1)\cdots(k-m+1),$$

and we conclude that $\langle f^{(m)}\phi_\lambda, (Y-\lambda)^l\rangle = \delta_{lm}$. Any $\phi \in H^\circ$ can be written as

$$\phi = \sum_{m\geq 0} e_m f^{(m)}\,\phi_\lambda.$$

It follows that those ϕ satisfying $\langle\phi, \big((Y-\lambda)^r\big)\rangle = 0$ are of the form

$$\phi = \sum_{m=0}^{r-1} e_m f^{(m)}\,\phi_\lambda; \tag{4.6}$$

we can think of them as linear recursive sequences [26]. In general, $p(Y) = \prod_{i=1}^{s}(Y-\lambda_i)^{r_i}$, and the ϕ satisfying $\langle\phi, (p(Y))\rangle = 0$ will be linear combinations of terms as in (4.6). Thus

$$H^\circ = \Big\{\sum e_{ij}\, f^{(i)}\,\phi_{\lambda_j} \,:\, e_{ij},\,\lambda_j \in \mathbb{C}\Big\}.$$

Furthermore, since $\phi_\lambda(Y^n) = \lambda^n = \phi_\lambda(Y)^n$, it ensues that $\phi_\lambda \in \mathrm{Hom}_{\mathrm{alg}}(H,\mathbb{C}) = G(H^\circ)$. Conversely, if ϕ is grouplike and $\phi(Y) = \lambda$, then

$$\langle\phi, Y^2\rangle = \langle\Delta^t\phi, Y\otimes Y\rangle = \langle\phi\otimes\phi, Y\otimes Y\rangle = \big(\phi(Y)\big)^2 = \lambda^2,$$

and so on. It follows that $\phi = \phi_\lambda$, in other words the grouplike elements of H° are precisely the exponentials $\phi_\lambda = \exp(\lambda f)$, and since $\phi_\lambda\phi_\mu = \phi_{\lambda+\mu}$, we conclude that $G(\mathbb{C}[Y]^\circ) \cong (\mathbb{C}, +)$. In summary, H° can be rewritten as

$$\mathbb{C}[Y]^\circ = H' \otimes G(\mathbb{C}[Y]^\circ).$$

An interpretation of $\mathbb{C}[Y]^\circ$ as the space of proper rational functions was given in [27].

In general, when H is commutative, $H' = \mathcal{U}\big(P(H^\circ)\big) = \mathcal{U}(\operatorname{Der}_\eta H)$. This is an instance of the Milnor–Moore theorem [23], on which we shall dwell a bit in Section 14. There are no grouplike elements in H', apart from η. The characters of a graded connected commutative bialgebra H can be recovered as the set of grouplike elements in the completion H° of the algebra $H' = \mathcal{U}(\operatorname{Der}_\eta H)$. The sets $\sum_{k \geq m} (\operatorname{Der}_\eta H)^k$, for $m = 1, 2, \dots$ form a basis of neighbourhoods of 0 for a vector space topology on $\mathcal{U}(\operatorname{Der}_\eta H)$; the grading properties mean that the Hopf algebra operations are continuous for this H_+-adic topology. An element of the completion of $\mathcal{U}(\operatorname{Der}_\eta H)$ is a series $\sum_{k \geq 0} z_k$ with $z_k \in (\operatorname{Der}_\eta H)^k$ for each k. As $\delta^{m+1}(a) = 0$ if $a \in (\ker \eta)^m$, the element $\exp(\delta) \in H^\circ$ makes sense for each $\delta \in \operatorname{Der}_\eta H$; and $\exp \delta \in G(H^\circ)$ since

$$\Delta \exp \delta = \exp \Delta\delta = \exp(\eta \otimes \delta + \delta \otimes \eta) = \exp \delta \otimes \exp \delta,$$

by continuity of Δ. This exponential map is a bijection between $\operatorname{Der}_\eta H$ and $G(H^\circ)$ [28, 29], with inverse

$$\log \mu = -\sum_{n=1}^{\infty} \frac{(\eta - \mu)^n}{n}.$$

That is, the group $G(H^\circ)$ is a pro-unipotent Lie group, and one regards the commutative Hopf algebra H as an algebra of affine coordinates on that group. As a general paradigm, the dual of $\mathcal{R}(G)$, where G is a Lie group with Lie algebra $\mathfrak{g}$, is of the form $\mathcal{U}(\mathfrak{g}) \otimes \mathbb{C}G$ as a coalgebra; as an algebra, it is the smash product of $\mathcal{U}(\mathfrak{g})$ and $\mathbb{C}G$. Smash products will be briefly discussed in Section 9; here they are just a fancy name for the adjoint action of G on $\mathfrak{g}$.

► Finally we can come back to our introductory remarks in the second half of Section 1. The whole idea of harmonic analysis is to "linearize" the action of a group —or a Lie algebra— on a space X by switching the attention, as it is done in noncommutative geometry, from X itself to spaces of functions on X, and the corresponding operator algebras. Now, linear representations of groups and of Lie algebras can be tensor multiplied, whereas general representations of algebras cannot. Thus from the standpoint of representation theory, the main role of the coproduct is to ensure that the action on H-modules propagates to their tensor products. To wit, if a bialgebra H acts on V and W, then it will also act on $V \otimes W$ by

$$h_\otimes \cdot (v \otimes w) = h_{(1)} \cdot v \otimes h_{(2)} \cdot w,$$

for all $h \in H, v \in V, w \in W$. In other words, if $\phi_V : H \otimes V \to V, \phi_W : H \otimes W \to W$ denote the actions, then

$$\phi_{V \otimes W} := (\phi_V \otimes \phi_W)(\operatorname{id} \otimes \tau \otimes \operatorname{id})(\Delta \otimes \operatorname{id} \otimes \operatorname{id}). \tag{4.7}$$

Indeed,

$$h_\otimes \cdot (k_\otimes \cdot (v \otimes w))) = (h_{(1)} k_{(1)}) \cdot v \otimes (h_{(2)} k_{(2)}) \cdot w = (hk)_\otimes \cdot (v \otimes w),$$

and moreover

$$1_\otimes \cdot (v \otimes w) := 1 \cdot v \otimes 1 \cdot w = v \otimes w$$

as it should. In this view, the product structure on the module of all representations of a group comes from the comultiplication: $g \mapsto g \otimes g$, for $g \in G$; in the case of representations of Lie algebras, where a $\mathfrak{g}$-module is the same as a module for the bialgebra $\mathcal{U}(\mathfrak{g})$, there is analogously a product. Note that $V \otimes W \not\simeq W \otimes V$ in an arbitrary H-module category.

Definition 4.1. Dually, we envisage *corepresentations* of coalgebras C. A right corepresentation or coaction of C on a vector space V is a linear map $\gamma : V \to V \otimes C$ such that $(\mathrm{id} \otimes \Delta)\gamma = (\gamma \otimes \mathrm{id})\gamma$ and $\mathrm{id} = (\mathrm{id} \otimes \eta)\gamma$. These conditions are expressed by the commutativity of the diagrams

$$
\begin{tikzcd}
V \otimes C \otimes C & V \otimes C \arrow[l, "\mathrm{id}\otimes\Delta"'] \\
V \otimes C \arrow[u, "\gamma\otimes\mathrm{id}"] & V, \arrow[l, "\gamma"'] \arrow[u, "\gamma"']
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, leftrightarrow] & V \otimes \mathbb{C} \\
V \arrow[u, "\mathrm{id}"] \arrow[r, "\gamma"] & V \otimes C, \arrow[u, "\mathrm{id}\otimes\eta"']
\end{tikzcd}
$$

obtained by the process of reversing arrows from the axioms of a left representation of an algebra $\phi :\ A \otimes V \to V$:

$$
\begin{tikzcd}
A \otimes A \otimes V \arrow[r, "m\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\phi"'] & A \otimes V \arrow[d, "\phi"] \\
A \otimes V \arrow[r, "\phi"] & V,
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, leftrightarrow] & \mathbb{C} \otimes V \arrow[d, "u\otimes\mathrm{id}"] \\
V \arrow[u, "\mathrm{id}"] & A \otimes V. \arrow[l, "\phi"']
\end{tikzcd}
$$

We use the convenient notation

$$\gamma(v) = \sum v^{(\underline{1})} \otimes v^{(2)},$$

with $v^{(\underline{1})} \in V$, $v^{(2)} \in C$. So, for instance, the first defining relation becomes

$$\sum v^{(\underline{1})(\underline{1})} \otimes v^{(\underline{1})(2)} \otimes v^{(2)} = \sum v^{(\underline{1})} \otimes v^{(2)}_{(1)} \otimes v^{(2)}_{(2)}.$$

Representations of algebras come from corepresentations of their predual coalgebras: if γ is a corepresentation as above, then

$$h \cdot v := (\mathrm{id} \otimes h)\gamma(v) \quad \text{or} \quad h \cdot v := \sum v^{(\underline{1})} h(v^{(2)})$$

defines a representation of C^*. Indeed

$$\begin{aligned}
h_1 \cdot (h_2 \cdot v) = h_1 \cdot \Big(\sum v^{(\underline{1})} h_2(v^{(2)})\Big) &= \sum v^{(\underline{1})(\underline{1})} h_1(v^{(\underline{1})(2)}) h_2(v^{(2)}) \\
= \sum v^{(\underline{1})} h_1(v^{(2)}_{(1)}) h_2(v^{(2)}_{(2)}) &= \sum v^{(\underline{1})} h_1 h_2(v^{(2)}) = (h_1 h_2) \cdot v.
\end{aligned}$$

Now we use the product structure: if a bialgebra H coacts on V and W, it coacts on the tensor product $V \otimes W$ by

$$\gamma_\otimes(v \otimes w) = \sum v^{(\underline{1})} \otimes w^{(\underline{1})} \otimes v^{(2)} w^{(2)}, \quad v \in V, w \in W;$$

that is

$$\gamma_{V\otimes W} := (\mathrm{id} \otimes \mathrm{id} \otimes m)(\mathrm{id} \otimes \tau \otimes \mathrm{id})(\gamma_V \otimes \gamma_W),$$

in complete parallel to (4.7). The required corepresentation properties are easily checked as well. For instance,

$$\begin{aligned}
&\sum (v \otimes w)^{(\underline{1})(\underline{1})} \otimes (v \otimes w)^{(\underline{1})(2)} \otimes (v \otimes w)^{(2)} \\
&\quad = \sum v^{(\underline{1})(\underline{1})} \otimes w^{(\underline{1})(\underline{1})} \otimes v^{(\underline{1})(2)} w^{(\underline{1})(2)} \otimes v^{(2)} w^{(2)} \\
&\quad = \sum \big(v^{(\underline{1})(\underline{1})} \otimes 1 \otimes v^{(\underline{1})(2)} \otimes v^{(2)}\big) \cdot \big(1 \otimes w^{(\underline{1})(\underline{1})} \otimes w^{(\underline{1})(2)} \otimes w^{(2)}\big) \\
&\quad = \sum \big(v^{(\underline{1})} \otimes 1 \otimes v^{(2)}_{(1)} \otimes v^{(2)}_{(2)}\big) \cdot \big(1 \otimes w^{(\underline{1})} \otimes w^{(2)}_{(1)} \otimes w^{(2)}_{(2)}\big) \\
&\quad = \sum v^{(\underline{1})} \otimes w^{(\underline{1})} \otimes v^{(2)}_{(1)} w^{(2)}_{(1)} \otimes v^{(2)}_{(2)} w^{(2)}_{(2)} \\
&\quad = \sum (v \otimes w)^{(\underline{1})} \otimes (v \otimes w)^{(2)}_{(1)} \otimes (v \otimes w)^{(2)}_{(2)}.
\end{aligned}$$

These simple observations prove decisive to our reconstruction of the Connes–Moscovici Hopf algebra in Section 9.

5 Antipodes

Definition 5.1. A *skewgroup* or *Hopf algebra* is a bialgebra H together with a (necessarily unique) convolution inverse S for the identity map id. Thus,

$$\mathrm{id} * S = m(\mathrm{id} \otimes S)\Delta = u\eta, \qquad S * \mathrm{id} = m(S \otimes \mathrm{id})\Delta = u\eta,$$

which boils down to the commutativity of the diagram

$$\begin{tikzcd}
H \otimes H \arrow[d, "\mathrm{id}\otimes S"'] & H \arrow[l, "\Delta"'] \arrow[r, "\Delta"] & H \otimes H \arrow[d, "S\otimes\mathrm{id}"] \\
H \otimes H \arrow[r, "m"] & H \arrow[u, "u\circ\eta"'] & H \otimes H. \arrow[l, "m"']
\end{tikzcd}$$

In terms of elements this means

$$\sum a_{(1)} S a_{(2)} = \eta(a) \quad \text{and} \quad \sum S a_{(1)}\, a_{(2)} = \eta(a).$$

The map S is usually called the *antipode* or *coinverse* of H. The notion of Hopf algebra occurred first in the work by Hopf in algebraic topology [30].

Uniqueness of the antipode can be seen as follows. Let S, S' be two antipodes on a bialgebra. Then

$$S'a = S'a_{(1)} \eta(a_{(2)}) = S'a_{(1)}\, a_{(2)(1)} S a_{(2)(2)} = S'a_{(1)(1)}\, a_{(1)(2)} S a_{(2)} = \eta(a_{(1)}) S a_{(2)} = Sa.$$

We have used counity in the first equality, and successively the antipode property for S, coassociativity, the antipode property for S' and counity again.

A bialgebra morphism between two Hopf algebras H, K is automatically a Hopf algebra morphism, i.e., it exchanges the antipodes: $\ell S_H = S_K \ell$. For that, it is enough to prove that these maps are one-sided convolution inverses for ℓ in $\mathrm{Hom}(H,K)$. Indeed, since the identity in $\mathrm{Hom}(H,K)$ is $u_K\eta_H$, it is enough to notice that

$$\ell S_H * \ell = \ell(S_H * \mathrm{id}_H) = \ell u_H \eta_H = u_K \eta_H = u_K \eta_K \ell = (\mathrm{id}_K * S_K)\ell = \ell * S_K \ell; \tag{5.1}$$

associativity of convolution then yields

$$S_K \ell = u_K \eta_H * S_K \ell = \ell S_H * \ell * S_K \ell = \ell S_H * u_K \eta_H = \ell S_H.$$

The antipode is an antimultiplicative and anticomultiplicative map of H. This means

$$Sm = m\tau(S \otimes S), \quad S1 = 1 \quad \text{and} \quad \tau\Delta S = (S \otimes S)\Delta, \quad \eta S = S.$$

The first relation, evaluated on $a \otimes b$, becomes the familiar antihomomorphism property $S(ab) = SbSa$. For the proof of it we refer to [24, Lemma 1.26]; the second relation is a similar exercise.

A grouplike element g of a Hopf algebra H is always invertible with $g^{-1} = Sg$. Indeed,

$$1 = u\eta(g) = m(\mathrm{id} \otimes S)\Delta g = gSg = m(S \otimes \mathrm{id})\Delta g = (Sg)g.$$

Often the antipode S is *involutive* (thus invertible); that is, $S^2 = \mathrm{id}_H$.

Proposition 5.1. *S is involutive if and only if*

$$Sa_{(2)}a_{(1)} = a_{(2)}Sa_{(1)} = \eta(a). \tag{5.2}$$

Proof. The relation $Sa_{(2)}\, a_{(1)} = \eta(a)$ implies $S * S^2 a = Sa_{(1)}\, S^2 a_{(2)} = S\big(Sa_{(2)}\, a_{(1)}\big) = S\eta(a) = \eta(a)$. Hence $S * S^2 = S^2 * S = u\eta$, which entails $S^2 = \mathrm{id}$. Reciprocally, if $S^2 = \mathrm{id}$, then

$$Sa_{(2)}\, a_{(1)} = Sa_{(2)}\, S^2 a_{(1)} = S\big(Sa_{(1)}\, a_{(2)}\big) = S\eta(a) = \eta(a),$$

and analogously $a_{(2)}Sa_{(1)} = \eta(a)$. □

In other words, the coinverse S is involutive when it is still the inverse of the identity for the new operation obtained from $*$ by twisting it with the flip map. Property (5.2) clearly holds true for Hopf algebras that are commutative or cocommutative. The antipode for a commutative and cocommutative Hopf algebra is an involutive bialgebra morphism.

A *Hopf subalgebra* of H is a vector subspace E that is a Hopf algebra with the restrictions of the antipode, product, coproduct and so on, the inclusion $E \hookrightarrow H$ being a bialgebra morphism. A Hopf ideal is a biideal J such that $SJ \subseteq J$; the quotient H/J gives a Hopf algebra.

A glance at the defining conditions for the antipode shows that, if H is a Hopf algebra, then H^{copp} is also a Hopf algebra with the same antipode. However, the bialgebras H^{opp}, H^{cop} are Hopf algebras if and only if S is invertible, and then the antipode is precisely S^{-1} [31, Section 1.2.4]. We prove this for H^{cop}. Assume S^{-1} exists. It will be an algebra antihomomorphism. Hence

$$S^{-1}a_{(2)}\, a_{(1)} = S^{-1}(Sa_{(1)}\, a_{(2)}) = S^{-1}\eta(a)1 = \eta(a)1;$$

similarly $m(\mathrm{id} \otimes S^{-1})\Delta_{\mathrm{cop}} = u\eta$. Reciprocally, if H^{cop} has an antipode S', then

$$SS' * Sa = \sum SS'a_{(1)}\, Sa_{(2)} = \sum S(a_{(2)}\, S'a_{(1)}) = S\big(\eta(a)1\big) = u\eta(a).$$

Therefore $SS' = \mathrm{id}$, so S' is the inverse of S under composition.

The duality nonsense of Section 4 is immediately lifted to the Hopf algebra category. The dual of $(H, m, u, \Delta, \eta, S)$ becomes $(H^\circ, \Delta^t, \eta^t, m^t, u^t, S^t)$; to equations (4.2) we add the condition

$$\langle S_H a, f\rangle = \langle a, S_K f\rangle,$$

which is actually redundant.

As for examples, the bialgebra $\mathbb{C}G$ is a Hopf algebra with coinverse $Sx = x^{-1}$; the bialgebra $\mathcal{R}(G)$ is a Hopf algebra with coinverse

$$Sf(x) = f(x^{-1}). \tag{5.3}$$

▶ The antipode is a powerful inversion machine. If H is a Hopf algebra, both algebra homomorphisms on an algebra A and coalgebra morphisms on a coalgebra C can be inverted in the convolution algebra. In fact, by going back to reinspect (5.1), we see already that in the case of algebra morphisms fS is a left inverse for f; also, using (2.3),

$$f * fS = f(\mathrm{id} * S) = fu_H\eta_H = u_A\eta_H.$$

In the case of coalgebra maps in (5.1) we see that Sf is a right inverse for f; and similarly it is a left inverse. Recall that $\mathrm{Hom}_{\mathrm{alg}}(H, A)$ denotes the convolution monoid of multiplicative morphisms on an algebra A with neutral element $u_A\eta_H$. The 'catch' is that fS does not belong to $\mathrm{Hom}_{\mathrm{alg}}(H, A)$ in general; as remarked before, it will if A is commutative (a moment's reflection reassures us that although S is antimultiplicative, fS indeed is multiplicative). In that case $\mathrm{Hom}_{\mathrm{alg}}(H, A)$ becomes a group (an abelian one if H is cocommutative). In particular, that is the case of the set $\mathrm{Hom}_{\mathrm{alg}}(H, \mathbb{C})$ of multiplicative functions or characters, and of $\mathrm{Hom}_{\mathrm{alg}}(H, H)$, when H is commutative. (One also gets a group when H is cocommutative, and considers the coalgebra morphisms from H to a coalgebra C.)

In the first of the examples given in Section 2, the group of real characters of $\mathcal{R}(G)$ reconstructs G in its full topological glory: this is Tannaka–Kreĭn duality —see [24, Ch. 1] and [28]. Characters of connected graded Hopf algebras have special properties, exhaustively studied in [32].

A pillar of wisdom in Hopf algebra theory: connected bialgebras are *always* Hopf. There are at least two 'grand strategies' to produce the antipode $S : H \to H$ in a connected bialgebra. One is to exploit its very definition as the convolution inverse of the identity in H, via a geometric series:

$$S := \mathrm{id}^{*-1} = S_G := (u\eta - (u\eta - \mathrm{id}))^{*-1} := u\eta + (u\eta - \mathrm{id}) + (u\eta - \mathrm{id})^{*2} + \cdots$$

Proposition 5.2. *The geometric series expansion of Sa with $a \in H_n$ has at most $n + 1$ terms.*

Proof. If $a \in H_0$ the claim holds since $(u\eta - \mathrm{id})1 = 0$. It also holds in H_{+1} because these elements are primitive. Assume that the statement holds for elements in H_{+n-1}, and let $a \in H_{+n}$; then

$$\begin{aligned}(u\eta - \mathrm{id})^{*(n+1)}(a) &= (u\eta - \mathrm{id}) * (u\eta - \mathrm{id})^{*n}(a) \\ &= m[(u\eta - \mathrm{id}) \otimes (u\eta - \mathrm{id})^{*n}]\Delta a \\ &= m[(u\eta - \mathrm{id}) \otimes (u\eta - \mathrm{id})^{*n}](a \otimes 1 + 1 \otimes a + \Delta' a).\end{aligned}$$

The first two terms vanish because $(u\eta - \mathrm{id})1 = 0$. By the induction hypothesis each of the summands of the third term are also zero. □

In view of (3.5), on H_+ we can write for $k \geq 1$

$$(u\eta - \mathrm{id})^{*k+1} = (-1)^{k+1} m^k \Delta'^k. \tag{5.4}$$

There is then a fully explicit expression for the antipode for elements without degree 0 components (recall $S1 = 1$), in terms of the product and the reduced coproduct

$$S_G = -\mathrm{id} + \sum_{k=1}^{\infty} (-1)^{k+1} m^k \Delta'^k. \tag{5.5}$$

All this was remarked in [16].

The second canonical way to show that a connected bialgebra is a Hopf algebra amounts to take advantage of the equation $m(S \otimes \mathrm{id})\Delta a = 0$ for $a \in H_+$. For $a \in H_{+n}$ and $n \geq 1$, one ushers in the recursive formula:

$$S_B(a) := -a - \sum S_B a'_{(1)} a'_{(2)}, \tag{5.6}$$

using the notation in (3.3).

Proposition 5.3. *If H is a connected bialgebra, then $S_G a = S_B a$.*

Proof. The statement holds, by a direct check, if $a \in H_{+1}$. Assume that $S_G b = S_B b$ whenever $b \in H_{+n}$, and let $a \in H_{+n+1}$. Then

$$\begin{aligned}S_G a &= (u\eta - \mathrm{id})a + \sum_{i=1}^{n} (u\eta - \mathrm{id})^{*i} * (u\eta - \mathrm{id})a \\ &= -a + m\bigg(\sum_{i=1}^{n} (u\eta - \mathrm{id})^{*i} \otimes (u\eta - \mathrm{id})\bigg)\Delta a \\ &= -a + m\sum_{i=1}^{n} (u\eta - \mathrm{id})^{*i} \otimes (u\eta - \mathrm{id})(a \otimes 1 + 1 \otimes a + \Delta' a) \\ &= -a + \sum_{i=1}^{n} (u\eta - \mathrm{id})^{*i} a'_{(1)}\, (u\eta - \mathrm{id}) a'_{(2)} = -a - \sum_{i=1}^{n} (u\eta - \mathrm{id})^{*i} a'_{(1)}\, a'_{(2)} \\ &= -a - S_B a'_{(1)}\, a'_{(2)} = S_B a,\end{aligned}$$

where the penultimate equality uses the induction hypothesis. □

Taking into account the alternative expression

$$S_G a = (u\eta - \mathrm{id})a + \sum_{i=1}^{n} (u\eta - \mathrm{id}) * (u\eta - \mathrm{id})^{*i} a,$$

it follows that the twin formula

$$S'_B a := -a - \sum a'_{(1)} S'_B a'_{(2)}$$

provides as well a formula for the antipode. The subindices B in S_B, S'_B reminds us that this second strategy corresponds precisely to Bogoliubov's formula for renormalization in quantum field theory.

The geometric series leading to the antipode can be generalized as follows. Consider, for H a connected bialgebra and A an arbitrary algebra, the set of elements $f \in \mathrm{Hom}(H, A)$ fulfilling $f(1) = 1$. They form a convolution monoid, with neutral element $u_A \eta_H$, as $f * g(1) = 1$, if both f and g are of this type. Moreover we can repeat the inversion procedure:

$$f^{*-1} := (u\eta - (u\eta - f))^{*-1} := u\eta + (u\eta - f) + (u\eta - f)^{*2} + \cdots$$

Then $f^{*-1}(1) = 1$, and for any $a \in H_+$

$$(u\eta - f)^{*k+1} a = (-1)^{k+1} m^k (f \otimes \cdots \otimes f) {\Delta'}^k a,$$

vanish for $a \in H_n$ when $n \leq k$. Therefore the series stops. The convolution monoid then becomes a group; of which, as we already know, the set $\mathrm{Hom}_{\mathrm{alg}}(H, A)$ of multiplicative morphisms is a subgroup when A is commutative.

▶ The foregoing indicates that, associated to any connected bialgebra, there is a natural filtering —we call it *depth*— where the order $\delta(a)$ of a generator $a \in H_+$ is $k > 0$ when H is the smallest integer such that $a \in \ker({\Delta'}^k p)$; we then say a is k-primitive. Whenever $a \in H_n$, it holds $\delta(a) \leq n$. On account of (3.5), $\ker(U_k) := \ker({\Delta'}^k p) = (\Delta^k)^{-1}(H_R^{k+1})$. In other words, for those bialgebras, depth is the coradical filtering. Antipodes are automatically filtered.

On H_+ one has $(u\eta - \mathrm{id})^{*k+1} a = 0$, in view of (5.4), if ${\Delta'}^k a = 0$; but of course the converse is not true.

Definition 5.2. For H commutative, we say a indecomposable is *quasiprimitive* if

$$Sa = -a; \quad \text{this implies} \quad (u\eta - \mathrm{id})^{*2} a = 0. \tag{5.7}$$

Obviously primitive elements are quasiprimitive. In Section 15 we give examples of elements that are quasiprimitive, but not primitive, and discuss their relevance. We will also see that a basis for H can be found such that, for any element b of it, $Sb = -b$ or b.

Following [33], we conclude this section with a relatively short proof of that δ indeed is a filtering, in the framework of connected graded bialgebras of finite type, comprising our main examples. Let us denote in the reminder of this section $H_k = \mathbb{C}1 \oplus \ker {\Delta'}^k = \{\, a \in H : \delta(a) \leq k \,\}$, for $k \geq 0$. This is certainly a linear filtering.

Proposition 5.4. *Let* $H'_+ := \bigoplus_{l\geq 1} H^{(l)^*} \subset H'$. *Then* $H_k^\perp = {H'_+}^{k+1}$ *—thus* $H_k = ({H'_+}^{k+1})^\perp$.

Proof. (Derivations are typical elements of $1^\perp$.) The assertion is true and obvious for $k = 0$. Consider $(\mathrm{id}-u\eta)^t$, the projection on H'_+ with kernel $1_{H'}$. Let $\kappa_1, \dots, \kappa_{k+1} \in H'_+$ and $a \in H_k$. We have

$$\begin{aligned}\langle \kappa_1 \cdots \kappa_{k+1}, a\rangle &= \langle \kappa_1 \otimes \cdots \otimes \kappa_{k+1}, \Delta^k a\rangle \\ &= \langle (\mathrm{id} - u\eta)^{t\otimes(k+1)}(\kappa_1 \otimes \cdots \otimes \kappa_{k+1}), \Delta^k a\rangle \\ &= \langle \kappa_1 \otimes \cdots \otimes \kappa_{k+1}, U_k a\rangle = 0.\end{aligned}$$

Therefore ${H'_+}^{k+1} \subseteq H_k^\perp$. Now, let $a \in ({H'_+}^{k+1})^\perp$. Then again

$$\langle \kappa_1 \otimes \cdots \otimes \kappa_{k+1}, U_k a\rangle = \langle \kappa_1 \cdots \kappa_{k+1}, a\rangle = 0.$$

Therefore $U_k a \in H_+^{\otimes(k+1)} \cap \big({H'_+}^{\otimes(k+1)}\big)^\perp = (0)$. Consequently $({H'_+}^{k+1})^\perp \subseteq H_k$, which implies $H_k^\perp \subseteq {H'_+}^{k+1}$. □

Given any augmented graded algebra A, connected in the sense that $A^{(0)} \simeq 0$ with $A_+ = \bigoplus_{i\geq 1} A^{(i)}$, it is not too hard to see that

$$\sum_{i+j>k} A_+^i \otimes A_+^j = \bigcap_{l+m=k} \big(A_+^{l+1} \otimes A + A \otimes A_+^{m+1}\big).$$

As a corollary:

Proposition 5.5. *The filtering by the* H_k *is a coalgebra filtering.*

Proof. We have

$$\begin{aligned}\sum_{l+m=k} H_l \otimes H_m &= \sum_{l+m=k} ({H'_+}^{l+1})^\perp \otimes ({H'_+}^{m+1})^\perp = \sum_{l+m=k} ({H'_+}^{l+1} \otimes H' + H' \otimes {H'_+}^{m+1})^\perp \\ &= \Bigg(\bigcap_{l+m=k} \big({H'_+}^{l+1} \otimes H' + H' \otimes {H'_+}^{m+1}\big)\Bigg)^\perp = \Bigg(\sum_{i+j>k} {H'_+}^i \otimes {H'_+}^j\Bigg)^\perp.\end{aligned}$$

Now, let $a \in H_k$ and $\kappa_1 \in {H'_+}^i, \kappa_2 \in {H'_+}^j, i + j > k$. Then $\kappa_1\kappa_2 \in {H'_+}^{k+1} = H_k^\perp$. Thus

$$0 = \langle a, \kappa_1\kappa_2\rangle = \langle \Delta a, \kappa_1 \otimes \kappa_2\rangle.$$

This means $\Delta a \in \sum_{l+m=k} H_l \otimes H_m$, and so H_k is a coalgebra filtering as claimed. □

Proposition 5.6. *The filtering by the* H_k *is an algebra filtering.*

Proof. If $a \in H_k$, then ${\Delta'}^k a = 0$, and the counit property entails ${\Delta'}^{k-1} a \in P(H)^{\otimes k}$. Let now $a \in H_l, b \in H_m$ with

$${\Delta'}^{l-1} a = \sum a'_{(1)} \otimes a'_{(2)} \otimes \cdots \otimes a'_{(l)}, \quad {\Delta'}^{m-1} b = \sum b'_{(1)} \otimes \cdots \otimes b'_{(m)}.$$

By (3.5) once again

$$\begin{aligned}
{\Delta'}^{l+m-1}(ab) = U_{l+m-1}(ab) &= (\mathrm{id} - u\eta)^{\otimes\, l+m}(\Delta^{l+m-1}a\Delta^{l+m-1}b) \\
&= \sum \sum_{\sigma \in S_{l+m,l}} a'_{(\sigma(1))} \otimes \cdots \otimes a'_{(\sigma(l))} \otimes b'_{(\sigma(l+1))} \otimes \cdots \otimes b'_{(\sigma(l+m))} \\
&=: \sum \sum_{\sigma \in S_{l+m,l}} \sigma \cdot (a'_{(1)} \otimes \cdots \otimes a'_{(l)} \otimes b'_{(1)} \otimes \cdots \otimes b'_{(m)}), \qquad (5.8)
\end{aligned}$$

where $S_{n,p}$ denotes the set of $(p, n-p)$-shuffles; a (p,q)-shuffle is an element of the group of permutations S_{p+q} of $\{1, 2, \dots p+q\}$ in which $\sigma(1) < \sigma(2) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$.

