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Excluding a Long Double Path Minor

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The "height" of a graph G is defined to be the number of steps to construct G by two simple graph operations. Let B_n be the graph obtained from an n-edge path by doubling each edge in parallel. Then, for any minor-closed class $\mathscr{G}$ of graphs, the following are proved to be equivalent: (1) Some B_n is not in $\mathscr{G}$; (2) There is a number h such that every graph in $\mathscr{G}$ has height at most h; (3) $\mathscr{G}$ is well-quasi-ordered by the topological minor relation; (4) There is a polynomial function $p(\cdot)$ such that the number of paths of every graph G in $\mathscr{G}$ is at most $p(|V(G)|+|E(G)|)$.

1. INTRODUCTION

All graphs in this paper are finite and may have loops or multiple edges. A graph H is called a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. A class $\mathscr{G}$ of graphs is *minor-closed* if G belongs to $\mathscr{G}$ whenever G is isomorphic to a minor of a member of $\mathscr{G}$. For a positive integer n, a *double path* B_n is the graph obtained from an n-edge path by doubling each edge in parallel. The main results of this paper are Theorem (1.5) and Theorem (1.6), which characterize those graphs that do not have a "long" double path as a minor.

This research was motivated by a conjecture of Robertson (a detailed discussion is given in Section 3). By using Theorem (1.6), we prove that Robertson's conjecture is true if we only consider minor-closed classes of graphs. In Section 4, we discuss another application of Theorem (1.6). It is proved, for every minor-closed class $\mathscr{G}$ of graphs, that some B_n is not in $\mathscr{G}$ if and only if there is a polynomial function $p(\cdot)$ such that the number of paths of every graph G in $\mathscr{G}$ is at most $p(|V(G)|+|E(G)|)$.

To formulate our main results, we need some preparations. A *rooted graph* is a connected graph, together with a specified vertex called the *root*. Let G be an *r-rooted graph*. That is, G is a rooted graph with root r. We shall say that G has B_n as a *rooted minor* if G has B_n as a minor while the root r of G is contracted to an end-vertex of B_n. We first make the following obvious observation.

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(1.1) Let G be a rooted graph. If B_n is not a minor of G, then B_n is not a rooted minor of G. On the other hand, if B_n is not a rooted minor of G, then B_{2n-1} is not a minor of G.

It follows from this observation that characterizing those graphs which do not have a "long" double path as a minor is equivalent to characterizing those rooted graphs which do not have a "long" double path as a rooted minor. In the following, we shall concentrate on rooted graphs since they are easier to work with.

Let G be a rooted graph. A *single-extension* of G is a rooted graph obtained from G by adding a new vertex r, which will be the root, and then, adding loops incident with r and adding new edges between r and $V(G)$ in an arbitrary way. It is easy to see from (1.1) that the following proposition holds.

(1.2) If G does not have B_n as a rooted minor, then a single-extension of G does not have B_{2n} as a rooted minor.

Let k be a positive integer and let $G_1, ..., G_k$ be mutually disjoint rooted graphs with roots $r_1, ..., r_k$ respectively. A *tree-connection* of these graphs is a rooted graph defined as follows. Take an r-rooted tree T disjointing from every G_i. Then, for each G_i, identify r_i with a vertex of T, where we allow more than one r_i to identify with a vertex of T. Finally, we define the root of the resulting graph to be r. A good feature of this operation is given by the following lemma. We leave the proof to the reader since it is easy.

(1.3) Let G be a tree-connection of rooted graphs $G_1, ..., G_k$. Then G has B_n as a rooted minor if and only if some G_i has B_n as a rooted minor.

Clearly, every rooted graph can be constructed from graphs with one vertex by a sequence of single-extension and tree-connection operations. This observation suggests that we may define the *height* of a rooted graph as follows. The height of a graph with one vertex is zero. Then, for a positive integer h, a rooted graph has height at most h if it is a single-extension of a rooted graph of height at most $h-1$, or it is a tree-connection of rooted graphs of height at most $h-1$. It is clear from (1.2) and (1.3) that the following holds.

(1.4) If the height of a rooted graph G is at most h, then G does not have B_{2^h} as a rooted minor.

Conversely, we have the following theorem, the rooted version of our main result.

(1.5) THEOREM. *There is a function $h(n)$ such that every rooted graph without B_n as a rooted minor has height at most $h(n)$.*

We shall prove this theorem in Section 2, where we show that $h(n) \leqslant 32n^4$.

