Inventiones math. 56, 167–189 (1980)

Inventiones mathematicae

Combinatorics and Topology of Complements of Hyperplanes

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1. Introduction

Let $V=\mathbb{C}^\ell$ and let $\mathbf{A}$ be a finite set of hyperplanes in V, all containing the origin. If $A\in\mathbf{A}$ let φ_A be a linear form with kernel A. Let $M=V-\bigcup_{A\in\mathbf{A}} A$ be the complement of the union of the hyperplanes. Define holomorphic differential forms ω_A on M by $\omega_A=d\varphi_A/2\pi i\varphi_A$ and let $[\omega_A]$ denote the corresponding deRham cohomology class. Let $\mathscr{R}=\bigoplus_{p=0}^{\ell}\mathscr{R}_p$ be the graded $\mathbb{C}$-algebra of holomorphic differential forms on M generated by the ω_A and the identity. Arnold [1] conjectured that the natural map $\eta\to[\eta]$ of $\mathscr{R}\to H^*(M)=H^*(M,\mathbb{C})$ is an isomorphism of graded algebras. This was proved by Brieskorn [5, Lemma 5] who showed in fact that the $\mathbb{Z}$-subalgebra of $\mathscr{R}$ generated by the forms ω_A and the identity is isomorphic to the singular cohomology $H^*(M,\mathbb{Z})$.

Let $z_1,\dots,z_\ell$ be coordinate functions on V. In case the linear forms are $\varphi_{jk}=z_j-z_k$, Arnold [1] found the formula

(1.1) $\quad P_M(t)=(1+t)(1+2t)\dots(1+(\ell-1)t)$

for the Poincaré polynomial of M. He also gave a presentation for the algebra $\mathscr{R}$ which may be described as follows. Let $\mathscr{E}$ be the exterior algebra of the vector space which has a basis consisting of elements labeled e_{jk} $1\leqq j<k\leqq\ell$. Let $\mathscr{I}$ be the ideal of $\mathscr{E}$ generated by all elements $e_{ij}e_{jk}+e_{jk}e_{ki}+e_{ki}e_{ij}$. Then the map $e_{jk}\to\omega_{jk}=d\varphi_{jk}/2\pi i\varphi_{jk}$ defines an algebra isomorphism $\mathscr{E}/\mathscr{I}\simeq\mathscr{R}$.

In this paper we extend these results in several ways. We give a general formula for the Poincaré polynomial of M and we give a presentation for the algebra $\mathscr{R}\simeq H^*(M)$ which agrees with Arnold in his special case. If G is a subgroup of $\mathbf{GL}(V)$ which permutes the set $\mathbf{A}$ of hyperplanes, then G has a representation on $H^*(M)$ and we compute the character of this representation. This allows us to compute the Poincaré polynomial of the orbit space M/G.

* This work was supported in part by the National Science Foundation

0020-9910/80/0056/0167/$04.60

We prove these results using certain algebraic and combinatorial constructions with a lattice L. This part of our work occupies Sect. 2–4 and involves no topology. If $\mathbf{A}$ is a set of hyperplanes we choose L to be the collection of subspaces of V of the form $X = A_1 \cap \ldots \cap A_p$ where $A_i \in \mathbf{A}$. We partially order L by reverse inclusion so that L has V as its unique minimal element. Then L is a finite geometric lattice [3, p. 80] with $\mathbf{A}$ as its set of atoms.

In general let L be a finite geometric lattice and let $\mathbf{A}$ be its set of atoms. Let $\mathscr{E} = \bigoplus_{p \geq 0} \mathscr{E}_p$ be the exterior algebra of the vector space which has a basis consisting of elements e_a in one to one correspondence with the elements $a \in \mathbf{A}$. In Sect. 2 we construct, in a functorial way, a graded anticommutative $\mathbb{C}$-algebra $\mathscr{A} = \mathscr{E}/\mathscr{I}$ whose Poincaré polynomial

$$(1.2) \quad P_{\mathscr{A}}(t) = \sum_{x \in L} \mu(x)(-t)^{r(x)}$$

is given by Theorem 2.6. Here $r(x)$ is the rank of x and $\mu(x) = \mu(\mathbf{0}, x)$ where μ is the Möbius function of L and $\mathbf{0}$ is the minimal element of L. Thus the Poincaré polynomial of $\mathscr{A}$ is essentially the characteristic polynomial $\chi_L(t) = \sum_{x \in L} \mu(x) t^{\ell - r(x)}$ in Rota's sense [13, p. 343], where ℓ is the rank of L.

Let G be a group which acts as a group of automorphisms of L. Then G has a representation on each space $\mathscr{A}_p$. To compute the character of this representation we introduce, in Sect. 3, a second graded anticommutative $\mathbb{C}$-algebra $\mathscr{B} = \bigoplus_{p=0}^{\ell} \mathscr{B}_p$. The elements of $\mathscr{B}$ are certain linear combinations of p-tuples $(x_1 < \ldots < x_p)$ of elements of L and the multiplication is defined using a shuffle product. The main result, Theorem 3.7, of Sect. 3 asserts that $\mathscr{A}$ and $\mathscr{B}$ are G-isomorphic algebras. Thus the trace computations may be done in $\mathscr{B}$. There is a close connection between $\mathscr{B}$ and the homology of L in the sense of Rota [13] and Folkman [9]. Thus the Hopf trace formula for a finite simplicial complex may be used to obtain the trace formula of Theorem 4.8:

$$(1.3) \quad \sum_{p=0}^{\ell} \operatorname{tr}(g|\mathscr{A}_p) t^p = \sum_{x \in L^g} \mu_g(x)(-t)^{r(x)} \qquad g \in G.$$

Here L^g is the subset of L fixed by g and μ_g is its Möbius function.

Baclawski [2] has defined a (co)homology theory for a geometric lattice L whose Poincaré polynomial is $P_{\mathscr{A}}(t) = P_{\mathscr{B}}(t)$. It turns out that our algebra $\mathscr{B}$ consists of cycles in Baclawski's homology $\mathscr{H}$, and in fact the natural map sending each element of $\mathscr{B}$ to its homology class is a vector space isomorphism $\mathscr{B} \simeq \mathscr{H}$.

In Sect. 5 we combine our combinatorial theorems with topological theorems of Brieskorn [5]. Let $\mathbf{A}$ be a finite set of hyperplanes in $V = \mathbb{C}^\ell$, let L be the corresponding lattice and let $M = V - \bigcup_{A \in \mathbf{A}} A$. We prove in Theorem 5.2 that the map $e_A \to [\omega_A]$ from $\mathscr{E}$ to $H^*(M)$ defines an algebra isomorphism $\mathscr{E}/\mathscr{I} \simeq H^*(M)$. This gives a presentation for the cohomology ring of M. Moreover, M and $\mathscr{E}/\mathscr{I}$ have the same Poincaré polynomial

$$(1.4) \quad P_M(t) = \sum_{X \in L} \mu(X)(-t)^{r(X)}.$$

Our isomorphism is G-equivariant for any subgroup G of $\mathbf{GL}(V)$ which permutes $\mathbf{A}$. Thus the combinatorial trace formula (4.8) yields the trace of G on $H^*(M)$.

In case the hyperplanes are defined by linear forms with real coefficients there is a remarkable coincidence between the dimension of $H^*(M)$ and the number of connected components of the corresponding real configuration, computed by Zaslavsky [18, p. 18]: if $\mathbf{A}_{\mathbb{R}}$ is a finite set of hyperplanes in $\mathbb{R}^\ell$ and $\mathbf{A}=\{A\otimes\mathbb{C}\mid A\in\mathbf{A}_{\mathbb{R}}\}$ then the number of connected components of $\mathbb{R}^\ell - \bigcup_{A\in\mathbf{A}_{\mathbb{R}}} A$ is equal to $\dim_{\mathbb{C}} H^*(M)$.

Since $\mathscr{H}\simeq\mathscr{B}$ as vector spaces and $\mathscr{B}\simeq H^*(M)$ as algebras the deRham cohomology of M is isomorphic, as vector space, to Baclawski's (co)homology of the lattice L. One might expect that $H^*(M)$ is algebra isomorphic to Baclawski's cohomology but we have not checked this.

Suppose $\mathbf{A}$ is a set of affine hyperplanes in $\mathbb{C}^\ell$ which do not all contain the origin. Here the corresponding poset L consisting of intersections $A_1\cap\ldots\cap A_p$ is not a geometric lattice (even if we adjoin the empty set as a maximal element). Nevertheless the Poincaré polynomial of M is given by the formula $P_M(t) = \sum_{X\in L}\mu(X)(-t)^{r(X)}$. Since L is not a geometric lattice our algebra $\mathscr{A}$ is not defined. However, an algebra like $\mathscr{A}$ may be defined for any finite poset with a unique minimal element and we intend to study its homological properties in a sequel to this paper.