Equation (5.8) implies that ${\Delta'}^{l+m-1}(ab) \in P(H)^{\otimes\, l+m}$, and hence ${\Delta'}^{l+m}(ab) = 0$. ☐

We retain as well the following piece of information:

$${\Delta'}^{n-1}(p_1 \dots p_n) = \sum_{\sigma \in S_n} p_{\sigma(1)} \otimes \cdots \otimes p_{\sigma(n)},$$

for primitive elements $p_1, \dots, p_n$. The proof is by induction. Obviously we have in particular:

$$\Delta'(p_1 p_2) = p_1 \otimes p_2 + p_2 \otimes p_1.$$

Assuming

$${\Delta'}^{n-2}(p_1 \dots p_{n-1}) = \sum_{\sigma \in S_{n-1}} p_{\sigma(1)} \otimes \cdots \otimes p_{\sigma(n-1)},$$

equation (5.8) tells us that

$${\Delta'}^{n-1}(p_1 \cdots p_n) = \sum_{\tau \in S_{n,1}} \sum_{\sigma \in S_{n-1}} \tau \cdot (p_{\sigma(1)} \otimes \cdots \otimes p_{\sigma(n-1)} \otimes p_n) = \sum_{\sigma \in S_n} p_{\sigma(1)} \otimes \cdots \otimes p_{\sigma(n)}.$$

6 Symmetric Algebras

In our second example in Section 2, we know $SX = -X$ for $X \in \mathfrak{g}$, since X is primitive, and $S(XY) = YX$; but the concrete expression in terms of a basis given a priori can be quite involved. Consider, however, the universal enveloping algebra corresponding to the trivial Lie algebra structure on V. This is clearly a commutative and cocommutative Hopf algebra, nothing else than the familiar symmetric, free commutative algebra or, in physics, boson algebra $B(V)$ over V of quantum field theory. Given V, a complex vector space, $B(V)$ is defined as $\bigoplus_{n=0}^{\infty} V^{\vee n}$, where $V^{\vee n}$ is the complex vector space algebraically generated by the symmetric products

$$v_1 \vee v_2 \vee \cdots \vee v_n := \frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)},$$

with $V^{\vee 0} = \mathbb{C}$ by convention. On $B(V)$ a coproduct and counit are defined respectively by $\Delta v := v \otimes 1 + 1 \otimes v$ and $\eta(v) = 0$, for $v \in V$, and then extended by the homomorphism property. In general,

$$\Delta(a^1 \vee a^2) = \sum a^1_{(1)} \vee a^2_{(1)} \otimes a^1_{(2)} \vee a^2_{(2)},$$

for $a^1, a^2 \in B(V)$. Another formula is:

$$\Delta(v_1 \vee v_2 \vee \cdots \vee v_n) = \sum_I U(I) \otimes U(I^c), \tag{6.1}$$

with sum over all the subsets $I \subseteq \{v_1, v_2, \ldots v_n\}$, $I^c = \{v_1, v_2, \ldots v_n\} \setminus I$, and $U(I)$ denotes the $\vee$ product of the elements in I. Thus, if $u = v_1 \vee v_2 \vee \cdots \vee v_n$, then

$$\Delta' u = \sum_{p=1}^{n-1} \sum_{\sigma \in S_{n,p}} v_{\sigma(1)} \vee \cdots \vee v_{\sigma(p)} \otimes v_{\sigma(p+1)} \vee \cdots \vee v_{\sigma(n)}.$$

Here we are practically repeating the calculations at the end of the previous section. In the particularly simple case when V is one dimensional, the Hopf algebra $B(V) \simeq \mathcal{U}(\mathbb{C})$ is just the binomial bialgebra. Finally $Sa = -a$ for elements of $B(V)$ of odd degree and $Sa = a$ for even elements; the reader can amuse himself checking how (5.5) works here.

It should be clear that an element of $B(V)$ is primitive iff it belongs to V. For a direct proof, let $\{e_i\}$ be a basis for V, any $a \in B(V)$ can be represented as

$$a = \alpha 1 + \sum_{k \geq 1} \sum_{i_1 \leq \cdots \leq i_k} \alpha_{i_1,\ldots,i_k} e_{i_1} \vee \cdots \vee e_{i_k}.$$

for some complex numbers $\alpha_{i_1,\ldots,i_k}$. Now, if a is primitive then $\alpha = \eta(a) = 0$, and

$$\Delta a = a \otimes 1 + 1 \otimes a = \sum \alpha_{i_1,\ldots,i_k} (e_{i_1} \vee \cdots \vee e_{i_k} \otimes 1 + 1 \otimes e_{i_1} \vee \cdots \vee e_{i_k}).$$

but also

$$\begin{aligned}
\Delta a &= \sum_k \sum \alpha_{i_1,\ldots,i_k} (e_{i_1} \otimes 1 + 1 \otimes e_{i_1}) \cdots (e_{i_k} \otimes 1 + 1 \otimes e_{i_k}) \\
&= \sum_k \sum \alpha_{i_1,\ldots,i_k} (e_{i_1} \vee \cdots \vee e_{i_k} \otimes 1 + 1 \otimes e_{i_1} \vee \cdots \vee e_{i_k} + \sum_J e_J \otimes e_{J^c}),
\end{aligned}$$

where the last sum runs over all nonempty subsets $J = \{i_1, \ldots, i_l\}$ of $[k] := \{\, 1, \ldots, k \,\}$ with at most $k-1$ elements, J^c denotes the complement of J in $[k]$, and $e_J := e_{i_1} \vee \cdots \vee e_{i_l}$. A comparison of the two expressions gives

$$\sum_{k \geq 2} \sum_{i_1 \leq \cdots \leq i_k} \alpha_{i_1,\ldots,i_k} \sum_{\emptyset \neq J \subsetneq [k]} e_J \otimes e_{J^c} = 0.$$

This forces $\alpha_{i_1,\ldots,i_k} = 0$ for $k \geq 2$, so $a \in V$.

Definition 6.1. A Hopf algebra H is said to be *primitively generated* when the smallest subalgebra of H containing all its primitive elements is H itself. Cocommutativity of H is clearly a necessary condition for that. It is plain that $B(V)$ is primitively generated.

▶ Symmetric algebras have the following universal property: any morphism of (graded, if you wish) vector spaces $\psi : V \to H$, where $V^{(0)} = (0)$ and H is a unital graded connected *commutative* algebra, extends uniquely to a unital graded algebra homomorphism $B\psi : B(V) \to H$; one says $B(V)$ is free over V. Note the isomorphism $B(V \oplus \tilde{V}) \simeq B(V) \otimes B(\tilde{V})$, implemented by $V \oplus \tilde{V} \ni (v, \tilde{v}) \mapsto v \otimes 1 + 1 \otimes \tilde{v}$, extended by linear combinations of products. In this perspective, the comultiplication Δ on $B(V)$ is the extension Bd induced by the diagonal map $d : V \mapsto V \oplus V$.

Proposition 6.1. *Let H be a graded connected commutative Hopf algebra, and denote by ψ_H the inclusion $P(H) \hookrightarrow H$. The universal property gives a graded algebra map $B\psi_H : B(P(H)) \to H$. This is a graded Hopf algebra morphism.*

Proof. The coproduct $\Delta : H \to H \otimes H$ gives a linear map $P\Delta : P(H) \to P(H \otimes H)$ such that $\Delta\psi_H = \psi_{H\otimes H}\, P\Delta$. The universal property of symmetric algebras then gives us maps $B\psi_H, B\psi_{H\otimes H}$ and $BP\Delta$ such that $\Delta B\psi_H = B\psi_{H\otimes H} BP\Delta$. By (3.6),

$$B\big(P(H \otimes H)\big) \cong B\big(P(H) \oplus P(H)\big) \cong B\big(P(H)\big) \otimes B\big(P(H)\big); \quad B\psi_{H\otimes H} \simeq B\psi_H \otimes B\psi_H.$$

Therefore $BP\Delta$ is identified to the coproduct of $B(P(H))$ and

$$\Delta B\psi_H = (B\psi_H \otimes B\psi_H)\Delta_{B(P(H))}.$$

In a similar fashion it is seen that $B\psi_H$ respects counity and coinverse. □

If $V \subseteq P(H)$, then the subalgebra $\mathbb{C}[V]$ generated by V is a primitively generated Hopf subalgebra of H. The inclusion $\iota : V \hookrightarrow H$ induces a morphism of graded algebras, indeed of Hopf algebras, $B\iota : B(V) \to H$ whose range is $\mathbb{C}[V]$. In particular, $\mathbb{C}[P(H)]$ is the largest primitively generated subalgebra of H and $B\psi_H : B(P(H)) \to H$ is a morphism of Hopf algebras *onto* $\mathbb{C}[P(H)]$; therefore H is primitively generated only if the underlying algebra is generated by $P(H)$. All these statements follow from the previous proof and the simple observation that $v \in V$ is primitive in both $\mathbb{C}[V]$ and $B(V)$.

Proposition 6.2. *The morphism $B\psi_H : B(P(H)) \to H$ is injective. In particular, if H is primitively generated, then $H = B(P(H))$.*

Proof. The vector space $P(H)$ can be regarded as the direct limit of all its finite-dimensional subspaces V, hence $B\big(P(H)\big)$ is the direct limit of all $B(V)$ —tensor products commute with direct limits— and the map $B\psi_H$ is injective if and only if its restriction to each of the algebras $B(V)$ is injective. Thus it is enough to prove the proposition for V a finite-dimensional subspace of $P(H)$.

We do that by induction on $m = \dim V$. For $m = 0$, there is nothing to prove. Assume, then, that the claim holds for all subspaces W of $P(H)$ with $\dim W \leq m - 1$, and let V be a subspace of $P(H)$ with $\dim V = m$. Let W be an $(m-1)$-dimensional subspace of V, then $B\psi : B(W) \to \mathbb{C}[W]$ is an isomorphism of Hopf algebras. Take $Y \in V \setminus \mathbb{C}[W]$ homogeneous of minimal degree, so $V = W \oplus \mathbb{C}Y$. Then $B(V) \cong B(W) \otimes B(\mathbb{C}Y) \cong \mathbb{C}[W] \otimes B(\mathbb{C}Y)$. Now, $\mathbb{C}[V] \cong \mathbb{C}[W \oplus \mathbb{C}Y] \cong \mathbb{C}[W] \otimes \mathbb{C}[Y]$. The remarks prior to the statement imply that $B\big(P(\mathbb{C}[Y])\big) \cong \mathbb{C}\big[P(\mathbb{C}[Y])\big]$, and since $Y \in V \subset P(H)$, clearly $P(\mathbb{C}[Y]) = \mathbb{C}Y$, so $B(\mathbb{C}Y) \cong \mathbb{C}[Y]$. It follows that

$$B(V) \cong \mathbb{C}[W] \otimes B(\mathbb{C}Y) \cong \mathbb{C}[W] \otimes \mathbb{C}[Y] \cong \mathbb{C}[V],$$

which completes the induction. □

A similar argument allows us to take up some unfinished business: the converse of Proposition 3.3.

Theorem 6.3. *The relation*

$$P(H) \cap H_+^2 = (0), \tag{6.2}$$

or equivalently $q_H : P(H) \to Q(H)$ is one-to-one, holds if H is commutative.

Proof. Suppose H commutative has a unique generator. Then a moment's reflection shows that H has to be the binomial algebra, and then $P(H) \simeq Q(H)$. Suppose now that the proposition is proved for algebras with less than or equal to n generators. Let the elements $a_1, \dots, a_{n+1}$ be such that their images by the canonical projection $H \to Q(H)$ form a basis of $Q(H)$. Leave out the element of highest degree among them, and consider B, the Hopf subalgebra of H generated by the other n elements. We form $\mathbb{C} \otimes_B H = H/B_+H$. This is seen to be a Hopf algebra with one generator. Moreover, $H \simeq B \otimes \mathbb{C} \otimes_B H$. Then (3.6) implies $P(H) = P(B) \oplus P(\mathbb{C} \otimes_B H)$. By the induction hypothesis, $q_H = q_B \oplus q_{\mathbb{C}\otimes_B H}$ is injective. The proposition is then proved for finitely generated Hopf algebras. Hopf algebras of finite type are clearly direct limits of finitely generated Hopf algebras; and direct limits preserve the functors P, Q and injective maps; so (6.2) holds true for Hopf algebras of finite type as well. Finally, by the result of [23] already invoked in Section 3, the proposition holds for all commutative connected graded Hopf algebras without exception. □

Yet another commutative and cocommutative Hopf algebra is the polynomial algebra $\mathbb{C}[Y_1, Y_2, Y_3, \dots]$ with coproduct given by

$$\Delta Y_n = \sum_{j=0}^{n} Y_j \otimes Y_{n-j};$$

to be studied later, using symmetric algebra theory. We denote this algebra by H_ℓ, and call it the *ladder Hopf algebra*; it is related to nested Feynman graphs and to the Hopf algebra of rooted trees, considered in Section 12.

▶ To finish, we come back to the universal property of $B(V)$ as an algebra. For that, the coalgebra structure previously considered does not come into play, and one could ask whether other coproducts are available, that also give a Hopf algebra structure on $B(V)$; the answer is yes, but the one exhibited here is the only one that makes $B(V)$ a *graded* Hopf algebra. More to the point: dually, given graded cocommutative coalgebras and linear maps into vector spaces $\varphi : H \to V$, there is a universal 'cofree cocommutative' coalgebra, say $Q_{\mathrm{cocom}}(V)$, 'cogenerated' by V, together with a unique coalgebra map $Q_{\mathrm{cocom}}\varphi : H \to Q_{\mathrm{cocom}}(V)$ restricting to φ by the projection $Q_{\mathrm{cocom}}(V) \to V$. Now, $Q_{\mathrm{cocom}}(V)$ is nothing but the vector space underlying $B(V)$ with the coalgebra structure already given here; and there is a unique algebra structure on $Q_{\mathrm{cocom}}(V)$ making it a graded Hopf algebra.

In many contexts it is important to waive the (co)commutativity requisite on B and on Q_{cocom}; this respectively leads to the tensor or free graded algebra (already touched upon in Example 2.2) and to the cotensor or cofree graded coalgebra, for which the most natural Hopf algebra structure is Ree's [34] and Chen's [35] *shuffle* algebra. In some detail: already in Section 2 we touched upon the tensor algebra $T(V)$, which is made into a (graded, connected, cocommutative) Hopf algebra by the algebra morphisms

$$\Delta 1 = 1 \otimes 1, \quad \Delta X := X \otimes 1 + 1 \otimes X; \qquad \eta(X) := 0, \quad \eta(1) = 1,$$

for $X \in V$. It is useful to think of the basis of V as an alphabet, whose elements are letters, and to write the tensor product simply as a *concatenation*. Then we have, as before,

$$\Delta(X_1 X_2 \cdots X_n) = \sum_{p=0}^{n} \sum_{\sigma \in S_{n,p}} X_{\sigma(1)} \cdots X_{\sigma(p)} \otimes X_{\sigma(p+1)} \cdots X_{\sigma(n)}, \tag{6.3}$$

with $S_{n,p}$ denoting the $(p, n-p)$-shuffles. It should be clear that primitive elements of $T(V)$ constitute a free Lie algebra [36]. For instance, if there are two letters $\{X, Y\}$ in V, a basis is made of

$$X,\ Y,\ [X,Y],\ [X,[X,Y]],\ [[X,Y],Y],\ [X,[X,[X,Y]]],\ [X,[[X,Y],Y]],$$

and so on. The graded dual of $T(V)$ —which at least in the finite type case is isomorphic to $T(V)$ as a graded vector space— is the shuffle Hopf algebra $Sh(V)$, a most interesting object that appears in many contexts. We find its (necessarily associative and commutative) product, denoted by $\pitchfork$, by dualizing (6.3): for the words $u = d_1 d_2 \cdots d_p$ and $v = d_{p+1} d_{p+2} \cdots d_n$ with each d_i in V, their product is

$$u \pitchfork v = \sum_{\sigma \in S_{n,p}} d_{\sigma(1)} \cdots d_{\sigma(n)}.$$

For instance:

$$1 \pitchfork d = d \pitchfork 1 = d; \ d_1 \pitchfork d_2 = d_2 \pitchfork d_1 = d_1 d_2 + d_2 d_1; \ d_1 d_2 \pitchfork d_3 = d_3 d_1 d_2 + d_1 d_3 d_2 + d_1 d_2 d_3.$$

There is a recursive definition given by

$$d_1 d_2 \cdots d_p \pitchfork d_{p+1} d_{p+2} \cdots d_n = d_1 \big(d_2 \cdots d_p \pitchfork d_{p+1} d_{p+2} \cdots d_n\big) + d_{p+1} \big(d_1 d_2 \cdots d_p \pitchfork d_{p+2} \cdots d_n\big).$$

We have indeed

$$\langle u \pitchfork v, X_1 X_2 \cdots X_n \rangle = \langle u \otimes v, \Delta(X_1 X_2 \cdots X_n) \rangle.$$

On can easily show, using the universal property, that $Sh(V)$ is the unique commutative graded Hopf algebra structure on the cofree graded subalgebra $Q(V)$ [37]. The coproduct on the latter is given by

$$\langle \Delta u, X_1 X_2 \cdots X_l \otimes Y_1 Y_2 \cdots Y_k \rangle = \langle u, X_1 X_2 \cdots X_l Y_1 Y_2 \cdots Y_k \rangle,$$

that is, *deconcatenation*:

$$\Delta u = \sum_{u = vw} v \otimes w.$$

Part II: The Faà di Bruno Bialgebras

7 Partitions, Bell polynomials and the Faà di Bruno algebras

In a "moral" sense, epitomized by Example 1 of Section 2, the discussion around equation (5.3) and the consideration of $G(H^\circ)$ in Section 4, commutative skewgroups are equivalent to groups. Now, we would like to deal with relatively complicated groups, like diffeomorphism groups. Variants of Hopf algebra theory generalizing categories of (noncompact in general) topological groups do exist [38]. It is still unclear how to handle diffeomorphism groups globally, though: the interplay between topology and algebra becomes too delicate [39]. We settle for a "perturbative version". Locally, one can think of orientation preserving diffeomorphisms of $\mathbb{R}$ leaving a fixed point as given by formal power series like

$$f(t) = \sum_{n=0}^{\infty} f_n \frac{t^n}{n!}, \tag{7.1}$$

with $f_0 = 0, f_1 > 0$. (Orientation preserving diffeomorphisms of the circle are just periodic ones of $\mathbb{R}$, and locally there is no difference.) Among the functions on the group G of diffeomorphisms the coordinate functions on jets

$$a_n(f) := f_n = f^{(n)}(0), \quad n \geq 1,$$

single out themselves. The product of two diffeomorphisms is expressed by series composition; to which just like in Example 1 we expect to correspond a coproduct for the a_n elements. As the a_n are representative, it is unlikely that this reasoning will lead us astray.

Let us then work out $f \circ g$, for f as above and g of the same form with coordinates $g_n = a_n(g)$. This old problem is solved with the help of the *Bell polynomials*. The (partial, exponential) Bell polynomials $B_{n,k}(x_1, \ldots, x_{n+1-k})$ for $n \geq 1$, $1 \leq k \leq n$ are defined by the series expansion:

$$\exp\biggl(u \sum_{m\geq 1} x_m \frac{t^m}{m!}\biggr) = 1 + \sum_{n\geq 1} \frac{t^n}{n!}\biggl[\sum_{k=1}^{n} u^k B_{n,k}(x_1, \ldots, x_{n+1-k})\biggr]. \tag{7.2}$$

The first Bell polynomials are readily found: $B_{n,1} = x_n$, $B_{n,n} = x_1^n$; $B_{2,1} = x_2$, $B_{3,2} = 3x_1x_2$, $B_{4,2} = 3x_2^2 + 4x_1x_3$, $B_{4,3} = 6x_1^2x_2$, $B_{5,2} = 10x_2x_3 + 5x_1x_4$, $B_{5,3} = 10x_1^2x_3 + 15x_1x_2^2$, $B_{5,4} = 10x_1^3x_2$, … Each Bell polynomial $B_{n,k}$ is homogeneous of degree k.

We claim the following: if $h(t) = f \circ g(t)$, then

$$h_n = \sum_{k=1}^{n} f_k B_{n,k}(g_1, \ldots, g_{n+1-k}). \tag{7.3}$$

One can actually allow here for $f_0 \neq 0$; then the same result holds, together with $h_0 = f_0$.

The proof is quite easy: it is clear that the h_n are linear in the f_n:

$$h_n = \sum_{k=1}^{n} f_k A_{n,k}(g).$$

In order to determine the $A_{n,k}$ we choose the series $f(t) = e^{ut}$. This entails $f_k = u^k$ and

$$h = f \circ g = e^{ug} = \exp\biggl(u \sum_{m\geq 1} g_m \frac{t^m}{m!}\biggr) = 1 + \sum_{n\geq 1} \frac{t^n}{n!}\biggl[\sum_{k=1}^{n} u^k B_{n,k}(g_1, \dots, g_{n+1-k})\biggr],$$

from which at once there follows

$$A_{n,k}(g) = B_{n,k}(g_1, \dots, g_{n+1-k}). \tag{7.4}$$

So $h_1 = f_1 g_1, h_2 = f_1 g_2 + f_2 g_1^2, h_3 = f_1 g_3 + 3 f_2 g_1 g_2 + f_3 g_1^3$, and so on. Francesco Faà di Bruno (beatified in 1988) gave a formula equivalent to (7.4) about a hundred and fifty years ago [40]. There are older instances of it: see our comment at the beginning of Section 16. Lest the reader think we are dealing with purely formal results, we remark that, if g is real analytic on an open interval I_1 of the real line and takes values on another open interval I_2, on which f is analytic as well, then $f \circ g$ given by (7.3) is analytic on I_1, too [41]. To obtain explicit formulae for the $B_{n,k}$, one can proceed directly from the definition. We shall only need the multinomial identity

$$(\beta_1 + \beta_2 + \cdots + \beta_r)^k = \sum_{c_1+c_2+\cdots+c_r=k} \frac{k!}{c_1! c_2! \cdots c_r!} \beta_1^{c_1} \beta_2^{c_2} \cdots \beta_r^{c_r},$$

that generalizes directly the binomial identity. To see that, note that if $c_1 + c_2 + \cdots + c_r = k$ then the multilinear coefficient $\binom{k}{c_1, c_2, \dots, c_r}$ of $\beta_1^{c_1}\beta_2^{c_2} \cdots \beta_r^{c_r}$ is the number of ordered r-tuples of mutually disjoint subsets $(S_1, S_2, \dots S_r)$ with $|S_i| = c_i$ whose union is $\{1, 2, \dots, k\}$. Then, since S_1 can be filled in $\binom{k}{c_1}$ different ways, and once S_1 is filled, S_2 can be filled in $\binom{n-n_1}{n_2}$ ways, and so on:

$$\binom{k}{c_1, \dots, c_r} = \binom{k}{c_1}\binom{k-c_1}{c_2}\binom{k-c_1-c_2}{c_3} \cdots \binom{c_r}{c_r} = \frac{k!}{c_1!\, c_2! \cdots c_r!}.$$

Now, we can expand

$$\begin{aligned}\sum_{k\geq 0} \frac{u^k}{k!}\biggl(\sum_{m\geq 1} x_m \frac{t^m}{m!}\biggr)^k &= \sum_{k\geq 0} \frac{u^k}{k!}\biggl(\sum_{c_1+c_2+\cdots=k} \frac{k!}{c_1! c_2! \cdots}(x_1 t)^{c_1}(x_2 t^2/2!)^{c_2} \cdots\biggr) \\ &= \sum_{c_1, c_2, \dots \geq 0} \frac{u^{c_1+c_2+c_3+\cdots}\ t^{c_1+2c_2+3c_3+\cdots}}{c_1! c_2! \cdots (1!)^{c_1}(2!)^{c_2} \cdots} x_1^{c_1} x_2^{c_2} \cdots.\end{aligned}$$

Taking the coefficients of $u^k t^n/n!$ in view of (7.2), it follows that

$$B_{n,k}(x_1, \dots, x_{n+1-k}) = \sum \frac{n!}{c_1! c_2! c_3! \cdots (1!)^{c_1}(2!)^{c_2}(3!)^{c_3} \cdots} x_1^{c_1} x_2^{c_2} x_3^{c_3} \cdots \tag{7.5}$$

where the sum is over the sets of positive integers $c_1, c_2, \dots, c_n$ such that $c_1+c_2+c_3+\cdots+c_n = k$ and $c_1 + 2c_2 + 3c_3 + \cdots + nc_n = n$.

It is convenient to introduce the notations

$$\binom{n}{\lambda; k} := \frac{n!}{\lambda_1! \lambda_2! \cdots \lambda_n! (1!)^{\lambda_1} (2!)^{\lambda_2} \dots (n!)^{\lambda_n}},$$

where λ is the sequence $(1, 1, \dots; 2, 2, \dots; \dots)$, better written $(1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3} \dots)$, of λ_1 1's, λ_2 2's and so on; and $x^\lambda := x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3} \cdots$; obviously some of the λ_i may vanish, and certainly λ_n is at most 1.

▶ The coefficients $\binom{n}{\lambda;k}$ also have a combinatorial meaning. We have already employed the concept of *partition* of a *set*: if S is a finite set, with $|S| = n$, a partition $\{A_1, \dots, A_k\}$ is a collection of $k \leq n$ nonempty, pairwise disjoint subsets of S, called *blocks*, whose union is S. It is often convenient to think of S as of $[n] := \{1, 2, \dots, n\}$. Suppose that in a partition of $[n]$ into k blocks there are λ_1 singletons, λ_2 two-element subsets, and so on, thereby precisely $\lambda_1 + \lambda_2 + \lambda_3 + \cdots = k$ and $\lambda_1 + 2\lambda_2 + 3\lambda_3 + \cdots = n$; sometimes k is called the length of the partition and n its weight. We just saw that the number of *ordered* $\lambda_1, \dots, \lambda_r$-tuples of subsets partitioning $[n]$ is

$$\frac{n!}{(1!)^{\lambda_1}(2!)^{\lambda_2} \dots (r!)^{\lambda_r}}.$$

Making the necessary permutations, we conclude that $[n]$ possesses $\binom{n}{\lambda;k}$ partitions of class λ. Also

$$B_{n,k}(1, \dots, 1) = \sum_\lambda \binom{n}{\lambda; k} = |\Pi_{n,k}|,$$

with $\Pi_{n,k}$ standing for the set of all partitions of $[n]$ into k subsets; the $|\Pi_{n,k}|$ are the so-called Stirling numbers of the second kind.

Later, it will be convenient to consider partitions of an *integer* n, a concept that should not be confused with partitions of the set $[n]$. A partition of n is a sequence of positive integers $(f_1, f_2, \dots, f_k)$ such that $f_1 \geq f_2 \geq f_3 \geq \dots$ and $\sum_{i=1}^k f_i = n$. The number of partitions of n is denoted $p(n)$. Now, consider a partition π of $[n]$ of type $\lambda(\pi) = (1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3} \dots)$, and let m be the largest number for which λ_m does not vanish; we put $f_1 = f_2 = \cdots = f_{\lambda_m} = m$; then we take for f_{λ_m+1} the largest number such that $\lambda_{f_{\lambda_m+1}}$ among the remaining λ's does not vanish, and so on. The procedure can be inverted, and it is clear that partitions of n can be indexed by the sequence $(f_1, f_2, \dots, f_k)$ of their definition, or by λ. The number of partitions π of $[n]$ for which $\lambda(\pi)$ represents a partition of n is precisely $\binom{n}{\lambda;k}$. To take a simple example, let $n = 4$. There are the following partitions of 4: (4)$\equiv (4^1)$, corresponding to one partition of [4]; (3,1)$\equiv (1^1, 3^1)$, corresponding to four partitions of [4]; (2,2)$\equiv (2^2)$, corresponding to three partitions of [4]; (2,1,1)$\equiv (1^2, 2^1)$, corresponding to six partitions of [4]; (1,1,1,1)$\equiv (1^4)$, corresponding to one partition of [4]. In all $p(4) = 5$, whereas the number B_4 of partitions of [4] is 15. We have $p(5) = 7$, whereas the number of partitions of [5] is $B_5 = 52$.

The results (7.3) and (7.5) are so important that to record a slightly differently worded argument to recover them will do no harm: let f, g, h be power series as above; notice that

$$h(t) = \sum_{k=0}^\infty \frac{f_k}{k!} \left(\sum_{l=1}^\infty \frac{g_l}{l!} t^l \right)^k.$$

To compute the n-th coefficient h_n of $h(t)$ we only need to consider the partial sum up to $k = n$, since the other products contain powers of t higher than n, on account of $g_0 = 0$.

Then for $n \geq 1$, from Cauchy's product formula

$$h_n = \sum_{k=1}^{n} \frac{f_k}{k!} \sum_{l_1+\cdots+l_k=n,\, 1\leq l_i} \frac{n!\, g_{l_1} \cdots g_{l_k}}{l_1! \cdots l_k!}.$$

Now, each sum $l_1 + \cdots + l_k = n$ can be written in the form $\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n$ for a unique vector $(\alpha_1, \ldots, \alpha_n)$, satisfying $\alpha_1 + \cdots + \alpha_n = k$; and since there are $k!/\alpha_1! \cdots \alpha_n!$ ways to order the g_l of each term, it again follows that

$$h_n = \sum_{k=1}^{n} \frac{f_k}{k!} \sum_{\alpha} \frac{n!k!}{\alpha_1! \cdots \alpha_n!} \frac{g_1^{\alpha_1} \cdots g_n^{\alpha_n}}{(1!)^{\alpha_1}\,(2!)^{\alpha_2} \cdots (n!)^{\alpha_n}} = \sum_{k=1}^{n} f_k\, B_{n,k}(g_1, \ldots, g_{n+1-k}),$$

where the second sum runs over the vectors fulfilling the conditions just mentioned.