The *height* of a graph G is at most h if all connected components of G can be rooted in such a way that the heights of these rooted graphs are at most h. Immediately, we conclude from (1.1), (1.4), and (1.5) the following theorem, the unrooted version of our main result.

(1.6) THEOREM. *Let $\mathcal{G}$ be a minor-closed class of graphs. Then some B_n is not in $\mathcal{G}$ if and only if there is a number h such that the height of every graph in $\mathcal{G}$ is at most h.*

Theorem (1.6) can be viewed in different ways. For instance, we may say that a long double path minor is the only obstacle to small height. We may also say that a graph has a long double path minor if and only if its height is big.

However, we need to point out that height is not a parameter preserved under taking minors. In another words, there exist graphs G such that the height of a minor of G exceeds the height of G. For example, let G be the graph obtained from a path on five vertices by adding a new vertex r and making it adjacent to all the five vertices of the path. Since the height of the path is one and G is a single-extension of the path, it follows that the height of G is two. Now let G' be the graph obtained from G by contracting the edge between r and the middle vertex of the path. Then it is easy to see that the height of G' is three, which is larger than the height of G.

If one is interested in a parameter that is preserved under taking minors, one may define the *m-height* (modified height) of a graph in the same way as height is defined, except allowing G to be the disjoint union of arbitrarily many rooted graphs in the definition of single-extension (this new operation will be called *modified single-extension*). It is not difficult to verify the following proposition.

(1.7) The m-height of a minor of a graph G is at most the m-height of G.

Observe that a modified single-extension of a graph G can also be obtained by a tree-connection of single-extensions of the connected components of G. Thus the following must hold.

(1.8) If the height and the m-height of a graph are h and m, respectively, then $m \leqslant h \leqslant 2m$.

Therefore, Theorem (1.5) and Theorem (1.6) still hold if height is replaced by m-height. We choose to use height in our discussion because single-extension is more elementary than the modified single-extension, which makes our theorems slightly stronger. Besides, even though a minor may have greater height, from (1.7) and (1.8) we conclude that the height of a minor cannot be arbitrarily large.

(1.9) The height of a minor of a graph G is at most twice the height of G.

2. Proving Theorem (1.5)

Let G be a graph and let G' be the graph obtained from G by deleting its loops. It is not difficult to see that G and G' have the same height. Thus, we shall only consider loopless graphs in this section. We begin with a corollary of Dilworth's theorem [1], which will be used in our proof.

(2.1) A poset with at least $(m-1)(n-1)+1$ elements must have a chain of size m or an antichain of size n.

A graph H is called a *topological minor* of a graph G if a subdivision of H is a subgraph of G. Let n be a positive integer and let L_n, C_n be graphs as illustrated in Fig. 1. Then we have the following lemma.

(2.2) Let n be a positive integer and let G be a cubic hamiltonian graph. Suppose G has at least $2(n-1)(n^2-1)+1$ vertices. Then G has a topological minor isomorphic to either L_n or C_n.

Proof. Let C be a hamiltonian circuit of G. Let $x_1, ..., x_s$ be vertices of G such that $\{x_1, x_2\}$, $\{x_2, x_3\}$, ..., $\{x_{s-1}, x_s\}$, and $\{x_s, x_1\}$ are the edges of C. Let F be the set of other edges of G. Then F is a perfect matching of G.

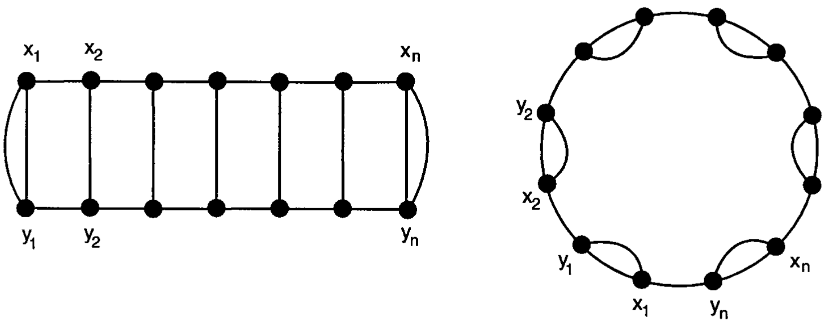

Fig. 1. A "ladder" L_n and a "circuit" C_n.