If L is the lattice of subspaces of an ℓ-dimensional vector space over $\mathbb{F}_q$ and $G=\mathbf{GL}_\ell(\mathbb{F}_q)$ our algebras $\mathscr{A}\simeq\mathscr{B}$ have Poincaré polynomial

$$(1.5)\quad (1+t)(1+qt)\ldots(1+q^{\ell-1}t)=\sum_{k=0}^{\ell}\begin{bmatrix}\ell\\ k\end{bmatrix} q^{k(k-1)/2}t^k$$

where $\begin{bmatrix}\ell\\ k\end{bmatrix}$ is the Gaussian q-binomial coefficient. Thus we have constructed an algebra $\mathscr{A}$ which plays the same role for the q-binomial theorem as the exterior algebra does for the ordinary binomial theorem where $q=1$. In this case the G-module $\mathscr{A}_1$ is the Steinberg module of $\mathbf{GL}_\ell(\mathbb{F}_q)$ and the isomorphism $\mathscr{A}_1\simeq\mathscr{B}_1$ gives its homological interpretation [16]. The map (3.3) and its anti-symmetrization are similar to constructions used by Steinberg [17, §§2–3]. Some of the ideas in Sect. 4 were used in work of Solomon [15]. The exact sequence (2.19) is closely related to a sequence used by Lusztig [10, p. 11] in his work on the discrete series of $GL_\ell(\mathbb{F}_q)$. Lusztig uses it to give a recursive formula for the Steinberg character [10, p. 22].

If G is a finite irreducible subgroup of $\mathbf{GL}_\ell(\mathbb{R})$ generated by reflections and $\mathbf{A}$ consists of the complexified reflecting hyperplanes, Brieskorn [5, Theorem 6] found the formula

$$(1.6)\quad P_M(t)=(1+m_1t)\ldots(1+m_\ell t)$$

where the m_i are the exponents of G. In a sequel to this paper we will show that if G is a finite irreducible subgroup of $\mathbf{GL}_\ell(\mathbb{C})$ generated by unitary reflections and $\mathbf{A}$ consists of the reflecting hyperplanes then

(1.7) $P_M(t)=(1+n_1 t)\dots(1+n_\ell t)$

where the $n_i \in \mathbb{N}$ are certain generalized exponents. In case G is real, $n_i = m_i$ so our formula agrees with Brieskorn's.

Our notational conventions are as follows. All vector spaces are over $\mathbb{C}$ and all (co)homology has coefficients in $\mathbb{C}$. In the course of the argument we define certain functors $\mathscr{E}, \mathscr{A}, \dots$ on a category of lattices. To simplify notation we usually suppress the dependence of the objects $\mathscr{E}(L), \mathscr{A}(L), \dots$ on L and write $\mathscr{E} = \mathscr{E}(L)$, $\mathscr{A} = \mathscr{A}(L), \dots$. Similarly for functions on L we write $\mu = \mu_L$, $r = r_L, \dots$ and exhibit the dependence on L only when necessary. The lattice terminology is standard as used by Birkhoff [3].

We would like to thank D. Zagier for a suggestion which simplified our proof of Theorem 5.2.

2. Geometric Lattices and Exterior Algebra

Let L be a finite poset (partially ordered set) with a unique minimal element $\mathbf{0}$. The Möbius function μ is an integer valued function on $L \times L$ defined recursively as follows: $\mu(x,x)=1$,

(2.1) $\sum_{x \leqq z \leqq y} \mu(x,z)=0 \quad \text{if } x<y$

and $\mu(x,y)=0$ otherwise. We agree to write $\mu(x)=\mu(\mathbf{0},x)$. Our aim in this section is to associate to a geometric lattice L a graded finite dimensional anticommutative $\mathbb{C}$-algebra $\mathscr{A} = \bigoplus_{p \geqq 0} \mathscr{A}_p$, and to show that the dimension of $\mathscr{A}_p$ may be computed in terms of μ.

We recall the definition of a geometric lattice. A poset L satisfies the chain condition if for each $x \in L$ all maximal linearly ordered subsets $\mathbf{0} = x_0 < x_1 < \dots < x_p = x$ have the same cardinality. The integer p is called the rank of x, and will be written $r(x)$. The rank $r(L)$ of L is the maximum of the ranks of its elements. Elements of rank 1 are called atoms. A finite lattice L is said to be geometric if (i) it satisfies the chain condition, (ii) every element in $L - \mathbf{0}$ is a join of atoms, and (iii) the rank function satisfies the inequality

(2.2) $r(x \wedge y) + r(x \vee y) \leqq r(x) + r(y), \qquad x, y \in L.$

Henceforth we assume that L is a finite geometric lattice. We let $\mathbf{1}$ denote the unique maximal element of L and assume that $\mathbf{0} \neq \mathbf{1}$.

Let $\mathbf{A}$ be the set of atoms of L. Let $\mathbf{S}_p$ be the set of all p-tuples $S=(a_1, \dots, a_p)$ where $a_i \in \mathbf{A}$. For $p=0$ we agree that $\mathbf{S}_0$ consists of the empty tuple ( ). Let $\mathbf{S} = \bigcup_{p \geqq 0} \mathbf{S}_p$. If $S=(a_1, \dots, a_p)$ write $\bigvee S = a_1 \vee \dots \vee a_p$ and for $p=0$ write $\bigvee(\,) = \mathbf{0}$. If $x \in L$ let $\mathbf{S}_x$ consist of all $S \in \mathbf{S}$ with $\bigvee S = x$. We introduce an arbitrary linear order $\prec$ on the set $\mathbf{A}$ and say that $S=(a_1, \dots, a_p)$ is standard if $a_1 \prec \dots \prec a_p$. This linear order is introduced for notational purposes and has nothing to do with the partial order $\leqq$ in L, since no two elements of $\mathbf{A}$ are comparable in L. If $S \in \mathbf{S}_p$ then (2.2) implies that $r(\bigvee S) \leqq p$. We say that $S \in \mathbf{S}_p$ is independent if $r(\bigvee S) = p$, and dependent if $r(\bigvee S) < p$. We agree that the empty tuple is standard and independent.

Let $\mathscr{E} = \bigoplus_{p \geqq 0} \mathscr{E}_p$ be the exterior algebra of the vector space which has a basis consisting of elements e_a in one to one correspondence with the elements $a \in \mathbf{A}$. If $S = (a_1, \ldots, a_p)$ let $e_S = e_{a_1} \ldots e_{a_p}$. Thus $\mathscr{E}$ has a basis consisting of all e_S with S standard. If $S = (\ )$ write $e_S = 1$. Define a $\mathbb{C}$-linear map $\partial: \mathscr{E} \to \mathscr{E}$ by $\partial 1 = 0$, $\partial e_a = 1$ and for $S = (a_1, \ldots, a_p)$

$$\partial e_S = \sum_{k=1}^{p} (-1)^{k-1} e_{a_1} \ldots \hat{e}_{a_k} \ldots e_{a_p}.$$

Then $\partial^2 = 0$. If $S \in \mathbf{S}_p$ and $T \in \mathbf{S}$ then

(2.3) $\quad \partial(e_S e_T) = (\partial e_S) e_T + (-1)^p e_S (\partial e_T)$.

Let $\mathscr{I}$ be the ideal of $\mathscr{E}$ generated by all elements ∂e_S where S is dependent. Let $\mathscr{A} = \mathscr{E}/\mathscr{I}$. Since $\mathscr{I}$ is generated by homogeneous elements $\mathscr{A}$ is a graded anticommutative $\mathbb{C}$-algebra. If L' is another geometric lattice and $f: L' \to L$ is a map satisfying

(2.4) $\quad$ (i) $f(\mathbf{A}_{L'}) \subseteq \mathbf{A}_L$; $\quad$ (ii) $fx \vee fy = f(x \vee y)$ $\quad$ (iii) $r_L(fx) \leqq r_{L'}(x)$

then the map $e_a \to e_{fa}$ defines a homomorphism $f_{\mathscr{E}}: \mathscr{E}(L') \to \mathscr{E}(L)$ of graded $\mathbb{C}$-algebras. It follows from (2.4) that f maps dependent S to dependent $f(S)$ and thus $f_{\mathscr{E}}$ induces a homomorphism $f_{\mathscr{A}}: \mathscr{A}(L') \to \mathscr{A}(L)$ of graded $\mathbb{C}$-algebras. In fact $\mathscr{E}$ and $\mathscr{A}$ are covariant functors from the category of geometric lattices and maps satisfying (2.4) to the category of graded anticommutative $\mathbb{C}$-algebras.

(2.5) *Remark.* If L' is the lattice of subsets of $\mathbf{A}_L$ then $\mathscr{I}(L') = 0$ so $\mathscr{A}(L') = \mathscr{E}(L')$. The map $f: L' \to L$ defined by $f\{a_1, \ldots, a_p\} = a_1 \vee \ldots \vee a_p$ satisfies (2.4). If we identify $\mathbf{A}_{L'}$ with $\mathbf{A}_L$ and hence $\mathscr{E}(L')$ with $\mathscr{E}(L)$ then the induced map $f_{\mathscr{A}}$ may be identified with the natural homomorphism $\mathscr{E}(L) \to \mathscr{A}(L)$.

(2.6) **Theorem.** *Let L be a finite geometric lattice. Then the graded algebra $\mathscr{A}$ has Poincaré polynomial*

$$P_{\mathscr{A}}(t) = \sum_{x \in L} \mu(x)(-t)^{r(x)}.$$

The rest of this section contains the proof of Theorem 2.6. Let $\mathscr{J} = \sum \mathbb{C} e_S$ where the sum is over all dependent $S \in \mathbf{S}$.

(2.7) **Lemma.** $\mathscr{I} = \mathscr{J} + \partial \mathscr{J}$.