The (complete, exponential) Bell polynomials Y_n are defined by $Y_0 = 1$ and

$$Y_n(x_1, \ldots, x_n) = \sum_{k=1}^{n} B_{n,k}(x_1, \ldots, x_{n+1-k});$$

that is, taking $u = 1$ in (7.2):

$$\exp\left(\sum_{m\geq 1} x_m \frac{t^m}{m!}\right) = \sum_{n\geq 0} \frac{t^n}{n!} Y_n(x_1, \ldots, x_n);$$

and the *Bell numbers* by $B_n := Y_n(1, \ldots, 1)$. It is clear that the Bell numbers coincide with cardinality of the set Π_n of all partitions of $\{1, 2, \ldots, n\}$; a fact already registered in our notation. Some amusing properties can be now derived: in formula (7.2), take $u = x_m = 1$. We get

$$\sum_{n=0}^{\infty} \frac{B_n t^n}{n!} = \exp(e^t - 1) \quad \text{or} \quad \log \sum_{n=0}^{\infty} \frac{B_n t^n}{n!} = e^t - 1.$$

Differentiating $n + 1$ times both sides, it ensues the recurrence relation:

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

The same relation is of course established by combinatorial arguments. For consider the partitions of $[n]$ as starting point to determine the number of partitions of $[n + 1]$. The number $n + 1$ must lie in a block of size $k + 1$ with $0 \leq k \leq n$, and there are $\binom{n}{k}$ choices for such a block. Once the block is chosen, the remaining $n - k$ numbers can be partitioned in B_{n-k} ways. Summing over k, one sees

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_{n-k},$$

which is the same formula.

▶ The analytical smoke has cleared, and now we put the paradigm of (2.6) in Example 1 to work. Taking our cue from Example 1, we have the right to expect that the formula

$$\Delta a_n(g,f) := a_n(f \circ g) = a_{n(1)}(f)\, a_{n(2)}(g)$$

give rise to a coproduct for the polynomial algebra generated by the coordinates a_n for n going from 1 to ∞. In other words,

$$\Delta a_n = \sum_{k=1}^{n} \sum_{\lambda} \binom{n}{\lambda; k} a^{\lambda} \otimes a_k = \sum_{k=1}^{n} B_{n,k}(a_1, \ldots, a_{n+1-k}) \otimes a_k \tag{7.6}$$

must yield a bialgebra, which is commutative but clearly not cocommutative. The quirk in defining $\Delta a_n(g,f)$ by $a_n(f \circ g)$ rather than by $a_n(g \circ f)$ owns to the wish of having the linear part of the coproduct standing on the right of the $\otimes$ sign, and not on the left.

The first few values for the coproduct will be

$$\begin{aligned}
\Delta a_1 &= a_1 \otimes a_1, \\
\Delta a_2 &= a_2 \otimes a_1 + a_1^2 \otimes a_2, \\
\Delta a_3 &= a_3 \otimes a_1 + a_1^3 \otimes a_3 + 3a_2 a_1 \otimes a_2, \\
\Delta a_4 &= a_4 \otimes a_1 + a_1^4 \otimes a_4 + 6a_2 a_1^2 \otimes a_3 + (3a_2^2 + 4a_3 a_1) \otimes a_2, \\
\Delta a_5 &= a_5 \otimes a_1 + a_1^5 \otimes a_5 + 10a_2 a_1^3 \otimes a_4 + (10a_3 a_1^2 + 15a_2^2 a_1) \otimes a_3 + (5a_4 a_1 + 10a_2 a_3) \otimes a_2.
\end{aligned} \tag{7.7}$$

The Hopf algebras of rooted trees and of Feynman graphs introduced in QFT by Kreimer and Connes [3,42], as well as the Connes–Moscovici Hopf algebra [5], are of the same general type, with a linear part of the coproduct standing on the right of the $\otimes$ sign and a polynomial one on the left. The kinship is also manifest in that, as conclusively shown in [4] —see also [43]— one can use Feynman diagrams to obtain formulae of the type of (7.6). In what follows, we shall clarify the relations, and show how all those bialgebras fit in the framework and machinery of incidence bialgebras. But before doing that, we plan to explore at leisure the obtained bialgebra and some of its applications.

We do not have a connected Hopf algebra here. Indeed, since a_1 is *grouplike*, it ought to be invertible, with inverse Sa_1. Besides, if $f^{(-1)}$ denotes the reciprocal series of f, then, according to the paradigm followed, S should be given by (5.3):

$$Sa_1 = a_1(f^{(-1)}) = a_1^{-1}(f).$$

To obtain a connected Hopf algebra it is necessary to set $a_1 = 1$; in other words, to consider only formal power series (7.1) of the form $f(t) = t + \sum_{n \geq 2} f_n\, t^n/n!$. The resulting graded connected bialgebra is hereinafter denoted $\mathcal{F}$ and called the Faà di Bruno algebra (terminology due to Joni and Rota [1]); the degree is given by $\#(a_n) = n - 1$, with the degree of a product given by definition as the sum of the degrees of the factors. If G_1 is the subgroup of diffeomorphisms of $\mathbb{R}$ such that $f(0) = 0$ and $df(0) = \mathrm{id}$, we could denote $\mathcal{F}$ by $\mathcal{R}^{\mathrm{cop}}(G_1)$. The coproduct formula is accordingly simplified as follows:

$$\Delta a_n = \sum_{k=1}^{n} \sum_{\lambda} \binom{n}{\lambda; k} a_2^{\lambda_2} a_3^{\lambda_3} \cdots \otimes a_k = \sum_{k=1}^{n} B_{n,k}(1, \ldots, a_{n+1-k}) \otimes a_k.$$

Now we go for the antipode in $\mathcal{F}$. Formula (5.5) applies, and in this context reduces to

$$Sa_n = -a_n + \sum_{j=2}^{n-1}(-1)^j \sum_{1<k_{j-1}<\cdots<k_1<n} B_{n,k_1}\, B_{k_1,k_2}\, \cdots\, B_{k_{j-2},k_{j-1}}\, a_{k_{j-1}}. \tag{7.8}$$

In (7.8) the arguments of the Bell polynomials have been suppressed for concision. In particular,

$$\begin{aligned} Sa_2 &= -a_2, \\ Sa_3 &= -a_3 + 3a_2^2, \\ Sa_4 &= -a_4 - 15a_2^3 + 10a_2a_3. \end{aligned}$$

Let us give the details in computing Sa_5:

$$\begin{aligned} Sa_5 = -a_5 &+ (B_{5,2}a_2 + B_{5,3}a_3 + B_{5,4}a_4) - (B_{5,4}B_{4,3}a_3 + B_{5,4}B_{4,2}a_2 + B_{5,3}B_{3,2}a_2) \\ &+ B_{5,4}B_{4,3}B_{3,2}a_2 \\ = -a_5 &+ 15a_2a_4 + 10a_3^2 + 25a_2^2a_3 - (130a_2^2a_3 + 75a_2^4) + 180a_2^4 \\ = -a_5 &+ 15a_2a_4 + 10a_3^2 - 105a_2^2a_3 + 105a_2^4. \end{aligned} \tag{7.9}$$

The computation using instead S_B runs as follows

$$\begin{aligned} Sa_5 &= -a_5 - 10a_2S_Ba_4 - (10a_3 + 15a_2^2)S_Ba_3 - (5a_4 + 10a_2a_3)S_Ba_2 \\ &= -a_5 + 10a_2(a_4 + 6a_2S_Ba_3 + (3a_2^2 + 4a_3)S_Ba_2) + (10a_3 + 15a_2^2)(a_3 + 3a_2S_Ba_2) \\ &\quad + 5a_2a_4 + 10a_2^2a_3 \\ &= -a_5 + 10a_2a_4 - 60a_2^2(a_3 + 3a_2S_Ba_2) - 30a_2^4 - 40a_2^2a_3 + 10a_3^2 - 30a_2^2a_3 + 15a_2^2a_3 \\ &\quad - 45a_2^4 + 5a_2a_4 + 10a_2^2a_3 \\ &= -a_5 + 10a_2a_4 - 60a_2^2a_3 + 180a_2^4 - 30a_2^4 - 40a_2^2a_3 + 10a_3^2 - 30a_2^2a_3 + 15a_2^2a_3 \\ &\quad - 45a_2^4 + 5a_2a_4 + 10a_2^2a_3 \\ &= -a_5 + 15a_2a_4 + 10a_3^2 - 105a_2^2a_3 + 105a_2^4. \end{aligned}$$

Note that in both procedures there are the same cancellations, although the expansions do not coincide term-by-term. However, since this Faà di Bruno algebra (vaguely) looks of the same general type as a Hopf algebra of Feynman graphs, this is a case where we would expect a formula à la Zimmermann, that is, without cancellations, to compute the antipode. Such a formula indeed exists, as mentioned in the introduction [19]. It leads to:

$$Sa_n = \sum_{k=1}^{n-1}(-1)^k B_{n-1+k,k}(0, a_2, a_3 \dots).$$

The elegance of this equation is immediately appealing. Using a standard identity of the Bell polynomials it can be further simplified:

$$Sa_n = \sum_{k=1}^{n-1}(-1)^k(n-1+k)\cdots n\, B_{n-1,k}\Big(\frac{a_2}{2}, \frac{a_3}{3}, \dots\Big). \tag{7.10}$$

For instance, (7.9) is recovered at once with no cancellations, the coincidence of (7.8) and the last formula actually gives nonstandard identities for Bell polynomials.

► We cannot omit a description, however brief, of the graded dual $\mathcal{F}'$. Consider the primitive elements $a'_n \in \mathcal{F}'$ given by $\langle a'_n, a_m \rangle = \delta_{nm}$, then also

$$\langle a'_n, a_p a_q \rangle = \langle \Delta a'_n, a_p \otimes a_q \rangle = \langle a'_n \otimes 1 + 1 \otimes a'_n, a_p \otimes a_q \rangle = 0,$$

if $p \geq 1$ and $q \geq 1$ —the a'_n kill nontrivial products of the a_q generators. On the other hand,

$$\begin{aligned}\langle a'_n a'_m, a_q \rangle &= \langle a'_n \otimes a'_m, \Delta a_q \rangle \\ &= \sum_{k=1}^{q} \langle a'_n \otimes a'_m, B_{q,k}(1, a_2, \dots, a_{q+1-k}) \otimes a_k \rangle \\ &= \sum_{k=1}^{q} \langle a'_n, B_{q,k}(1, a_2, \dots, a_{q+1-k}) \rangle \langle a'_m, a_k \rangle \\ &= \langle a'_n, B_{q,m}(1, a_2, \dots, a_{q+1-m}) \rangle.\end{aligned}$$

The polynomial $B_{q,m}$ is homogeneous of degree m, and the only monomial in it giving a nonvanishing contribution is the $x_1^{m-1} x_n$ term. Its coefficient is $\binom{q}{\lambda;m}$ with λ the sequence $(1^{m-1}, 0, \dots, n^1, 0, \dots)$, satisfying $q = \lambda_1 + 2\lambda_2 + \cdots + q\lambda_q = m - 1 + n$. Thus

$$\langle a'_n a'_m, a_q \rangle = \begin{cases} \binom{m+n-1}{n} & \text{if } q = m + n - 1 \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand,

$$\Delta(a_q a_r) = a_q a_r \otimes 1 + 1 \otimes a_q a_r + a_q \otimes a_r + a_r \otimes a_q + R,$$

where R is either vanishing or a sum of terms of the form $b \otimes c$ with b or c a monomial in $a_2, a_3, \dots$ of degree greater than 1. Therefore

$$\langle a'_n a'_m, a_q a_r \rangle = \langle a'_n \otimes a'_m, \Delta(a_q a_r) \rangle = \begin{cases} 1 & \text{if } n = q \neq m = r \text{ or } n = r \neq m = q, \\ 2 & \text{if } m = n = q = r, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, it is clear that all the terms of the coproduct of three or more a's are the tensor product of two monomials where at least one of them is of order greater than 1. So $\langle a'_n a'_m, a_{q_1} a_{q_2} a_{q_3} \cdots \rangle = 0$. All together gives

$$a'_n a'_m = \binom{m-1+n}{n} a'_{n+m-1} + \big(1 + \delta_{nm}\big)(a_n a_m)'.$$

In particular,

$$[a'_n, a'_m] := a'_n a'_m - a'_m a'_n = (m-n) \frac{(n+m-1)!}{n!m!} a'_{n+m-1}.$$

Therefore, taking $b'_n := (n+1)!a'_{n+1}$, we get the simpler looking

$$[b'_n, b'_m] = (m-n)b'_{n+m}. \tag{7.11}$$

The Milnor–Moore theorem implies that $\mathcal{F}'$ is isomorphic to the universal enveloping algebra of the Lie algebra defined by the last equation. The algebra $\mathcal{F}'$ can be realized by vector fields [5]. As we saw in Chapter 4, $\mathcal{F}^\circ$ is bigger and contains grouplike elements $f, g \dots$ with $g * f = f \circ g$.

We shall come back to the structure of the Faà di Bruno algebra in Section 12. Also, primitivity in the Faà di Bruno algebra is thoroughly examined in Section 15.

8 Working with the Faà di Bruno Hopf algebra

The Faà di Bruno formulae (7.2), (7.3) and (7.5), and the algebra $\mathcal{F}$ are ubiquitous in quantum field theory. We give a couple of examples and then we turn to a famous example in a combinatorial-algebraic context.

Example 8.1. Consider charged fermions in an external field. The complex space $\mathcal{H}$ of classical solutions of the corresponding Dirac equation is graded by $E_+ - E_-$, where E_+, E_- project on the particle and antiparticle subspaces; we write $\mathcal{H}^\pm := E_\pm \mathcal{H}$. Operators on $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}^-$ can be presented in block form,

$$A = \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix}.$$

In particular, a unitary operator

$$S = \begin{pmatrix} S_{++} & S_{+-} \\ S_{-+} & S_{--} \end{pmatrix}$$

corresponds to a classical scattering matrix if and only if

$$\begin{gathered} S_{++}S_{++}^\dagger + S_{+-}S_{+-}^\dagger = E_+, \quad S_{--}S_{--}^\dagger + S_{-+}S_{-+}^\dagger = E_-, \\ S_{++}S_{-+}^\dagger + S_{+-}S_{--}^\dagger = S_{--}S_{+-}^\dagger + S_{-+}S_{++}^\dagger = 0. \end{gathered}$$

By the Shale–Stinespring theorem [24, Ch. 6], this operator is implemented in Fock space iff S_{+-}, S_{-+} are Hilbert–Schmidt. Assuming this is the case, the quantum scattering matrix $\mathbb{S}$ is obtained through the spin representation [44] and it can be proved that (the square of the absolute value of) the vacuum persistence amplitude is of the form

$$|\langle 0_{\mathrm{in}}, 0_{\mathrm{out}}\rangle|^2 = |\langle 0_{\mathrm{in}}, \mathbb{S}0_{\mathrm{in}}\rangle|^2 = \det(1 - S_{+-}S_{+-}^\dagger).$$

The determinant exists because of the Shale–Stinespring condition. With A of trace class and of norm less than 1 one can use the development

$$\det(1-A) = \exp(\operatorname{Tr}\log(1-A)) =: \exp\biggl(-\sum_{k=1}^\infty \frac{\sigma_k}{k}\biggr),$$

with $\sigma_k := \operatorname{Tr} A^k$. We may want to reorganize this series, for instance to expand the exact result for $\langle 0_{\rm in}, \mathbb{S}0_{\rm in}\rangle$ in terms of coupling constants. Say

$$1 + \sum_{n=1}^{\infty} b_n := \exp\biggl(-\sum_{k=1}^{\infty}\frac{\sigma_k}{k}\biggr).$$

Our formula (7.2) gives

$$b_n = \frac{1}{n!}\sum_{k=1}^{n}(-1)^k B_{n,k}(\sigma_1, \sigma_2, 2\sigma_3, \dots, (n-k)!\sigma_{n+1-k}); \tag{8.1}$$

so, for example,

$$b_1 = -\sigma_1, \quad b_2 = \tfrac12(\sigma_1^2 - \sigma_2), \quad b_3 = -\tfrac16(\sigma_1^3 - 3\sigma_1\sigma_2 + 2\sigma_3), \dots$$

One finds [45] that

$$b_n = \frac{(-1)^n}{n!}\det\begin{pmatrix} \sigma_1 & n-1 & 0 & \dots & 0 \\ \sigma_2 & \sigma_1 & n-2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{n-1} & \sigma_{n-2} & \sigma_{n-3} & \dots & 1 \\ \sigma_n & \sigma_{n-1} & \sigma_{n-2} & \dots & \sigma_1 \end{pmatrix}.$$

Example 8.2. The first generating functional in QFT is usually defined by the expression

$$Z[j] = 1 + \sum_{n=1}^{\infty}\frac{1}{n!}\Bigl(\frac{i}{\hbar}\Bigr)^n \int d^4x_1 \cdots \int d^4x_n\, G(x_1, \dots, x_n) j(x_1)\cdots j(x_n).$$

This is often taken to be a formal expression, but actually it makes sense perturbatively if by the Green function $G(x_1, \dots, x_n)$ we understand a renormalized chronological product à la Epstein–Glaser. The second generating functional is defined:

$$W[j] := -i\hbar \log Z[j].$$

We assert that

$$W[j] = \sum_{m=1}^{\infty}\frac{1}{m!}\Bigl(\frac{i}{\hbar}\Bigr)^{m-1} \int d^4x_1 \cdots \int d^4x_m\, G_c(x_1, \dots, x_m)\, j(x_1)\cdots j(x_m);$$

where G_c denotes the *connected* Green functions [46, Ch. 5]:

$$G(x_1, \dots, x_n) := \sum_{k=1}^{n}\sum_{\lambda\in\Pi_{n,k}} G_c^{\lambda_1}(x_1) G_c^{\lambda_2}(x_1, x_2)\cdots$$

with $(1^{\lambda_1}, 2^{\lambda_2}, \dots)$ as before a partition of $[n]$ in k blocks. To prove our assertion directly, one can use the Faà di Bruno formula

$$\exp\biggl(\sum_{m\ge1}\frac{x_m}{m!}\biggr) = \sum_{n\ge1}\frac{1}{n!}\biggl[\sum_{k=1}^{n}\sum_{\lambda_1,\lambda_2\dots}\frac{n!}{\lambda_1!\lambda_2!\lambda_3!\cdots(1!)^{\lambda_1}(2!)^{\lambda_2}(3!)^{\lambda_3}\cdots}x_1^{\lambda_1}x_2^{\lambda_2}x_3^{\lambda_3}\cdots\biggr]$$

Therefore

$$\begin{aligned}\exp\Big(\frac{i}{\hbar}W[j]\Big) &= \sum_{c_1=0}^{\infty}\frac{1}{c_1!}\bigg[\frac{i}{\hbar}\int d^4x_1\,G_c(x_1)j(x_1)\bigg]^{c_1}\\ &\quad\times\sum_{c_2=0}^{\infty}\frac{1}{c_2!}\bigg[\Big(\frac{i}{\hbar}\Big)^2\frac{1}{2!}\int d^4x_1\,d^4x_2 G_c(x_1,x_2)j(x_1)j(x_2)\bigg]^{c_2}\\ &\quad\times\dots,\end{aligned}$$

which is precisely $Z[j]$.

The proof is rather spectacular; but only a simple counting principle, well known in combinatorics [47, Section 5.1], is involved here. The same principle plays a role in the cluster expansion of the $\mathbb{S}$-matrix [48, Ch. 4]: the sum of operators associated with Wick diagrams is equal to the normally ordered exponential of the sum of operators associated with the connected Wick diagrams.

The most important application in QFT concerns the *renormalization group*; but this would take us too far afield. See our comments in Section 12.

Example 8.3. In (8.1) we might need to solve for the σ's instead of the b's. Notice that to study diffeomorphisms as formal power series of the type $f(t) = t + \sum_{n\geq 2} a_n(f)t^n/n!$ it is not indispensable to use the coordinate functions a_n. Instead, one can use for example the new set of coordinates $d_n := a_{n+1}/(n+1)!$, for $n \geq 1$. Consider then

$$1+\sum_{n=1}^{\infty} d_n := \exp\bigg(\sum_{k=1}^{\infty}\omega_n\bigg) \quad\text{or}\quad \sum_{k=1}^{\infty}\omega_n = \log\bigg(1+\sum_{n=1}^{\infty} d_n\bigg),$$

to be solved for the ω_n. Since $\log(1+t) = \sum_{k\geq 1}(-1)^{k-1}(k-1)!t^k/k!$, from the very Faà di Bruno formula (7.3) it follows that

$$\omega_n = \frac{1}{n!}\sum_{k=1}^{n}(-1)^{k-1}(k-1)!B_{n,k}(d_1, 2d_2, \dots, (n+1-k)!d_{n+1-k}) =: S_n(d_1,\dots,d_n).$$

The polynomials S_n are the so-called Schur polynomials; they are well known from the theory of symmetric functions. The first four are:

$$S_1(d_1) = d_1;\quad S_2(d_1,d_2) = d_2 - \frac{d_1^2}{2};\quad S_3(d_1,d_2,d_3) = d_3 - d_1d_2 + \frac{d_1^3}{3};$$
$$S_4(d_1,d_2,d_3,d_4) = d_4 - \frac{d_2^2}{2} - d_1d_3 + d_1^2d_2 - \frac{d_1^4}{4}.$$

In view of (8.1), the inverse map expressing the d_n in terms of the ω_n is given by

$$\begin{aligned} d_n &= \frac{1}{n!}\sum_{k=1}^{n} B_{n,k}(\omega_1, 2\omega_2, \dots, (n+1-k)!\omega_{n+1-k})\\ &= \sum_{c_1+2c_2+\dots+nc_n=n}\frac{\omega_1^{c_1}\cdots\omega_n^{c_n}}{c_1!\cdots c_n!} =: AS_n(\omega_1,\dots,\omega_n);\end{aligned}$$

in the second equality the formula (7.5) has been used.

Now, we can construct an isomorphism between H_ℓ and $B(V)$, where V is the vector space with a denumerable basis $Y_1, Y_2, \dots$, by the algebraic correspondence

$$l_i \mapsto S_i(Y_1, \dots, Y_i),$$

with inverse

$$Y_i \mapsto AS_i(l_1, \dots, l_i).$$

An easy computation [49] allows one to verify that

$$\Delta_{B(V)}[AS_i(Y_1, \dots, Y_i)] = \left[\sum_{j=0}^{n} AS_j \otimes AS_{n-j}\right](Y_1, \dots, Y_i).$$

This is enough to conclude as well that the elements

$$S_i(l_1, \dots, l_i)$$

are all primitive, and constitute a basis for the space of primitive elements of H_ℓ, which is thus primitively generated. We shall come back to the isomorphism between H_ℓ and $B(V)$ in Section 14, where it plays an important role.

Example 8.4. Among the techniques for solving differential equations of the type

$$\frac{dY}{dt} = A(t)Y(t), \qquad Y(t) = 1,$$

with the variable Y being an operator valued function, there is the Magnus technique [50], where the solution is sought in the form $\exp\big(\sum_{k=1}^{\infty} \Omega_k(t)\big)$ —and an explicit form for the Ω_k is given— and the (Feynman–Dyson) time-ordered exponential, which is in fact an iterative solution of the form

$$1 + \sum_{l=1}^{\infty} P_l,$$

where the P_l are time-ordered products. Relations between both types of solutions have been rather painfully obtained in applied science papers [51, 52]. Here they are transparent; of course, it does not reduce to $\Omega_k = S_k(P_1, \dots, P_k)$, because now the P_l need not commute. It is however possible to introduce symmetrized Schur polynomials like:

$$\begin{aligned} \tilde{S}_3(d_1, d_2, d_3) &= d_3 - \tfrac{1}{2}(d_1 d_2 + d_2 d_1) + \frac{d_1^3}{3}; \\ \tilde{S}_4(d_1, d_2, d_3, d_4) &= d_4 - \tfrac{1}{2}d_2^2 - \tfrac{1}{2}(d_1 d_3 + d_3 d_1) + \frac{d_1^2 d_2 + d_1 d_2 d_1 + d_2 d_1^2}{3} - \frac{d_1^4}{4}, \end{aligned}$$

and so on; and then certainly

$$\Omega_k = \tilde{S}_k(P_1, \dots, P_k).$$

More details on this will be given in [53].

Example 8.5. Formulae for reversion of formal power series, going back to Lagrange [54], have enchanted generations of mathematicians and physicists, and there is an immense literature on them. It is very easy to prove that an element $f(x) = ax + bx^2 + cx^3 + \dots$ of $x\mathbb{C}[[x]]$ like the ones we have used to study diffeomorphisms has a reciprocal or compositional inverse $f^{(-1)}(x)$ such that $f(f^{(-1)}(x)) = x$ and $f^{(-1)}(f(x)) = x$ if and only if $a \neq 0$, in which case the reciprocal is unique; and any left or right compositional inverse must coincide with it. For computing it, many methods are available. One finds in the handbook of mathematical functions [55] the recipe: given

$$y = ax + bx^2 + cx^3 + dx^4 + ex^5 + \dots$$

then

$$x = Ay + By^2 + Cy^3 + Dx^4 + Ey^5 + \dots$$

where

$$\begin{aligned} aA &= 1, \\ a^3B &= -b \\ a^5C &= 2b^2 - ac \\ a^7D &= 5abc - a^2d - 5b^3 \\ a^9E &= 6a^2bd + 3a^2c^2 + 14b^4 - a^3e - 21ab^2c \\ &\dots \end{aligned} \tag{8.2}$$

Let us translate the search for $f^{(-1)}$ into algebraic-combinatorial terms; everything that follows should be pretty obvious. As $\mathcal{F} \simeq \mathcal{R}^{\mathrm{cop}}(G)$, at least 'morally' speaking, one expects to get back to the group G (or rather G^{opp}) by means of the Tannaka–Kreĭn paradigm. Since $\mathbb{R}$ is commutative, the set $\mathrm{Hom}_{\mathrm{alg}}(\mathcal{F}, \mathbb{R})$ of all algebra morphisms is a group under convolution. Now, the action of $f \in \mathrm{Hom}_{\mathrm{alg}}(\mathcal{F}, \mathbb{R})$ is determined by its values on the a_n. The map

$$f \mapsto f(t) = \sum_{n=1}^{\infty} f_n \frac{t^n}{n!},$$

where $f_n := \langle f, a_n \rangle$, establishes a bijection from $\mathrm{Hom}_{\mathrm{alg}}(\mathcal{F}, \mathbb{R})$ onto the set of formal (exponential) power series over the reals such that $f_0 = 0$ and $f_1 = 1$. We know that such series form a group under the operation of functional composition.

This correspondence is an *anti-isomorphism* of groups: there is really nothing to prove, but we may go through the motions again. Indeed, let $f, g \in \mathrm{Hom}_{\mathrm{alg}}(\mathcal{F}, \mathbb{R})$; then

$$f * g(a_n) = m(f \otimes g)\Delta a_n = \sum_{k=1}^{n} g_k B_{n,k}(1, f_2, \dots, f_n),$$

where we took in consideration that $f_1 = \langle f, 1 \rangle = 1$. This is the same as the n-th coefficient of $h(t) = g(f(t))$.

In other words, $G = \mathrm{Hom}_{\mathrm{alg}}(\mathcal{F}^{\mathrm{cop}}, \mathbb{R})$. By our discussion in Section 5, the antipode of $\mathcal{F}^{\mathrm{cop}}$ is $S^{-1} = S$. Therefore, making allowance for our prior choice of $a = 1$ —an almost trivial matter on which we shall reflect later— and the use of ordinary instead of exponential series, whenever f and g are formal exponential power series with $f_0 = 0$ and $f_1 = 1$ verifying $f \circ g(x) = x$ or $g \circ f(x) = x$, the formulae (8.2) correspond to

$$a_n(g) = Sa_n(f).$$

Indeed $Sa_2 = -a_2$ gives $B = -b$; $Sa_3 = -a_3 + 3a_2^2$ gives $3!C = -3!c + 3(2!b)^2$, that is $C = 2b^2 - c$; $Sa_4 = -a_4 - 15a_2^3 + 10a_2a_3$ gives $4!D = -4!d + 10(2!b)(3!c) - 15(2!b)^3$, that is $D = 5bc - d - 5b^3$; and so on.

Lagrange reversion is usually proved by Cauchy's theorem in the context of analytic functions, where it is known as the Lagrange–Bürmann formula; or by matrix algebraic methods. Its derivation by Hopf algebraic methods in [19] is particularly elegant, but it is worthwhile to note that the equivalent of formula (7.10) that they use was known prior to that derivation.

9 From Faà di Bruno to Connes–Moscovici

The reader familiar with the "noncommutative geometry" Connes–Moscovici algebras will have noticed the similitude between the commutative Hopf subalgebra of $\mathcal{H}_{\mathrm{CM}}(1)$ —the simplest of those— and the Faà di Bruno algebra $\mathcal{F}$. In fact, they are one and the same. To see this, remember that to study diffeomorphisms as formal power series it is not necessary to use the coordinates a_n. Without losing information, a description of $\mathcal{F}$ can be done in terms of the new set of coordinates

$$\delta_n(f) := \big[\log f'(t)\big]^{(n)}(0), \quad n \geq 1.$$

Consider then

$$h(t) := \sum_{n\geq 1} \delta_n(f)t^n/n! = \log f'(t) = \log\big(1 + \sum_{n\geq 1} a_{n+1}(f)t^n/n!\big).$$

From the formula (7.3) it follows that

$$\delta_n = \sum_{k=1}^{n} (-1)^{k-1}(k-1)!B_{n,k}(a_2, \dots, a_{n+2-k}) =: L_n(a_2, \dots, a_{n+1}). \tag{9.1}$$

The polynomials L_n (closely related to the Schur polynomials) are called *logarithmic polynomials* in combinatorics: they give the successive derivatives of (exponential) series of the form $\log\big(f(t)\big)$.

On the other hand, from the series expression of $\exp(h(t)) = f'(t)$ we see that

$$a_{n+1} = \sum_{k=1}^{n} B_{n,k}(\delta_1, \dots, \delta_{n+1-k}) =: Y_n(\delta_1, \dots, \delta_n), \tag{9.2}$$

and we have inverted (9.1). Recall that

$$Y_n(\delta_1, \dots, \delta_n) = \sum_{\lambda\in\Pi_n} \delta_1^{\lambda_1} \cdots \delta_n^{\lambda_n},$$

where as usual λ_j is the number of blocks of size j in a partition λ of $[n]$.