Let us define a binary relation $\preccurlyeq$ on F as follows. For any two members $e = \{x_i, x_j\}$ and $e' = \{x_{i'}, x_{j'}\}$ of F, let $e \preccurlyeq e'$ if $e = e'$ or $\max\{i, j\} < \min\{i', j'\}$. It is clear that $Q = (F, \preccurlyeq)$ is a poset. Moreover, if Q has a chain P of size n, then the subgraph of G induced by $P \cup E(C)$ is isomorphic to a subdivision of C_n.

Since F is perfect matching of G and G has at least $2(n-1)(n^2-1)+1$ vertices, it follows that F has at least $(n-1)(n^2-1)+1$ elements. Thus, by (2.1), we may assume that Q has an antichain F' of size n^2. Now we define a binary relation $\preccurlyeq'$ on F'. For any two members $e = \{x_i, x_j\}$ and $e' = \{x_{i'}, x_{j'}\}$ of F', let $e \preccurlyeq' e'$ if either $e = e'$ or i and j are between i' and j'. It is clear that $Q' = (F', \preccurlyeq')$ is a poset. Moreover, if Q' has a chain P' of size n, then the subgraph of G induced by $P' \cup E(C)$ is isomorphic to a subdivision of L_n.

Therefore, by (2.1) again, we may assume that Q' has an antichain A of size $n+2$. Suppose $\{x_{i_1}, x_{j_1}\}, ..., \{x_{i_{n+2}}, x_{j_{n+2}}\}$ are the members of A. Without loss of generality, let us assume that $i_k < j_k$ for all k, and $i_1 < i_2 < \cdots < i_{n+2}$. Since A is an antichain of both Q and Q', it is clear that we must have $i_{n+2} < j_1 < \cdots < j_{n+2}$. Let the two paths of C induced by $\{x_i : i_1 \leqslant i \leqslant i_{n+2}\}$ and $\{x_j : j_1 \leqslant j \leqslant j_{n+2}\}$ be C_i and C_j, respectively. Then it is easy to see that the subgraph of G induced by $A \cup E(C_i) \cup E(C_j)$ is isomorphic to a subdivision of L_n. Thus (2.2) is proved. ▮

Our next lemma is a result on how to estimate the height of a rooted graph.

(2.3) Let G be an r-rooted graph and let X be a subset of $V(G)$ with $r \in X$. If the height of every connected component of $G - X$ is at most h, then the height of G is at most $h + 2\,|X|$.

Proof. We prove the result by induction on $|X|$. If $|X| = 1$, then $X = \{r\}$. Let the connected components of $G - X$ be $G'_1, ..., G'_k$. For each G'_i, let G_i be the subgraph of G induced by $X \cup V(G_i)$. Since G is connected, it is clear that each G_i is also connected. Let the root of each G_i be r. Then we conclude that each G_i is a single-extension of G'_i. Consequently, the height of each G_i is at most $h+1$. Let T be the tree with vertex set X. Then it is clear that G is a tree-connection of $G_1, ..., G_k$ (over T). Therefore, the height of G is at most $h+2$.

Now we assume that $|X| > 1$. Let x be a vertex in $X - \{r\}$ and let X' be $X - \{x\}$. Let G' be the connected component of $G - X'$ that contains x. Since all the connected components of $G' - \{x\}$ are connected components of $G - X$, it follows that the heights of these graphs are at most h. Let the root of G' be x. Then we conclude from our induction hypothesis that the height of G' is at most $h+2$. Notice that all the other connected components of $G - X'$ are connected components of $G - X$. Thus the height of every connected component of $G-X'$ is at most $h+2$. By our induction hypothesis, it is clear that the height of G is at most $(h+2)+2|X'|$, which is $h+2|X|$, as required. ∎

A rooted graph G is *well-rooted* if its root r is contained in a circuit and $G-\{r\}$ is connected.

(2.4) Every rooted graph G is either a rooted tree or a tree-connection of well-rooted graphs.

Proof. Let r be the root of G and let E' be the set of edges e of G that do not belong to any circuit of G. Then $(V(G), E')$ must be a forest. Let T be the connected component of this forest that contains r. If $G=T$, then we are done. If $G\neq T$, since T is an induced subgraph of G, it follows that $V(G)\neq V(T)$. Thus there is a positive integer k such that $G'_1, ..., G'_k$ are the connected components of $G-V(T)$. From the choice of T it is clear that, for each G'_i, there is a vertex r_i in $V(T)$ such that the edges between r_i and $V(G'_i)$ are exactly the edges between $V(T)$ and $V(G'_i)$. For $i=1, ..., k$, let G_i be the subgraph of G induced by $V(G'_i)\cup\{r_i\}$. Then, from the choice of T again, we conclude that each G_i is well rooted if we choose r_i to be its root, and that G is the tree-connection of $G_1, ..., G_k$ (over T). The proof is complete. ∎

To prove our last lemma, we need a result of Gallai [3]. Let G be a graph and let S be a subset of $V(G)$. Then an *S-path* is a path P of G such that $E(P)\neq\varnothing$ and $S\cap V(P)$ consists of the two end-vertices of P.