Proof. If $S \in \mathbf{S}$ is arbitrary and $T \in \mathbf{S}$ is dependent then the formula $\partial(e_S(\partial e_T)) = (\partial e_S)(\partial e_T)$ shows that $\partial \mathscr{J} \subseteq \mathscr{I}$. Suppose S is dependent. Choose $a \in \mathbf{A}$. Then $T = (a, S)$ is dependent and since $e_T = e_a e_S$ it follows from (2.3) that $e_S = \partial e_T + e_a \partial e_S \in \mathscr{I}$. Thus $\mathscr{J} \subseteq \mathscr{I}$. Since $\partial \mathscr{J}$ contains the generators of $\mathscr{I}$ it suffices to show that $\mathscr{J} + \partial \mathscr{J}$ is an ideal. If $a \in \mathbf{A}$ then $e_a \mathscr{J} \subseteq \mathscr{J}$ and the last equation above shows that $e_a \partial \mathscr{J} \subseteq \mathscr{J} + \partial \mathscr{J}$. $\square$

Define a Hermitian inner product $\langle\ ,\ \rangle$ on $\mathscr{E}$ by requiring that the standard basis elements e_S form an orthonormal basis. If $u\in\mathscr{E}$ we define the support of u, supp(u), as follows. Write u uniquely in the form $u=\sum c_S e_S$ where $c_S\in\mathbb{C}$ and the S are *standard*. Then supp(u) is defined to be the set of S with $c_S\neq 0$. The support, supp($\mathscr{M}$) of a subspace $\mathscr{M}$ of $\mathscr{E}$ is the union of the supports of its elements. Two subspaces with disjoint supports are orthogonal. If $x\in L$ let $\mathscr{E}_x=\sum_{S\in\mathbf{S}_x}\mathbb{C}e_S$. Thus $\mathscr{E}=\bigoplus_{x\in L}\mathscr{E}_x$. The subspaces $\mathscr{E}_x$ have pairwise disjoint supports and thus they are pairwise orthogonal.

If $S\in\mathbf{S}$ is dependent and $S=(a_1,\ldots,a_p)$ let $\Phi(S)$ be the set of all indices $k=1,\ldots,p$ such that $S_k=(a_1,\ldots,\hat{a}_k,\ldots,a_p)$ is independent. Since $\bigvee S_k\leqq\bigvee S$ for all k, it follows that for $k\in\Phi(S)$ we have $p-1=r(\bigvee S_k)\leqq r(\bigvee S)\leqq p-1$ and thus $\bigvee S_k=\bigvee S$. Let $f_S=\sum_{k\in\Phi(S)}(-1)^{k-1}e_{S_k}$. Then $f_S\equiv\partial e_S \bmod \mathscr{J}$. If $x\in L$ let $\mathscr{K}_x=\sum\mathbb{C}f_S$ where the sum is over all dependent $S\in\mathbf{S}_x$. Since $\bigvee S=\bigvee S_k$ whenever $k\in\Phi(S)$ we have $e_{S_k}\in\mathscr{E}_x$ and thus $f_S\in\mathscr{E}_x$. Therefore $\mathscr{K}_x\subseteq\mathscr{E}_x$.

(2.8) **Lemma.** $\mathscr{I}=\mathscr{J}+\sum_{x\in L}\mathscr{K}_x$, *direct sum of pairwise orthogonal subspaces.*

Proof. By Lemma 2.7 $\mathscr{I}=\mathscr{J}+\partial\mathscr{J}$. Since $\partial e_S\equiv f_S \bmod \mathscr{J}$ for any dependent S, we have $\mathscr{I}=\mathscr{J}+\sum\mathscr{K}_x$. Since $\mathscr{K}_x\subseteq\mathscr{E}_x$ the subspaces $\mathscr{K}_x$ are pairwise orthogonal. Since supp($\mathscr{K}_x$) consists of independent $S\in\mathbf{S}_x$ and supp($\mathscr{J}$) consists of dependent $S\in\mathbf{S}$, $\mathscr{J}$ is orthogonal to every $\mathscr{K}_x$. □

(2.9) **Lemma.** *Let π be the orthogonal projection of $\mathscr{E}$ onto $\mathscr{I}^\perp$. If $S\in\mathbf{S}_x$ is independent then $\pi e_S\in\mathscr{E}_x$.*

Proof. If $\mathscr{M}$ is any subspace of $\mathscr{E}$, let $\pi(e_S,\mathscr{M})$ be the orthogonal projection of e_S on $\mathscr{M}$. By Lemma 2.8 we have

$$\pi(e_S,\mathscr{I})=\pi(e_S,\mathscr{J})+\sum_{y\in L}\pi(e_S,\mathscr{K}_y).$$

If T is dependent then $T\neq S$ because S is independent so $\langle e_S,e_T\rangle=0$. Thus $\pi(e_S,\mathscr{J})=0$. If $y\neq x$ we prove that $\pi(e_S,\mathscr{K}_y)=0$. By definition $\mathscr{K}_y=\sum\mathbb{C}f_T$ summed over dependent $T\in\mathbf{S}_y$. If $k\in\Phi(T)$ then $\bigvee T_k=\bigvee T=y$ so $T_k\neq S$ and

$$\langle e_S,f_T\rangle=\sum_{k\in\Phi(T)}(-1)^{k-1}\langle e_S,e_{T_k}\rangle=0$$

so $\pi(e_S,\mathscr{K}_y)=0$. Thus $\pi(e_S,\mathscr{I})=\pi(e_S,\mathscr{K}_x)\in\mathscr{E}_x$. Since $e_S\in\mathscr{E}_x$ we have $\pi e_S=e_S-\pi(e_S,\mathscr{I})\in\mathscr{E}_x$. □

Let $\alpha_S=\varphi e_S$ and let $\mathscr{A}_x=\varphi\mathscr{E}_x$ where $\varphi\colon\mathscr{E}\to\mathscr{A}$ is the natural homomorphism.

(2.10) **Proposition.** $\mathscr{A}=\bigoplus_{x\in L}\mathscr{A}_x$.

Proof. Since $\mathscr{E}=\bigoplus_{x\in L}\mathscr{E}_x$ we have $\mathscr{A}=\sum_{x\in L}\mathscr{A}_x$. Define a positive definite Hermitian inner product $\langle\ ,\ \rangle$ on $\mathscr{A}$ by $\langle\varphi u,\varphi v\rangle=\langle\pi u,\pi v\rangle$ for $u,v\in\mathscr{E}$. We prove that the sum $\sum\mathscr{A}_x$ is direct by showing that the subspaces $\mathscr{A}_x$ are pairwise orthogonal with respect to this inner product. Suppose $x\neq y$ and $S\in\mathbf{S}_x$ and $T\in\mathbf{S}_y$. If S is dependent then $e_S\in\mathscr{I}$ so $\alpha_S=0$ and $\langle\alpha_S,\alpha_T\rangle=0$. Similarly if T is dependent.

Suppose S and T are independent. By Lemma 2.9 $\pi e_S \in \mathscr{E}_x$ and $\pi e_T \in \mathscr{E}_y$ so $\langle \alpha_S, \alpha_T \rangle = \langle \pi e_S, \pi e_T \rangle = 0$ because $\mathscr{E}_x$ and $\mathscr{E}_y$ are orthogonal. $\square$

Remark. If $S \in \mathbf{S}$ is dependent then $\alpha_S = 0$. If $S \in \mathbf{S}_x$ is independent then $\alpha_S \in \mathscr{A}_p$ where $p = r(x)$. Thus

(2.11) $\mathscr{A}_p = \bigoplus_{r(x)=p} \mathscr{A}_x.$

Our next aim is to show that $\dim \mathscr{A}_x = (-1)^{r(x)} \mu(x)$. We will do this by induction on the rank of L after several lemmas. Here it is important to make explicit the dependence on L of all the spaces we have constructed. Let $\mathbf{A}'$ be any subset of the set $\mathbf{A}$ of atoms of L. Let L' be the set of all $\bigvee S$ where S ranges over the set $\mathbf{S}'$ consisting of all sequences $(a_1, \ldots, a_p)$ with $a_i \in \mathbf{A}'$. Then L' is a geometric lattice and $\mathbf{A}'$ is its set of atoms. We view $\mathscr{E}(L')$ as a subalgebra of $\mathscr{E}(L)$. Thus $\mathscr{E}(L') = \bigoplus \mathbb{C} e_S$ where the sum is over all standard $S \in \mathbf{S}'$. Since the rank function of L' is the restriction of the rank function of L, a sequence $S \in \mathbf{S}'$ is dependent in L' if and only if it is dependent in L. Thus

(2.12) $\mathscr{J}(L) \cap \mathscr{E}(L') = \mathscr{J}(L').$

If $x \in L'$ then $\mathscr{E}_x(L) \subseteq \mathscr{E}(L')$. If $x \notin L'$ then $\mathscr{E}_x(L) \cap \mathscr{E}(L') = 0$. Since $\mathscr{K}_x(L) \subseteq \mathscr{E}_x(L)$ this implies

(2.13) $\mathscr{K}_x(L) \cap \mathscr{E}(L') = \mathscr{K}_x(L) = \mathscr{K}_x(L') \quad \text{if } x \in L'$

and $\mathscr{K}_x(L) \cap \mathscr{E}(L') = 0$ otherwise.

