

De Bruijn Sequences and a Mathematical Card Trick

By Prof. Sara Billey

Contact info: billey@math.washington.edu

History: [Nicolaas Govert de Bruijn](#) is a Dutch mathematician born in 1918. He studied sequences of 0's and 1's such that every window of size k contains a unique binary word. For example, if $k=3$ then

0 0 0 1 0 1 1 1

is a de Bruijn sequence. Notice that every window is unique even if your window wraps around from the end of the sequence to the beginning. It is a theorem of Camille Flye Saint-Marie from 1894 that there are exactly

$$2^{2^{k-1}-k}$$

different de Bruijn sequences of length k with 2^k entries.

Encoding cards: In order to use de Bruijn sequences for our card trick, we need to make a secret code for cards using 0's and 1's. My secret code works like this. I only use the cards with face value Ace=1 through 8. First I encode the suit using two bits called ab :

Clubs	becomes	00
Spades	becomes	01
Diamonds	becomes	10
Hearts	becomes	11

Second, I encode the face value of the card using three bits called cde :

Ace	becomes	001	5	becomes	101
2	becomes	010	6	becomes	110
3	becomes	011	7	becomes	111
4	becomes	100	8	becomes	000

Putting the two parts together we get five bits $abcde$ which encode the card. For example, the card 4 of Spades gets encoded at 01100 because the first two bits are 01 for Spades and the last three bits are 100 which means 4 in this code.

Practice Here: See if you can encode all of these cards.

6 of Spades	becomes	<u>0</u> <u>1</u> <u>1</u> <u>1</u> <u>0</u>	2 of Diamonds becomes	___
4 of Spades	becomes	___	3 of Diamonds becomes	___
7 of Diamonds	becomes	___	Ace of Clubs becomes	___
8 of Clubs	becomes	___	5 of Spades becomes	___
Ace of Hearts	becomes	___	2 of Hearts becomes	___

Decoding cards: If you see a 5-letter word in 0's and 1's, then you know that the first two bits encode a suit and the last three bits encode a face value between 1 and 8. So you can figure out which card it is supposed to be. Practice decoding cards here:

0 1 1 1 0 means	<u>6 of Spades</u>	0 0 0 0 0 means	_____
1 1 0 1 1 means	_____	1 0 1 1 1 means	_____
0 0 0 0 1 means	_____	1 1 0 1 0 means	_____
1 0 1 0 0 means	_____	0 0 1 0 1 means	_____

The Trick Revealed: Before doing the trick, I have ordered the deck according to a de Bruin sequence based on the colors of the cards. Black cards have been encoded so their secret code all starts with 0's and the red cards have codes that all start with 1's. To order the cards in the sequence that I used in the trick, start with any card with face value 1 through 8 except for the 8 of Clubs. Encode your starting card in 0's and 1's like we practiced. Say the code is $abcde$ then to get the next bit called f , you add together $a + c$. If $a + c$ is even then $f = 0$, and if $a + c$ is odd then $f = 1$. We say $f = a + c \pmod{2}$. Now, to get the next card you just have to decode the 5-letter word corresponding to the shifted code $bcdef$. Repeat until you have used all 31 cards with face value 1 through 8 except for the 8 of Clubs.

For example, if I start with the 6 of Spades, then I know the code is 0 1 1 1 0. Here $a + c = 0 + 1$ is odd so $f = 1$. Adding a 1 on the end of 0 1 1 1 0 and dropping the first bit, we get 1 1 1 0 1, which means the 5 of Hearts. Now starting with 1 1 1 0 1 we can compute f again to get the next card (hint: it is the last card on Page 1). On the next page is a table to help you compute the whole list.

Helpful hint: As you are setting up the deck, if you find that you have already used the card that is supposed to be the next card, then you made a mistake. No problem! Just pick up a new card that hasn't been used and start computing the sequence from there. It will eventually take you to a card from the first try. This will help you find your mistake.

Performing the Trick: Ask someone to cut the deck. Lay out the top 5 cards face down. Ask them to look at the cards and somehow get them to tell you the color of each card. From the colors, you get the secret code of the first card. Knowing the first card, you compute the next card just the way you did when you set up the deck. Practice a few times on yourself first!