(2.5) The maximum number of vertex-disjoint S-paths of G equals

$$\min_{Z\subseteq V(G)} |Z| + \sum_{i=1}^{k} \left\lfloor \frac{|S\cap V(G_i)|}{2} \right\rfloor,$$

where $G_1, ..., G_k$ are the connected components of $G-Z$.

The following is an immediate corollary of (2.5).

(2.6) Let G be a graph and let S be a subset of $V(G)$. Let m be a nonnegative integer. Then there exist either $m+1$ vertex-disjoint S-paths or a subset X of $V(G)$ such that $|X|\leqslant 2m$ and each connected component of $G-X$ has at most one vertex in S.

For every positive integer n, let $g(n)=(2n-1)(4n^2-1)+1$ and $h(n)=4(g(1)+\cdots+g(n))$.

(2.7) If G is a well-rooted graph without B_n as a rooted minor, then the height of G is less than $h(n)$.

Proof. Let r be the root of G and let C be a circuit of G containing r. Let $S = V(C)$ and let $G' = G - E(C)$. Then G' does not have $g(n)$ vertex-disjoint S-paths. Because otherwise, if H is the union of C and these S-paths, then H is a subdivision of a cubic hamiltonian graph on $2g(n)$ vertices. It follows from (2.2) that H, and hence G, has either L_{2n} or C_{2n} as a topological minor. But in both of these two cases, G has B_n as a rooted minor, a contradiction. Therefore, by (2.6), there is a subset X' of $V(G')$ such that $|X'| \leqslant 2(g(n)-1)$ and every connected component of $G' - X'$ has at most one vertex in S.

Let $X = X' \cup \{r\}$. Then $|X| \leqslant 2g(n) - 1$ and every connected component of $G' - X$ has at most one vertex in S. Let H be a connected component of $G' - X$. Since $G - \{r\}$ is connected, there is a path P_H between S and $V(H)$ without using r. Let us choose this path so that $|V(P_H)|$ is minimized. Let s_H in S and r_H in $V(H)$ be the end-vertices of P_H. It is clear that $s_H = r_H$ if H has a vertex in S. Let G_H be the union of C, P_H, and H. Let the root of G_H be r and let the root of H be r_H. Since G, and hence G_H, does not have B_n as a rooted minor, it follows that H does not have B_{n-1} has a rooted minor.

Let J be a connected component of $G - X$. Since $G - X$ can be obtained from $G' - X$ by adding the edges of $C - X$, and since each connected component of $G' - X$ has at most one vertex in S, it follows that J is a tree-connection of connected components of $G' - X$ (over a path). From (1.3) and the discussion in the last paragraph, we conclude that J can be rooted so that it does not have B_{n-1} as a rooted minor. Thus, by our induction hypothesis, the height of J is at most $h(n-1)$. Consequently, by (2.3), the height of G is at most $h(n-1) + 2(2g(n)-1)$, which is less than $h(n)$. Therefore, (2.7) is proved. ∎

Proof of (1.5). We shall prove, by induction on n, that the function h in (2.7) satisfies the requirement. First, we consider the case when n is one. If a rooted graph G does not have B_1 as a rooted minor, then G must be a rooted tree. It follows that the height of G is at most one, which is clearly less than $h(1)$.

Next we consider the case when n exceeds one. Clearly, we may assume that G is not a tree. Thus we conclude from (2.4) that G is a tree-connection of well-rooted graphs $G_1, ..., G_k$. It follows from (1.3) that G_i does not have B_n as a rooted minor for all i. Now, by (2.7), the height of each G_i is at most $h(n) - 1$. Therefore the height of G is at most $h(n)$, as required. ∎

Remark. From the definition of g we deduce that $g(n) \leqslant 8n^3$. Therefore, $h(n) \leqslant 32n^4$.

3. Well-quasi-ordering

A binary relation $\leqslant$ on a set Q is a *quasi-order* if $\leqslant$ is reflexive and transitive. An infinite sequence $q_1, q_2, \ldots$ of members of Q is *bad* (with respect to $\leqslant$) if $i \geqslant j$ whenever $q_i \leqslant q_j$. We call $(Q, \leqslant)$ a *well-quasi-order* (or a *wqo* for brevity) if there is no bad sequence.