(2.14) **Lemma.** $\mathscr{I}(L) \cap \mathscr{E}(L') = \mathscr{I}(L').$

Proof. We begin with a general remark. Let $\mathscr{M}_1, \mathscr{M}_2, \ldots$ be any subspaces of $\mathscr{E}(L)$ with pairwise disjoint supports. If $u_i \in \mathscr{M}_i$ then $\operatorname{supp}(\sum u_i) = \bigcup \operatorname{supp}(u_i)$. Thus if $\sum u_i \in \mathscr{E}(L')$ then $\operatorname{supp}(u_i) \in \operatorname{supp} \mathscr{E}(L')$ so that $u_i \in \mathscr{E}(L')$. Thus $(\sum \mathscr{M}_i) \cap \mathscr{E}(L') = \sum (\mathscr{M}_i \cap \mathscr{E}(L'))$ whenever the $\mathscr{M}_i$ are subspaces of $\mathscr{E}(L)$ with pairwise disjoint supports. We showed in the proof of Lemma 2.8 that the subspaces $\mathscr{J}(L)$ and $\mathscr{K}_x(L)$ have pairwise disjoint supports. Thus by Lemma 2.8 together with (2.12) and (2.13) we conclude that

$$\begin{aligned} \mathscr{I}(L) \cap \mathscr{E}(L') &= (\mathscr{J}(L) \cap \mathscr{E}(L')) + \sum_{x \in L} (\mathscr{K}_x(L) \cap \mathscr{E}(L')) \\ &= \mathscr{J}(L') + \sum_{x \in L'} \mathscr{K}_x(L') \\ &= \mathscr{I}(L') \end{aligned}$$

where the last equality follows from Lemma 2.8 applied to L'. $\square$

Let i: $L' \to L$ be the inclusion and let φ': $\mathscr{E}(L') \to \mathscr{A}(L')$ be the natural map. Then we have a commutative diagram.

$$\begin{array}{ccc} \mathscr{E}(L') & \xrightarrow{i_{\mathscr{E}}} & \mathscr{E}(L) \\ \Big\downarrow{\scriptstyle \varphi'} & & \Big\downarrow{\scriptstyle \varphi} \\ \mathscr{A}(L') & \xrightarrow[i_{\mathscr{A}}]{} & \mathscr{A}(L). \end{array}$$

(2.15) **Proposition.** *The map $i_{\mathscr{A}}$: $\mathscr{A}(L')\to\mathscr{A}(L)$ is injective. Thus $\mathscr{A}(L')$ is isomorphic to the subalgebra of $\mathscr{A}(L)$ generated by the identity and all φe_a where $a\in\mathbf{A}'$. Furthermore for each $x\in L'$ the map $i_{\mathscr{A}}$: $\mathscr{A}_x(L')\to\mathscr{A}_x(L)$ is an isomorphism.*

Proof. Clear from (2.14) and the diagram. □

(2.16) **Lemma.** $\sum_{p=0}^{\ell}(-1)^p \dim\mathscr{A}_p=0$ where $\ell=r(L)$.

Proof. Our convention $\mathbf{0}\neq\mathbf{1}$ implies that ℓ is not zero. Note from (2.7) that $(\mathscr{I},\partial)$ is a chain complex. Let $e=\sum_{a\in\mathbf{A}} e_a$ and define ∂^*: $\mathscr{E}\to\mathscr{E}$ by $\partial^* u=eu$. Then $(\partial\partial^*+\partial^*\partial)u=nu$ for all $u\in\mathscr{E}$ where $n=|\mathbf{A}|$ is the number of atoms. The maps ∂, ∂^* carry $\mathscr{I}$ to $\mathscr{I}$ and thus induce maps σ, σ^*: $\mathscr{A}\to\mathscr{A}$. Since $\sigma\sigma^*+\sigma^*\sigma=n\cdot\mathrm{id}_{\mathscr{A}}$ the map σ^* is a contracting homotopy for the chain complex $(\mathscr{A},\sigma)$ which is therefore acyclic. In particular the Euler characteristic of $(\mathscr{A},\sigma)$ is zero. □

(2.17) **Lemma.** $\dim\mathscr{A}_x=(-1)^{r(x)}\mu(x)\qquad x\in L.$

Proof. We argue by induction on the rank $\ell=r(L)$. If $\ell=1$ then $L=\{\mathbf{0},\mathbf{1}\}$ and the assertion is clear.

Suppose $x\in L$ and $x\neq\mathbf{1}$. Let $\mathbf{A}_x=\{a\in\mathbf{A}\,|\,a\leqq x\}$ and let L_x be the geometric lattice consisting of $\mathbf{0}$ and joins of elements of $\mathbf{A}_x$. The rank and Möbius functions of L_x are the restrictions of the corresponding functions of L. By Proposition 2.15 $\mathscr{A}_x(L_x)\simeq\mathscr{A}_x(L)$. Since the rank of L_x is less than the rank of L the induction hypothesis shows that $\dim\mathscr{A}_x=(-1)^{r(x)}\mu(x)$ if $x\neq\mathbf{1}$. For $x=\mathbf{1}$ we note that by (2.11) $\mathscr{A}_p=\bigoplus_{r(x)=p}\mathscr{A}_x$ and then the vanishing of the Euler characteristic for $\ell>0$ gives

$$\sum_{x\in L}(-1)^{r(x)}\dim\mathscr{A}_x=\sum_{x\neq\mathbf{1}}\mu(x)+(-1)^{\ell}\dim\mathscr{A}_{\mathbf{1}}=0.$$

On the other hand, by the definition (2.1) of μ we have $\sum_{x\in L}\mu(x)=0$, so $\dim\mathscr{A}_{\mathbf{1}}=(-1)^{\ell}\mu(\mathbf{1})$. □

This completes the proof of Theorem 2.6. Since $(\mathscr{A},\sigma)$ is acyclic we have an exact sequence

$$(2.18)\quad 0\to\mathscr{A}_{\mathbf{1}}\to\bigoplus_{r(x)=\ell-1}\mathscr{A}_x\to\bigoplus_{r(x)=\ell-2}\mathscr{A}_x\to\dots\to\mathscr{A}_{\mathbf{0}}\to 0.$$

3. An Algebra Defined by Shuffles

Let L be a finite geometric lattice of rank ℓ. In this section we construct an algebra $\mathscr{B}$ whose elements are certain $\mathbb{C}$-linear combinations of ordered subsets of L with multiplication defined using a shuffle product. The aim of this section is to prove that $\mathscr{A}$ and $\mathscr{B}$ are isomorphic algebras. In fact $\mathscr{B}$ is also a functor from the category of geometric lattices and maps satisfying (2.4) to the category of graded anticommutative $\mathbb{C}$-algebras, and our construction yields a natural transformation between the functors $\mathscr{A}$ and $\mathscr{B}$.

Define vector spaces $\mathcal{T}_p$ for $p \geqq 0$ as follows. For $p>0$ let $\mathcal{T}_p$ have basis consisting of all p-tuples $(x_1, \dots, x_p)$ where $x_i \in L - \mathbf{0}$. For $p=0$ let $\mathcal{T}_0 = \mathbb{C}$. Let $\mathrm{Sym}(p)$ be the symmetric group on the letters $1, \dots, p$. If $\pi \in \mathrm{Sym}(p)$ and $u = (x_1, \dots, x_p)$ let $\pi u = (x_{\pi^{-1}1}, \dots, x_{\pi^{-1}p})$. This makes $\mathcal{T}_p$ a $\mathrm{Sym}(p)$-module. Let $\mathcal{T} = \bigoplus_{p \geqq 0} \mathcal{T}_p$. Define a product $\mathcal{T} \times \mathcal{T} \to \mathcal{T}$, written $*$, as follows. If $u = (x_1, \dots, x_p)$ and $v = (y_1, \dots, y_q)$ let $w = (z_1, \dots, z_{p+q}) = (x_1, \dots, x_p, y_1, \dots, y_q)$ and define

$$u * v = \sum \operatorname{sgn} \pi \cdot \pi w$$

where the sum is over all (p, q)-shuffles π of $1, \dots, p+q$. Recall [11, p. 243] that a (p, q)-shuffle of $1, \dots, p+q$ is a permutation $\pi \in \mathrm{Sym}(p+q)$ such that $\pi i < \pi j$ whenever $i < j \leqq p$ or $p < i < j$. This makes $\mathcal{T}$ into an associative graded anticommutative $\mathbb{C}$-algebra with identity. Let $\eta \colon \mathcal{T} \to \mathcal{T}$ be the antisymmetrizer defined for $u = (x_1, \dots, x_p)$ by

$$\text{(3.1)} \quad \eta u = \sum (\operatorname{sgn} \pi)\, \pi u = \sum (\operatorname{sgn} \pi)\, \pi^{-1} u$$

summed over all $\pi \in \mathrm{Sym}(p)$. It follows by induction that

$$\text{(3.2)} \quad \eta(x_1, \dots, x_p) = (x_1) * \dots * (x_p).$$

Define a $\mathbb{C}$-linear map $\lambda \colon \mathcal{T} \to \mathcal{T}$ by $\lambda 1 = 1$ and

$$\text{(3.3)} \quad \lambda(x_1, \dots, x_p) = (x_1, x_1 \vee x_2, \dots, x_1 \vee x_2 \vee \dots \vee x_p).$$