Now, from (9.1) we get $\delta_1 = a_2, \delta_2 = a_3 - a_2^2, \delta_3 = a_4 - 3a_2a_3 + 2a_2^3$, and $\delta_4 = a_5 - 3a_3^2 - 4a_2a_4 + 12a_2^2a_3 - 6a_2^4$, and since the coproduct is an algebra morphism, by use of (7.7) we obtain the coproduct in the Connes–Moscovici coordinates. For instance, for the first few generators,

$$
\begin{aligned}
\Delta\delta_1 &= \delta_1 \otimes 1 + 1 \otimes \delta_1, \\
\Delta\delta_2 &= \delta_2 \otimes 1 + 1 \otimes \delta_2 + \delta_1 \otimes \delta_1, \\
\Delta\delta_3 &= \delta_3 \otimes 1 + 1 \otimes \delta_3 + 3\delta_1 \otimes \delta_2 + (\delta_1^2 + \delta_2) \otimes \delta_1, \\
\Delta\delta_4 &= \delta_4 \otimes 1 + 1 \otimes \delta_4 + 6\delta_1 \otimes \delta_3 + (7\delta_1^2 + 4\delta_2) \otimes \delta_2 + (3\delta_1\delta_2 + \delta_1^3 + \delta_3) \otimes \delta_1.
\end{aligned}
$$

In conclusion, the commutative subalgebra of the Connes–Moscovici Hopf algebra is but an avatar of the Faà di Bruno algebra: algebras that differ only by the basis presentation we of course do not distinguish.

The task is now to assemble the whole of $\mathcal{H}_{\mathrm{CM}}(1)$ from $\mathcal{F}$. For that, it is required to look closer at the structure of G, and just a bit more of Hopf algebra theory.

A *matched pair* of groups is by definition a group G with two subgroups G_1, G_2 such that $G = G_2G_1, G_1 \cap G_2 = 1$. The affine subgroup G_2 of transformations in $\mathrm{Diff}^+(\mathbb{R})$ given by $t \to \beta + \alpha t$, with $\beta \in \mathbb{R}, \alpha \in \mathbb{R}^+$ and the group G_1 of transformations tangent to the identity, considered in Section 8, give such a bijection onto G.

Since matched pairs are conspicuous by their absence in most textbooks, we discuss the situation in some detail. The definition implies that there is a natural action of G (and in particular of G_2) on the *homogeneous space* $G_1 \simeq G_2\backslash G$, and vice versa of G (thus of G_1) on $G_2 \simeq G/G_1$. Both actions are given by composition. In detail: given $\psi \in G$, we decompose it as $\psi = kf$, with $k \in G_2$, $f \in G_1$, and

$$
\beta = \psi(0), \quad \alpha = \psi'(0), \quad f(t) = \frac{\psi(t) - \psi(0)}{\psi'(0)}.
$$

The right action of $\psi \in G$ on $f \in G_1$ is given by

$$
[f \triangleleft \psi](t) = \frac{f(\psi(t)) - f(\psi(0))}{\psi'(0) f'(\psi(0))};
$$

note $[f \triangleleft \psi](0) = 0, [f \triangleleft \psi]'(0) = 1$; and in particular

$$
[f \triangleleft (\beta, \alpha)](t) = \frac{f(\alpha t + \beta) - f(\beta)}{\alpha f'(\beta)}. \tag{9.3}
$$

The left action of G on G_2 is seen from consideration of $\psi(\beta + \alpha t)$, and we obtain in particular for $\psi = f \in G_1$:

$$
f \triangleright (\beta, \alpha) = (f(\beta), \alpha f'(\beta)). \tag{9.4}
$$

Note that $(k_1f_1)(k_2f_2) = k_1(f_1 \triangleright k_2)(f_1 \triangleleft k_2)f_2$ and $(kf)^{-1} = (f^{-1} \triangleright k^{-1})(f^{-1} \triangleleft k^{-1})$, as general equalities of the theory of matched pairs.

We ought to translate the previous considerations into Hopf-algebraic terms. In what follows we consider bialgebras acting on algebras from the left. On $\mathbb{C}$ they are assumed to act by $h \cdot 1 = \eta(h)\, 1$.

Definition 9.1. A (left, Hopf) *H-module algebra* A is an algebra which is a left module for the algebra H such that the defining maps $u : \mathbb{C} \to A$ and $m : A \otimes A \to A$ intertwine the action of H:

$$u(h \cdot 1) = h \cdot u(1); \quad m(h_{\otimes} \cdot (a \otimes b)) = h \cdot m(a \otimes b) \tag{9.5}$$

that is, $h \cdot 1_A = \eta(h)\, 1_A$ and $h \cdot (ab) = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot b)$ whenever $a, b \in A$ and $h \in H$.

The first condition is actually redundant [57]. It is easy to see that the definition corresponds to the usual notions of groups acting by automorphisms, or Lie algebras acting by derivations. If h is a *primitive* element of H, then (9.5) entails that $h \cdot 1_A = 0$ and that $h \cdot (ab) = (h \cdot a)b + a(h \cdot b)$: as already observed, primitive elements act by derivations. Therefore (9.5) may be regarded as a generalized Leibniz rule.

(Analogously one defines H-module coalgebras, H-module bialgebras and so on; we have no use for them in this article.)

When there is an algebra covariant under a group or Lie algebra, one constructs in a standard way the semidirect product algebra. In the same spirit [59]:

Definition 9.2. Let H be a Hopf algebra and A a (left) Hopf H-module algebra. The (untwisted) *smash product* or *crossed product* algebra, denoted $A \# H$ or $A \rtimes H$, is the vector space $A \otimes H$ endowed with unit $1 \otimes 1$ and the product

$$(a \otimes h)(b \otimes k) := \sum a(h_{(1)} \cdot b) \otimes h_{(2)} k.$$

We verify associativity of the construction:

$$\begin{aligned}
(a \otimes h)\big((b \otimes k)(c \otimes l)\big) &= (a \otimes h)(b(k_{(1)} \cdot c) \otimes k_{(2)} l) \\
&= a\Big(h_{(1)} \cdot \big(b(k_{(1)} \cdot c)\big)\Big) \otimes h_{(2)} k_{(2)} l \\
&= a(h_{(1)} \cdot b)(h_{(2)} k_{(1)} \cdot c) \otimes h_{(3)} k_{(2)} l \\
&= (a(h_{(1)} \cdot b) \otimes h_{(2)} k)(c \otimes l) \\
&= \big((a \otimes h)(b \otimes k)\big)(c \otimes l).
\end{aligned}$$

To alleviate the notation, we can identify $a \equiv a \otimes 1$ and $h \equiv 1 \otimes h$. Then $ah = (a \otimes 1)(1 \otimes h)$, whereas

$$ha = (1 \otimes h)(a \otimes 1) = h_{(1)} \cdot a \otimes h_{(2)} = (h_{(1)} \cdot a) h_{(2)}.$$

Both A and H are subalgebras of $A \rtimes H$. A very simple example is given by $H = U(\mathfrak{g})$, where $\mathfrak{g}$ is the 1-dimensional Lie algebra acting as $\frac{d}{dx}$ on $A = \mathbb{C}[x]$. Then $A \# H$ is the Weyl algebra. In our case H will be the enveloping algebra $\mathcal{U}(\mathfrak{g}_2)$ of the affine group Lie algebra $\mathfrak{g}_2$; and the left module algebra is none other than $\mathcal{F}$.

We exhibit then the action of $\mathcal{U}(\mathfrak{g}_2)$ on $\mathcal{F}$, and check that the Faà di Bruno algebra is a Hopf $\mathcal{U}(\mathfrak{g}_2)$-module algebra. Consider the action on G_1 given by (9.3). We proceed step by step. With $\beta = 0$, one gathers

$$a_n([f \triangleleft (0, \alpha)]) := [f \triangleleft (0, \alpha)]^{(n)}(0) = \alpha^{n-1} a_n(f),$$

for $n \geq 2$. At the infinitesimal level, with $(0, \alpha) =: \exp sY$, we conclude the existence of an action:

$$Y \cdot a_n = (n-1)a_n. \tag{9.6}$$

Similarly

$$Y \cdot \delta_n = n\delta_n. \tag{9.7}$$

Note that $Y\cdot$ is the grading operator for $\mathcal{F}$. At this stage, one could construct already the crossed product of $\mathcal{F}$ by the grading.

We proceed to the other generator. With $\alpha = 1$, equation (9.3) now gives

$$[f \triangleleft (\beta, 1)]^{(n)}(0) = \left(\frac{f(t+\beta) - f(\beta)}{f'(\beta)}\right)^{(n)}(0).$$

At the infinitesimal level, with $(\beta, 1) =: \exp \beta X$, we conclude:

$$X \cdot a_n = a_{n+1} - a_2 a_n. \tag{9.8}$$

At this point, if not before, one sees the wisdom of Connes and Moscovici's logarithmic coordinates, as then (9.3) mutates into

$$\log[f \triangleleft (\beta, \alpha)]'(t) = \log f'(\beta + \alpha t),$$

up to a constant; from which one infers the simpler action

$$X \cdot \delta_n = \delta_{n+1}. \tag{9.9}$$

Both X and Y act by derivation; thus $\mathcal{F}$ is a $\mathcal{U}(\mathfrak{g}_2)$-module algebra. Here the relations $Za = Z \cdot a + aZ$ hold, for $Z = X$ or Y. Note that $\mathcal{F}$ is *not* a $\mathcal{U}(\mathfrak{g}_2)$-module bialgebra. By inspection of [5], and summarizing so far, we conclude that the algebra $\mathcal{H}_{\mathrm{CM}}(1)$ is the smash product $\mathcal{F} \rtimes \mathcal{U}(\mathfrak{g}_2)$ of the enveloping algebra $\mathcal{U}(\mathfrak{g}_2)$ of the affine Lie algebra by the Faà di Bruno Hopf algebra; it is generated just by $Y, X, a_2 = \delta_1$.

We can as well regard the algebra $\mathcal{H}_{\mathrm{CM}}(1)$ as the enveloping algebra of the extension $\mathcal{L}'$ of $\mathfrak{g}_2$ by the abelian Lie algebra $\mathcal{L}$ spanned by the a_n, which is a derivation $\mathfrak{g}_2$-module. Recall that an extension of this type is an exact sequence

$$0 \to \mathcal{L} \xrightarrow{i} \mathcal{L}' \xrightarrow{h} \mathfrak{g}_2 \to 0,$$

where $[x, i(a)] := i(h(x) \cdot a)$, for $x \in \mathcal{L}', a \in \mathcal{L}$; this lifts to an exact sequence of enveloping algebras, in our case

$$0 \to \mathcal{F} \to \mathcal{H}_{\mathrm{CM}}(1) \to \mathcal{U}(\mathfrak{g}_2) \to 0.$$

It is permissible to write $[X, \delta_n]$ for $X \cdot \delta_n$, and so on; but we do not use that notation.

▶ The burning question now is, what is the "good" coproduct on the smash product algebra $\mathcal{F} \# \mathcal{U}(\mathfrak{g}_2)$? But before tackling that, we pause to redeliver (9.6) from Connes and

Moscovici's (9.7) and (9.8) from (9.9) by the combinatorial argument. Since X, Y are derivations

$$Y \cdot (\delta_1^{c_1} \cdots \delta_n^{c_n}) = (1c_1 + 2c_2 + \cdots + nc_n)\delta_1^{c_1} \cdots \delta_n^{c_n}, \tag{9.10}$$

$$X \cdot (\delta_1^{c_1} \cdots \delta_n^{c_n}) = \sum_{j=1}^{n} c_j \delta_1^{c_1} \cdots \delta_j^{c_j - 1} \delta_{j+1}^{c_{j+1}+1} \cdots \delta_n^{c_n}. \tag{9.11}$$

Thus, using (9.2) and (9.10)

$$\begin{aligned} Y \cdot a_{n+1} = Y \cdot Y_n(\delta_1, \ldots, \delta_n) &= \sum_{c \in \Pi_n} Y \cdot (\delta_1^{c_1} \cdots \delta_n^{c_n}) \\ &= \sum_{c \in \Pi_n} (1c_1 + 2c_2 + \cdots + nc_n)\, \delta_1^{c_1} \cdots \delta_n^{c_n} \\ &= nY_n(\delta_1, \ldots, \delta_n) = na_{n+1}. \end{aligned}$$

We have verified (9.6). Similarly, (9.11) entails

$$X \cdot a_{n+1} = \sum_{c \in \Pi_n} X \cdot (\delta_1^{c_1} \cdots \delta_n^{c_n}) = \sum_{c \in \Pi_n} \sum_{j=1}^{n} c_j\, \delta_1^{c_1} \cdots \delta_j^{c_j - 1} \delta_{j+1}^{c_{j+1}+1} \cdots \delta_n^{c_n}.$$

Notice that

$$1c_1 + 2c_2 + \cdots + j(c_j - 1) + (j+1)(c_{j+1} + 1) + (j+2)c_{j+2} + \cdots + nc_n = n + 1;$$

therefore we must think of partitions of $[n+1]$. Now, by deleting the number $n+1$ such partitions give a partition of $[n]$. Furthermore, if c is a partition of $[n]$, then the partitions of $[n+1]$ that give rise to c by dropping $n+1$, are obtained by either adding the singleton $\{n+1\}$ or by inserting $n+1$ in one of the blocks of c. Conversely, all partitions of $[n+1]$ are obtained from one of $[n]$ by either of these two procedures. Moreover, if c is of class $1^{c_1}2^{c_2} \cdots n^{c_n}$, and $n+1$ is included in a block of size j of c we obtain a partition of $[n+1]$ of class $1^{c_1}2^{c_2} \cdots j^{c_j - 1}(j+1)^{c_{j+1}+1} \cdots n^{c_n}$, and there are as many of these partitions as blocks of size j; namely c_j. Thus

$$\begin{aligned} a_{n+2} = Y_{n+1}(\delta_1, \ldots, \delta_{n+1}) &= \sum_{e \in \Pi_{n+1}} \delta_1^{e_1} \cdots \delta_{n+1}^{e_{n+1}} \\ &= \sum_{c \in \Pi_n} \sum_{j=1}^{n} c_j \delta_1^{c_1} \cdots \delta_j^{c_j - 1} \delta_{j+1}^{c_{j+1}+1} \cdots \delta_n^{c_n} + \delta_1 \sum_{c \in \Pi_n} \delta_1^{c_1} \cdots \delta_n^{c_n} \\ &= X \cdot a_{n+1} + \delta_1\, a_{n+1}, \end{aligned}$$

which is precisely (9.8).

We return to the search for a *compatible* coalgebra structure on $\mathcal{F} \rtimes \mathcal{U}(\mathfrak{g}_2)$. Given a Hopf algebra H, one considers on $\mathbb{C}$ the coaction given by $\gamma(1) = u(1)$.

Definition 9.3. A (right) Hopf H*-comodule coalgebra* C is a coalgebra which is a right comodule for the coalgebra H such that the counit map and the coproduct on C intertwine the coaction of H:

$$(\eta \otimes \mathrm{id})\gamma = \gamma\eta; \quad (\Delta \otimes \mathrm{id})\gamma = \gamma_\otimes \Delta.$$

In this context, the *smash coproduct* or crossed coproduct coalgebra $H \ltimes C$ is defined as the vector space $H \otimes C$ endowed with the counit $\eta_H \otimes \eta_C$ and the coproduct

$$\Delta(a \otimes c) = \sum a_{(1)} \otimes c^{(1)}_{(\underline{1})} \otimes a_{(2)} c^{(2)}_{(1)} \otimes c_{(2)}.$$

Now comes "the revenge of the Faà di Bruno coalgebra", since for the identification of the Connes–Moscovici Hopf algebra we need $H = \mathcal{F}$ to coact on the coalgebra $C = \mathcal{U}(\mathfrak{g}_2)$. One ought to be careful here, as we are about to use the coalgebra structure of $\mathcal{F}$ for the first time, and actually $\mathcal{F} \not\simeq \mathcal{R}(G_1)$, but $\mathcal{F} \simeq \mathcal{R}^{\mathrm{cop}}(G_1)$!

To see why $\mathcal{F}$ naturally coacts on $\mathcal{U}(\mathfrak{g}_2)$, note that, to implement the action (9.4) on G_2 of the diffeomorphisms tangent to the identity, and reasoning as above, we would have an action of $\mathcal{F}'^{\mathrm{opp}}$ on a suitable algebra $C(G_2)$ of functions on G_2. This we can choose to regard as a coaction of $\mathcal{F}^{\mathrm{cop}}$ on $C(G_2)$ [29, Section 1.2], or better on its dual coalgebra $U(\mathfrak{g}_2)$. At the infinitesimal level, if $(\beta, \alpha) = \exp(\beta X)\exp(\log\alpha Y)$ we see that $f \cdot Y = Y$ for all f, but

$$f \cdot X = \frac{d}{d\beta}\bigg|_{\alpha=1,\beta=0} [f \triangleleft (\alpha, \beta)] = X + f''(0)Y = X + a_2(f)Y.$$

The astuteness of Connes and Moscovici's definition of $\mathcal{H}_{\mathrm{CM}}(1)$ is precisely that this information on the structure of the diffeomorphism group is made patent. To the last expressions does correspond the coaction $\gamma : U(\mathfrak{g}_2) \to U(\mathfrak{g}_2) \otimes \mathcal{F}^{\mathrm{cop}}$ given by $\gamma(Y) = Y \otimes 1, \gamma(X) = X \otimes 1 + Y \otimes a_2$.

The smash coproduct structure $\mathcal{F}^{\mathrm{cop}} \ltimes \mathcal{U}(\mathfrak{g}_2)$ is given by

$$\begin{aligned} \Delta^{\mathrm{cop}} X &:= \Delta(1 \otimes X) = 1 \otimes 1 \otimes 1 \otimes X + 1 \otimes X \otimes 1 \otimes 1 + 1 \otimes Y \otimes a_2 \otimes 1 \\ &= 1 \otimes X + X \otimes 1 + Y \otimes a_2, \end{aligned} \tag{9.12}$$

the other pertinent coproducts staying unchanged. It is time to return to the use of $\mathcal{F}$, instead of $\mathcal{R}(G_1)$, and to account for that it is enough to turn (9.12) round; therefore we finally have

$$\Delta X = X \otimes 1 + 1 \otimes X + a_2 \otimes Y.$$

In conclusion, we have proved the following result.

Theorem 9.1. *The Connes–Moscovici bialgebra is the crossed product algebra and coalgebra of the enveloping algebra of the Lie algebra of the affine group and the Beatus Faà di Bruno bialgebra.*

This conclusion is restated in [58]. The construction is similar to Majid's bicrossedproduct bialgebra, although with important differences of detail. In [59] and [60] the canonical link to the matched pair of groups situation is also highlighted; and one might borrow the notation $\mathcal{F} \bowtie \mathcal{U}(\mathfrak{g}_2)$ for $\mathcal{H}_{\mathrm{CM}}(1)$. Furthermore, $\mathcal{F} \bowtie \mathcal{U}(\mathfrak{g}_2)$ is automatically a Hopf algebra; in fact

$$0 = m(\mathrm{id} \otimes S)\Delta X = m(X \otimes 1 + 1 \otimes SX + a_2 \otimes SY) = X + SX - a_2 Y$$

entails that the only nontrivial formula for the good antipode is given by $SX = -X + a_2Y$. The same result is obtained from the formula $S(aZ) = \sum SZ^{(1)}S(aZ^{(2)})$ in [59, Ch. 6], with $a = 1, Z = X$. Note that S is of infinite order, as

$$S^{2n}X = X + na_2 \quad \text{and} \quad S^{2n+1}X = -X + a_2Y - na_2.$$

The natural pairing between $\mathcal{H}_{\mathrm{CM}}(1)$ and $C(G_2) \rtimes G_1$ given by

$$\langle aZ_k, gf\rangle = g(k)a(f),$$

where g is a function on G_2 and Z_k is the element of $\mathfrak{g}_2$ that gives $k \in G_2$ by exponentiation, is also transparent.

Similarly, $\mathcal{H}_{\mathrm{CM}}(n) \simeq \mathcal{F}(n) \bowtie (\mathbb{R}^n \rtimes \mathfrak{gl}(n;\mathbb{R}))$, with $\mathcal{F}(n)$ the n-coloured Faà di Bruno Hopf algebra (on which something will be said later).

Given the role accorded in this work to the Faà di Bruno algebras, a reference to the somewhat checkered past of the Beatus Faà di Bruno formula might not be out of place. Consult the excellent article by Johnson [61], where, besides a nice derivation of the formula very much in the spirit of our Section 7, it is recalled that there is no proof for (7.4) in [40], and explained that formulae equivalent to it were present in the mathematical literature from the beginning of the nineteenth century. Also the further historical investigation [62] deserves a look.

Part III: Hopf Algebras of Graphs and Distributive Lattices

10 Hopf algebras of Feynman graphs

We start this part by an invitation, too: some readers will want to see a worked out example from perturbative quantum field theory, where the combinatorics of renormalization leads to a finite 'renormalized' graph by means of local counterterms, from a given Feynman graph. The analytical complications militate against developing an example that is really challenging from the combinatorial viewpoint; but this cannot be helped. Needless to say, the experts can skip the following discussion.

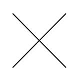
To make matters as simple as possible, our example will be taken from the (Ginzburg–Landau) φ_4^4 scalar model in Euclidean field theory. We work in the dimensional regularization scheme in momentum space [63, 64] and display the counterterms *ab initio*. The free energy functional is given by

$$E_{\text{free}}[\varphi] = \int d^D x \left[\frac{1}{2}\varphi\Box\varphi + \frac{1}{2}m^2\varphi^2\right].$$

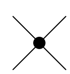
Here $\Box$ denotes the D-dimensional Laplacian. The interaction part is extended by counterterms, of the form

$$E_{\text{int}}[\varphi] = \int d^D x \left[\frac{\mu^\varepsilon \tilde{g}}{4!}\varphi^4 + c_g\frac{\mu^\varepsilon \tilde{g}}{4!}\varphi^4 + \frac{c_\varphi}{2}\varphi\Box\varphi + \frac{c_{m^2}}{2}m^2\varphi^2\right]. \tag{10.1}$$

Denote $\varepsilon := 4 - D$. The definition of the original vertex includes the mass parameter μ, introduced to make $\tilde{g}$ dimensionless:

$$g = \tilde{g}\mu^\varepsilon =$$  .

The counterterms produce additional vertices in the diagrammatic expansion. In particular:

$$c_g\,\tilde{g}\mu^\varepsilon =$$  .

Dimensional analysis indicates that all the counterterms are dimensionless; therefore they can only depend on $\varepsilon, \tilde{g}$ or combinations like $m^2/\mu^2, k^2/\mu^2$. It turns out that the last one appears just in intermediate stages as $\log(k^2/\mu^2)$ and that these nonlocal terms cancel in the final expressions —this is the key to the whole affair. Thus c_{m^2}, c_φ, c_g depend only on $\varepsilon, \tilde{g}, m^2/\mu^2$ (and the dependence on m^2/μ^2 can be made to disappear).

The quantities $\varphi, m, \tilde{g}$ in the previous displays are the *renormalized* field, mass and coupling constant. The original form of the theory is recovered by multiplicative renormalization:

$$Z_\varphi = 1 + c_\varphi, \quad Z_{m^2} = 1 + c_{m^2}, \quad Z_g = 1 + c_g.$$

The total energy functional becomes

$$E[\varphi] = \int d^D x \left[\frac{1}{2}Z_\varphi\,\varphi\Box\varphi + \frac{1}{2}Z_{m^2}\,m^2\varphi^2 + \frac{\mu^\varepsilon\tilde{g}}{4!}Z_g\,\varphi^4\right],$$

with the ε-dependent coefficients. Introduction of the bare field and the bare mass and coupling:

$$\varphi_B = \sqrt{Z_\varphi}\,\varphi, \quad m_B^2 = \frac{Z_{m^2}}{Z_\varphi}, \quad \tilde{g}_B = \frac{Z_g}{Z_\varphi^2}\,\mu^\varepsilon \tilde{g},$$

brings the energy functional to the standard form

$$E[\varphi_B] = \int d^D x \left[\frac{1}{2}\,\varphi_B \Box \varphi_B + \frac{1}{2}\,m_B^2 \varphi_B^2 + \frac{\tilde{g}_B}{4!}\,\varphi_B^4\right].$$

The bare quantities here are functions of the renormalized quantities $\tilde{g}$, m, the mass scale μ, and ε.

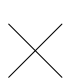
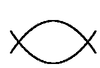
The Feynman rules for the model are next recalled; instead of using directly the modified Feynman rules including the counterterms, we will 'discover' the latter in the process of renormalization. We plan to concentrate on the proper (1PI) vertex function $\Gamma^{(4)}(k_1, \dots, k_4)$, which is defined only for $k_1 + \dots + k_4 = 0$ and is represented by *amputated* diagrams. This means that we only need:

- A propagator factor $\dfrac{1}{p_j^2 + m^2}$, with $j \in \{1, \dots, I = 2p - 2\}$, where p is the approximation order (number of vertices), for each internal line. Each internal momentum p_j is expressed in terms of the loop momenta and the four external momenta.
- An integration $(2\pi)^{-D} \int d^4 l_m$ over each (independent) loop momentum with the index $m \in \{1, \dots, L = I - p + 1 = p - 1\}$.
- The factors $\tilde{g}\mu^\varepsilon$, one for each vertex.
- The weight factor of the graph.

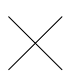
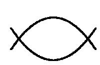
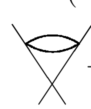
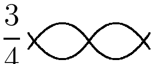
We also briefly recall the counting of ultraviolet divergences. According to the rules, a Feynman integral I_Γ with p vertices and four external lines contains $L = p - 1$ loop integrations and thus in the numerator of the integrand a power $D(p-1)$ of the momentum appears. Each of the internal lines contributes a propagator. Thus there are altogether

$$\omega(\Gamma) := D(p-1) - 2(2p-2) = \varepsilon(1-p)$$

powers of momentum in such a Feynman integral. This power $\omega(\Gamma)$ is the *superficial degree of divergence* of Γ. As $\varepsilon \downarrow 0$, all four-point proper graphs are superficially logarithmically *divergent*. A graph is said to have *subdivergences* if it contains a superficially divergent subdiagram γ, that is to say, with $\omega(\gamma) \geq 0$ in that limit. The only possibly divergent subintegrations are those of the two- and four-point subdiagrams. Up to one loop, we have

$\Gamma^{(4)}(k_1, \dots, k_4) =$  $+ \dfrac{3}{2}$  $+$ counterterm $+ O(g^3)$.

Actually '$\frac{3}{2}$ ' stands for three integrals of the same form, in terms of the 'Mandelstam' variables $s = (k_1 + k_2)^2$, $t = (k_1 + k_3)^2$ and $u = (k_1 + k_4)^2$. In the next to leading order:

$\Gamma^{(4)}(k_1, \dots, k_4) =$  $+ \dfrac{3}{2}$  $+ 3$  $+ \dfrac{3}{4}$  $+$ counterterms $+ O(g^4)$.

A similar comment applies in relation to the s, t, u variables. (We ignore a fish-cum-tadpole graph that would easily be taken into account anyway; only at three-loop order would we have to handle in parallel the renormalization of two-point diagrams in earnest.)

We compute the fish graph :

$$I_{\rm fish}(k) = \tilde{g}^2\mu^{2\varepsilon}\int \frac{d^Dp}{(2\pi)^D}\,\frac{1}{p^2+m^2}\,\frac{1}{(p+k)^2+m^2},$$

where $k = k_1 + k_2$, say. Using Feynman's formula

$$\frac{1}{A^aB^b} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 dx\,\frac{x^{a-1}(1-x)^{b-1}}{[Ax+B(1-x)]^{a+b}},$$

we obtain

$$\begin{aligned}
I_{\rm fish}(k) &= g^2\int_0^1 dx\int\frac{d^Dp}{(2\pi)^D}\,\frac{1}{\{(p^2+m^2)(1-x)+[(p+k)^2+m^2]x\}^2}\\
&= g^2\int_0^1 dx\int\frac{d^Dp}{(2\pi)^D}\,\frac{1}{(p^2+2pkx+k^2x+m^2)^2}\\
&= \frac{g^2\,\Gamma(2-D/2)}{(4\pi)^{D/2}}\int_0^1 dx\,\frac{1}{(sx(1-x)+m^2)^{2-D/2}}\\
&= \frac{\tilde{g}^2\mu^\varepsilon\Gamma(\varepsilon/2)}{(4\pi)^2}\int_0^1 dx\left[\frac{4\pi\mu^2}{(sx(1-x)+m^2)}\right]^{\varepsilon/2}.
\end{aligned}$$

Expanding now partially in powers of ε, this yields

$$I_{\rm fish}(s) = \tilde{g}\mu^\varepsilon\frac{\tilde{g}}{(4\pi)^2}\left[\frac{2}{\varepsilon}+\psi(1)+\int_0^1 dx\,\log\frac{4\pi\mu^2}{sx(1-x)+m^2}+O(\varepsilon)\right]. \tag{10.2}$$

Here $\psi(z) := \Gamma'(z)/\Gamma(z)$ is the digamma function. Some comments are already in order. The divergence happens for D an even integer greater than or equal to 4 (so it is clearly a logarithmic one). The remaining integral is finite as long as $m^2 \neq 0$. In equation (10.2) we have separated $\tilde{g}\mu^\varepsilon$, that will become the coupling constant, and is not expanded in powers of ε; only the expression multiplying it contributes to the renormalization constant Z_g with a pole term independent of the free mass scale μ. That parameter appears only in the finite part, and the arbitrariness of its choice exhibits a degree of ambiguity in the regularization procedure.

Now comes the renormalization prescription. Suppose we wanted the value of $\Gamma^{(4)}(k_i)$ at $k_i = 0$ to be (finite and) equal, at least at the present order, to the renormalized coupling constant

$$\Gamma^{(4)}(0) = g. \tag{10.3}$$

This would lead us to take

$$Z_g = 1 + \tfrac{3}{2}\,\tilde{g}\,I_{\rm fish}(0),$$

that is, a counterterm given by

$$c_g(\varepsilon, \tilde{g}, m^2/\mu^2) = -\frac{\tilde{g}}{(4\pi)^2}\left[\frac{6}{\varepsilon} + 3\psi(1) - 3\log(4\pi m^2/\mu^2)\right].$$

However, other choices like $c_g = -6\tilde{g}/(4\pi)^2\varepsilon$, not 'soaking up' the finite terms, would be admissible. We do not bother to write the (finite, if a bit involved) result for $\Gamma^{(4)}(s,t,u)$ in any of those cases.

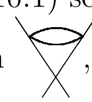
Let us tackle now the two-loop diagrams of the four-point function. The bikini graph  is uncomplicated:

$$I_{\text{bikini}}(k) = -g^3 \int \frac{d^Dp}{(2\pi)^D} \frac{1}{((p-k)^2+m^2)(p^2+m^2)} \int \frac{d^Dq}{(2\pi)^D} \frac{1}{((q-k)^2+m^2)(q^2+m^2)},$$

a product of two independent integrals. Here k again denotes any one of k_1+k_2, k_1+k_3, or k_1+k_4. We obtain

$$\begin{aligned} I_{\text{bikini}} &= -\tilde{g}\mu^\varepsilon \frac{\tilde{g}^2}{(4\pi)^4}\left[\frac{2}{\varepsilon} + \psi(1) + \int_0^1 dx\, \log\frac{4\pi\mu^2}{k^2x(1-x)+m^2} + O(\varepsilon)\right]^2 \\ &= -\tilde{g}\mu^\varepsilon \frac{\tilde{g}^2}{(4\pi)^4}\left[\frac{4}{\varepsilon^2} + \frac{4}{\varepsilon}\,\psi(1) + \frac{4}{\varepsilon}\int_0^1 dx\, \log\frac{4\pi\mu^2}{k^2x(1-x)+m^2} + O(\varepsilon^0)\right]. \end{aligned}$$

For the first time, we face a combinatorial problem. It would *not* do to square and subtract (one third of) the previously obtained vertex counterterm, as this procedure gives rise to a nonlocal divergence of the form $(\log k^2)/\varepsilon$. The solution is easy enough: we take into account the counterterms corresponding to the subdivergences by substituting $\big(I_{\text{fish}}(k) - I_{\text{fish}}(0)\big)^2$ for $I_{\text{bikini}}(k)$. This happens to accord with the renormalization prescription and both the double and the single pole then cancel out. In other words, renormalization 'factorizes', and the *three* counterterms that come from taking the functional (10.1) seriously do appear.