For any two graphs H and G, let $H \leqslant_t G$ if H is isomorphic to a topological minor of G. Clearly, $\leqslant_t$ is a quasi-order on the class of all graphs. However, $\leqslant_t$ is not a wqo, as shown by the bad sequence $A_1, A_2, \ldots$, where A_n is the graph as illustrated in Fig. 2.

There are many other bad sequences with respect to $\leqslant_t$ (see [2]), but all these known examples, in some way, involve double paths of arbitrary length. Based on this observation, Robertson made the following (unpublished) conjecture.

(3.1) *Robertson's Conjecture. Let $\mathcal{G}$ be a class of graphs. If there is a positive integer n such that no graph in $\mathcal{G}$ contains B_n as a topological minor, then $(\mathcal{G}, \leqslant_t)$ is a well-quasi-order.*

We will show that Robertson's conjecture is true if $\mathcal{G}$ is minor-closed. We formulate this result in the following form.

(3.2) Let $\mathcal{G}$ be a minor-closed class of graphs. Then $(\mathcal{G}, \leqslant_t)$ is a well-quasi-order if and only if the intersection of $\mathcal{G}$ and $\mathcal{A}$ is finite, where $\mathcal{A} = \{A_1, A_2, \ldots\}$.

Notice that (3.2) implies the following well-known result of Mader [6].

(3.3) Mader's Theorem. *For every positive integer n, $(\mathcal{M}_n, \leqslant_t)$ is a well-quasi-order if $\mathcal{M}_n$ is the class of graphs that do not have n vertex-disjoint circuits.*

In fact, the result we are going to prove is stronger than (3.2), as explained below. Let H and G be graphs with $H \leqslant_t G$. Then there is an isomorphism ρ from a subdivision H' of H to a subgraph G' of G. The *homomorphism* from H to G is a mapping σ defined as follows. For each vertex v of H, let $\sigma(v) = \rho(v)$; for each edge e of H, let $\sigma(e)$ be the path of G' that corresponds to the subdivision of e (if e is a loop, $\sigma(e)$ is actually a circuit). If H and G are rooted graphs with roots r_H and r_G, respectively, then a *homomorphism* from H to G is a homomorphism σ from H to G, where H and G are viewed as unrooted graphs, such that $\sigma(r_H) = r_G$.

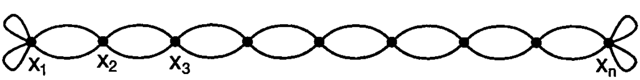

Fig. 2. The graph A_n.

Let Q be a set and let G be a graph, which might be rooted. A *Q-labeling* of G is a mapping from $V(G)$ to Q. If $\mathscr{G}$ is a class of graphs or a class of rooted graphs, then we denote by $\mathscr{G}(Q)$ the class of all pairs (G, f) such that G is in $\mathscr{G}$ and f is a Q-labeling of G. Suppose now that $\preccurlyeq$ is a quasi-order on Q. Then, for any two members (G, f) and (G', f') of $\mathscr{G}(Q)$, we define $(G, f) \preccurlyeq_l (G', f')$ if there is a homomorphism σ from G to G' such that $f(v) \preccurlyeq f'(\sigma(v))$ for all v in $V(G)$. Now we may state the main result of this section.

(3.4) THEOREM. *Let $(Q, \preccurlyeq)$ be a wqo and let $\mathscr{G}$ be a minor-closed class of graphs. Then $(\mathscr{G}(Q), \preccurlyeq_l)$ is a wqo if and only if $\mathscr{G} \cap \mathscr{A}$ is finite.*

Clearly, by taking Q to be a single-element set, we deduce (3.2) from (3.4) immediately. We shall prove (3.4) at the end of this section. We first establish a lemma, (3.5), which is the major part of the proof of (3.4). Let n be a nonnegative integer and let $\mathscr{G}_n$ be the class of rooted graphs of height at most n.

(3.5) For every nonnegative integer n, if $(Q, \preccurlyeq)$ is a wqo, then $(\mathscr{G}_n(Q), \preccurlyeq_l)$ is a wqo.