(3.4) **Lemma.** *If* $u, v \in \mathcal{T}$ *then*

$$\lambda(\lambda u * \lambda v) = \lambda(u * v).$$

Proof. It suffices to check this for $u = (x_1, \dots, x_p)$ and $v = (y_1, \dots, y_q)$. Then $\lambda u = (x'_1, \dots, x'_p)$ and $\lambda v = (y'_1, \dots, y'_q)$ where $x'_i = x_1 \vee \dots \vee x_i$ and $y'_j = y_1 \vee \dots \vee y_j$. Write $(z_1, \dots, z_{p+q}) = (x_1, \dots, x_p, y_1, \dots, y_q)$ and $(z'_1, \dots, z'_{p+q}) = (x'_1, \dots, x'_p, y'_1, \dots, y'_q)$. It follows from the idempotence $z = z \vee z$ of the lattice join that $z'_{\pi 1} \vee \dots \vee z'_{\pi i} = z_{\pi 1} \vee \dots \vee z_{\pi i}$ for all $i = 1, \dots, p+q$ and all permutations π of $1, \dots, p+q$. Thus

$$\begin{aligned} \lambda(\lambda u * \lambda v) &= \sum_{\pi} \operatorname{sgn} \pi \cdot \lambda(z'_{\pi 1}, \dots, z'_{\pi(p+q)}) \\ &= \sum_{\pi} \operatorname{sgn} \pi \cdot \lambda(z_{\pi 1}, \dots, z_{\pi(p+q)}) \\ &= \lambda(u * v). \quad \square \end{aligned}$$

Let $\mathcal{U} = \lambda(\mathcal{T})$. Then $\mathcal{U}$ inherits a grading from $\mathcal{T}$. Since λ is idempotent $\mathcal{U}$ is spanned by the identity and all $(x_1, \dots, x_p)$ with $x_1 \leqq \dots \leqq x_p$. Define a product in $\mathcal{U}$ by

$$u v = \lambda(u * v) \qquad u, v \in \mathcal{U}.$$

This multiplication is associative: if $u, v, w \in \mathcal{U}$ then $\lambda w = w$ so Lemma 3.4 shows

$$(u v) w = \lambda(u v * w) = \lambda(\lambda(u * v) * \lambda w) = \lambda((u * v) * w).$$

Thus $\mathscr{U}$ is an associative, anticommutative algebra with identity. We may view each element $S \in \mathbf{S}$ as an element of $\mathscr{T}$. If $S=(\,)$ let $\beta_S=1$ and for $S=(a_1,\ldots,a_p)$ define $\beta_S \in \mathscr{U}$ by

(3.5) $\beta_S=\lambda(\eta S)=\sum_{\pi} \operatorname{sgn}\pi(a_{\pi 1}, a_{\pi 1}\vee a_{\pi 2},\ldots, a_{\pi 1}\vee a_{\pi 2}\vee\ldots\vee a_{\pi p})$.

(3.6) **Lemma.** *Let S, $T\in\mathbf{S}$. Then $\beta_S\beta_T=\beta_{(S,T)}$.*

Proof. Let $S=(a_1,\ldots,a_p)$ and $T=(b_1,\ldots,b_q)$ where a_i, $b_j\in\mathbf{A}$. Then using (3.2) and (3.4) we have

$$\begin{aligned}\beta_S\beta_T&=\lambda(\beta_S*\beta_T)\\&=\lambda(\lambda(\eta S)*\lambda(\eta T))\\&=\lambda(\eta S*\eta T)\\&=\lambda((a_1)*\ldots*(a_p)*(b_1)*\ldots*(b_q))\\&=\lambda\eta(a_1,\ldots,a_p,b_1,\ldots,b_q)\\&=\beta_{(S,T)}.\quad\square\end{aligned}$$

Let $\mathscr{B}=\sum_{S\in\mathbf{S}}\mathbb{C}\beta_S$. Lemma 3.6 shows that $\mathscr{B}=\bigoplus_{p\geqq 0}\mathscr{B}_p$ is a graded subalgebra of $\mathscr{U}$. If $f\colon L'\to L$ satisfies (2.4) then the induced map $f_{\mathscr{T}}\colon \mathscr{T}(L')\to\mathscr{T}(L)$ commutes with λ and η and thus $f_{\mathscr{T}}$ induces a homomorphism $f_{\mathscr{B}}\colon \mathscr{B}(L')\to\mathscr{B}(L)$ of graded $\mathbb{C}$-algebras, which is functorial.

(3.7) **Theorem.** *Let L be a finite geometric lattice. There exists an isomorphism $\theta\colon\mathscr{A}\to\mathscr{B}$ of algebras such that $\theta\alpha_S=\beta_S$. The map $\theta\colon\mathscr{A}\to\mathscr{B}$ defines a natural transformation of functors.*

Theorem 3.7 is a consequence of Lemmas 3.8 and 3.9. If $S=(a_1,\ldots,a_p)$ and $\pi\in\operatorname{Sym}(p)$ then $\eta\pi S=(\operatorname{sgn}\pi)\eta S$ so $\beta_{\pi S}=(\operatorname{sgn}\pi)\beta_S$. Thus there exists a $\mathbb{C}$-linear map $\psi\colon\mathscr{E}\to\mathscr{B}$ such that $\psi e_S=\beta_S$. Since $e_Se_T=e_{(S,T)}$ it follows from (3.6) that ψ is a homomorphism of algebras.

(3.8) **Lemma.** *If $S\in\mathbf{S}$ is dependent then $\beta_S=0$.*

Proof. Let $S=(a_1,\ldots,a_p)$. If $S_k=(a_1,\ldots,\hat{a}_k,\ldots,a_p)$ is dependent for some $k=1,\ldots,p$ then $\beta_S=(-1)^{k-1}\beta_{(a_k,S_k)}=(-1)^{k-1}\beta_{a_k}\beta_{S_k}$ and we are done by induction. Thus we may assume that S_k is independent for each k. Then as in the paragraph preceding Lemma 2.8 we see that $\bigvee S_k=\bigvee S$ for all k. If $\pi\in\operatorname{Sym}(p)$ let ζ be the permutation defined by $\zeta k=\pi k$ for $k=1,\ldots,p-2$, $\zeta(p-1)=\pi p$ and $\zeta(p)=\pi(p-1)$. Then $\operatorname{sgn}\zeta=-\operatorname{sgn}\pi$ and the terms corresponding to π and ζ in (3.5) cancel. $\square$

Note that Lemma 3.8 implies $\beta_S=0$ if $|S|>\ell$ so that $\mathscr{B}=\bigoplus_{p=0}^{\ell}\mathscr{B}_p$. Define a $\mathbb{C}$-linear map $\tau\colon\mathscr{T}\to\mathscr{T}$ by $\tau 1=0$, $\tau(x)=1$ for $x\in L-\mathbf{0}$, and $\tau(x_1,\ldots,x_p)=(-1)^{p-1}(x_1,\ldots,x_{p-1})$ for $p\geqq 2$ and $x_i\in L-\mathbf{0}$. The computation in the next lemma shows in particular that $\tau\mathscr{B}_p\subseteq\mathscr{B}_{p-1}$.

(3.9) **Lemma.** *The diagram below commutes.*

$$\begin{array}{ccc} \mathscr{E}_p & \xrightarrow{\partial} & \mathscr{E}_{p-1} \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} \\ \mathscr{B}_p & \xrightarrow[\tau]{} & \mathscr{B}_{p-1} \end{array}$$

Proof. If $S \in \mathbf{S}_p$ then

$$\begin{aligned} \psi\,\partial\, e_S &= \lambda\left(\sum_{k=1}^{p} (-1)^{k-1} \eta S_k\right) \\ &= \lambda\left(\sum_{k=1}^{p} \sum_{\zeta \in W_k} (-1)^{k-1} (\operatorname{sgn}\zeta)(a_{\zeta 1}, \ldots, \hat{a}_{\zeta k}, \ldots, a_{\zeta p})\right) \end{aligned}$$

where W_k is the group of permutations of $1, \ldots, \hat{k}, \ldots, p$. On the other hand since $\tau\lambda = \lambda\tau$ we have

$$\tau\psi\, e_S = \lambda\left(\sum_{\pi} \operatorname{sgn}\pi (a_{\pi 1}, \ldots, a_{\pi(p-1)})\right)$$

where π ranges over $\operatorname{Sym}(p)$. If $\pi \in \operatorname{Sym}(p)$ and $\pi p = k$ define $\zeta \in W_k$ by

$$(a_{\pi 1}, \ldots, a_{\pi(p-1)}) = (a_{\zeta 1}, \ldots, a_{\zeta(k-1)}, a_{\zeta(k+1)}, \ldots, a_{\zeta p}).$$

Then $\operatorname{sgn}\pi = (-1)^{p-k} \operatorname{sgn}\zeta$ and the sums $\psi\,\partial\, e_S$ and $\tau\psi\, e_S$ are equal term for term. □

Now we may complete the proof of (3.7). If S is dependent then $\psi\,\partial\, e_S = \tau\psi\, e_S = \tau\beta_S = 0$ so that $\partial e_S \in \ker\psi$. Thus $\mathscr{I} \subseteq \ker\psi$ and ψ induces a surjective map θ: $\mathscr{A} \to \mathscr{B}$ such that $\theta\alpha_S = \beta_S$. Since ψ is an algebra homomorphism so is θ. Recall that ∂, ∂^*: $\mathscr{E} \to \mathscr{E}$ defined in Sect. 2, carry $\mathscr{I}$ to $\mathscr{I}$ and thus induce maps σ, σ^*: $\mathscr{A} \to \mathscr{A}$. We have a commutative diagram

