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A Characterization of Perfect Graphs

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It is shown that a graph is perfect iff maximum clique · number of stability is not less than the number of vertices holds for each induced subgraph. The fact, conjectured by Berge and proved by the author, follows immediately that the complement of a perfect graph is perfect.

Throughout this note, graph means finite, undirected graph without loops and multiple edges. $\bar{G}$ and $|G|$ denote the complement and the number of vertices of G, respectively. Let $\mu(G)$ denote the maximum cardinality of a clique in the graph G, and let $\chi(G)$ be the chromatic number of G. Obviously

$$\chi(G) \geqslant \mu(G).$$

A graph G is called *perfect* if

$$\chi(G') = \mu(G')$$

for every induced subgraph G' of G. Berge [1] formulated two conjectures in connection with this notion:

(A) *A graph is perfect iff neither it nor its complement contains an odd circuit without diagonals.*

(B) *The complement of a perfect graph is perfect.*

Obviously, (A) is stronger than (B). In [3] (B) was proved. This result also follows from the theory of anti-blocking polyhedra, developed by Fulkerson [2].

In the present paper a theorem stronger than (B) but weaker than (A) is proved. This possibility of sharpening of (B) was raised by A. Hajnal.

THEOREM. *A graph G is perfect if and only if*

$$\mu(G')\,\mu(\bar{G}') \geqslant |\, G' \,|$$

for every induced subgraph G' of G.

Proof. Part "only if" is trivial. To prove part "if" we use induction on $|\, G \,|$. Thus we may assume that any proper induced subgraph of G, as well as its complement, is perfect.

Let *multiplication* of a vertex x by h ($h \geqslant 0$) mean substituting for it h independent vertices, joined to the same set of vertices as x. This notion is closely related to the notion of *pluperfection*, introduced by D. R. Fulkerson.

(I) As a first step of the proof we show that if G_0 arises from G by multiplication of its vertices then G_0 satisfies

$$\mu(G_0)\,\mu(\bar{G}_0) \geqslant |\, G_0 \,|.$$

Assume this is not the case and consider a G_0 failing to have this property and with minimum number of vertices. Obviously, there is a vertex y of G which is multiplied by $h \geqslant 2$; let $y_1 ,..., y_h$ be the corresponding vertices of G_0. Then

$$\mu(G_0 - y_1)\,\mu(\bar{G}_0 - y_1) \geqslant |\, G_0 \,| - 1$$

by the minimality of G_0; hence

$$\mu(G_0) = \mu(G_0 - y_1) = p, \qquad \mu(\bar{G}_0) = \mu(\bar{G}_0 - y_1) = r$$

and

$$|\, G_0 \,| = pr + 1.$$

Put $G_1 = G_0 - \{y_1 ,..., y_h\}$. Then G_1 arises from $G - y$ by multiplication of its vertices, hence by [1, Theorem 1], $\bar{G}_1$ is perfect. Thus, $\bar{G}_1$ can be covered by $\mu(\bar{G}_1) \leqslant \mu(\bar{G}_0) = r$ disjoint cliques of G_1; let $C_1 ,..., C_r$ be these cliques, $|\, C_1 \,| \geqslant |\, C_2 \,| \geqslant \cdots \geqslant |\, C_r \,|$.

Obviously, $h \leqslant r$. Since $|\, G_1 \,| = |\, G_0 \,| - h = pr + 1 - h$,

$$|\, C_1 \,| = \cdots = |\, C_{r-h+1} \,| = p.$$

Let G_2 be the subgraph of G_0 induced by $C_1 \cup \cdots \cup C_{r-h+1} \cup \{y_1\}$, then

$$|\, G_2 \,| = (r - h + 1)p + 1 < |\, G_0 \,|;$$

thus, by the minimality of G_0,

$$\mu(G_2)\,\mu(\bar{G}_2) \geqslant |\,G_2\,|.$$

Since $\mu(G_2) \leqslant \mu(G_0) = p$, this implies

$$\mu(\bar{G}_2) \geqslant r - h + 2.$$

Let F be a stable set of $r - h + 2$ vertices of G_2; then $|\,F \cap C_i\,| \leqslant 1$ $(1 \leqslant i \leqslant r - h + 1)$, hence $y_1 \in F$. This implies that $F \cup \{y_2, \ldots, y_h\}$ is stable in G_0. On the other hand

$$|\,F \cup \{y_2, \ldots, y_h\}| = r + 1 > \mu(\bar{G}_0),$$

a contradiction.

(II) We show that $\chi(G) = \mu(G)$. It is enough to find a stable set F such that $\mu(G - F) < \mu(G)$ since then, by the induction hypothesis, $G - F$ can be colored by $\mu(G) - 1$ colors and, adding F as a further one, we obtain a $\mu(G)$-coloring of G.

Assume indirectly that $G - F$ contains a $\mu(G)$-clique C_F for any stable set F in G. Let, for $x \in G$, $h(x)$ denote the number of C_F's containing x. Let G_0 arise from G by multiplying each x by $h(x)$.

Then, by Part I above,

$$\mu(G_0)\,\mu(\bar{G}_0) \geqslant |\,G_0\,|.$$

On the other hand, obviously

$$|\,G_0\,| = \sum_x h(x) = \sum_F |\,C_F\,| = pf,$$

where f denotes the number of all stable sets in G_0, and

$$\mu(G_0) \leqslant \mu(G) = p,$$
$$\mu(\bar{G}_0) = \max_F \sum_{x \in F} h(x) = \max_F \sum_{F'} |\,F \cap C_{F'}\,| \leqslant \max_F \sum_{F' \neq F} 1 = f - 1,$$

a contradiction.

REMARK. The condition given in the theorem is strictly related to the max-max inequality given by Fulkerson [2]. Multiplication of a vertex is the same as what he calls *pluperfection*.

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