To prove (3.5), we need some preparations. For a quasi-order $(Q, \preccurlyeq)$, let Q^* be the set of all finite sequences of members of Q. Suppose that $p = [p_1, ..., p_s]$ and $q = [q_1, ..., q_t]$ are members of Q^*. Then we define $p \preccurlyeq^* q$ if there exist indices $i_1, ..., i_s$ such that $1 \leqslant i_1 < \cdots < i_s \leqslant t$ and $p_1 \preccurlyeq q_{i_1}, ..., p_s \preccurlyeq q_{i_s}$. Observe that, if p is the sequence with no term, then $p \preccurlyeq^* q$ for all q in Q^*. The following classical result is due to Higman [4].

(3.6) HIGMAN'S THEOREM. *$(Q^*, \preccurlyeq^*)$ is a wqo if $(Q, \preccurlyeq)$ is.*

Let $(Q_1, \preccurlyeq_1)$ and $(Q_2, \preccurlyeq_2)$ be quasi-orders and let $Q = Q_1 \times Q_2$. For any two members $p = (p_1, p_2)$ and $q = (q_1, q_2)$ of Q, we define $p \preccurlyeq q$ if $p_1 \preccurlyeq_1 q_1$ and $p_2 \preccurlyeq_2 q_2$. We define $\preccurlyeq_1 \times \preccurlyeq_2$ to be $\preccurlyeq$. It is not difficult to show (see [4], for instance) that

(3.7) $(Q_1 \times Q_2, \preccurlyeq_1 \times \preccurlyeq_2)$ is a wqo if both $(Q_1, \preccurlyeq_1)$ and $(Q_2, \preccurlyeq_2)$ are.

We also need the following result of Kruskal [5].

(3.8) KRUSKAL'S THEOREM. *Let $\mathscr{K}$ be the class of rooted trees and let $(Q, \preccurlyeq)$ be a wqo. Then $(\mathscr{K}(Q), \preccurlyeq_l)$, is a wqo.*

Proof of (3.5). We proceed by induction on n. First, we consider the case when $n = 0$. For any member (G, f) of $\mathscr{G}_0(Q)$, let $q(G, f) = (f(v), k)$, where v is the only vertex of G and k is the number of loops of G. Clearly, $q(G, f)$ is a member of $Q \times N$, where N is the set of nonnegative integers.

In addition, for any two members (G, f) and (G', f') of $\mathscr{G}_0(Q)$, it is easy to see that $(G, f) \leqslant_l (G', f')$ if and only if $q(G, f) \leqslant' q(G', f')$, where $\leqslant'$ is $\leqslant \times \leqslant$. Therefore, $(\mathscr{G}_0(Q), \leqslant_l)$ is isomorphic to $(Q \times N, \leqslant')$, and thus we conclude from (3.7) that (3.5) holds for $n = 0$. Next, we consider the case when n is positive. We shall assume that $(\mathscr{G}_{n'}(Q'), \leqslant_l)$ is a wqo for every wqo $(Q', \leqslant')$ and for every nonnegative integer n' less than n.

Let $\mathscr{G}^s$ be the class of rooted graphs G in $\mathscr{G}_n$ such that G is a single-extension of a rooted graph in $\mathscr{G}_{n-1}$. Then we prove that

(3.9) $(\mathscr{G}^s(Q), \leqslant_l)$ is well-quasi-order.

Let (G, f) be a member of $\mathscr{G}^s(Q)$ and let r be the root of G. Then $G - \{r\}$, which will be denoted by G^s, belongs to $\mathscr{G}_{n-1}$. For each vertex v of G^s, let $f^s(v) = (f(v), k_v)$, where k_v is number of edges between r and v. Then we define $q(G, f) = (f(r), k, (G^s, f^s))$, where k is the number of loops incident with r. Clearly, $q(G, f)$ belongs to $Q \times N \times \mathscr{G}_{n-1}(Q \times N)$. Moreover, for any two members (G, f) and (G', f') of $\mathscr{G}^s(Q)$, it is easy to see that $q(G, f) \leqslant' q(G', f')$ implies $(G, f) \leqslant_l (G', f')$, where $\leqslant'$ is $\leqslant \times \leqslant \times \leqslant_l$. Thus (3.9) follows from (3.7) and our induction hypothesis.

Let $\mathscr{G}^t$ be the class of rooted graphs G in $\mathscr{G}_n$ such that G is a tree-connection of rooted graphs in $\mathscr{G}_{n-1}$. Then we prove that

(3.10) $(\mathscr{G}^t(Q), \leqslant_l)$ is a wqo.

