Math 582: Foundations of Combinatorics – Graph Theory
Lecturer: Prof. Sara Billey

Problem Set #6

due Friday, February 16, 2018

Reading: Read Chapter 2 of Diestel and "How to apply de Bruijn graphs to genome assembly" by Phillip Compeau, Pavel Pevzner and Glenn Tesler. For additional information on the ring of symmetric functions, you can start with the Wikipedia page

https://en.wikipedia.org/wiki/Ring_of_symmetric_functions

Homework Problems: For each of the problems below, explain your answer fully. No credit will be given for a simple numerical answer.

(1) Find an expansion of h_λ in the m_μ basis.

(2) Using the expansion of e_λ in the m_μ basis stated in class, prove that the $\{e_\lambda : \lambda \vdash n\}$ form a basis for Λ_n, the vector space of homogeneous degree n symmetric functions.

(3) Expand the power series $\prod_{i>=1}(1+x_i+x_i^2)$ in terms of elementary symmetric functions.

(4) Show that a graph G contains k independent edges if and only if $q(G-S) \leq |S|+|G|-2k$ for all subsets $S \subset V(G)$. (Hint: consider the graph $G * K^{|G|-2k}$ and apply Tutte's 1-factor theorem.)

(5) The head master at Hogwarts has 5 new children which she needs to pair up with 5 different ghosts. They have all met and ranked their choices. The head master has to keep the ghosts happy or they cause trouble and the pairs have to stick with their assignments all year. How would you recommend the head master pair them up with the following preference table?

	C_1	C_2	C_3	C_4	C_5
G_1	$(3,1)$	$(5,5)$	$(4,3)$	$(1,3)$	$(2,5)$
G_2	$(5,3)$	$(4,4)$	$(1,4)$	$(3,4)$	$(2,4)$
G_3	$(2,5)$	$(5,3)$	$(3,5)$	$(1,1)$	$(4,1)$
G_4	$(3,4)$	$(5,2)$	$(1,1)$	$(4,5)$	$(2,2)$
G_5	$(2,2)$	$(5,1)$	$(1,2)$	$(4,2)$	$(3,3)$

where the ghosts' choices go along the rows in the first entry and the children's' choices go down columns in the second entry.

(6) In the student-optimal stable matching with n students and hospitals, how many students could possibly be assigned their last choice hospital?

(7) For each statement below, determine the smallest n such that the following hold for each $k \geq 4$.
   (a) There exists a k-regular graph with n vertices.
   (b) There exists at least two non-isomorphic k-regular graphs with n vertices.

(8) Let G be a connected graph of order at least four such that every edge belongs to a 1-factor of G. Show that G is 2-connected. Show also that if $|G| \geq 2k$ and every set of $k-1$ independent edges is contained in a 1-factor, then G is k-connected.

(9) (Bonus) Prove Gessel's functional equation for B,

$$\frac{(1+\bar{\rho}B)(1+\bar{\lambda}B)}{(1+\rho B)(1+\lambda B)} = e^{[(\bar{\rho}\bar{\lambda}-\rho\lambda)B+(\bar{\rho}+\bar{\lambda}-\rho-\lambda)]x}.$$