(3.10)
$$\begin{array}{ccc} \mathscr{A}_p & \xrightarrow{\sigma} & \mathscr{A}_{p-1} \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} \\ \mathscr{B}_p & \xrightarrow[\tau]{} & \mathscr{B}_{p-1} \end{array}$$

Let $\beta = \sum_{a \in \mathbf{A}} \beta_a$. If $u \in \mathscr{B}$ then $\beta u \in \mathscr{B}$. Define τ^*: $\mathscr{B} \to \mathscr{B}$ by $\tau^* u = \beta u$. Then $\psi\,\partial^* e_S = \psi\left(\sum_{a \in \mathbf{A}} a e_S\right) = \sum_{a \in \mathbf{A}} (\psi a)(\psi e_S) = \sum_{a \in \mathbf{A}} \beta_a \beta_S = \tau^* \psi e_S$. Thus the diagram

(3.11)
$$\begin{array}{ccc} \mathscr{A}_p & \xleftarrow{\sigma^*} & \mathscr{A}_{p-1} \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} \\ \mathscr{B}_p & \xleftarrow[\tau^*]{} & \mathscr{B}_{p-1} \end{array}$$

commutes. Since σ^* is a contracting homotopy for the chain complex $(\mathscr{A}, \sigma)$, τ^* is a contracting homotopy for $(\mathscr{B}, \tau)$ so $(\mathscr{B}, \tau)$ is also acyclic. Thus (2.16) holds with $\mathscr{B}$ in place of $\mathscr{A}$. Define L' as we did following (2.11). Then (2.15) holds with $\mathscr{B}$ in place of $\mathscr{A}$. The proof of this fact does not require the same effort as (2.15): there is a natural inclusion $\mathscr{T}(L') \to \mathscr{T}(L)$, and because joins in L' are the same as joins in L there is a natural inclusion $\mathscr{U}(L') \to \mathscr{U}(L)$ and hence $\mathscr{B}(L') \to \mathscr{B}(L)$.

If $x \in L$ let $\mathscr{B}_x = \sum_{S \in \mathbf{S}_x} \mathbb{C}\beta_S$. Then $\mathscr{B} = \bigoplus_{x \in L} \mathscr{B}_x$. We may choose $L' = L_x$ and apply the remarks of the preceding paragraph to conclude that $\mathscr{B}_x(L_x) \simeq \mathscr{B}_x(L)$. This allows us to prove (2.17) with $\mathscr{B}$ in place of $\mathscr{A}$. Thus $\dim \mathscr{A}_x = \dim \mathscr{B}_x$ so the surjective map θ: $\mathscr{A}_x \to \mathscr{B}_x$ is an isomorphism. Thus θ: $\mathscr{A} \to \mathscr{B}$ is an isomorphism of algebras. Since all our constructions are functorial, this completes the proof of Theorem 3.7. □

Since $\mathscr{B}_p = \bigoplus_{r(x)=p} \mathscr{B}_x$ we have a commutative diagram of exact sequences

$$
\text{(3.12)} \qquad
\begin{array}{ccccccccc}
0 \to \mathscr{A}_1 \to & \bigoplus_{r(x)=\ell-1} \mathscr{A}_x \to & \bigoplus_{r(x)=\ell-2} \mathscr{A}_x \to \dots \to \mathscr{A}_0 \to 0 \\
\downarrow \qquad & \downarrow & \downarrow \qquad\qquad \downarrow \\
0 \to \mathscr{B}_1 \to & \bigoplus_{r(x)=\ell-1} \mathscr{B}_x \to & \bigoplus_{r(x)=\ell-2} \mathscr{B}_x \to \dots \to \mathscr{B}_0 \to 0
\end{array}
$$

where the vertical maps are isomorphisms. The Poincaré polynomial of $\mathscr{B}$ is

$$\text{(3.13)} \quad P_{\mathscr{B}}(t) = \sum_{x \in L} \mu(x)(-t)^{r(x)}.$$

Baclawski [2] has shown that there corresponds to the finite geometric lattice L a (co)chain complex whose (co)homology has Poincaré polynomial equal to $P_{\mathscr{B}}(t)$. We show here that the elements $\beta_S \in \mathscr{B}$ are cycles in Baclawski's homology $\mathscr{H}$, and that the map ρ: $\mathscr{B} \to \mathscr{H}$ sending β_S to its homology class $[\beta_S]$ is an isomorphism of vector spaces. We recall Baclawski's definitions, slightly altered to suit our purposes in that we shift dimensions up by one in order to agree with the earlier results in this section and the topological applications in Sect. 5.

Define subspaces $\mathscr{C}_p$ of $\mathscr{T}_p$ for $p = 0, \dots, \ell$ as follows. For $p = 0$ let $\mathscr{C}_0 = \mathbb{C}$. For $p > 0$ let $\mathscr{C}_p$ have basis consisting of all p-tuples $(x_1, \dots, x_p)$ where $x_i \in L - \mathbf{0}$ and $x_1 < \dots < x_p$. Let $\mathscr{C} = \bigoplus_{p=0}^{\ell} \mathscr{C}_p$. Define a $\mathbb{C}$-linear map δ: $\mathscr{C} \to \mathscr{C}$ by $\delta 1 = 0$, $\delta(x) = 0$ for $x \in L - \mathbf{0}$ and

$$\delta(x_1, \dots, x_p) = \sum_{k=1}^{p-1} (-1)^{k-1} (x_1, \dots, \hat{x}_k, \dots, x_p)$$

for $p = 2, \dots, \ell$. Note that δ differs from the usual boundary operator in that x_p is never deleted. Nevertheless $\delta^2 = 0$ so $(\mathscr{C}, \delta)$ is a chain complex. There is an ancestor of this complex in work of Deheuvels [6, § 10]. Let $\mathscr{H}$ be the homology of $(\mathscr{C}, \delta)$.

(3.14) **Theorem** (Baclawski). *Let L be a finite geometric lattice. The Poincaré polynomial of $\mathscr{H}$ is* $P_{\mathscr{H}}(t) = \sum_{x \in L} \mu(x)(-t)^{r(x)}$.

Baclawski's proof uses results of Rota [13] and Folkman [9] on the homology of geometric lattices. We give a direct argument along the lines of (2.17) as follows. For $x \in L-\mathbf{0}$ let $\mathscr{C}_x$ be the subspace of $\mathscr{C}$ spanned by all $(x_1, \ldots, x_p)$ with $x_p = x$ and let $\mathscr{C}_{\mathbf{0}} = \mathbb{C}$. Then $\mathscr{C} = \bigoplus_{x \in L} \mathscr{C}_x$ and $\delta\colon \mathscr{C}_x \to \mathscr{C}_x$. Thus $(\mathscr{C}_x, \delta)$ is a subcomplex and $\mathscr{H} = \bigoplus_{x \in L} \mathscr{H}_x$ where $\mathscr{H}_x$ is the homology of $(\mathscr{C}_x, \delta)$. Let $\mathscr{C}'_0 = \mathbb{C}$. For $p>0$ let $\mathscr{C}'_p \subseteq \mathscr{C}_p$ be the subspace spanned by all $(x_1, \ldots, x_p)$ with $x_p \neq \mathbf{1}$ and let $\mathscr{C}''_p \subseteq \mathscr{C}_p$ be the subspace spanned by all $(x_1, \ldots, x_p)$ with $x_p = \mathbf{1}$. Then $\mathscr{C}_p = \mathscr{C}'_p \oplus \mathscr{C}''_p$. The $\mathbb{C}$-linear map defined by $1 \to (\mathbf{1})$ and $(x_1, \ldots, x_p) \to (x_1, \ldots, x_p, \mathbf{1})$ establishes an isomorphism $\mathscr{C}'_p \simeq \mathscr{C}''_{p+1}$ of vector spaces, for $p = 0, \ldots, \ell-1$. Thus the Euler characteristic of $\mathscr{C}$ is zero. Since $\mathscr{H} = \bigoplus_{x \in L} \mathscr{H}_x$ and the natural identification $\mathscr{C}_x(L_x) = \mathscr{C}_x(L)$ implies $\mathscr{H}_x(L_x) \simeq \mathscr{H}_x(L)$, we may argue as in the proof of (2.17) that $\dim \mathscr{H}_x = (-1)^{r(x)} \mu(x)$, which proves the assertion.

4. Character Formulas

Let L be a finite geometric lattice. Let G be a group which acts as a group of automorphisms of L. This means that G acts as a permutation group on L and preserves the partial order. We do not assume that the action is effective. If $g \in G$ then $g(x \vee y) = gx \vee gy$ and $r(gx) = r(x)$. Thus G induces maps which satisfy the conditions (2.4) and hence, by functoriality the group G is represented by linear transformations of the graded vector spaces $\mathscr{A}$ and $\mathscr{B}$. In this section we compute the character of the representation of G on $\mathscr{B}$, and hence, by Theorem 3.7, on $\mathscr{A}$. In Sect. 5 we will see that the character formulas on $\mathscr{A}$ admit a topological interpretation.