Let (G, f) be a member of $\mathscr{G}^t(Q)$. Then there exists a rooted tree T_G and a set $\mathscr{J}$ of rooted graphs in $\mathscr{G}_{n-1}$ such that G is the tree-connection of the members of $\mathscr{J}$ (over T_G). Let v be a vertex of T_G and let $G_1, ..., G_k$ be all the v-rooted graphs in $\mathscr{J}$ (k can be zero here). Then, for each G_i, let f_i be the restriction of f to $V(G_i)$. Now we define $F_G(v)$ to be the sequence $((G_1, f_1), ..., (G_k, f_k))$. Clearly, F_G is a $(\mathscr{G}_{n-1}(Q))^*$-labeling of T_G. Moreover, for any two members (G, f) and (G', f') of $\mathscr{G}^t(Q)$, it is easy to see that $(T_G, F_G) \leqslant_l (T_{G'}, F_{G'})$ implies $(G, f) \leqslant_l (G', f')$. Thus (3.10) follows from (3.6), (3.8), and our induction hypothesis.

From the definition of height it is clear that $\mathscr{G}_n$ is the union of $\mathscr{G}^s$ and $\mathscr{G}^t$. Consequently, (3.9) and (3.10) imply that $(\mathscr{G}_n(Q), \leqslant_l)$ is a wqo. Therefore, the induction is completed and so (3.5) is proved. ∎

Proof of (3.4). The "only if" part is obvious and so we need only prove the "if" part. Let n be a positive integer such that A_n is not in $\mathscr{G}$. Since $\mathscr{G}$ is minor-closed and A_n is a minor of B_{n+4}, it follows that B_{n+4} is not in $\mathscr{G}$. As a consequence, no graph in $\mathscr{G}$ has B_{n+4} as a minor. Let $\mathscr{C}$ be the class of connected graphs in $\mathscr{G}$. Then we deduce from (1.6) and (3.5) that $(\mathscr{C}(Q), \leqslant_l)$ is a wqo. Observe that all connected components of a graph in $\mathscr{G}$ must belong to $\mathscr{C}$. Therefore, if we view every graph in $\mathscr{G}$ as a sequence (in any order) of members of $\mathscr{C}$, then (3.4) follows from (3.6) immediately. ▮

4. Bounding the Number of Paths of a Graph

For any graph G, let $a(G)$ denote the number of subgraphs of G that are isomorphic to paths. For instance, if G is a tree with m edges, then $a(G) = (m+1)(m+2)/2$. The main result in this section is the following.

(4.1) Theorem. *Let $\mathscr{G}$ be a minor-closed class of graphs. Then there is a polynomial function $p(\cdot)$ such that $a(G) \leqslant p(|V(G)| + |E(G)|)$ for all graphs G in $\mathscr{G}$ if and only if some B_n is not in $\mathscr{G}$.*

Let $\mathscr{G}$ be a minor-closed class of graphs. Suppose that one is interested in finding an optimal path, with respect to certain criteria, for every graph in $\mathscr{G}$. Depending on the criteria, one may design different kinds of algorithms to solve this problem. But the following is an algorithm that is independent of the criteria. For every input graph G in $\mathscr{G}$, list all paths of G and then find the best one. In general, this algorithm is not very efficient. However for some special $\mathscr{G}$ (for example, the class of all forests), this is a polynomial time algorithm (here we assume that the criterion can be tested in polynomial time for any given path). Theorem (4.1) actually is a characterization of the classes of graphs for which this algorithm runs in polynomial time.

To prove (4.1), we first prove a lemma. Let G be a graph. If $G \neq K_1$, we define $s(G)$ to be the number of edges of G. If $G = K_1$, then we define $s(G) = 1$. Let n be a nonnegative integer and let $\mathscr{H}_n$ be the class of connected loopless graphs of height at most n.

(4.2) $a(G) \leqslant 3^{2^{n-1}} s^{2^{n+1}-2}$ for all graphs G in $\mathscr{H}_n$, where $s = s(G)$.

Proof. We proceed by induction on n. If $n = 0$, since K_1 is the only graph in $\mathscr{H}_0$, (4.2) holds obviously. Next, we assume that n is positive. Let G be a graph in $\mathscr{H}_n$. We shall consider two cases, G is a single-extension of a graph of height at most $n-1$ or G is a tree-connection of graphs of height at most $n-1$.