Things are slightly more complicated for the wine-cup graph , both analytically and combinatorially. We find the integral

$$-g^3 \int \frac{d^Dp}{(2\pi)^D}\frac{d^Dq}{(2\pi)^D} \frac{1}{((k_1+k_2-p)^2+m^2)(p^2+m^2)} \frac{1}{(q^2+m^2)((p-q+k_3)^2+m^2)},$$

or

$$I_{\text{winecup}}(k) = -g^3 \int \frac{d^Dp}{(2\pi)^D} \frac{1}{((k_1+k_2-p)^2+m^2)(p^2+m^2)}\, I_{\text{fish}}(p+k_3).$$

(We have taken the loop momentum of the integral on the loop with two sides as q, and p for the loop momentum of the integral on the loop with three sides, and have used $k_1+k_2 = -k_3-k_4$.) Therefore,

$$I_{\text{winecup}}(k) = \frac{\tilde{g}^3\mu^{2\varepsilon}\,\Gamma(\varepsilon/2)}{(4\pi)^2}\int_0^1 dx \int \frac{d^Dp}{(2\pi)^D} \frac{\left[\dfrac{4\pi\mu^2}{(k_3+p)^2x(1-x)+m^2)}\right]^{\varepsilon/2}}{((k_1+k_2-p)^2+m^2)(p^2+m^2)}.$$

Combining the denominators in the usual way, after integrating with respect to p, we obtain

$$I_{\rm winecup}(k) = \frac{\tilde g^3 \mu^\varepsilon (4\pi\mu^2)^\varepsilon \Gamma(\varepsilon)}{(4\pi)^4} \int_0^1 dx\,[x(1-x)]^{-\varepsilon/2} \int_0^1 dy\, y(1-y)^{\varepsilon/2-1} \int_0^1 dz$$
$$\times \biggl[yz(1-yz)(k_1+k_2)^2 + y(1-y)k_3^2 - 2yz(1-y)k_3(k_1+k_2) + m^2\biggl(y + \frac{1-y}{x(1-x)}\biggr)\biggr]^{-\varepsilon}.$$

There is a pole term coming from the endpoint singularity in the integral at $y = 1$. Around this point the square bracket can be expanded:

$$\Bigl[\cdots\Bigr] = \biggl[1 - \varepsilon \log\biggl(z(1-z) + \frac{m^2}{s}\biggr) + O(\varepsilon^2)\biggr].$$

We are left with

$$\frac{\tilde g^3 \mu^\varepsilon (4\pi\mu^2)^\varepsilon}{(4\pi)^4} \frac{1}{\varepsilon} (1+\varepsilon\psi(1)) \int_0^1 dx\,[x(1-x)]^{-\varepsilon/2} \int_0^1 dy\, y(1-y)^{\varepsilon/2-1}$$
$$\times \int_0^1 dz \biggl[1 - \varepsilon \log\biggl(z(1-z) + \frac{m^2}{s}\biggr) + O(\varepsilon^2)\biggr]. \tag{10.4}$$

Now,

$$\int_0^1 dx\,[x(1-x)]^{-\varepsilon/2} = \frac{\Gamma^2(1-\varepsilon/2)}{\Gamma(2-\varepsilon)}.$$

Similarly for the integral with respect to y. We reckon that the integral in (10.4) is

$$\frac{\Gamma^2(1-\varepsilon/2)\Gamma(\varepsilon)}{\Gamma(2-\varepsilon)\Gamma(2+\varepsilon/2)} \biggl[1 - \varepsilon \int_0^1 dx\, \log\biggl(x(1-x) + \frac{m^2}{s}\biggr)\biggr].$$

After some work,

$$\frac{\Gamma^2(1-\varepsilon/2)\Gamma(\varepsilon)}{\Gamma(2-\varepsilon)\Gamma(2+\varepsilon/2)} = \frac{2}{\varepsilon} \frac{1}{(1-\varepsilon)(1+\varepsilon/2)} [1 + O(\varepsilon)],$$

and in conclusion

$$I_{\rm winecup}(s) = \tilde g \mu^\varepsilon \frac{\tilde g^2}{(4\pi)^4} \frac{2}{\varepsilon^2} \biggl[1 + \frac{\varepsilon(1+2\psi(1))}{2} - \varepsilon \int_0^1 dx\, \log\biggl(\frac{sx(1-x)+m^2}{4\pi\mu^2}\biggr)\biggr] + O(\varepsilon^0).$$

There is no way to contemplate the renormalization of this without getting rid of the dreaded nonlocal divergence. This can be done precisely by taking account of the subdivergence in the wine-cup diagram, that is, subtracting the term (one half of) $I_{\rm fish}(s) I_{\rm fish}(0)$. The nonlocal divergence in this expression is seen to exactly cancel the nonlocal divergence in the previous display!

The result of such a subtraction is still divergent, but it contains only terms independent of momentum, that now can be reabsorbed in the subtraction of the overall divergence.

Although the combinatorial resources required so far are trivial, it is pretty clear that at a higher approximation order every detail of renormalization computations becomes nightmarish. It makes sense, then, also from the viewpoint of general renormalizability theorems, to clarify the combinatorial aspect as much as possible independently of the analytical ones. This was accomplished to a large extent by the precursors, culminating in the Zimmermann forest formula [13]. A step further was taken when Kreimer recognized the Hopf algebra structure lurking behind it. This is what eventually allows us to extract the general combinatorial meaning of the forest formula in this paper.

Thus motivated, we turn to the Connes–Kreimer paradigm.

▶ Bialgebras of Feynman graphs, encoding the combinatorics of renormalization, were introduced in [3]. The precise definition we use in this paper was first given in [65]. It is appropriate for massless scalar models in configuration space; in the examples we shall always envisage the φ^4_4 model. Nevertheless, similar constructions hold in any given quantum field theory, such as the massive φ^3_6 model considered in [3]. Just like in the original paper by Connes and Kreimer and its follow-ups, we avoid cumbersome formalism by relying on the pictorial intuition provided by the diagrams themselves. The reader is by now supposed thoroughly familiar with the concept of (superficial) degree of divergence.

We recall that a *graph* or diagram Γ of the theory is specified by a set $V(\Gamma)$ of *vertices* and a set $L(\Gamma)$ of *lines* (propagators) among them; *external* lines are attached to only one vertex each, *internal* lines to two. Diagrams with no external lines ('vacuum diagrams') will not be taken into account; in φ^4_4 theory only graphs with an even number of external lines are to be found. We are also excluding diagrams with a single vertex, and thus it is desirable to banish tadpole diagrams, in which a line connects a vertex to itself, as well.

Given a graph Γ, a *subdiagram* γ of Γ is specified by a subset of at least two elements of $V(\Gamma)$ and a subset of the lines that join these vertices in Γ. By exception, the empty subset $\emptyset$ will be admitted as a subdiagram of Γ. As well as Γ itself. Clearly, the external lines for a subdiagram γ include not only a subset of original incident lines, but also some internal lines of Γ not included in γ.

The connected pieces of Γ are the maximal connected subdiagrams. A diagram is called *proper* (or 1PI) when the number of its connected pieces would not increase on the removal of a single internal line; otherwise it is *improper*. An improper graph is the union of proper components plus subdiagrams containing a single line. A diagram is called 1VI when the number of its connected pieces can increase upon the removal of a single vertex.

A *subgraph* of a proper graph is a subdiagram that contains all the elements of $L(\Gamma)$ joining its vertices in the whole graph; as such, it is determined solely by the vertices. A subgraph of an improper graph Γ, distinct from Γ itself, is a proper subdiagram each of whose components is a subgraph with respect to the proper components of Γ.

We write $\gamma \subseteq \Gamma$ if and only if γ is a subgraph of Γ as defined (not just a subdiagram), although practically everything we have to say would work the same with a less restrictive definition. (For renormalization in configuration space, it is more convenient to deal with subgraphs than with more general subdiagrams. Moreover, Zimmermann showed long ago that only subtractions corresponding to subgraphs that are renormalization parts need to be used [66] in renormalization, and this dispenses us from dealing with a more general definition.) When a subdiagram contains several connected pieces, each one of them being a subgraph, we still call it a subgraph. For example, Figure 1 illustrates the case of a subgraph of the φ^4_4 model, made out of two connected pieces, which in spite of containing all the vertices, does not coincide with the whole graph.

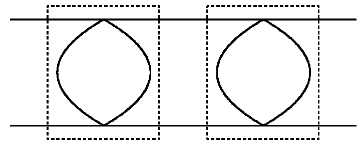


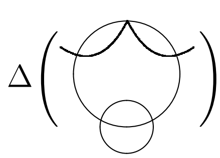
Figure 1: The "roll" in φ_4^4 theory

Two subgraphs γ_1, γ_2 of Γ are said to be *nonoverlapping* when $\gamma_1 \cap \gamma_2 = \emptyset$ (that is, γ_1 and γ_2 have no common vertices) or $\gamma_1 \subseteq \gamma_2$ or $\gamma_2 \subseteq \gamma_1$; otherwise they are overlapping. Given $\gamma \subseteq \Gamma$, the quotient graph or cograph Γ/γ (reduced graph in Zimmermann's [13] parlance) is defined by shrinking each *connected component* of γ in Γ to a point, that is to say, each piece of γ (bereft of its external lines) is considered as a vertex of Γ, and all the lines in Γ not belonging to γ belong to Γ/γ. This is modified in the obvious way when γ represents a propagator correction (i.e., has two external lines). A nonempty Γ/γ will be proper iff Γ is proper. The graphs Γ and Γ/γ have the same external structure, that is, structure of external lines. It is useful to think of the external structure as a kind of colour: although subgraphs of a given graph Γ may have colours different from Γ, cographs cannot.

► Now, a bialgebra $\mathcal{H}$ is defined as the polynomial algebra generated by the empty set $\emptyset$ and the connected Feynman graphs that are (superficially) divergent and/or have (superficially) divergent subgraphs (renormalization parts in Zimmermann's parlance), with set union as the product operation. Hence $\emptyset$ is the unit element $1 \in \mathcal{H}$. The counit is given by $\eta(\Gamma) := 0$ on any generator, except $\eta(\emptyset) = 1$. The really telling operation is the coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$; as it is to be a homomorphism of the algebra structure, we need only define it on connected diagrams. By definition, the (reduced) coproduct of Γ is given by

$$\Delta'\Gamma := \sum_{\emptyset \subsetneq \gamma \subsetneq \Gamma} \gamma \otimes \Gamma/\gamma. \tag{10.5}$$

The sum is over all divergent, proper, not necessarily connected subgraphs of Γ (not including $\emptyset$ and Γ) such that *each piece* is divergent, and such that Γ/γ is not a tadpole part. We put $\Gamma/\Gamma = 1$. When appropriate, the sum runs also over different types of local counterterms associated to γ [3]; this is not needed in our model example, because it corresponds to a massless theory, where propagators carry only one type of counterterms. It is then suggestive that the concept of primitive element for Δ coincides with that of primitive diagram in QFT. We show in Figure 2 how appearance of tadpole parts in Γ/γ can happen. The cograph corresponding to the "bikini" subgraph in the upper part of the graph in Figure 2 is a tadpole correction. It gives rise to a one-vertex reducible subgraph that can be suppressed [67].



would contain the term

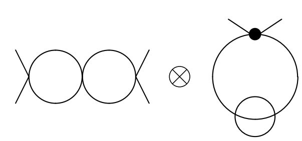


Figure 2: Cograph which is a tadpole part

For the proof of the bialgebra properties of $\mathcal{H}$, we refer to [24]; for graphical examples of coproducts, see [65]. From now on, we restrict consideration to the nontrivial subbialgebra of proper and superficially divergent graphs —the situation considered in [3]— that we still refer to as $\mathcal{H}$. Actually $\mathcal{H}$ is a graded bialgebra. Obvious grading operators are available: if $\#(\Gamma)$ denotes the number of vertices in Γ (i.e., the coupling order), then we define the degree of a generator (connected element) Γ as $\nu(\Gamma) := \#(\Gamma) - 1$; the degree of a product is the sum of the degrees of the factors. This grading is compatible with the coproduct, and clearly scalars are the only degree 0 elements. Other gradings are by the number $I(\Gamma)$ of internal lines in Γ and by loop number $l(\Gamma) := I(\Gamma) - \nu(\Gamma)$. For the φ_4^4 model, $l(\Gamma) = \nu(\Gamma) + 1$ for two-point graphs and $l(\Gamma) = \nu(\Gamma)$ for four-point graphs.

By our remarks in Section 5, $\mathcal{H}$ is a Hopf algebra, and we have at least two formulae for the antipode: S_G and S_B. Nevertheless, in this context it is preferable to write the antipode in a more pictorial language, taking advantage of geometrical intuition.

Definition 10.1. A *chain* $\mathcal{C}$ of a proper, connected graph Γ is a sequence $\gamma_1 \subsetneq \gamma_2 \subsetneq \cdots \subsetneq \gamma_k$ of proper, divergent, *not necessarily connected* subgraphs of Γ, not including $\emptyset$ and Γ. We denote by $\mathrm{Ch}(\Gamma)$ the set of chains of Γ. The *length* of a chain $\mathcal{C}$ is the number of 'links' $l(\mathcal{C}) = k+1$, and we write $\Omega(\mathcal{C}) := \gamma_1\,(\gamma_2/\gamma_1)\cdots(\gamma_k/\gamma_{k-1})\,(\Gamma/\gamma_k)$. We allow also the empty set to be a chain.

With this notation we can define the antipode as follows:

$$S_{\mathrm{DS}}(\Gamma) := \sum_{\mathcal{C}\in\mathrm{Ch}(\Gamma)} (-1)^{l(\mathcal{C})}\,\Omega(\mathcal{C}). \tag{10.6}$$

This definition corresponds, on the one hand, to the correct version of the Dyson–Salam formula for renormalization. On the other hand, formula (10.6) looks similar to the explicit expression for the antipode given by Schmitt for his incidence Hopf algebras [14]; we shall come back to that.

We show that S_{DS} is nothing but a reformulation of S_G; in other words:

Theorem 10.1. *The expansion of S_{DS} coincides identically with the expansion of S_G.*

Proof. First, given a proper, connected graph Γ, we rewrite

$$S_{\mathrm{DS}}(\Gamma) := \sum_k (-1)^{k+1} \sum_{\mathcal{C}\in\mathrm{Ch}_k(\Gamma)} \Omega(\mathcal{C}),$$

where $\mathrm{Ch}_k(\Gamma)$ denote the set of chains of length $k+1$. Thus, it is enough to prove that

$$(-1)^{k+1} \sum_{\mathcal{C}\in\mathrm{Ch}_k(\Gamma)} \Omega(\mathcal{C}) = (u\eta - \mathrm{id})^{*(k+1)}(\Gamma).$$

For this purpose, first notice that

$$\sum_{\emptyset\subsetneq\gamma_1\subsetneq\gamma_2\cdots\subsetneq\gamma_k} \gamma_1\otimes\gamma_2/\gamma_1\otimes\cdots\otimes\gamma_k/\gamma_{k-1}\otimes\Gamma/\gamma_k = \Delta'_k\cdots\Delta'_2\Delta'\Gamma. \tag{10.7}$$

Indeed, by definition of the coproduct the statement is true for $k = 1$. Moreover, if (10.7) holds for $k - 1$, then

$$\begin{aligned}
\sum_{\emptyset \subsetneq \gamma_1 \subsetneq \gamma_2 \cdots \subsetneq \gamma_k \subsetneq \Gamma} & \gamma_1 \otimes \gamma_2/\gamma_1 \otimes \cdots \otimes \gamma_k/\gamma_{k-1} \otimes \Gamma/\gamma_k \\
&= \sum_{\emptyset \subsetneq \gamma_2 \subsetneq \gamma_3 \cdots \subsetneq \gamma_k \subsetneq \Gamma} \Delta'_k(\gamma_2 \otimes \gamma_3/\gamma_2 \otimes \cdots \otimes \gamma_k/\gamma_{k-1} \otimes \Gamma/\gamma_k) \\
&= \Delta'_k\Bigg(\sum_{\emptyset \subsetneq \gamma_1 \subsetneq \gamma_2 \cdots \subsetneq \gamma_{k-1} \subsetneq \Gamma} \gamma_1 \otimes \gamma_2/\gamma_1 \otimes \cdots \otimes \gamma_{k-1}/\gamma_{k-2} \otimes \Gamma/\gamma_{k-1}\Bigg) \\
&= \Delta'_k \Delta'_{k-1} \cdots \Delta'_2 \Delta' \Gamma.
\end{aligned}$$

Thus, by (5.4)

$$\begin{aligned}
(-1)^{k+1} \sum_{\mathcal{C} \in \mathrm{Ch}_k(\Gamma)} \Omega(\mathcal{C}) &= (-1)^{k+1} \sum_{\emptyset \subsetneq \gamma_1 \subsetneq \gamma_2 \cdots \subsetneq \gamma_k \subsetneq \Gamma} \gamma_1 \left(\gamma_2/\gamma_1\right) \cdots \left(\gamma_k/\gamma_{k-1}\right) \left(\Gamma/\gamma_k\right) \\
&= (-1)^{k+1} m\, m_2 \cdots m_k\, \Delta'_k \cdots \Delta'_2 \Delta'(\Gamma) = (u\eta - \mathrm{id}) {*} (k+1)(\Gamma). \qquad \square
\end{aligned}$$

The proof shows how the chains are generated from the coproduct.

For illustration, now that we are at that, we record (5.6) in the language of the algebra of graphs:

$$S_B \Gamma := -\Gamma - \sum_{\emptyset \subsetneq \gamma \subsetneq \Gamma} (S_B \gamma)\, \Gamma/\gamma.$$

Let us illustrate as well the construction of the graded dual Hopf algebra H', for Hopf algebras of Feynman graphs. Each connected element Γ gives a derivation or element $Z_\Gamma : \mathcal{H} \to \mathbb{C}$ of the Lie algebra of infinitesimal characters on H, defined by

$$\begin{aligned}
\langle Z_\Gamma, \Gamma_1 \cdots \Gamma_k \rangle &= 0 \quad \text{unless } k = 1 \text{ and } \Gamma_1 = \Gamma; \\
\langle Z_\Gamma, \Gamma \rangle &= 1.
\end{aligned}$$

Also, $\langle Z_\Gamma, 1 \rangle = 0$ since $Z_\Gamma \in \mathrm{Der}_\eta \mathcal{H}$. Clearly any derivation δ vanishes on the ideal generated by products of two or more connected elements. Therefore, derivations are determined by their values on the subspace spanned by single graphs, and reduce to linear forms on this subspace.

11 Breaking the chains: the formula of Zimmermann

Definition 11.1. A (normal) *forest* $\mathcal{F}$ of a proper, connected graph Γ is a set of proper, divergent and connected subgraphs, not including $\emptyset$ and Γ, such that any pair of elements are nonoverlapping. $F(\Gamma)$ denotes the set of forests of Γ. The *density* of a forest $\mathcal{F}$ is the number $d(\mathcal{F}) = |\mathcal{F}| + 1$, where $|\mathcal{F}|$ is the number of elements of $\mathcal{F}$. We allow also the empty set to be a forest. Given $\gamma \in \mathcal{F} \cup \{\Gamma\}$ we say that γ' is a *predecessor* of γ in $\mathcal{F}$ if $\gamma' \subsetneq \gamma$ and there is no element γ'' in $\mathcal{F}$ such that $\gamma' \subsetneq \gamma'' \subsetneq \gamma$; that is to say, γ covers γ'. Let

$$\Theta(\mathcal{F}) := \prod_{\gamma \in \mathcal{F} \cup \{\Gamma\}} \gamma/\tilde{\gamma},$$

where $\tilde{\gamma}$ denote the disjoint union of all predecessors of γ in $\mathcal{F}$. When γ is minimal, $\tilde{\gamma} = \emptyset$, and $\gamma/\tilde{\gamma} = \gamma$.

Notice that if a forest $\mathcal{F}$ is a chain, then $\Theta(\mathcal{F}) = \Omega(\mathcal{F})$, and conversely if a chain $\mathcal{C}$ is a forest, $\Omega(\mathcal{C}) = \Theta(\mathcal{C})$. Obviously not every forest is a chain; but also not every chain is a forest, because product subgraphs can occur in chains and cannot in forests.

Zimmermann's version for the antipode is given by

$$S_Z(\Gamma) := \sum_{\mathcal{F} \in F(\Gamma)} (-1)^{d(\mathcal{F})} \Theta(\mathcal{F}). \tag{11.1}$$

We assert that S_Z provides yet another formula for the antipode of $\mathcal{H}$. A standard proof of this, except for minor combinatorial details, is just the mechanical one of checking that a given formula yields a convolution inverse for id [17]. It is not entirely uninformative; but gives little insight into why there should be no cancellations in the actual computation; and cancellations in (11.1) there are not, since in order to have $\Theta(\mathcal{F}) = \Theta(\mathcal{F}')$ it is required that $d(\mathcal{F}) = d(\mathcal{F}')$! Instead of repeating such a proof, and with a view to generalizations, here we give a longer, but elementary and much more instructive treatment that (we hope) clarifies how chains of different length give cancelling contributions.

Of course the assertion means that in general there are fewer forests than chains. Experience with Feynman graphs indicates that overlapping tends to reduce the cancellation phenomenon. There are "very overlapping" diagrams like the one in the φ_4^4 model pictured in Figure 3, for which the sets of chains and forests coincide.

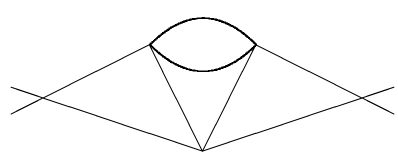


Figure 3: Diagram Γ without extra cancellations in $S_Z(\Gamma)$ with respect to $S_{DS}(\Gamma)$

Despite examples like this one, it is apparent that Zimmermann's formula is, from the combinatorial viewpoint, altogether subtler than those of Bogoliubov or Dyson and Salam. We plunge now into showing how all the cancellations implicit in the Dyson–Salam approach for graphs are taken into account and *suppressed*. To each chain $\mathcal{C}$ of Γ we associate the forest $\mathcal{F}_\mathcal{C}$ consisting of all the connected components of the different elements of $\mathcal{C}$.

Lemma 11.1. *Let $\mathcal{C}$ be a chain; then $\Omega(\mathcal{C}) = \Theta(\mathcal{F}_\mathcal{C})$.*

Proof. Clearly the statement is true for chains with only one element. Assume the result holds for chains in $\bigcup_{i=2}^{k-1} \mathrm{Ch}_i(\Gamma)$ where Γ is an arbitrary graph in $\mathcal{H}$, and let $\mathcal{C} := \gamma_1 \subset \gamma_2 \subset \cdots \subset \gamma_k$ be a chain in $\mathrm{Ch}_k(\Gamma)$. Then $\mathcal{D} := \gamma_1 \subset \gamma_2 \subset \cdots \subset \gamma_{k-1}$ belongs to $\mathrm{Ch}_{k-1}(\gamma_k)$, and if $\lambda_1, \ldots, \lambda_s$ are the connected components of γ_k, then $\mathcal{F}_\mathcal{C} = \mathcal{F}_\mathcal{D} \cup \{\lambda_1, \ldots, \lambda_s\}$. Now, clearly

$\lambda_1, \ldots, \lambda_s$ are the maximal elements of $\mathcal{F}_\mathcal{C}$, therefore, in $\mathcal{F}_\mathcal{C}$, $\Gamma/\gamma_k = \Gamma/\lambda_1 \cdots \lambda_s = \Gamma/\tilde{\Gamma}$, thus by the induction hypothesis

$$\Omega(\mathcal{C}) = \Omega(\mathcal{D})\big(\Gamma/\gamma_k\big) = \Theta(\mathcal{F}_\mathcal{D})\big(\Gamma/\gamma_k\big) = \prod_{\gamma \in \mathcal{F}_\mathcal{C} \bigcup \{\Gamma\}} \gamma/\tilde{\gamma} = \Theta(\mathcal{F}_\mathcal{C}). \qquad \square$$

We can as well associate chains to forests in such a way that, if $\mathcal{C}$ has been associated to $\mathcal{F}$, then $\mathcal{F}_\mathcal{C} = \mathcal{F}$. If a forest $\mathcal{F}$ is a chain, then $\mathcal{F}$ itself is the only chain so associated to $\mathcal{F}$, and clearly $d(\mathcal{F}) = \ell(\mathcal{F})$. Each forest $\mathcal{F}$ has associated at least one chain. Indeed, let $L_1 := \{\lambda_1^1, \ldots, \lambda_{n_1}^1\}$ be the list of maximal elements of $\mathcal{F}$, let $L_2 := \{\lambda_1^2, \ldots, \lambda_{n_2}^2\}$ be the set of predecessors in $\mathcal{F}$ of elements in L_1, let L_3 be the collection of predecessors in $\mathcal{F}$ of elements in L_2, and so on. If the listing of the elements of $\mathcal{F}$ is completed in the k-th iteration, we let $\mathcal{C}_\mathcal{F}$ be the chain

$$\mathcal{C}_\mathcal{F} := \lambda_1^k \cdots \lambda_{n_k}^k \subset \cdots \subset \lambda_1^2 \cdots \lambda_{n_2}^2 \subset \lambda_1^1 \cdots \lambda_{n_1}^1;$$

which by definition is associated to $\mathcal{F}$. We denote by $\mathrm{Ch}(\mathcal{F})$ the set of chains (of different length in general) associated to a given forest $\mathcal{F}$ of Γ.

So far, we find

$$S_{DS}(\Gamma) = \sum_{\mathcal{C} \in \mathrm{Ch}(\Gamma)} (-1)^{\ell(\mathcal{C})} \Omega(\mathcal{C}) = \sum_{\mathcal{C} \in \mathrm{Ch}(\Gamma)} (-1)^{\ell(\mathcal{C})} \Theta(\mathcal{F}_\mathcal{C}) = \sum_{\mathcal{F} \in F(\Gamma)} \bigg(\sum_{\mathcal{C} \in \mathrm{Ch}(\mathcal{F})} (-1)^{\ell(\mathcal{C})} \bigg) \Theta(\mathcal{F}).$$

Theorem 11.2. *For each proper, connected graph Γ, $S_{DS}(\Gamma) = S_Z(\Gamma)$.*

Proof. To conclude that $S_{DS}(\Gamma) = S_Z(\Gamma)$ it remains to prove that for each forest $\mathcal{F}$, $\sum_{\mathcal{C} \in \mathrm{Ch}(\mathcal{F})} (-1)^{\ell(\mathcal{C})} = (-1)^{d(\mathcal{F})}$. For this the idea is to enumerate the chains associated to a forest with $k+1$ elements from a list of the chains associated to the forest obtained by deleting one element. Thus we proceed by induction on the density of the forests. By the previous remarks the statement holds if $\mathcal{F}$ is a chain, in particular if $\mathcal{F}$ has only one element. Suppose that the claim holds whenever $\mathcal{F}$ has at most k elements, and let $\mathcal{G}$ be a forest with $k+1$ elements, say $\mathcal{G} = \mathcal{F} \cup \{\gamma\}$. Let $\mathcal{C}_1, \ldots, \mathcal{C}_r$ be the collection of all chains in $\mathrm{Ch}(\mathcal{F})$. If $\mathcal{C}_i$ is the chain $\gamma_1^i \subset \gamma_2^i \subset \cdots \subset \gamma_{k_i}^i$, let $\gamma_{t_i}^i$ be the first (from the left) term of the chain $\mathcal{C}_i$ such that a connected component of it covers γ. If no term satisfies that condition, set $t_i = k_i + 1$. On the other hand, let $\gamma_{s_i}^i$ be the last (from the right) link of the chain with (one or several) connected components contained in γ, not appearing in a previous link of the chain. If no γ_j^i has connected components contained in γ, take $s_i = 0$. We shall denote by $\widehat{\gamma}_j^i$ the element of $\mathcal{H}$ obtained from γ_j^i by replacing the product of all its connected components contained in γ by γ itself. Then we can construct the following $2(t_i - s_i) - 1$ chains in $\mathrm{Ch}(\mathcal{G})$:

$$\begin{aligned}
\mathcal{C}_i^0 &:= \gamma_1^i \subset \cdots \subset \gamma_{s_i}^i \subset \widehat{\gamma}_{s_i}^i \subset \widehat{\gamma}_{s_i+1}^i \subset \cdots \subset \widehat{\gamma}_{t_i-1}^i \subset \gamma_{t_i}^i \subset \cdots \subset \gamma_{k_i}^i \\
\mathcal{C}_i^1 &:= \gamma_1^i \subset \cdots \subset \gamma_{s_i}^i \subset \widehat{\gamma}_{s_i+1}^i \subset \widehat{\gamma}_{s_i+2}^i \subset \cdots \subset \widehat{\gamma}_{t_i-1}^i \subset \gamma_{t_i}^i \subset \cdots \subset \gamma_{k_i}^i \\
\mathcal{C}_i^2 &:= \gamma_1^i \subset \cdots \subset \gamma_{s_i+1}^i \subset \widehat{\gamma}_{s_i+1}^i \subset \widehat{\gamma}_{s_i+2}^i \subset \cdots \subset \widehat{\gamma}_{t_i-1}^i \subset \gamma_{t_i}^i \subset \cdots \subset \gamma_{k_i}^i \\
&\;\;\vdots \\
\mathcal{C}_i^{2j-1} &:= \gamma_1^i \subset \cdots \subset \gamma_{s_i+j-1}^i \subset \widehat{\gamma}_{s_i+j}^i \subset \cdots \subset \widehat{\gamma}_{t_i-1}^i \subset \gamma_{t_i}^i \subset \cdots \subset \gamma_{k_i}^i \\
\mathcal{C}_i^{2j} &:= \gamma_1^i \subset \cdots \subset \gamma_{s_i+j}^i \subset \widehat{\gamma}_{s_i+j}^i \subset \cdots \subset \widehat{\gamma}_{t_i-1}^i \subset \gamma_{t_i}^i \subset \cdots \subset \gamma_{k_i}^i \\
&\;\;\vdots \\
\mathcal{C}_i^{2(t_i-s_i)-3} &:= \gamma_1^i \subset \cdots \subset \gamma_{t_i-2}^i \subset \widehat{\gamma}_{t_i-1}^i \subset \gamma_{t_i}^i \subset \cdots \subset \gamma_{k_i}^i \\
\mathcal{C}_i^{2(t_i-s_i)-2} &:= \gamma_1^i \subset \cdots \subset \gamma_{t_i-1}^i \subset \widehat{\gamma}_{t_i-1}^i \subset \gamma_{t_i}^i \subset \cdots \subset \gamma_{k_i}^i \,.
\end{aligned}$$

Notice that $\ell(\mathcal{C}_i^{2u}) = \ell(\mathcal{C}_i) + 1$ for $u = 0, 1, \dots, (t_i - s_i) - 1$, whereas $\ell(\mathcal{C}_i^{2v-1}) = \ell(\mathcal{C}_i)$ for $v = 1, 2, \dots, (t_i - s_i) - 1$. Since every chain in $\mathrm{Ch}(\mathcal{G})$ is equal to $\mathcal{C}_i^j$ for some pair (i, j), it follows, by the induction hypothesis, that

$$\begin{aligned}
\sum_{\mathcal{C} \in \mathrm{Ch}(\mathcal{G})} (-1)^{\ell(\mathcal{C})} &= \sum_{i=1}^{r} \sum_{j=0}^{2(t_i-s_i)-2} (-1)^{\ell(\mathcal{C}_i^j)} = \sum_{i=1}^{r} (-1)^{\ell(\mathcal{C}_i^0)} \\
&= -\sum_{i=1}^{r} (-1)^{\ell(\mathcal{C}_i)} = -(-1)^{d(\mathcal{F})} = (-1)^{d(\mathcal{G})}. \qquad \square
\end{aligned}$$

12 Incidence Hopf algebras

The kinship between the Connes–Moscovici algebra, the Kreimer Hopf algebra [2] of renormalization and Connes–Kreimer algebras of rooted trees and Feynman graphs is by now well known; it was the discovery of this kinship what gave the current impulse to the subject. Yet the "classical" Faà di Bruno algebra appears to be of the same kind, and it is known to fit in the framework of incidence bialgebras of lattices and posets, that goes back to the pioneering work by Rota, together with Joni, Doubilet and Stanley [1, 68]. Rota introduced bialgebras in combinatorics, with coproducts becoming natural tools to systematize decompositions of posets and other combinatorial objects. This has been developed by Schmitt [14, 69]. The lecture notes by Dür [70] also touch upon the subject.