Let L be a finite poset with unique minimal element $\mathbf{0}$ and maximal element $\mathbf{1}$. Define a simplicial complex $K = K(L)$ as follows. The vertices of K are the elements of $L - \{\mathbf{0}, \mathbf{1}\}$ and $(x_1, \ldots, x_p)$ is a $(p-1)$-simplex if $\mathbf{0} < x_1 < \ldots < x_p < \mathbf{1}$. If L is a geometric lattice of rank $\ell \geqq 2$ let $\tilde{K}$ be the augmented complex obtained from K by adjoining a simplex of dimension -1 on which G acts trivially. Then the reduced homology $\tilde{H}(K)$ of K is the homology $H(\tilde{K})$ of $\tilde{K}$. According to Folkman [9] and Rota [13] the homology of $\tilde{K}$ is given by

$$(4.1)\quad \begin{aligned} &\dim H_p(\tilde{K}) = 0 \quad \text{if } p \neq \ell-2 \\ &\dim H_{\ell-2}(\tilde{K}) = (-1)^\ell \mu(\mathbf{1}). \end{aligned}$$

If L is a Boolean algebra on atoms $a_1, \ldots, a_\ell$, $\ell \geqq 2$, then K is the barycentric subdivision of the boundary of the $(\ell-1)$-simplex with vertices $a_1, \ldots, a_\ell$. The group $H_{\ell-2}(\tilde{K})$ is generated by the cycle $z(a_1, \ldots, a_\ell) = \tau \beta_{(a_1, \ldots, a_\ell)}$ where $\tau\colon \mathscr{T}_\ell \to \mathscr{T}_{\ell-1}$ defined by $\tau(x_1, \ldots, x_\ell) = (-1)^{\ell-1}(x_1, \ldots, x_{\ell-1})$ is the map used in (3.9). Thus

$$(4.2)\quad z(a_1, \ldots, a_\ell) = \sum_{\pi \in \mathrm{Sym}(\ell)} (-1)^{\ell-1} \operatorname{sgn} \pi (a_{\pi 1}, a_{\pi 1} \vee a_{\pi 2}, \ldots, a_{\pi 1} \vee \ldots \vee a_{\pi(\ell-1)}).$$

Note that a G-action on L induces a simplicial G-action on K, so that $H_{\ell-2}(\tilde{K})$ is a G-module.

(4.3) **Theorem.** *Let L be a finite geometric lattice of rank $\ell \geqq 2$. Then $\mathcal{B}_1$ and $H_{\ell-2}(\tilde{K})$ are isomorphic G-modules.*

Proof. If $\ell=2$ then we have G-isomorphisms

$$\mathcal{B}_1 \simeq (\bigoplus_{a \in \mathbf{A}} \mathbb{C}a)/\mathbb{C}(\sum_{a \in \mathbf{A}} a) \simeq H_0(\tilde{K}).$$

Assume $\ell \geqq 3$. Since there are no $\ell-2$ boundaries we have $H_{\ell-2}(\tilde{K}) = H_{\ell-2}(K) = Z_{\ell-2}(K)$. We identify the cycle group $Z_{\ell-2}(K)$ with a subspace of $\mathcal{T}_{\ell-1}$. Note that $\mathcal{B}_1 \subseteq \mathcal{T}_\ell$. We show that $\tau\mathcal{B}_1 \subseteq Z_{\ell-2}(K)$. Let $S=(a_1, \ldots, a_\ell) \in \mathbf{S}_\ell$. If S is dependent then (3.8) shows that $\tau\beta_S = \tau 0 = 0$. Suppose S is independent. Let L_S be the sublattice of L generated by $a_1, \ldots, a_\ell$ and $\mathbf{0}$. Then L_S is a Boolean algebra on $a_1, \ldots, a_\ell$. Let K_S be the corresponding complex. From (3.5) and (4.2) we see that $\tau\beta_S = z(a_1, \ldots, a_\ell)$ is a generating cycle of $Z_{\ell-2}(K_S) \subseteq Z_{\ell-2}(K)$. Thus $\tau\mathcal{B}_1 \subseteq Z_{\ell-2}(K)$. Let $\mathcal{T}_\ell'$ be the subspace of $\mathcal{T}_\ell$ spanned by all $(x_1, \ldots, x_\ell)$ with $x_\ell = \mathbf{1}$. If $S \in \mathbf{S}_\ell$ is independent then $\bigvee \pi S = \bigvee S = \mathbf{1}$ for all $\pi \in \mathrm{Sym}(\ell)$ so $\mathcal{B}_1 \subseteq \mathcal{T}_\ell'$. But τ: $\mathcal{T}_\ell' \to \mathcal{T}_{\ell-1}$ is a monomorphism and thus τ: $\mathcal{B}_1 \to Z_{\ell-2}(K)$ is a monomorphism. Since $\dim \mathcal{B}_1 = (-1)^\ell \mu(\mathbf{1}) = \dim H_{\ell-2}(K)$ and τ: $\mathcal{T}_\ell \to \mathcal{T}_{\ell-1}$ is a G-module homomorphism, τ: $\mathcal{B}_1 \to H_{\ell-2}(K)$ is a G-isomorphism. □

The G-isomorphisms θ: $\mathcal{A} \to \mathcal{B}$ and τ: $\mathcal{B}_1 \to H_{\ell-2}(\tilde{K})$ allow us to compute the character of the representation of G on $\mathcal{A}$. Let $R(G)$ be the representation ring of G. If M is a G-module we let $[M]$ denote its image in $R(G)$. If H is a subgroup of G and N is an H-module we let $\mathrm{Ind}_H^G N$ denote the induced G-module and write $\mathrm{Ind}_H^G [N] = [\mathrm{Ind}_H^G N]$. For simplicity of notation we write $[M] = [G/H]$ if $M = \mathbb{C}[G/H]$. If $x \in L$ and $r(x) \geqq 2$ the complex $\tilde{K}(L_x)$ is defined. Let $G_x = \{g \in G \mid gx = x\}$. If σ is a simplex of $\tilde{K}(L)$ let $d(\sigma)$ denote its dimension, let $G_\sigma = \{g \in G \mid g\sigma = \sigma\}$ and let $G_{x,\sigma} = G_x \cap G_\sigma$.

(4.4) **Theorem.** *Let L be a finite geometric lattice of rank ℓ and let G act as a group of automorphisms of L. Then*

$$[\mathcal{A}_0] = [\mathbb{C}],$$
$$[\mathcal{A}_1] = \sum_{r(x)=1} |G:G_x|^{-1} [G/G_x]$$

and for $p = 2, \ldots, \ell$

$$[\mathcal{A}_p] = (-1)^p \sum_{r(x)=p} \sum_{\sigma \in \tilde{K}(L_x)} (-1)^{d(\sigma)} |G:G_{x,\sigma}|^{-1} [G/G_{x,\sigma}].$$

Proof. The assertion of the theorem is clear for $\mathcal{A}_0$ and $\mathcal{A}_1$, since $\mathcal{A}_1 \simeq \mathcal{E}_1$ is the G-module defined by permuting the atoms. This proves the assertion if $\ell = 1$. Henceforth assume $\ell \geqq p \geqq 2$. The Hopf trace formula says

$$\sum_{q=0}^{\ell-2} (-1)^q [H_q(K)] = \sum_{q=0}^{\ell-2} (-1)^q [C_q(K)]$$

where $C_q(K)$ is the group of q-chains of K. It follows from (4.1) and (4.3) that

$$\sum_{q=0}^{\ell-2}(-1)^q[H_q(K)]=[\mathbb{C}]+(-1)^\ell[\mathscr{B}_1].$$

If $\sigma\in K(L)$ then, since G preserves the orientation of σ, the modules $\mathbb{C}[G\sigma]$ and $\mathbb{C}[G/G_\sigma]$ are isomorphic. Thus we have

$$\sum_{q=0}^{\ell-2}(-1)^q[C_q(K)]=\sum_{\sigma\in K(L)}(-1)^{d(\sigma)}|G:G_\sigma|^{-1}[G/G_\sigma].$$

Putting these facts together shows that

$$[\mathbb{C}]+(-1)^\ell[\mathscr{B}_1]=\sum_{\sigma\in K(L)}(-1)^{d(\sigma)}|G:G_\sigma|^{-1}[G/G_\sigma].$$

Now replacing $K(L)$ by the augmented complex $\tilde{K}(L)$ we absorb the term $[\mathbb{C}]$ on the right hand side so we get

$$(4.5)\quad (-1)^\ell[\mathscr{B}_1(L)]=\sum_{\sigma\in\tilde{K}(L)}(-1)^{d(\sigma)}|G:G_\sigma|^{-1}[G/G_\sigma].$$

If $x\in L$ and $r(x)=p\geqq 2$ then we may apply (4.5) to L_x and the group G_x operating on L_x to conclude that

$$(4.6)\quad (-1)^{r(x)}[\mathscr{B}_x(L_x)]=\sum_{\sigma\in\tilde{K}(L_x)}(-1)^{d(\sigma)}|G_x:G_{x,\sigma}|^{-1}[G_x/G_{x,\sigma}]$$

in the representation ring $R(G_x)$. Let $\mathcal{O}$ be a G-orbit on the set of elements of L of rank p. Let $\mathscr{B}_\mathcal{O}=\bigoplus_{x\in\mathcal{O}}\mathscr{B}_x(L)$. Thus $\mathscr{B}_p=\bigoplus_{\mathcal{O}}\mathscr{B}_\mathcal{O}$, sum over all orbits. Since $g\mathscr{B}_x(L)=\mathscr{B}_{gx}(L)$ we have $\mathscr{B}_\mathcal{O}\simeq\operatorname{Ind}_{G_x}^G\mathscr{B}_x(L)$ for any fixed $x\in\mathcal{O}$. Since $\mathscr{B}_x(L)\simeq\mathscr{B}_x(L_x)$ as G_x-modules we have $[\mathscr{B}_\mathcal{O}]=\operatorname{Ind}_{G_x}^G[\mathscr{B}_x(L_x)]$. Now it follows from (4.6) and transitivity of induction that

$$[\mathscr{B}_p]=(-1)^p\sum_{r(x)=p}\sum_{\sigma\in K(L_x)}(-1)^{d(\sigma)}|G:G_{x,\sigma}|^{-1}[G/G_{x,\sigma}].$$