Suppose that G is a single-extension of H and the height of H is at most $n-1$. Let r be the vertex of G that is not in H. If $H = K_1$, then

$$a(G) = |E(G)| + 2 \leqslant 3\,|E(G)|^2 \leqslant 3^{2^{n-1}} s^{2^{n+1}-2},$$

as required. Therefore we may assume that $H \neq K_1$. Let D be the set of edges of G that are incident with r. Let $d = |D|$ and let p be the number of paths of H. For $i=0, 1, 2$, let p_i be the number of paths of G that use exactly i edges of D. Then

$$p_0=1+p, \qquad p_1 \leqslant dp, \qquad p_2 \leqslant \binom{d}{2}\binom{p}{2}.$$

It follows that

$$a(G)=p_0+p_1+p_2 \leqslant (d^2+d^2p)+d^2p+d^2p(p-1)/4 \leqslant d^2p^2,$$

where the last inequality holds since $p \geqslant 3$. Now, by our induction hypothesis, we have

$$a(G) \leqslant d^2p^2 \leqslant s^2(3^{2^{n-2}}\,|E(H)|^{2^{n-2}})^2 \leqslant 3^{2^{n-1}}s^{2^{n+1}-2},$$

as required.

Suppose now T is a tree, $\mathscr{J}=\{J_1, ..., J_k\}$ is a set of rooted graphs of height at most $n-1$, and G is a tree-connection of the rooted graphs in $\mathscr{J}$ (over T). For each vertex v of T, let k_v be the number of v-rooted graphs in $\mathscr{J}$ and let k'_v be the number of v-rooted graphs in $\mathscr{J}$ with a none-empty edge-set. By adding and removing copies of K_1 to and from $\mathscr{J}$, if necessary, we may assume for very vertex v of T that $k_v=k'_v$ whenever $k'_v>0$ and $k_v=1$ whenever $k'_v=0$. Since have shown that (4.2) holds for $n=0$, we may also assume that G is not in $\mathscr{H}_0$, that is, $G \neq K_1$. Finally, let $m=|E(T)|$ and $k'=\sum\{k'_v : v \in V(T)\}$.

If $G=T$ then $m \geqslant 1$ since $G \neq K_1$. Thus

$$a(G)=(m+1)(m+2)/2 \leqslant 3m^2 \leqslant 3^{2^{n-1}}s^{2^{n+1}-2},$$

as required. Therefore, k' is positive and thus $k-k' \leqslant |V(T)|-1=m$. For each J_i, let $s(J_i)$ be denoted by s_i. Then

$$\begin{aligned}\sum_{i=1}^{k} s_i &= \sum_{E(J_i)=\varnothing} s_i + \sum_{E(J_i)\neq\varnothing} s_i \\ &= k-k' + \sum_{E(J_i)\neq\varnothing} s_i \leqslant m + \sum_{E(J_i)\neq\varnothing} s_i = s.\end{aligned}$$

Let P be a path of G. If P is not a path of any J in $\mathscr{J}$, then it is easy to see that there exist J_i and J_j in $\mathscr{J}$ such that P is the concatenation of a path of J_i, a path of J_j, and the unique path of T between the roots of J_i and J_j. Thus we have

$$a(G) \leqslant \sum_{i, j \in \{1, ..., k\}} a(J_i)\, a(J_j) = \left(\sum_{i=1}^{k} a(J_i) \right)^2 \leqslant \left(\sum_{i=1}^{k} 3^{2^{n-2}} s_i^{2^n - 2} \right)^2$$

$$\leqslant 3^{2^{n-1}} \left(\sum_{i=1}^{k} s_i^{2^n - 1} \right)^2 \leqslant 3^{2^{n-1}} \left(\sum_{i=1}^{k} s_i \right)^{2^{n+1} - 2} \leqslant 3^{2^{n-1}} s^{2^{n+1} - 2},$$

as required. ▌

For every positive integer n, let $p_n(x) = (3x)^{2^{n+1}}$. Then the following is an immediate corollary of (4.2).

(4.3) Let n be a positive integer and let G be a graph of height at most n. Let k be the number of connected components C of G with $|V(C)| = 1$. Then $a(G) \leqslant k + p_n(|E(G)|)$.

Now we are ready to prove the main result of this section.

Proof of (4.1). The "only if" part is obvious and so we need only prove the "if" part. Let m be a positive integer such that B_m is not in $\mathcal{G}$. Then we conclude from (1.6) that there is a positive integer n such that the height of every graph in $\mathcal{G}$ is at most n. Now we verify that the function p_n in (4.3) satisfies our requirement. Let G be a graph in $\mathcal{G}$ and let k be as in (4.3). Then

$$a(G) \leqslant k + p_n(|E(G)|) \leqslant |V(G)| + p_n(|E(G)|) \leqslant p_n(|V(G)| + |E(G)|),$$

as required. ▌

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