We want to explore the relations between Connes–Kreimer–Moscovici algebras and the lore of incidence Hopf algebras, eventually culminating in the importation of Zimmermann's formula into the latter realm. Before that, we will obtain the Faà di Bruno algebra $\mathcal{F}$ (yes, the symbol $\mathcal{F}$ is overworked in this paper) anew as the incidence Hopf algebra on the family of partitions of finite sets.

A family $\mathcal{P}$ of finite intervals (that is, finite partially ordered sets, or *posets* for short, of the form $\{x, z\} := \{\, y : x \le y \le z \,\}$) is called (interval) *closed* if it contains all the subintervals of its elements; that is, if $x \le y \in P \in \mathcal{P}$, then $\{x, y\} \in \mathcal{P}$. We will write $x < y$

whenever $x \le y$ and $x \ne y$. Analogously to Section 11, we will say that y covers x whenever $x < y$ and there is no z such that $x < z < y$. If $P = \{x, y\} \in \mathcal{P}$, denote $0_P := x$ and $1_P := y$.

An *order-compatible equivalence* relation on a closed family $\mathcal{P}$ is an equivalence relation $\sim$ such that whenever the intervals P, Q are equivalent in $\mathcal{P}$, then there exists a bijection $\phi : P \to Q$ such that $\{0_P, x\} \sim \{0_Q, \phi(x)\}$ and $\{x, 1_P\} \sim \{\phi(x), 1_Q\}$ for all $x \in P$. Denote by $[P]$ the equivalence class of P in the quotient family $\mathcal{P}_\sim$; such equivalence classes are called *types*.

Definition 12.1. Let $C(\mathcal{P}_\sim)$ be the vector space generated by the types. It is a coalgebra under the maps $\Delta : C(\mathcal{P}_\sim) \to C(\mathcal{P}_\sim) \otimes C(\mathcal{P}_\sim)$ and $\eta : C(\mathcal{P}_\sim) \to \mathbb{C}$ defined by

$$\Delta[P] = \sum_{x \in P} [\{0_P, x\}] \otimes [\{x, 1_P\}], \tag{12.1}$$

and

$$\eta[P] = 1 \quad \text{if} \quad |P| = 1, \qquad \eta[P] = 0 \text{ otherwise}.$$

Since $\sim$ is order-compatible, Δ is well defined. The verifications are elementary. We call $C(\mathcal{P}_\sim)$ the *incidence coalgebra* of $\mathcal{P}$ modulo $\sim$.

It is also easily seen that all incidence coalgebras are quotients of the "full" incidence coalgebra, associated to the trivial (isomorphism) equivalence relation, by the coideals spanned by elements $P - Q$, when $P \sim Q$.

The family $\mathcal{P}$ is *hereditary* if it is closed as well under Cartesian products: $P \times Q \in \mathcal{P}$ for every pair P, Q in $\mathcal{P}$; the partial order $(x_1, y_1) \le (x_2, y_2)$ iff $x_1 \le x_2$ in P and $y_1 \le y_2$ in Q is understood. A *chain* in $\mathcal{P}$ is an interval $\mathcal{C}$ such that $x, y \in \mathcal{C}$ implies either $x \le y$ or $y \le x$. A chain $\mathcal{C}$ is said to have length $\ell(\mathcal{C}) = n$ if $|\mathcal{C}| = n + 1$. Clearly a chain of length n can be written as $x_0 < x_1 < \cdots < x_n$. The set of chains that are subintervals of an interval P, such that $x_0 = 0_P$ and $x_n = 1_P$, is denoted by $\mathrm{Ch}(P)$. We write $\Omega(\mathcal{C})$ for the type of the Cartesian product $\{x_0, x_1\} \times \{x_1, x_2\} \times \cdots \times \{x_{n-1}, x_n\}$. A chain is saturated if no further 'links' can be found between its elements; that is, for $1 \le i \le n$, $x_{i-1} \le y \le x_i$ implies $y = x_{i-1}$ or $y = x_i$.

Incidence coalgebras are filtered by length in a natural way. An interval P is said to have *length* n if the longest chain in P has length $\ell(P) = n$. If $P \sim Q$, then $l(P) = l(Q)$, so length is well defined on the set of types; also clearly $\mathrm{Ch}(P) \sim \mathrm{Ch}(Q)$ if $P \sim Q$. Let C_n be the vector subspace of $C(\mathcal{P}_\sim)$ generated by types of length n or less; then $C_0 \subseteq C_1 \subseteq \dots$ is a filtering of $C(\mathcal{P}_\sim)$. Indeed, for any interval P and x in P, the union of a chain of $\{0_P, x\}$ and a chain of $\{x, 1_P\}$ gives a chain of P, hence $l([\{0_P, x\}]) + l([\{x, 1_P\}]) \le l(P)$, and so $\Delta C_n \subseteq \sum_{k=0}^{n} C_k \otimes C_{n-k}$. An interval is *graded* when the length of all saturated chains between 0_P and 1_P is the same. Whenever all elements of the family $\mathcal{P}$ are graded, $C(\mathcal{P}_\sim)$ is a graded coalgebra.

Suppose furthermore that there is an order-compatible equivalence relation $\sim$ on a hereditary family $\mathcal{P}$ satisfying:

(i) if $P \sim Q$, then $P \times R \sim Q \times R$ and $R \times P \sim R \times Q$ for all $R \in \mathcal{P}$.

(ii) if Q has just one element, then $P \times Q \sim P \sim Q \times P$.

In particular, the second of these conditions imply that all one-point intervals are of the same type (hereinafter denoted by 1). If they both hold, then $C(\mathcal{P}_\sim)$ turns out to be a connected bialgebra, with the product induced by the Cartesian product of intervals. In this context, types that are not (nontrivial) Cartesian products are called *indecomposables*. The set of indecomposable types is denoted $\mathcal{P}^\circ_\sim$.

Theorem 12.1. *Let us rebaptize $C(\mathcal{P}_\sim)$ as $H(\mathcal{P}_\sim)$. In view of our remarks in Section 5 it is in fact a Hopf algebra, called an incidence Hopf algebra; and a formula for the antipode is*

$$S_{\mathrm{DS}}[P] = \sum_{\mathcal{C}\in \mathrm{Ch}([P])} (-1)^{l(\mathcal{C})}\Omega(\mathcal{C}). \tag{12.2}$$

Proof. We follow our argument of [17] for the Hopf algebras of Feynman graphs, which translates almost verbatim into general incidence algebras theory; whereby it is directly checked that S_{DS} is an inverse of id under convolution. By definition

$$\begin{aligned}(S_{\mathrm{DS}} * \mathrm{id})[P] &= \sum_{x\in P} S_{\mathrm{DS}}([\{0_P, x\}])\,[\{x, 1_P\}] = S_{\mathrm{DS}}([P]) + \sum_{0_P\le x<1_P} S_{\mathrm{DS}}([\{0_P, x\}])\,[\{x, 1_P\}] \\ &= S_{\mathrm{DS}}([P]) + \sum_{0_P\le x<1_P}\ \sum_{\mathcal{D}\in\mathrm{Ch}(\{0_P,x\})} (-1)^{l(\mathcal{D})}\Omega(\mathcal{D})\,[\{x, 1_P\}].\end{aligned}$$

Now, if $\mathcal{D}\in \mathrm{Ch}(\{0_P, x\})$, say $\mathcal{D} = x_0 < x_1 < \cdots < x_n = x$, then $\mathcal{C} = x_0 < x_1 < \cdots < x_n < x_{n+1} = 1_P \in \mathrm{Ch}(P)$. Moreover,

$$\Omega(\mathcal{C}) = \Omega(\mathcal{D})\,[\{x, 1_P\}], \quad \text{and} \qquad l(\mathcal{C}) = l(\mathcal{D}) + 1. \tag{12.3}$$

Reciprocally, given a chain $\mathcal{C} = x_0 < x_1 < \cdots < x_{n+1} \in \mathrm{Ch}(P)$, then $\mathcal{D} = x_0 < x_1 < \cdots < x_n =: x \in \mathrm{Ch}(\{0_P, x\})$, and (12.3) holds. Therefore

$$S_{\mathrm{DS}} * \mathrm{id}([P]) = S_{\mathrm{DS}}([P]) - \sum_{\mathcal{C}\in\mathrm{Ch}(P)} (-1)^{l(\mathcal{C})}\Omega(\mathcal{C}) = 0 = u\eta([P]);$$

in other words, S_{DS} is a left inverse for id, and therefore it is an antipode. ∎

It is still possible to have a Hopf algebra, albeit not a connected one, by weakening condition (ii) above: one requires that a neutral element 1 exist in $\mathcal{P}$ such that $P \sim P\times 1 \sim 1\times P$ for all $P\in\mathcal{P}$, and that all $P\in\mathcal{P}$ with $|P| = 1$ be invertible.

By now the reader should not be surprised to find that the expansions for S_{DS} and S_G are identical: one formally can reproduce exactly the argument of Theorem 10.1 to obtain

$$m^k\Delta'^k([P]) = \sum_{0_P=x_0<x_1<\cdots<x_{k+1}=1_P}\ \prod_{i=1}^{k+1}[\{x_{i-1}, x_i\}] = \sum_{\mathcal{C}\in\mathrm{Ch}_k(P)} \Omega(\mathcal{C}).$$

In summary, the (correct version of) the Dyson–Salam scheme for renormalization is analogous to the explicit expression for the antipode (12.2), typical of incidence Hopf algebras.

The dual convolution algebra $H^*(\mathcal{P}_\sim)$ of $H(\mathcal{P}_\sim)$ is called the incidence algebra of $\mathcal{P}$ modulo $\sim$ (unfortunately the standard terminology is a bit confusing). It can be identified to the set of maps from $\mathcal{P}_\sim$ to $\mathbb{C}$; and the group of multiplicative functions $\mathrm{Hom}_{\mathrm{alg}}(H(\mathcal{P}_\sim), \mathbb{C})$ can be identified with the set of functions whose domain is the set of indecomposable types. Explicitly,

$$\langle f * g, [P] \rangle = \sum_{x \in P} \langle f, [\{0_P, x\}] \rangle \, \langle g, [\{x, 1_P\}] \rangle.$$

In this context, the classical theory of the *Möbius function* is reformulated as follows. Incidence Hopf algebras come to the world with a distinguished character: the *zeta function* ζ of $\mathcal{P}$, given by $\langle \zeta, [P] \rangle = 1$ for all types; it is patently multiplicative. Its convolution inverse μ is the *Möbius character* or Möbius function of posets [71]. Therefore, $\mu = \zeta S$, and different formulae for the antipode S provide different ways to compute μ. For instance, (12.2) gives

$$\mu([P]) = -1 + c_2 - c_3 + \dots,$$

where c_n is the number of chains of length n. Rota defined in [71] the *Euler characteristic* of P as $E([P]) := 1 + \mu([P])$, and related it with the classical Euler characteristic in a suitable homology theory associated to P.

As for the examples: if one considers the family $\mathcal{P}$ of finite Boolean algebras and let $\sim$ be isomorphism, it is seen that $\mathcal{P}$ is hereditary; and if we denote by Y the isomorphism class of the poset of subsets of a one-element set, the associated incidence Hopf algebra is the binomial Hopf algebra with coproduct (4.5). Also, if we take for $\mathcal{P}$ the family of linearly ordered sets and $\sim$ the isomorphism relation again, and we let Y_n denote the type of the linearly ordered set of length n, we recover the ladder Hopf algebra H_ℓ, introduced after Theorem 6.3, as an incidence Hopf algebra.

▶ The similarity of the previous setup to the theory of algebras of Feynman graphs is striking. A dictionary for translating the family of Feynman graphs, reduced by isomorphism classes, into the language of incidence algebras would then seem to be quickly established, purporting to identify $\leq$ with $\subseteq$ and to show (10.5) as an avatar of (12.1). In order to underline the parallelism, one could try to depart somewhat from the usual notation: indeed, writing $\emptyset, \Gamma$ for any $0_P, 1_P$ respectively; x for $\{0_P, x\}$ and Γ/x for $\{x, 1_P\}$, would make the notations completely parallel. It is also clear that *connected* in the sense of graph types should translate into *indecomposable* in the incidence algebra framework.

However, the direct attempt to identify Connes–Kreimer algebras and the Rota incidence algebras does *not* work: $\mathcal{H}$ represents a particular class of incidence algebras, and it is important to recognize the traits that characterize it inside that framework: one can be badly misled by formal analogies. We show how this can happen by means of an example (see 12.1 further down) inside the commutative Hopf algebra of rooted trees.

An (undecorated) *rooted tree* is a connected poset in which each element (vertex) is covered by at most another element: this selects a unique distinguished uncovered element, called the root. An *admissible cut* c of a rooted tree T is a subset of its lines such that the path from the root to any other vertex includes at most one line of c. Deleting the cut branches produces several subtrees; the component containing the original root (the trunk) is denoted $R_c(T)$. The remaining branches also form rooted trees, where in each case the new root is the vertex immediately below the deleted line; $P_c(T)$ denotes the juxtaposition of these pruned branches.

Definition 12.2. The (Connes–Kreimer) Hopf algebra of *rooted trees* H_R is the commutative algebra generated by symbols T, one for each isomorphism class of rooted trees, plus a unit 1 corresponding to the empty tree; the product of trees is written as the juxtaposition of their symbols. A product of trees is called a (rooted) *wood* (we can hardly call it a forest). The counit $\eta : H_R \to \mathbb{R}$ is the linear map defined by $\eta(1) := 1_{\mathbb{R}}$ and $\eta(T_1T_2\cdots T_n) = 0$ if $T_1,\dots,T_n$ are trees. In [42] the coproduct on H_R is defined as a map $\Delta : H_R \to H_R \otimes H_R$ on the generators (extending it as an algebra homomorphism) as follows:

$$\Delta 1 := 1 \otimes 1; \quad \Delta T := T \otimes 1 + 1 \otimes T + \sum_{c \in C(T)} P_c(T) \otimes R_c(T). \tag{12.4}$$

Here $C(T)$ is the list of admissible cuts.

It should be clear that the ladder Hopf algebra H_ℓ above is a (cocommutative) Hopf subalgebra of H_R: the type of the linearly ordered set of length n is identified to the "stick" with n vertices.

Let the *natural growth* $N : H_R \to H_R$ be the unique derivation defined, for each tree T with vertices $v_1,\dots,v_n$, by

$$N(T) := T_1 + T_2 + \cdots + T_n,$$

where each T_j is obtained from T by sprouting one new leaf from each vertex v_j. In particular, let t_1, t_2 respectively denote the trees with one and two vertices, t_{31}, t_{32} the two trees with three vertices, where the root has respectively fertility one and two; for trees with four vertices, we shall denote by t_{41} the rooted tree where all vertices have fertility 1 (a stick), by t_{42} and t_{43} the four-vertex trees with 2 or 3 outgoing lines from the root (respectively a hook and a claw), and by t_{44} the tree whose root has fertility 1 and whose only vertex with length 1 has fertility 2 (a biped); this being the notation of [24, Ch. 14]. We obtain

$$\begin{aligned}
N(t_1) &= t_2, \\
N^2(t_1) &= N(t_2) = t_{31} + t_{32}, \\
N^3(t_1) &= N(t_{31} + t_{32}) = t_{41} + 3t_{42} + t_{43} + t_{44}.
\end{aligned}$$

From (12.4),

$$\begin{aligned}
\Delta N(t_1) &= N(t_1) \otimes 1 + 1 \otimes N(t_1) + t_1 \otimes t_1, \\
\Delta N^2(t_1) &= N^2(t_1) \otimes 1 + 1 \otimes N^2(t_1) + (t_1^2 + t_2) \otimes t_1 + 3t_1 \otimes t_2, \\
\Delta N^3(t_1) &= N^3(t_1) \otimes 1 + 1 \otimes N^3(t_1) + N^2(t_1) \otimes t_1 + 4t_2 \otimes t_2 \\
&\quad + 6t_1 \otimes N^2(t_1) + 7t_1^2 \otimes t_2 + 3t_1t_2 \otimes t_1 + t_1^3 \otimes t_1;
\end{aligned}$$

this the reader will find familiar, as these are completely analogous to the expressions for the coproduct of the $\delta_2, \delta_3, \delta_4$ in the Faà di Bruno algebra. This is no accident. The number of times that the tree T with n vertices appears in $N^{n-1}(t_1)$ is known, and the reader can consult for instance [42] or [24, Section 14.1] for a proof that indeed with the identifications $t_1 \equiv \delta_1, N^{n-1}(t_1) \equiv \delta_n$, the Faà di Bruno algebra $\mathcal{F}$ is a *Hopf subalgebra* of H_R. By the way, the identification of the natural growth subalgebra in H_R to $\mathcal{F}$ is not unexpected, in that there is a relation between differentials and rooted trees, that goes back to Cayley [72], giving precisely that correspondence; see [73], and also [24, Section 14.2] and the more recent [74].

Hopf algebras of rooted trees have extremely important normative properties, into which unfortunately we cannot go here. We refer the reader to [37] and to [75]. For the relation between the algebra of rooted trees and the shuffle algebra, consult [76].

Example 12.1. Consider now, for instance, the tree t_{32}. With this definition, its coproduct is

$$\Delta t_{32} = t_{32} \otimes 1 + 1 \otimes t_{32} + 2t_1 \otimes t_2 + t_1^2 \otimes t_1.$$

This is shown in Figure 4. Now, we can make every tree into an interval $\{0_T, 1_T\}$ by collating all its leaves to a notional element 0_T. But even so, if we think of a tree as just such a poset and identify $\{0_T, v\}$ with the tree generated by a cut just above v, and omit the reasonable doubt about the meaning of $\{v, 1_T\}$, this definition would *not* agree with (12.1): the last term certainly does not appear.

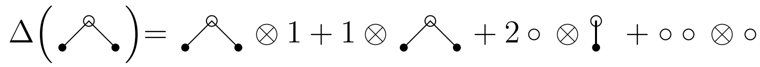


Figure 4: Subinterval rule $\neq$ cut rule

The point here is that the interval $\{0_T, 1_T\}$ does not contain all of its convex subposets. What "convex" means is recalled in Section 13. Once this is clarified, the proof of Zimmermann's formula in the previous section will be seen to adapt to a large class of incidence Hopf algebras. To do this in some generality, and to dispel the misunderstandings we alluded to above, a new bevy of concepts in incidence Hopf algebra theory is required. Before turning to that, let us obtain $\mathcal{F}$ anew, as advertised, in terms of the theory of this section.

12.1 The Beatus Faà di Bruno algebra as an incidence bialgebra

The Faà di Bruno algebra, it could be suspected by now, is an incidence bialgebra, corresponding to the case where the intervals belong to the family of posets that are partitions of finite sets; this was realized in [68]. We start by ordering the partitions of a finite set S. One says that $\{A_1, \dots, A_n\} = \pi \leq \tau = \{B_1, \dots, B_m\}$, or that π refines τ, if each A_i is contained in some B_j. The set of partitions $\Pi(S)$ of S is an interval, where the biggest element is the partition with just one block, and the smallest the partition whose blocks are all singletons.

Proposition 12.2. *The subinterval $\{\pi, \tau\}$ of $\Pi(S)$ is isomorphic to the poset $\Pi_1^{\lambda_1} \times \cdots \times \Pi_k^{\lambda_k} \cdots$, where λ_j is the number of blocks of τ that are the union of exactly j blocks of π (so $\sum \lambda_j = |\tau|$ and $\sum j\lambda_j = |\pi|$).*

Proof. Let $B_1^1, \dots, B_{\lambda_1}^1$ be the blocks of π that are also blocks of τ, and for each integer k let $\{B_1^{k1}, \dots, B_1^{kk}\}, \dots, \{B_{\lambda_k}^{k1}, \dots, B_{\lambda_k}^{kk}\}$ be the collections of k blocks of π that produce a block of τ. Clearly

$$\Pi(\{B_1^1\} \times \cdots \times \{B_1^{21}, B_1^{22}\} \times \cdots \times \{B_{\lambda_k}^{k1}, \dots, B_{\lambda_k}^{kk}\} \times \cdots) \cong \Pi_1^{\lambda_1} \times \cdots \times \Pi_k^{\lambda_k} \cdots. \quad (12.5)$$

If $\sigma \in \{\pi, \tau\}$, then σ is obtained by dividing some of the blocks of τ into pieces that are unions of blocks of π, which amounts to taking a partition of some of the products on the right hand side of (12.5). $\square$

It is natural therefore to assign to each interval $\{\pi, \tau\}$ the sequence $\lambda = (1^{\lambda_1}, \cdots, k^{\lambda_k}, \cdots)$ —only finitely many components are nonzero— and to declare two intervals in the family $\mathcal{P} = \{\, \Pi(S) : S \text{ is a finite set} \,\}$ to be equivalent when the corresponding vectors are equal; there are of course an infinite number of intervals in each equivalence class. It is immediate that $\mathcal{P}$, with this equivalence relation $\sim$, is interval closed and hereditary. Equivalently, one can think of the poset of finite partitions of a countable set (such that only one block is infinite) ordered by refinement. Furthermore, from the correspondences

$$[\{\pi, \tau\}] \longleftrightarrow \lambda \longleftrightarrow x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k} \cdots,$$

it is also natural to declare $\widetilde{\mathcal{P}}$ isomorphic, as an algebra, to the algebra of polynomials of infinitely many variables $\mathbb{C}[x_1, \dots, x_k, \dots]$.

To describe the associated coproduct explicitly it is enough to find what its action is on the indecomposable types $[\Pi_n]$. If $[\{\pi, \tau\}] = [\Pi_n]$ —which corresponds to the vector $(0, \dots, 0, 1, 0, \dots)$ with the 1 in the n-th place, or to the polynomial x_n— then π has n blocks and τ just one. Moreover, if $\sigma \in \{\pi, \tau\}$, then $[\{\sigma, \tau\}]$ corresponds to x_k for some $k \leq n$, whereas $[\{\pi, \sigma\}]$ goes with a vector α satisfying $\alpha_1 + \cdots + \alpha_n = k$, and $\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n$. We conclude from (12.1) that

$$\Delta x_n = \sum_{k=1}^{n} \sum_{\alpha} \binom{n}{\alpha; k} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \otimes x_k,$$

and we have recovered (7.6), with the identification $a_n \equiv [\Pi_n]$. Now, there is the proviso $[\Pi_1] = 1$, implicit in the main construction of Section 7. In consequence, $\mathcal{F} = H(\{\, \Pi(S) : S \text{ is a finite set} \,\}_{\sim})$.

► Several final remarks on Faà di Bruno algebras are in order.

A variant of the construction of $\mathcal{F}$ allows one to obtain an enlarged Faà di Bruno algebra, that we shall denote by $\mathcal{F}_{\text{enl}}$. For that we adjoin to $\mathbb{C}[x_2, \dots, x_k, \dots]$ the grouplike elements x_1 and x_1^{-1}. Obviously the coproduct formula (7.6) holds and the antipode generalizes as follows:

$$S_{\text{enl}} x_n = x_1^{-n} \sum_{k=1}^{n-1} (-1)^k \frac{(n-1+k)!}{(n-1)!} B_{n-1,k}\Big(\frac{x_2 x_1^{-1}}{2}, \frac{x_3 x_1^{-1}}{3}, \dots\Big),$$

yielding (together with the discussion in that section) formula (8.2) of Section 8. This can be shown to arise from a variation of the chain formula to take into account the one-point segments.

The following is taken from [19]. A (multi)coloured set is a finite set X with a map $\theta : X \to \{1, \dots, N\}$. The *colour* of $x \in X$ is the value of $\theta(x)$. If $X_r := \{\, x \in X : \theta(x) = r \,\}$, the size $|X|$ of X is the row vector $(|X_1|, \dots, |X_N|)$. A *coloured partition* of a coloured set X is a partition of X whose set of blocks is also coloured, with the condition $\theta(\{x\}) = \theta(x)$ for singletons. In what follows, take $N = 2$ for simplicity of notation. Let $|X| = (n_1, n_2)$.

Coloured partitions form a poset Π_{n_1,n_2}, with $\pi \leq \rho$ if $\pi \leq \rho$ as partitions and, for each block B of π which is also a block of ρ, $\theta_\pi(B) = \theta_\rho(B)$. We banish from the family of posets one of the maximal elements 1_X, so there are the two families $\Pi^r_{n_1,n_2}$ with $r = 1$ or 2 according to which maximal element is kept. On the hereditary classes of segments of such coloured partition posets under the relation of colour-isomorphism it is possible to define a Hopf algebra structure, called the coloured Faà di Bruno Hopf algebra $\mathcal{F}(N)$. There is a corresponding anti-isomorphism between the groups (by composition) of complex formal series in two variables of the form

$$f^r(x_1, x_2) = x_r + \sum_{n_1+n_2>1} f^r_{n_1 n_2} \frac{x_1^{n_1} x_2^{n_2}}{n_1! n_2!} \tag{12.6}$$

and the group of multiplicative functions f on coloured Faà di Bruno Hopf algebras, given by $f^r_{n_1 n_2} := \langle f, [\Pi^r_{n_1 n_2}] \rangle$. The antipode on $\mathcal{F}(N)$ provides Lagrange reversion in several variables; we refrain from going into that, but refer the reader to [77] for a classical kind of proof; and to [19, 43] in the spirit of this review.

For the coloured Faà di Bruno algebras, $G(\mathcal{F}(N)^\circ)$ reproduces the opposite G^{opp}_N of the formal diffeomorphism group in N variables; for undecorated trees, the dual group was identified by Brouder as the *Butcher group* of Runge-Kutta methods [73]; series of trees are composed by appropriate grafting. The dual group $G(\mathcal{H}^\circ)$ of $\mathcal{H}$ for a quantum field theory has been termed *diffeographism group* by Connes and Kreimer. The name is entirely appropriate, as one should regard also the apparatus of Feynman graphs as a (very sophisticated) approximation machinery for the computation of (quantum corrections to) the coupling constants of a physical theory. In view of the "main theorem of renormalization" [78], these are typically given by series like (12.6). In the outstanding paper a Hopf algebra homomorphism $\mathcal{F}(N) \to \mathcal{H}$ (with the coupling constants as colours) and its dual group morphism $G(\mathcal{H}^\circ) \to G^{\mathrm{opp}}_N$ are exhibited. This second morphism is first constructed at the level of the infinitesimal characters, and then lifted to the group; the transpose map from $\mathcal{F}(N)$ to $\mathcal{H}$ coincides with the one obtained when calculating the coupling constants in terms of Feynman graphs. For simple theories like massless φ^4_4, only the simple Faà di Bruno algebra and G^{opp}_1 are involved. The previous considerations authorize us to regard the diffeographism group as the 'general renormalization group'. Unfortunately, this is the place where we stop; but we can refer on this point, besides [4], to [79–81], in a different vein to [82][1]; and to the excellent review [22].

Schmitt [69] defines a *uniform family* as a hereditary family $\mathcal{P}$ of graded posets together with a relation $\sim$, giving rise to an incidence Hopf algebra, such that, moreover:

- The monoid $\mathcal{P}$ is commutative.
- If $[P]$ is indecomposable, then $\{y, 1_P\}$ for $y < 1_P$ is indecomposable.
- For all $n \geq 1$, there is exactly one type in $\mathcal{P}^\circ_\sim$ of degree n.

[1]This paper came out at an early stage of the subject, with the excellent idea of developing its Lie algebraic aspect in the context of triangular matrix representations; but its treatment of factorization for the Bogoliubov recursive procedure of renormalization is inconsistent

Obviously the families of posets giving rise, respectively, to the ladder Hopf algebra H_ℓ and to the Faà di Bruno algebra $\mathcal{F}$, are uniform. This allows nice determinant formulae for the antipode [61, 69], that need not concern us now. We may define a quasi-uniform family by dropping the last condition; the coloured posets giving rise to the $\mathcal{F}(N)$ do correspond to quasi-uniform families. The algebras H_R and $\mathcal{H}$ (even more so modulo the considerations in the next Section) do correspond to quasi-uniform families as well.

13 Distributive lattices and the general Zimmermann formula

This section is partly motivated by the closing remarks of [69]. Our goal is to employ the distributive lattice of order ideals associated with a general partially ordered set in order to interpret the combinatorics of cuts in the algebra of rooted trees, and further, to resolve the combinatorics of overlapping divergences. We need some definitions. We work with finite posets. An *antichain* in a poset P is a subset of P such that any two of its elements are incomparable in P. An *order ideal* is a subset I that includes all the descendants of its elements, that is, $x \in I$ whenever x is smaller than some $y \in I$. We include the empty set among the ideals of P. The *principal ideal* generated by $y \in P$ is the ideal $\Lambda_y := \{\, x \in P : x \leq y \,\}$. Let A be an antichain; the order ideal *generated by* A is the subset of P of those elements smaller than some $y \in A$. A subposet K is *convex* when it contains the intervals associated to any pair of its elements; in particular an interval is convex. Differences of order ideals are also convex.

Definition 13.1. A *lattice* is a particular class of poset, in which every pair of elements s, t has a greatest lower bound $s \wedge t$ ("meet") and a least upper bound $s \vee t$ ("join"). That is, there exist respectively an element $s \wedge t$ satisfying $s \wedge t \leq s$, $s \wedge t \leq t$ such that any other with the same property is smaller, and one element $s \vee t$ satisfying $s \vee t \geq s$, $s \vee t \geq t$, and any other element with the same property is greater. A lattice L is *distributive* if for any s, t, u in L

$$s \wedge (t \vee u) = (s \wedge t) \vee (s \wedge u) \quad \text{or equivalently} \quad s \vee (t \wedge u) = (s \vee t) \wedge (s \vee u).$$

Distributive lattices are always intervals. Sublattices of distributive lattices are distributive. The collection of all subsets of any set is a lattice, with meet $s \wedge t := s \cap t$ and join $s \vee t := s \cup t$. In this example, these are the usual distributive laws between unions and intersections. The sets of (finite) partitions of finite or countable sets are lattices; but already Π_3 is not distributive.

The main example of distributive lattices is the poset J_P of ideals of any finite poset P (ordered by inclusion). In fact, it is the only example. An important theorem by Birkhoff [18, Section 3.4] states that for every finite distributive lattice L there exists a unique (up to isomorphism) poset P, such that $L \cong J_P$. For instance, this correspondence sends an antichain A with n elements into the set of its subsets, ordered by inclusion, that is, the Boolean lattice of rank n. In general, the set P can be taken as the subset of *join irreducible* elements of L, that is those elements s that can not be written as $s = t \vee u$ with $t < s$ and $u < s$. An order ideal of P is join irreducible in J_P iff it is a principal order ideal. Thus there is a one-to-one correspondence $\Lambda_y \leftrightarrow y$ between the join irreducibles of J_P and the elements of P.

A poset P is *connected* if it cannot be written as the disjoint union of two nontrivial posets; in other words, given any two elements $x, y \in P$, one can find a sequence of elements $x_0 = x, x_1, \ldots, x_{n-1}, x_n = y$ such that any two successive elements of the sequence are related by $\leq$ or $\geq$. Now, to the family of all finite posets, or to any subfamily $\mathcal{P}$ of it closed under the formation of disjoint unions and containing all the convex subsets of its elements, we can associate the new family of posets $\mathcal{J}_{\mathcal{P}} := \{\, J_P : P \in \mathcal{P}\,\}$. An interval $\{I, I'\}$ in J_P, is isomorphic to $J_{I' \setminus I}$, where $I' \setminus I$ is regarded as a subposet of P. It is easy to see that distributive lattices are always graded, with the degree or 'rank' of J_P being precisely the cardinality $|P|$ of P. Consider the isomorphism equivalence relation both in $\mathcal{P}$ and $\mathcal{J}_{\mathcal{P}}$. Since $J_{P \cup Q} \cong J_P \times J_Q$, the set of indecomposable types $\mathcal{J}^{\circ}_{\mathcal{P}}$ of $\mathcal{J}_{\mathcal{P}}$ is precisely the set of isomorphism classes of connected posets of $\mathcal{P}$. Furthermore, by (12.1) the coproduct in the incidence Hopf algebra $H(\mathcal{J}_{\mathcal{P}\sim})$ is given by

$$\Delta[J_P] = \sum_{I \in J_P} [0_{J_P}, I\}] \otimes [\{I, 1_{J_P}\}] = \sum_{I \in J_P} [\{\emptyset, I\}] \otimes [\{I, P\}] = \sum_{I \in J_P} [J_I] \otimes [J_{P \setminus I}].$$

(Note that since both I and $P \setminus I$ are convex their types belong to $\mathcal{P}_{\sim}$.)

Motivated by all this, we introduce a new commutative Hopf algebra structure in $\mathcal{P}_{\sim}$, defined by the product

$$PQ = QP := P \cup Q \tag{13.1}$$

and the coproduct

$$\Delta[P] := \sum_{I \in J_P} [I] \otimes [P \setminus I], \tag{13.2}$$

with the obvious unit and counit. By construction, this Hopf algebra is isomorphic to $H(\mathcal{J}_{\mathcal{P}\sim})$. With some abuse of notation, in the remainder of this section we call $H(\mathcal{P}_{\sim})$ the Hopf algebra given by (13.1) and (13.2). In this setting a chain $\mathcal{C}$ of P is defined by a chain of J_P, that is, a sequence of order ideals $\emptyset = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_n = P$, and

$$\Omega(\mathcal{C}) = [\{I_0, I_1\}\{I_1, I_2\} \cdots \{I_{n-1}, I_n\}] = [J_{I_1 \setminus I_0} J_{I_2 \setminus I_1} \cdots J_{I_n \setminus I_{n-1}}].$$

When $\mathcal{C}$ is regarded as a subposet of P we write

$$\Omega(\mathcal{C}) = [I_1 \setminus I_0][I_2 \setminus I_1] \cdots [I_n \setminus I_{n-1}].$$

By Proposition 12.1 the antipode in $H(\mathcal{P}_{\sim})$ is given by

$$S_{\mathrm{DS}}[P] = \sum_{\mathcal{C} \in \mathrm{Ch}([P])} (-1)^{l(\mathcal{C})} \Omega(\mathcal{C}),$$

where $\mathrm{Ch}([P])$ denotes the set of chain of $[P]$.

In complete analogy to the Hopf algebra of Feynman graphs we consider the following.

Definition 13.2. A *forest* $\mathcal{F}$ of P is a collection of connected subposets of P such that if I_1 and I_2 are in $\mathcal{F}$, then either $I_1 \cap I_2 = \emptyset$, or $I_1 \subset I_2$ or $I_2 \subset I_1$. If $I \in \mathcal{F}$, then I' is a predecessor of I in $\mathcal{F}$ if there is no $I'' \in \mathcal{F}$ such that $I' \subsetneq I'' \subsetneq I$, and we denote by $\tilde{I}$ the disjoint union of all predecessors of I in $\mathcal{F}$. As in Section 11, we define

$$\Theta(\mathcal{F}) = \prod_{I \in \mathcal{F} \cup \{P\}} [I \setminus \tilde{I}].$$

We come to the main result: a general formula of the Zimmermann type for the antipode of the incidence Hopf algebras $H(\mathcal{P}_\sim)$.

Theorem 13.1. *Let $\mathcal{P}$ be a family of posets that contains all the convex subsets of its elements and is closed under the formation of disjoint unions, and consider on $\mathcal{P}$ the isomorphism equivalence relation. For any $[P] \in H(\mathcal{P}_\sim)$, let $F([P])$ be the set of all forests of $[P]$. Then*

$$S([P]) = S_Z([P]) := \sum_{\mathcal{F} \in F([P])} (-1)^{d(\mathcal{F})} \Theta([P]).$$

Proof. Our proof of Theorem 11.2, and its preliminaries, just go through with no more than notational change, and there is no point in repeating them. $\square$

We turn to the examples. In order to obtain (12.4) from the theory of incidence Hopf algebras, one can proceed as follows: consider the family W of posets constituted by all rooted woods; consider $\mathcal{J}_W := \{ J_P : P \in W \}$ modulo isomorphism. Then $\mathcal{J}_W$ is hereditary and we proceed with the construction of the incidence algebra associated to it; the indecomposable types are precisely the trees, and $H(W_\sim) \simeq H(\mathcal{J}_{W\sim})$ now denotes the polynomial algebra on them with the coproduct of above. This is seen to reproduce the good coproduct (12.4). For instance, in Example 12.1, as well as the trivial ideals for t_{32}, there are the ideals given by each leaf, and both leaves together. In conclusion, $H(W_\sim) \simeq H_R$. Clearly H_R is a quasi-uniform incidence algebra. Note that if a tree t is a stick with n vertices, then J_t is a stick with $n+1$ vertices.

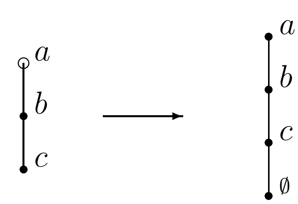


Figure 5: The distributive lattice for the tree t_{31}

In this way we are able to complement the graphic description of rooted trees by the Hasse diagrams of their distributive lattices of ideals (the Hasse diagram of a finite poset P is drawn by representing the elements of P by vertices and the cover relations by edges [83]). We illustrate with the accompanying figures the correspondence of (incidence Hopf algebras of) distributive lattices to (the Hopf algebra of) rooted trees, up to four vertices.

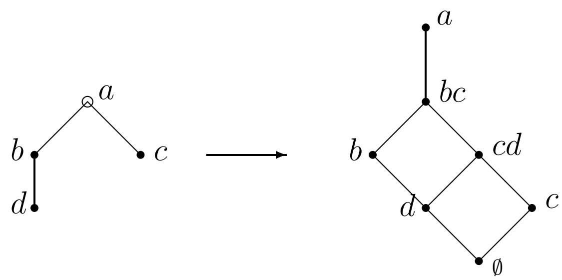


Figure 6: The distributive lattice for the tree t_{42}

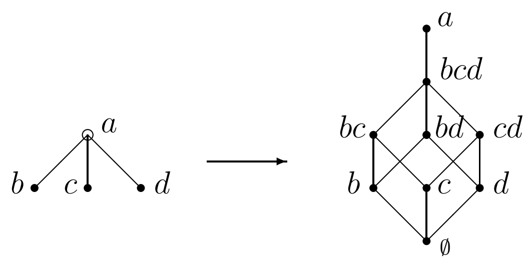


Figure 7: The distributive lattice for the tree t_{43}

With a bit of practice, one is able to read quickly the lattice paths; for instance, in figure 6, one sees that there are 12 chains and 8 forests associated to the tree t_{42}. There is no particular advantage in computing the antipode in this way; however on other matters the distributive lattice viewpoint is very helpful. For bialgebras of Feynman graphs, things are just a tad more difficult. We also illustrate with a few figures the correspondence of incidence Hopf algebras to some Feynman graphs for the four-point function in φ^4_4 theory, up to three loops. The reader will have noticed that Boolean lattices (and thus antichains) must be ascribed to articulate (one-vertex reducible) diagrams in φ^4_4 theory. This is in consonance with the fact, noted in Section 11, that renormalization factorizes for such graphs. The case of the cat's-eye diagram illustrates the possibility, for overlapping divergencies, of chooing a subposet of $\mathcal{J}_{\mathcal{P}}$, for $\mathcal{P}$ the divergence poset. Note the grading by length of Feynman graphs (that for connected graphs coincides as a filtering with the filtering by depth).

In conclusion for this part, we have understood the question on the relation between the Hopf algebras of renormalization of Connes and Kreimer and the incidence Hopf algebras of

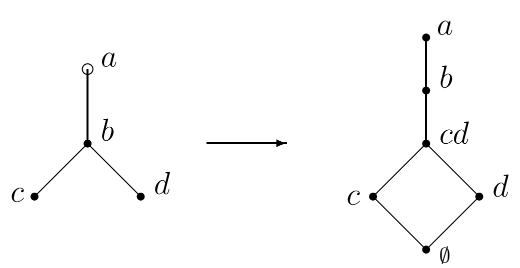


Figure 8: The distributive lattice for the tree t_{44}

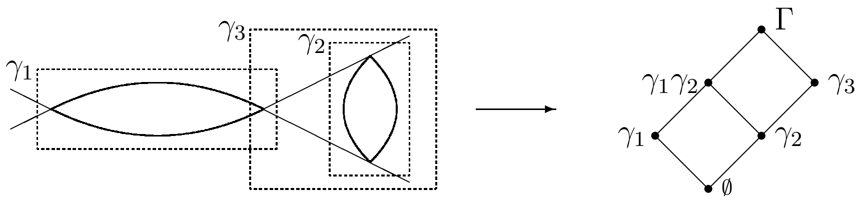


Figure 9: The distributive lattice for the feeding-shark diagram

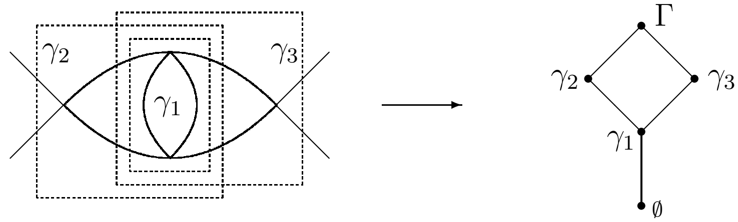


Figure 10: The distributive lattice for the cat's-eye diagram

Rota, Schmitt and others, in terms of the incidence Hopf algebra on the family of (sub)poset ideals $\mathcal{J}_{\mathcal{P}}$, where $\mathcal{P}$ denotes the family of posets made of Feynman graphs and subgraphs pertaining to a given model theory.

▶ If in the antipode formulae for connected elements T of H_R one formally substitutes 1 for all trees, one obtains zero whenever $T \neq t_1$. Concretely, the zeta function on trees is the character taking the value 1 on every tree (i.e., the 'geometric series' element of $G(H_T^\circ)$, see [74], and the Möbius function ζ^{*-1} sends 1 to 1, t_1 to -1 and any other tree to zero. This reflects the well-known fact that the Möbius function of any distributive lattice J is zero, except when J is the Boolean algebra of rank n, for which $\zeta^{*-1}(J) = (-1)^n$ —whereby on $H(\mathcal{P}_\sim)$ the Möbius function vanishes except for antichains; for the latter, $\zeta^{*-1}(A) = (-1)^{|A|}$ holds.

▶ The formula of Zimmermann is also valid under more general circumstances; for instance, for the Faà di Bruno algebra, although the partition lattices are not distributive. What is needed is a good description of the intervals in the algebra in terms of indecomposable types. This occurs in the Faà di Bruno algebra: see Proposition 12.2. The expression of S_Z then leads to (7.10); this was found in [19], as repeatedly indicated. After the first version of our paper appeared, in an interesting development, it has been found to work for the *free* Faà di Bruno algebra [84] as well. In our present framework, it is enough to remark that, as discussed in Section 12, the Faà di Bruno algebra $\mathcal{F}$ is a Hopf subalgebra of H_R, to which the distributive lattice paradigm applies.

Part IV: The General Structure Theorems

14 Structure of commutative Hopf algebras I

The title of the section might have been: 'why the boson Hopf algebra is important'. Indeed $B(V)$, one of the simplest commutative and cocommutative connected bialgebras, beyond its role in QFT will be seen in this section to play a normative role in the theory.

We have invoked a couple of times the Milnor–Moore theorem. This is the well known structure theorem for connected graded cocommutative Hopf algebras, stating that such a Hopf algebra H is necessarily isomorphic, as a Hopf algebra, to $\mathcal{U}(P(H))$. Remember $P(\mathcal{U}(P(H))) = P(H)$. For algebras of finite type, from our remarks on the filtering by depth at the end of Section 6, there is but a short step to a proof for the Milnor–Moore theorem, and the reader is invited to provide such proof. We refrain from giving any, as that is found in several places, including the book [24] by ourselves; we remit as well to [21] and the original paper [23].

Theorem 14.1. *Let H be a graded connected commutative skewgroup of finite type, and let V be a graded vector space such that $V \oplus H_+^2 = H_+$. Then there exists a graded algebra isomorphism between $B(V)$ and H.*

Proof. One has $H \simeq H'' \simeq \mathcal{U}(P(H'))'$ by the Milnor–Moore theorem. If $\{ e_i \}$ is a graded basis of $P(H')$, then consider the vector space V_0 linearly generated by the dual basis (defined by $f_j(e_i) = \delta_{ij}$); note that the space algebraically generated by V_0 is all of $\mathcal{U}(P(H'))'$ —this is essentially the Poincaré–Birkhoff–Witt theorem [85]. Thus the map $B\iota_{V_0} : B(V_0) \to \mathcal{U}(P(H'))'$ lifting $V_0 \ni f_j \to f_j \in \mathcal{U}(P(H'))'$ is an algebra isomorphism.

One has $V_0 \oplus H_+^2 = H_+$. This is so in view of

$$H = \mathcal{U}(P(H'))' = V_0 \oplus P(H')^\perp = \mathbb{C}1 \oplus V_0 \oplus H_+^2.$$

where (4.4) has been used in the last equality. Now, there is nothing very particular about V_0. Let V another supplementary graded vector subspace of H_+^2 in H_+, and denote as usual by $\mathbb{C}[V]$ the subalgebra of H generated by V. To prove that $\mathbb{C}[V] = H$ we (routinely) show by induction that $\bigoplus_{k=0}^n H^{(n)} \subset \mathbb{C}[V]$. Clearly this is true for $n = 0$. Assume the claim is true for all $k < n$, and let a be an element of degree n, that can be taken homogeneous. One has $a = v + b_1 b_2$ with $v \in V$ and $b_i \in \bigoplus_{k=0}^{n-1} H^{(k)}$; therefore $a \in \mathbb{C}[V]$, i.e., V indeed generates H. Moreover V_0 and V are isomorphic graded vector spaces; therefore $B(V), B(V_0)$ and H are isomorphic graded algebras.

As a consequence, H cannot have divisors of zero; also, the filtering by depth in connected graded commutative Hopf algebras (discussed extensively at the end of Section 6) has an associated *algebra* grading. □

It is useful to reflect more on Theorem 14.1. Borrowing the language of topology, we may say that $s : H_+/H_+^2 \to H_+$ is a section of $\pi : H_+ \to H_+/H_+^2$ if s is a right inverse for π. We have arrived at the conclusion that any section of π for a commutative connected graded bialgebra H induces an algebra isomorphism between the free commutative algebra over H_+/H_+^2 and H. This is Leray's theorem [23, Theorem 7.5], proved here at least for H of finite type.

(By the way, the theorem by Milnor and Moore, together with the Poincaré–Birkhoff–Witt theorem, incorporates the statement that a *cocommutative* connected graded bialgebra H is isomorphic as a coalgebra with the cofree cocommutative coalgebra $Q_{\mathrm{cocom}}(P(H))$ over the space of its primitive elements, with the isomorphism induced by any left inverse for the inclusion $P(H) \hookrightarrow H$.) Note as well that $H_+/H_+^2 \simeq P(H')^*$, and as $P(H')$ is a Lie algebra, H_+/H_+^2 inherits a natural Lie coalgebra structure, on which unfortunately we cannot dwell.

Theorem 14.1 does not give all we would want, as for instance we are still mostly in the dark about the primitive elements of the Hopf algebra, that seem to play such an important role in the Connes–Kreimer approach to renormalization theory. The Hopf algebra of rooted trees contains many primitive elements, but, not being cocommutative, cannot be primitively generated. Also Hopf algebras of Feynman graphs are not primitively generated; and the Faà di Bruno algebra, as we shall soon see, is very far from being primitively generated.

At least we know for sure, in view of (6.2), that primitive elements are indecomposable. The landscape thus emerging can be summarized as follows (look back at Section 6, and, if you can endure the category theory jargon, have a look at [86], too). Let $H(n) := \mathbb{C}\big[\bigoplus_{l=1}^n H^{(l)}\big]$. Each $H(n)$ is a graded subalgebra, and $H = \bigcup_{n=1}^\infty H(n)$. It is clear that $H(n)^{(n+1)} = H^{(n+1)} \cap H_+^2$. Therefore, $P(H) \cap H(n)^{(n+1)} = (0)$, and one can decompose $H^{(n+1)}$ as

$$H^{(n+1)} = H(n)^{(n+1)} \oplus P(H)^{(n+1)} \oplus W_H(n+1),$$

for some suitable (nonuniquely defined) supplementary vector space $W_H(n+1)$. Furthermore, since $H(n)$ is generated by $\sum_{l=1}^n H^{(l)}$, and the compatibility of the coproduct with the grading entails $\Delta H^{(n)} \subset H(n) \otimes H(n)$ for each n, we see that $H(n)$ is an ascending sequence of Hopf algebras. Thus, in any graded connected commutative bialgebra H there is a graded vector subspace W_H, with $W_H^{(0)} = W_H^{(1)} = 0$, such that the unique morphism of graded Hopf algebras $\nu : B(P(H)) \otimes B(W_H) \to H$, extending the morphism

$$\begin{aligned} (P(H) \otimes 1) \oplus (1 \otimes W_H) &\to P(H) \oplus W_H \subseteq H \\ p \otimes 1 + 1 \otimes w &\mapsto p + w \end{aligned} \tag{14.1}$$

is a graded algebra isomorphism. The quotient morphism $H \to H/H_+^2$ maps the graded subspace $P(H) + W_H$ isomorphically onto H/H_+^2. Moreover, the following statements are equivalent: (1) $q_H : P(H) \to H_+/H_+^2$ is an isomorphism; (2) $W_H = 0$; (3) H is primitively generated; (4) ν is an isomorphism of Hopf algebras.

Also, $B(P(H))$ is precisely the largest cocommutative Hopf subalgebra (subcoalgebra, subbialgebra) of H.

► To investigate the primitive elements (and primitivity degree) in important commutative algebras, in the few years after the introduction of the Connes–Kreimer paradigm, a few strategies, due to Broadhurst and Kreimer, Foissy, and Chryssomalakos and coworkers, were made available. The reader wishing to familiarize himself with the structure of graded connected commutative Hopf algebras ought to consult the original papers [9, 33, 49, 87]. In the last of these, so-called normal coordinate elements are introduced. We indicate the origin of the 'normal coordinate' terminology: the ψ-coordinates on $G(H^\circ)$ defined by $\psi_\delta(\exp \lambda\delta) = \lambda$ are elements of H that can be interpreted geometrically as 'normal coordinates'. We commend [87] for other geometrical insights, such as the identification of primitive elements in H with closed left invariant 1-forms on $G(H^\circ)$.

However, the computational algorithm in [87] for 'normal coordinate elements' —consult for instance their equation (25)— is painfully indirect. Our own strategy, inspired by work of Reutenauer [36], Patras [88, 89] and Loday [90, 91], is to settle for the knowledge of what in Section 5 we called quasiprimitive elements. They constitute a privileged canonical supplementary space of H_+^2 in H_+. Beautifully, and somewhat unexpectedly, it coincides with the space of normal coordinate elements. This treatment grants us the best grip on the reconstruction of H; it is deeply related to cyclic homology, but we will not go into that here.

As a rule, examples are more instructive than proofs. Therefore we illustrate our purpose by giving first the 'easy' example of the ladder algebra H_ℓ. The finer structure theorem is proved in the next section. The example of the Faà di Bruno algebra follows afterwards.

Example 14.1. As a corollary of Leray's theorem, if $P(H) \to H_+/H_+^2$ is an isomorphism, then $B\iota_{P(H)}$ is an isomorphism of Hopf algebras, and H is primitively generated. In particular, there will be a polynomial expression for the basis elements of H in terms of primitive elements. Now, in any Hopf algebra associated to a field theory there exist "ladder" subalgebras of diagrams with only completely nested subgraphs, and any of these is isomorphic to H_ℓ. The structure of H_ℓ is completely understood in terms of the isomorphism, already described in Example 8.3, between this bialgebra with its filtering by depth (which in this case give rise to a Hopf algebra grading), and a symmetric algebra.

As an incidence algebra, H_ℓ comes from a uniform family; therefore with the standard grading # there are exactly $p(n)$ linearly independent elements of degree n. Let us examine the first few stages. For $n = 1$ there is a basis with only the stick with one vertex l_1; for $n = 2$, there is a basis with l_2, the stick with two vertices, and l_1^2. Further bases are l_3, $\mathrm{l}_1\mathrm{l}_2$, l_1^3 for $n = 3$; l_4, $\mathrm{l}_1\mathrm{l}_3$, l_2^2, $\mathrm{l}_1^2\mathrm{l}_2$, l_1^4 for $n = 4$; l_5, $\mathrm{l}_1\mathrm{l}_4$, $\mathrm{l}_2\mathrm{l}_3$, $\mathrm{l}_1^2\mathrm{l}_3$, $\mathrm{l}_2^2\mathrm{l}_1$, $\mathrm{l}_1^3\mathrm{l}_2$, l_1^5 for $n = 5$.

On the other hand, there is an infinite number of primitives, given, as we know, by the Schur polynomials in the sticks, one at each #-order; it must be so because there is one indecomposable at each #-order. The first few are:

$$\begin{aligned}
&p_1 = \mathrm{l}_1; \quad p_2 = \mathrm{l}_2 - \tfrac{1}{2}\mathrm{l}_1^2; \quad p_3 = \mathrm{l}_3 - \mathrm{l}_1\mathrm{l}_2 + \tfrac{1}{3}\mathrm{l}_1^3;\\
&p_4 = \mathrm{l}_4 - \mathrm{l}_1\mathrm{l}_3 - \tfrac{1}{2}\mathrm{l}_2^2 + \mathrm{l}_1^2\mathrm{l}_2 - \tfrac{1}{4}\mathrm{l}_1^4;\\
&p_5 = \mathrm{l}_5 - \mathrm{l}_1\mathrm{l}_4 - \mathrm{l}_2\mathrm{l}_3 + \mathrm{l}_1^2\mathrm{l}_3 + \mathrm{l}_1\mathrm{l}_2^2 - \mathrm{l}_1^3\mathrm{l}_2 + \tfrac{1}{5}\mathrm{l}_1^5.
\end{aligned}$$

Following [9], we now introduce in the Hopf algebra H_ℓ the dimension $h_{n,k}$ of the space of homogeneous elements a such that $\#(a) = n$ and $\delta(a) = k$; we know that if a is not a scalar, $1 \le k \le n$ holds. We claimed already $h_{n,1} = 1$ for all n. And certainly $h_{n,n} \ge 1$, as for instance l_n has depth n —in fact, $h_{n,n} = 1$. To explore farther, in analogy to the definition of the antipode as a geometric series, we introduce

$$\log^* \mathrm{id} := \sum_{k\ge 1} \frac{(-1)^{k-1}}{k} (\mathrm{id} - u\eta)^{*k} = (\mathrm{id} - u\eta) - \frac{(\mathrm{id} - u\eta)^{*2}}{2} + \frac{(\mathrm{id} - u\eta)^{*3}}{3} - \cdots .$$

which, by the argument used in Proposition 5.2, is a finite sum in H_+; and its convolution powers. We easily find

$$
\begin{aligned}
\log^* \mathrm{id}\, \mathrm{l}_2 &= p_2; &\qquad \log^* \mathrm{id}\, \mathrm{l}_1^2 &= 0. \\
\log^* \mathrm{id}\, \mathrm{l}_3 &= p_3; &\qquad \log^* \mathrm{id}\, \mathrm{l}_1\mathrm{l}_2 = \log^* \mathrm{id}\, \mathrm{l}_1^3 &= 0; \\
\log^* \mathrm{id}\, \mathrm{l}_4 &= p_4; &\qquad \log^* \mathrm{id}\, \mathrm{l}_1\mathrm{l}_3 = \log^* \mathrm{id}\, \mathrm{l}_2^2 = \log^* \mathrm{id}\, \mathrm{l}_1^2\mathrm{l}_2 = \log^* \mathrm{id}\, \mathrm{l}_1^4 &= 0;
\end{aligned}
\tag{14.2}
$$

and so on, as $\log^* \mathrm{id}$ sends the indecomposables into the primitives, and kills products — see Proposition 15.2 below for details. Now

$$\frac{\log^{*2} \mathrm{id}}{2!}\, p_2 = 0; \quad \frac{\log^{*2} \mathrm{id}}{2!}\, \mathrm{l}_1^2 = \mathrm{l}_1^2.$$

This together with the first line in (14.2) means $h_{2,1} = h_{2,2} = 1$; it could not be otherwise.

Next,

$$\frac{\log^{*2} \mathrm{id}}{2!}\, p_3 = 0; \quad \frac{\log^{*2} \mathrm{id}}{2!}\, \mathrm{l}_1\mathrm{l}_2 = \mathrm{l}_1\mathrm{l}_2 - \tfrac{1}{2}\mathrm{l}_1^3 = \mathrm{l}_1 p_2; \quad \frac{\log^{*2} \mathrm{id}}{2!}\, \mathrm{l}_1^3 = 0.$$

At order 3, finally:

$$\frac{\log^{*3} \mathrm{id}}{3!} p_3 = 0; \quad \frac{\log^{*3} \mathrm{id}}{3!} \mathrm{l}_1 p_2 = 0; \quad \frac{\log^{*3} \mathrm{id}}{3!} \mathrm{l}_1^3 = \mathrm{l}_1^3.$$

Thus the 'natural basis' for $H_\ell^{(3)}$ is

$$(\mathrm{l}_3 - \mathrm{l}_1\mathrm{l}_2 + \tfrac{1}{3}\mathrm{l}_1^3, \mathrm{l}_1\mathrm{l}_2 - \tfrac{1}{2}\mathrm{l}_1^3, \mathrm{l}_1^3) = (p_3, p_1p_2, p_1^3),$$

with respective depths (primitivity degrees) $1, 2, 3$; thus $h_{3,1} = h_{3,2} = h_{3,3} = 1$.

Similarly, then

$$
\begin{aligned}
&\frac{\log^{*2} \mathrm{id}}{2!}\, \mathrm{l}_1\mathrm{l}_3 = \mathrm{l}_1\mathrm{l}_3 - \mathrm{l}_1^2\mathrm{l}_2 + \tfrac{1}{3}\mathrm{l}_1^4 = \mathrm{l}_1 p_3; \quad \frac{\log^{*2} \mathrm{id}}{2!}\, \mathrm{l}_2^2 = \mathrm{l}_2^2 - \mathrm{l}_1^2\mathrm{l}_2 + \tfrac{1}{4}\mathrm{l}_1^4 = p_2^2; \\
&\frac{\log^{*2} \mathrm{id}}{2!}\, \mathrm{l}_1^2\mathrm{l}_2 = 0; \quad \frac{\log^{*2} \mathrm{id}}{2!}\, \mathrm{l}_1^4 = 0.
\end{aligned}
$$

Therefore there is a space of subprimitive elements of dimension 2 in $H_\ell^{(4)}$, that is $h_{4,2} = 2$. Moreover, $\frac{\log^{*3} \mathrm{id}}{3!}$ kills p_4, $\mathrm{l}_1 p_3$, p_2^2 and l_1^4; also,

$$\frac{\log^{*3} \mathrm{id}}{3!}\, \mathrm{l}_1^2\mathrm{l}_2 = \mathrm{l}_1^2\mathrm{l}_2 - \tfrac{1}{2}\mathrm{l}_1^4 = \mathrm{l}_1^2 p_2; \quad \text{finally} \quad \frac{\log^{*4} \mathrm{id}}{4!} \mathrm{l}_1^4 = \mathrm{l}_1^4.$$

The natural basis for $H_\ell^{(4)}$ is:

$$\big(\mathrm{l}_4 - \mathrm{l}_1\mathrm{l}_3 - \tfrac{1}{2}\mathrm{l}_2^2 + \mathrm{l}_1^2\mathrm{l}_2 - \tfrac{1}{4}\mathrm{l}_1^4, \mathrm{l}_1\mathrm{l}_3 - \mathrm{l}_1^2\mathrm{l}_2 + \tfrac{1}{3}\mathrm{l}_1^4, \mathrm{l}_2^2 - \mathrm{l}_1^2\mathrm{l}_2 + \tfrac{1}{4}\mathrm{l}_1^4, \mathrm{l}_1^2\mathrm{l}_2 - \tfrac{1}{2}\mathrm{l}_1^4, \mathrm{l}_1^4\big) = (p_4, p_1p_3, p_2^2, p_1^2p_2, p_1^4).$$

Moreover $Sp_i = -p_i$ for all i. The reader is invited to examine the next homogeneous subspace. One sees that $h_{n,k}$ for H_ℓ is the number of partitions of n into k integers; this was proved in [9] using a powerful technique involving Hilbert series.

In summary, the algebra of ladder graphs or sticks H_ℓ offers a case in point for the simplest version of the result in (14.1), for which $W_{H_\ell} = 0$.

15 Structure of commutative Hopf algebras II

Consider, for H a graded connected commutative Hopf algebra, a suitable completion of the tensor product $H \otimes H'$, say $H \mathbin{\overline{\otimes}} H'$. This is a unital algebra, with product $m \otimes \Delta^t$ and unit $1 \otimes 1$. Now by Leray's theorem 14.1, our H is a boson algebra over a supplement V of H_+^2 in H_+; let A index a basis for V, let $\tilde{A}$ index the linear basis of H consisting of monomials X_u in elements of A, and let Z_u denote an element of the dual basis in H'; then the product on $H \mathbin{\overline{\otimes}} H'$ is given by the *double series* product:

$$\Bigl(\sum_{u,v\in\tilde{A}} \alpha_{uv} X_u \otimes Z_v\Bigr)\Bigl(\sum_{w,t\in\tilde{A}} \beta_{wt} X_w \otimes Z_t\Bigr) := \sum_{u,v,w,t\in\tilde{A}} \alpha_{uv}\beta_{wt}\, X_u X_w \otimes Z_v Z_t.$$

This is done just like in [36] for the shuffle–deconcatenation Hopf algebras of words. The linear embedding $\operatorname{End} H \to H \mathbin{\overline{\otimes}} H'$ given by

$$f \mapsto \sum_{u\in\tilde{A}} f(X_u) \otimes Z_u,$$

is really a convolution algebra embedding

$$(\operatorname{End} H, *) \to (H \mathbin{\overline{\otimes}} H', m \otimes \Delta^t).$$

Indeed,

$$\begin{aligned}
\Bigl(\sum_{u\in\tilde{A}} f(X_u) \otimes Z_u\Bigr)\Bigl(\sum_{v\in\tilde{A}} g(X_v) \otimes Z_v\Bigr) &= \sum_{u,v\in\tilde{A}} f(X_u)g(X_v) \otimes Z_u Z_v \\
&= \sum_{t\in\tilde{A}} \Bigl(\sum_{u,v\in\tilde{A}} f(X_u)g(X_v)\,\langle Z_u Z_v, X_t\rangle\Bigr) \otimes Z_t \\
&= \sum_{t\in\tilde{A}} \Bigl(\sum_{u,v\in\tilde{A}} f(X_u)g(X_v)\,\langle Z_u \otimes Z_v, \Delta X_t\rangle\Bigr) \otimes Z_t \\
&= \sum_{t\in\tilde{A}} f * g(X_t) \otimes Z_t. \qquad (15.1)
\end{aligned}$$

Notice that the identities $u\eta$ for convolution and id for composition in $\operatorname{End} H$ correspond respectively to

$$u\eta \mapsto 1 \otimes 1; \qquad \mathrm{id} \mapsto \sum_{u\in\tilde{A}} X_u \otimes Z_u$$

in the double series formalism.

On the other hand, let $\tilde{A}_+ := \tilde{A} \setminus 1$. Using the same idea as in (15.1), we get

$$\begin{aligned}
\log\Bigl(\sum_{u\in\tilde{A}} X_u \otimes Z_u\Bigr) &:= \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \Bigl(\sum_{u\in\tilde{A}_+} X_u \otimes Z_u\Bigr)^k \\
&= \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1,\dots,u_k\in\tilde{A}_+} X_{u_1}\cdots X_{u_k} \otimes Z_{u_1}\cdots Z_{u_k} \\
&= \sum_{w\in\tilde{A}} \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1,\dots,u_k\in\tilde{A}_+} \langle Z_{u_1}\cdots Z_{u_k}, X_w\rangle\, X_{u_1}\cdots X_{u_k} \otimes Z_w.
\end{aligned}$$

Therefore

$$\log\Bigl(\sum_{u\in\tilde{A}} X_u \otimes Z_u\Bigr) = \sum_{w\in\tilde{A}} \pi_1(X_w) \otimes Z_w, \tag{15.2}$$

where

$$\pi_1(X_w) := \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1,\dots,u_k\in\tilde{A}_+} \langle Z_{u_1}\cdots Z_{u_k}, X_w\rangle\, X_{u_1}\cdots X_{u_k}.$$

Now, by definition

$$\pi_1(X_w) = \log^* \mathrm{id}\, X_w.$$

We moreover consider the endomorphisms

$$\pi_n := \frac{\pi_1^{*n}}{n!},$$

so that, by (15.1):

$$\sum_{w\in A^*} \pi_n(X_w) \otimes Z_w = \frac{1}{n!}\Bigl(\sum_{v\in A^*} \pi_1(X_v) \otimes Z_v\Bigr)^n.$$

We may put $\pi_0 := u\eta$. Thus, if $a \in H$ is of order n, $\pi_m(a) = 0$ for $m > n$. Furthermore, for $n > 0$,

$$\mathrm{id}^{*l} a = \exp^*(\log^*(\mathrm{id}^{*l}))a = \sum_{m=1}^{n} \frac{(\log^*(\mathrm{id}^{*l} a))^m}{m!} = \sum_{m=1}^{n} l^m \frac{(\log^* \mathrm{id}\, a)^m}{m!} = \sum_{m=1}^{n} l^m \pi_m(a). \tag{15.3}$$

In particular $\mathrm{id} = \sum_{m\geq 0} \pi_m$.

Proposition 15.1. *For any integers n and k,*

$$\mathrm{id}^{*n}\, \mathrm{id}^{*k} = \mathrm{id}^{*nk} = \mathrm{id}^{*k}\, \mathrm{id}^{*n}. \tag{15.4}$$

Proof. The assertion is certainly true for $k = 1$ and all integers n, and if it is true for some k and all integers n, then taking into account that id is an algebra homomorphism, (2.3) and the induction hypothesis give

$$\mathrm{id}^{*n}\, \mathrm{id}^{*k+1} = \mathrm{id}^{*n}(\mathrm{id}^{*k} * \mathrm{id}) = \mathrm{id}^{*nk} * \mathrm{id}^{*n} = \mathrm{id}^{*n(k+1)}.$$

□

Substituting the final expression of (15.3) in (15.4), with very little work one obtains [90, Theorem 2.9.c]

$$\pi_m\pi_k = \delta_{mk}\,\pi_k. \tag{15.5}$$

In other words the π_k form a family of orthogonal projectors, and therefore the space H has the direct sum decomposition

$$H = \bigoplus_{n\geq 0} H^{(n)} := \bigoplus_{n\geq 0} \pi_n(H). \tag{15.6}$$

Moreover, from (15.3),

$$\mathrm{id}^{*l} H^{(n)} = l^n H^{(n)},$$

so the $H^{(n)}$ are the common eigenspaces of the operators id^{*l} with eigenvalues l^n. Thus, the decomposition (15.6) turns H into a graded algebra. Indeed, if $a \in H^{(r)}$ and $b \in H^{(s)}$, then

$$\mathrm{id}^{*l}(ab) = \mathrm{id}^{*l}a\,\mathrm{id}^{*l}b = l^{r+s}(ab),$$

and therefore m sends $H^{(r)} \otimes H^{(s)}$ into $H^{(r+s)}$.

If H were cocommutative, nearly all the previous arguments in this section would go through. For (15.4) one uses (2.4) instead of (2.3), and naturally instead of an algebra grading one gets a coalgebra grading. In particular, appealing to the identity (3.1), and since obviously $P(H)$ is contained in $\pi_1(H)$, we conclude that $\pi_1(H) = P(H)$ in this case.

Assume again that H is commutative, so H' is cocommutative. Our contention now is that the logarithm kills products.

Proposition 15.2. *On a commutative, connected Hopf algebra H, $\pi_1(H_+^2) = 0$ holds.*

Proof. Recall first that H_+^2 is orthogonal to $P(H')$, by (4.4). Now, to avoid confusion, denote in this proof by π_1' the first projector of H'. Since $(\mathrm{id} - u\eta)_H = (\mathrm{id} - u\eta)_{H'}^t$, clearly π_1 is the transpose of π_1', and then for an arbitrary u

$$\langle \pi_1(X_iX_j), Z_u\rangle = \langle X_iX_j, \pi_1'(Z_u)\rangle = 0,$$

since $\pi_1'(Z_u)$ is primitive. Therefore $\pi_1(X_iX_j)$ must vanish as claimed. (For the trained field theorist this was plausible, as he knows that logarithms identify 'connected' elements.) ☐

The grading associated to $\log^* \mathrm{id}$ does not, in general, coincide with the previous grading. At any rate, because of homogeneity of the convolution powers of id, we get an algebra bigrading of the Hopf algebra, becoming a bialgebra bigrading when H is cocommutative as well.

▶ Now turn to the so-called normal coordinate elements (canonical coordinates of the first kind would be a more appropriate name) introduced in [87] by the definition

$$\sum_{u\in\tilde{A}} X_u \otimes Z_u =: \exp\Bigg(\sum_{j\in A} \psi_j \otimes Z_j\Bigg).$$

Note that the sum on the right hand side is only on A (that is, only on 'letters', not on all 'words'). From this, the authors immediately conclude that any algebra basis element has a canonical decomposition as follows:

$$\begin{aligned} X_j &= \psi_j + \sum_{k\geq 2} \frac{\langle Z_{i_1}\cdots Z_{i_k}, X_j\rangle}{k!}\,\psi_{i_1}\ldots\psi_{i_k} \\ &= \psi_j + \sum_{k\geq 2} \frac{\langle Z_{i_1}\otimes\cdots\otimes Z_{i_k}, \Delta^k X_j\rangle}{k!}\,\psi_{i_1}\ldots\psi_{i_k}. \end{aligned}$$

The second form makes clear that only a finite number of terms intervene, corresponding to sequences $J = (i_1, \ldots, i_k)$ compatible with X_j in the sense that $\langle Z_{i_1}\otimes\cdots\otimes Z_{i_k}, \Delta^k X_j\rangle \neq 0$. Even so, to extract ψ_j from this change of infinite basis is a painful business. However, looking back at (15.2), and exponentiating both sides, we see that

$$\sum_{u\in\tilde{A}} X_u\otimes Z_u = \exp\biggl(\sum_{w\in\tilde{A}} \pi_1(X_w)\otimes Z_w\biggr) = \exp\biggl(\sum_{j\in A} \pi_1(X_j)\otimes Z_j\biggr);$$

since π_1 kills products, the sum of the right hand side extends only over A. Therefore $\psi_j = \pi_1(X_j)$.

All the properties claimed for H_R in [87] are seen as easy corollaries of the properties of the canonical projector π_1. For instance, the diagonality of the antipode in the ψ_j basis, or quasiprimitivity. Indeed, since $Sg = g^{-1}$ if g is grouplike, then

$$\langle g^{-1}, \psi_j\rangle = \langle Sg, \psi_j\rangle = \langle g, S\psi_j\rangle.$$

But, if

$$g = \exp\biggl(\sum_{k\in A} \alpha_k Z_k\biggr),$$

then $\langle g, \psi_j\rangle = \alpha_j$ and $\langle g^{-1}, \psi_j\rangle = -\alpha_j$, for all $j \in A$. The conclusion that $S\psi_j = -\psi_j$ follows.

For applications in QFT, further investigation of (depth and) quasiprimitivity in commutative algebras of the kind studied in this review seems paramount.

▶ Before reexamining our old friend $\mathcal{F}$, we make a skimpy remark on the algebra H_R of rooted trees, much studied by several authors [9, 33, 49, 87]. Even taken as a proxy for the Hopf algebras of QFT, its complexity is staggering. The number of rooted trees with n vertices is given by a famous sequence $r = (r_1 = 1, r_2 = 1, 2, 4, 9, 20, 48, 115, \ldots)$; then, with the standard grading by number of vertices, clearly $\dim H^{(n)} = r_{n+1}$. There are many primitives, beyond those contained in its Hopf subalgebra H_ℓ: for instance $h_{15,1} = 30135$. We wish to remark that the Faà di Bruno formulae are instrumental in finding $h_{n,k}$ in the context of H_R [49], again. The space W_{H_R} starts at level 3, since Figure 11 exhibits an indecomposable nonprimitive (which has depth 2).

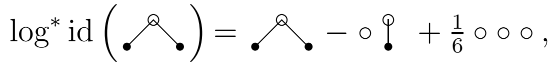


Figure 11: Nonprimitive combination of nonproduct woods

Example 15.1. Matters for the Beatus Faà di Bruno algebra are rather different from the ladder algebra. Of course $\dim \mathcal{F}^{(n)} = p(n)$ still holds. Consider $\mathcal{F}^{(2)}$, that is, the linear span of $a_2^2 = \delta_1^2$ and $a_3 = \delta_2 + \delta_1^2$ —remember that with our notation $\#(a_n) = n - 1$. Plainly, $a_2^2 \in \mathcal{F}_+^2$. Now, although a_3 in the indecomposable class is not primitive, it belongs to $\mathcal{F}_+^2 \oplus P(\mathcal{F})^{(2)}$, since

$$\log^* \mathrm{id}\, a_3 = a_3 - \tfrac{3}{2} a_2^2 = \delta_2 - \tfrac{1}{2}\delta_1^2 =: p_2$$

(the Schwarzian derivative) is primitive; the telltale sign was $\tau\Delta a_3 = \Delta a_3$.

Notice that no more primitives in $\mathcal{F}$, outside the space spanned by $p_1 := \delta_1$ and p_2, can possibly happen: equation (4.4) tells us

$$P(\mathcal{F}) = (\mathbb{C}1 \oplus \mathcal{F}'^{\,2}_{+})^{\perp}.$$

But then from (7.11) a dual basis of $\mathcal{F}'$ is made of products, except for two elements. Therefore $\dim P(\mathcal{F}) = 2$.

Also, the projector $\log^* \mathrm{id}$ will produce primitives only on indecomposable classes fulfilling $\Delta a = \tau\Delta a$; this cannot be contrived here. We obtain

$$\tilde{p}_3 := \log^* \mathrm{id}\, a_4 = a_4 - 5a_2a_3 + \tfrac{9}{2}a_2^3 = \delta_3 - 2\delta_1\delta_2 + \tfrac{1}{2}\delta_1^2,$$

and $\log^* \mathrm{id}$ kills products as usual. We know that $\tilde{p}_3$ is not primitive, rather it has depth 2, as it can easily be checked. It still is quasiprimitive, in that

$$m\Delta'\tilde{p}_3 = 0.$$

The algebra grading respectively induced by the convolution powers of the logarithm and by the depth filtering in this case do not coincide. We have as well

$$\frac{\log^{*2}\mathrm{id}}{2!}\, a_2a_3 = a_2a_3 - \tfrac{3}{2}a_2^3 = p_1p_2; \quad \frac{\log^{*2}\mathrm{id}}{2!}\, a_4 = 5(a_2a_3 - \tfrac{3}{2}a_2^3); \quad \frac{\log^{*2}\mathrm{id}}{2!}\, a_2^3 = 0.$$

Finally

$$\frac{\log^{*3}\mathrm{id}}{2!}\, a_2^3 = a_2^3 = p_1^3.$$

A suitable basis for $\mathcal{F}^{(3)}$ is then

$$(a_4 - 5a_2a_3 + \tfrac{9}{2}a_2^3, a_2a_3 - \tfrac{3}{2}a_2^3, a_2^3) = (\delta_3 - 2\delta_1\delta_2 + \tfrac{1}{2}\delta_1^2, \delta_1\delta_2 - \tfrac{1}{2}\delta_1^3, \delta_1^3) = (\tilde{p}_3, p_1p_2, p_1^3),$$

with respective depths (primitivity degrees) $2, 2, 3$.

In $\mathcal{F}^{(4)}$, we will have the linearly independent elements

$$p_2^2,\ p_1\tilde{p}_3,\ p_1^2p_2,\ p_1^4,$$

of respective depths $2, 3, 3, 4$. We seek a suitable representative for the missing indecomposable element. Proceeding systematically with the projectors $\frac{\log^{*k} \mathrm{id}}{k!}$, we obtain

$$\tilde{p}_4 := \log^* \mathrm{id}\, a_5 = a_5 - 15a_2a_4 - 5a_2^3 + \tfrac{185}{6}a_2^2a_3 - 20a_2^4,$$

next,

$$\frac{\log^{*2} \mathrm{id}}{2!}\, a_2^3 = p_2^2; \quad \frac{\log^{*2} \mathrm{id}}{2!}\, a_2a_4 = p_1\tilde{p}_3; \quad \frac{\log^{*3} \mathrm{id}}{3!}\, a_2^2a_3 = p_1^2p_2; \quad \frac{\log^{*4} \mathrm{id}}{4!}\, a_2^4 = p_1^4.$$

A suitable basis for $\mathcal{F}^{(4)}$ is then

$$(\tilde{p}_4,\ p_2^2,\ p_1\tilde{p}_3,\ p_1^2p_2,\ p_1^4).$$

Note that $S(p_1\tilde{p}_3) = p_1\tilde{p}_3, S(p_2^2) = p_2^2, S(p_1^2p_2) = -p_1^2p_2, S(\tilde{p}_4) = -\tilde{p}_4$; and so in general $S\frac{\log^{*n} \mathrm{id}}{n!}a = (-)^n\frac{\log^{*n} \mathrm{id}}{n!}a$ (we owe this remark to F. Patras). Going back to $\tilde{p}_4$, one can check quasiprimitivity, and the fact it has depth 3. The pattern repeats itself: at every order $\#$ in the original grading one nonproduct generator of depth $(\# - 1)$ —for which Broadhurst and Kreimer give a recipe in [9]— is found.

Concerning $W_{\mathcal{F}}$: we know a_3 belongs to $\mathcal{F}(1)^{(2)} \oplus P(\mathcal{F})^{(2)}$, since $a_3 - 3/2a_2^2$ is primitive. Therefore, in this case $W_{\mathcal{F}}^{(2)}$ is trivial. Now consider $\mathcal{F}^{(3)}$, the linear span of a_2^3, a_2a_3 and a_4. The element a_4 cannot be primitively generated, and so there is a one-dimensional $W_{\mathcal{F}}^{(3)}$. The same is true for $W_{\mathcal{F}}^{(n)}$, for every $n \geq 3$.

To summarize the general situation: $\mathrm{im}\, \pi_1$ is the 'good' supplementary space for H_+^2 we were looking for. The elements a and $\log^* \mathrm{id}\, a$ belong to the same class modulo H_+^2. The logarithms of the nonproduct elements, all quasiprimitives, provide an eminently suitable algebra basis for H. We can think of H, as a graded algebra, as the polynomial algebra on the $\log^* \mathrm{id}\, a$. Moreover, the projectors behave well in regard to the action of the antipode.

16 Coda: on twisting and other matters

We conclude provisionally these notes with pointers to two subjects of current interest: Hopf algebras of quantum fields, and investigation of 'twisted antipodes'. Both are indebted to Rota's work.

Consider now the theory of the free neutral scalar (for simplicity) field $\varphi(x)$. Then the space of quantum observables can be identified to the Fock boson algebra on $V \equiv$ the space of complex solutions of the Klein–Gordon (KG) equation

$$(\Box + m^2)v(x) := \frac{\partial^2 v(x)}{\partial t^2} - \Delta v(x) + m^2 v(x) = 0$$

on Minkowski space M_4. The algebra product is just the normal product of fields. Let D be the (Jordan–Pauli) distribution solving the Cauchy problem for the KG equation. The

Hilbert space of states is in turn a Fock space, built on the space of real solutions $V_{\mathbb{R}}$ made complex with the help of a complex structure, so that if

$$v_i(x) = \int D(x,y)h_i(y)\, d^4y \tag{16.1}$$

for $i = 1, 2$ are real solutions of the KG equation, then

$$\langle v_1 \mid v_2 \rangle = \int h_1(x)D_+(x,y)h_2(y)\, d^4x\, d^4y,$$

where D_+ is the standard Wightman function; the projection $h \mapsto v$ to field equation solutions in (16.1) corresponds to dividing out the unrestricted fields by the ideal generated by that equation, and will be implicitly used all the time. The quantum field is a V-valued distribution; it is defined by its action [92] by creating and annihilating a particle in the distributional state $D(\cdot - x)$:

$$\varphi(x) = \frac{1}{\sqrt{2}}[a(D(\cdot - x)) + a^\dagger(D(\cdot - x))].$$

This ensures that $(\Box + m^2)\varphi(h) := \varphi((\Box + m^2)h) = 0$, for any complex smearing function h. Note that

$$\langle 0, \varphi(h_1)\varphi(h_2)\, 0\rangle = \langle v_1, v_2\rangle. \tag{16.2}$$

On this concrete boson algebra we put the compatible cocommutative coalgebra structure already described; the counit η is at once identified with the vacuum expectation value

$$\eta(a) = \langle 0, a\, 0\rangle.$$

It is instructive to consider the Wick monomials in the field operator $\varphi(x)$. In regard to notation, to conform to usual practice, we write $:\!\varphi(x_1)\cdots\varphi(x_n)\!:$ instead of $\varphi(x_1)\vee\cdots\vee\varphi(x_n)$. The powers $\varphi(x)\vee\varphi(x)$ will be denoted simply $\varphi^2(x)$ in place of the standard $:\!\varphi^2(x)\!:$ (no other powers are defined). For the purpose one posits

$$\varphi^n(x) := \langle :\!\varphi(x)\varphi(x_2)\cdots\varphi(x_n)\!:, \delta(x - x_2)\cdots\delta(x - x_n)\rangle_{x_2\cdots x_n}, \tag{16.3}$$

or, if one wishes:

$$\varphi^n(h) := \langle :\!\varphi(x)\varphi(x_2)\cdots\varphi(x_n)\!:, h(x)\delta(x - x_2)\cdots\delta(x - x_n)\rangle_{x,x_2\cdots x_n}.$$

This is known to be well defined. The diagonalization map for Wick monomials can hardly be simpler: from (6.1) and (16.3) it is a couple of easy steps to obtain:

$$\Delta\varphi^n(x) = \sum_{i=0}^{n} \binom{n}{i} \varphi^i(x) \otimes \varphi^{n-i}(x),$$

i.e., the comultiplication is the binomial one (as befits honest-to-God monomials). One can alternatively use divided powers $\varphi^{(n)} := \varphi^n/n!$. More generally then:

$$\Delta^l(\varphi^{(n)}(x)) = \sum \varphi^{(m_1)}(x) \otimes \cdots \otimes \varphi^{(m_{l+1})}(x),$$

with sum over all combinations of $l+1$ nonnegative integers m_i such that $\sum_{i=1}^{l+1} m_i = n$. Very easy is also to check, from (16.3), that

$$\langle \varphi^n(x) \mid \varphi^m(y) \rangle = \frac{\delta_{nm}}{n!}\, D_+^n(x,y).$$

(Unlike other propagators, $D_\pm$ have powers of all orders.)

From this starting point, reference [7] proceeds by 'twisting' the Hopf algebra structure of $B(V)$ by suitable bilinear forms z on V. The twisted product is given by

$$a \bullet b = \sum z(a_{(1)}, b_{(1)})\, a_{(2)} \vee b_{(2)}.$$

Setting $z(1,1) = 1$ is understood. By cocommutativity, we equivalently have

$$a \bullet b = \sum a_{(1)} \vee b_{(1)} z(a_{(2)}, b_{(2)}) = \sum a_{(1)} \vee b_{(2)} z(a_{(2)}, b_{(1)}) = \sum a_{(2)} \vee b_{(1)} z(a_{(1)}, b_{(2)}).$$

Note that there is *no* restriction on the degrees of $a, b \in B(V)$. The authors consider two different twisted products, respectively corresponding to the operator and the time-ordered products of elements of $B(V)$. The associated bilinear forms are $z(v_1, v_2) = \langle v_1 \mid v_2 \rangle$ and the *symmetric* pairing $(\cdot \mid \cdot)$, given by

$$z(v_1, v_2) = (v_1 \mid v_2) = (v_2 \mid v_1) = \langle 0 \mid \mathrm{T}[\varphi(h_1)\varphi(h_2)]\, 0 \rangle = \int h_1(x) D_F(x,y) h_2(y);$$

we recall that the time-ordered product of free fields is defined by

$$\mathrm{T}[\varphi(x_1)\varphi(x_2)] = \mathrm{T}[\varphi(x_2)\varphi(x_1)] = \Theta(t_1 - t_2)\varphi(x_1)\varphi(x_2) + \Theta(t_2 - t_1)\varphi(x_2)\varphi(x_1),$$

where Θ is the Heaviside function (so, time increases from right to left) and that

$$\langle 0 \mid \mathrm{T}[\varphi(x_1)\varphi(x_2)]\, 0 \rangle = D_F(x_1 - x_2).$$

The resulting algebra in the first case can be called the Weyl algebra, since the canonical commutation relations

$$\langle 0 \mid [\varphi(h_1), \varphi(h_2)]\, 0 \rangle = s(v_1, v_2)$$

are satisfied; where the bilinear form s is given by the integral on the space $V_{\mathbb{R}}$ of solutions

$$s(v_1, v_2) := \int_\Sigma (v_1 \partial_\mu v_2 - v_2 \partial_\mu v_1)\, d\sigma^\mu = \int h_1(x) D(x,y) h_2(y)\, d^4x\, d^4y,$$

that does not depend on Σ itself, and defines a symplectic form, which is complexified in the standard way. In the second case we obtain a *commutative* algebra. By use on it of the above indicated comultiplication, recently Mestre and Oeckl have been able to show the relations between the different n-point functions of quantum field theory in a very economical manner [93].

An interesting property, generalizing (16.2), is

$$\eta(a \bullet b) = z(a, b).$$

In effect, by the defining property of η

$$\eta(a \bullet b) = \sum \eta(a_{(1)} \vee b_{(1)}) z(a_{(2)}, b_{(2)}) = \sum \eta(a_{(1)}) \eta(b_{(1)}) z(a_{(2)}, b_{(2)})$$
$$= z(\eta(a_{(1)}) a_{(2)}, \eta(b_{(1)}) b_{(2)}) = z(a, b).$$

The bilinear forms used are examples of *Laplace pairings* (a concept originally introduced by Rota [94]); in turn these are cocycles in a Hopf algebra cohomology. More general cocycles would seem to relate to interacting quantum fields, and to the passage to noncommutative field theory [95]. It is worth noting that the (much more difficult) Hopf algebra cohomology of the noncocommutative Hopf algebras H_R and $\mathcal{H}$ underlines Kreimer's program in the direction of using Hopf algebraic techniques to simplify the calculation of Feynman diagrams: see for instance [96].

In summary, the fundamentals of quantum field theory have been co-algebrized. The price is worth paying for the complete automation not only of the twisted product formulae, but of many indispensable calculations in field theory that are not found, or only haphazardly, in textbooks, and otherwise require a substantial amount of combinatorics (in this respect, the splendid little book by Caianello [97] still stands out). The Hopf algebra approach frees us from the perennial, tiresome recourse of decompositions in $\vee$-products of homogeneous elements of order 1 in every argument.

The benefits of this abstract framework were harvested in [6,7], as their results translate into strong versions of the Wick theorems of quantum field theory. As we said in the introduction, however, the approach from quantum theoretical first principles is still evolving (see [98] in this connection), and the passage from the Hopf algebras that Brouder, Oeckl and others have associated to quantum fields to the renormalization Hopf algebra of Connes and Kreimer is perhaps in the cards.

▶ The reader should be aware that, with respect to the Hopf algebra approach to renormalization, we have done less than to scratch the surface. The heart of this approach is a multiplicative map f (the "Feynman rule") of $\mathcal{H}$ into an algebra V of Feynman amplitudes: for instance, in dimensional regularization the character takes values in a ring of Laurent series in the regularization parameter ε. In physics, the Feynman rules are essentially fixed by the interpretation of the theory, and thus one tends to identify Γ with $f(\Gamma)$. Perhaps the main path-breaking insight of [2] is the introduction of the "twisted antipode". Let us then usher in the other personages of this drama. There is a linear map $T : V \to V$, which effects the subtraction of ultraviolet divergencies in each renormalization scheme. The twisted (or "renormalized") antipode $S_{T,f}$ is a map $\mathcal{H} \to V$ defined by $S_{T,f}(\emptyset) = 1; S_{T,f} = T \circ f \circ S$ for primitive diagrams, and then recursively:

$$S_{T,f}\Gamma = -[T \circ f]\Gamma - T\Bigg[\sum_{\emptyset \subsetneq \gamma \subsetneq \Gamma} S_{T,f}(\gamma)\, f(\Gamma/\gamma)\Bigg].$$

In other words, $S_{T,f}$ is the map that produces the counterterms in perturbative field theory. The Hopf algebra approach works most effectively because in many cases $S_{T,f}$ is multiplicative; for that, it is not necessary for T to be an endomorphism of the algebra of amplitudes V, but the following weaker condition [99] is sufficient:

$$T(hg) = T(T(h)g) + T(hT(g)) - T(h)T(g).$$

This endows V with the structure of a *Rota–Baxter algebra* of weight 1; it is fulfilled in the BPHZ formalism for massive particles, and the dimensional regularization scheme with minimal subtraction, for which the Connes–Kreimer paradigm is most cleanly formulated. By the way, here is where [82] fails. Finally, the renormalized amplitude $R_{T,f}$ is given by

$$R_{T,f} := S_{T,f} * f.$$

This map is also a homomorphism; compatibility with the coproduct operation is given by its very definition as a convolution. The (nonrecursive) forest formula for $R_{T,f}$ is precisely (the complete) Zimmermann's formula [13] of quantum field theory.

A Rota–Baxter algebra of weight θ is given by the condition

$$\theta T(hg) = T(T(h)g) + T(hT(g)) - T(h)T(g).$$

Shuffle algebras and Rota–Baxter algebras of weight 0 are essentially the same thing [35, 76]. The theory of Rota–Baxter algebras is examined in depth in [100] and [101].

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