The assertion of the theorem follows from the G-module isomorphism $\mathscr{A}_p\simeq\mathscr{B}_p$ of Theorem 3.7. □

By choosing representatives for the orbits we may write $[\mathscr{A}_p]$ as a $\mathbb{Z}$-linear combination of certain $[G/G_{x,\sigma}]$ and thereby put the formulas of Theorem 4.4 in a form more suitable for calculation. Let T_p be a set of representatives for the G-orbits on the set of elements of L of rank p. For each $x\in T_p$ let U_x be a set of representatives for G_x-orbits on the set of simplices of $\tilde{K}(L_x)$. Then

$$(4.7)\quad [A_1]=\sum_{x\in T_1}[G/G_x]$$

$$[A_p]=(-1)^p\sum_{x\in T_p}\sum_{\sigma\in U_x}(-1)^{d(\sigma)}[G/G_{x,\sigma}]\qquad p=2,\ldots,\ell.$$

(4.8) **Theorem.** *Let L be a finite geometric lattice of rank ℓ. Let G be a group which acts as an automorphism group of L. If $g \in G$ let $L^g = \{x \in L \mid gx = x\}$ and let μ_g be the Möbius function of the poset L^g. Then*

$$\sum_{p=0}^{\ell} \operatorname{tr}(g \mid \mathscr{A}_p) t^p = \sum_{x \in L^g} \mu_g(x)(-t)^{r(x)}$$

where r is the rank function in L.

Proof. The statement is clear for $p = 0, 1$. Assume $p \geqq 2$ and compute $\operatorname{tr}(g \mid \mathscr{B}_p)$. Since $g\mathscr{B}_x(L) = \mathscr{B}_{gx}(L)$ the terms with $gx \neq x$ do not contribute to the trace. Thus

$$\operatorname{tr}(g \mid \mathscr{B}_p) = \sum_{\substack{x \in L^g \\ r(x) = p}} \operatorname{tr}(g \mid \mathscr{B}_x(L)) = \sum_{\substack{x \in L^g \\ r(x) = p}} \operatorname{tr}(g \mid \mathscr{B}_x(L_x)).$$

Choose $x \in L^g$ with $r(x) = p$. Let $\mathcal{O}_1, \ldots, \mathcal{O}_s$ be the orbits of G_x on $\tilde{K}(L_x)$ and choose $\sigma_j \in \mathcal{O}_j$. Let $M_j = \mathbb{C}[\mathcal{O}_j] \simeq \mathbb{C}[G_x/G_{x,\sigma_j}]$. Now (4.6) gives

$$\begin{aligned}(-1)^p \operatorname{tr}(g \mid \mathscr{B}_x(L_x)) &= \sum_{j=1}^{s} (-1)^{d(\sigma_j)} \operatorname{tr}(g \mid M_j) \\ &= \sum_{j=1}^{s} (-1)^{d(\sigma_j)} |\mathcal{O}_j^g| = -1 + e(K(L_x)^g)\end{aligned}$$

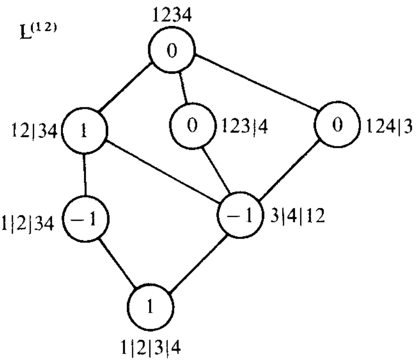
where e denotes the Euler characteristic. If L is any finite poset with $\mathbf{0}, \mathbf{1}$, μ is its Möbius function and K is the corresponding complex then Rota [14, Cor. 2] has shown that $e(K) = 1 + \mu(\mathbf{1})$. Since $K(L_x)^g = K(L_x^g)$ this gives, in our case, $(-1)^p \operatorname{tr}(g \mid \mathscr{B}_x(L_x)) = \mu_g(x)$. □

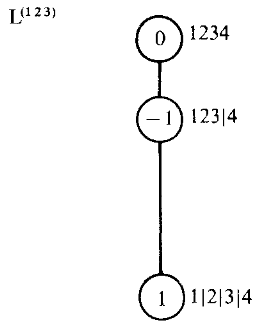
(4.9) *Remark.* Note that L^g is a lattice containing $\mathbf{0}, \mathbf{1}$ and joins of elements in L^g are the same as in L. But L^g need not satisfy the chain condition and the elements of L^g need not be joins of atoms of L^g.

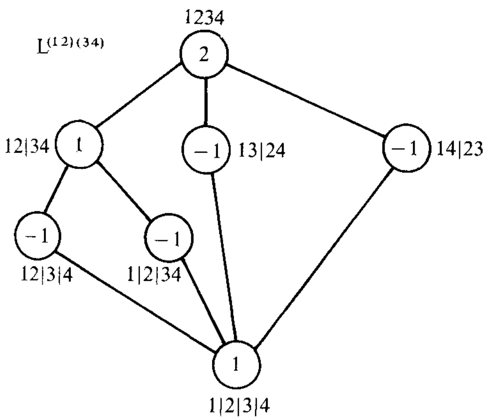
(4.10) *Example.* Let L be the lattice of partitions of the set $\{1, 2, 3, 4\}$ with refinement as the order relation. The elements of L listed by rank are

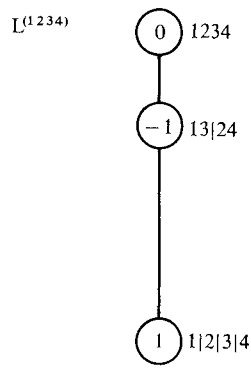
$r(x)$	x
0	1|2|3|4
1	12|3|4, 13|2|4, 14|2|3, 23|1|4, 24|1|3, 34|1|2
2	123|4, 124|3, 134|2, 234|1, 12|34, 13|24, 14|23
3	1234

Let $G = \operatorname{Sym}(4)$. Choose representatives $g = (1), (12), (123), (12)(34), (1234)$ for the conjugacy classes. The following diagrams, for $g \neq (1)$, picture L^g with the values of $\mu_g(x)$ inside the circles. A picture of $L = L^{(1)}$ appears in Birkhoff's book [3, p. 15].









The polynomials $P_{\mathscr{A}}(t, g) = \sum \operatorname{tr}(g|\mathscr{A}_p)\, t^p$ are:

g	$P_{\mathscr{A}}(t, g)$
(1)	$(1+t)(1+2t)(1+3t)$
(12)	$(1+t)(1+t)$
(123)	$(1-t)(1+t)$
(12)(34)	$(1-t)(1+t)(1+2t)$
(1234)	$(1-t)(1+t)$

The first line of this table is a special case of a known fact about the lattice of partitions of the set $\{1, \ldots, \ell\}$ where $P_{\mathscr{A}}(t, 1) = (1+t)(1+2t)\ldots(1+(\ell-1)t)$. This has a topological counterpart in work of Arnold [1]. The connection between the combinatorics and the topology will be given in Sect. 5. The formulas for $P_{\mathscr{A}}(t, g)$ with $g \neq (1)$ will be studied in a forthcoming paper.

5. Topology of Complements of Hyperplanes

Let $V = \mathbb{C}^\ell$ and let $\mathbf{A}$ be a finite set of hyperplanes in V. If $A \in \mathbf{A}$ let φ_A be a linear form with kernel A. Let $M = V - \bigcup_{A \in \mathbf{A}} A$ be the complement of the union of the hyperplanes. Define holomorphic differential forms ω_A on M by $\omega_A = d\varphi_A/2\pi i\varphi_A$ and let $[\omega_A]$ denote the corresponding deRham cohomology class. Let $\mathscr{R} = \bigoplus_{p=0}^{\ell} \mathscr{R}_p$ be the graded $\mathbb{C}$-algebra of holomorphic differential forms on M generated by the ω_A and the identity.

Let L be the collection of subspaces of V of the form $X = A_1 \cap \ldots \cap A_p$ where $A_i \in \mathbf{A}$. We partially order L by *reverse* inclusion; thus $X' \leqq X$ means $X' \supseteq X$. The poset L satisfies the chain condition, has a unique minimal element $\mathbf{0} = V$, and a unique maximal element $\mathbf{1} = \bigcap_{A \in \mathbf{A}} A$. The poset L is a lattice with $X \vee X' = X \cap X'$ and $X \wedge X' \supseteq X + X'$. Since $X + X'$ may not be in L the equality $\dim(X + X') + \dim(X \cap X') = \dim X + \dim X'$ becomes an inequality $r(X \wedge X') + r(X \vee X') \leqq r(X) + r(X')$. Thus L is a geometric lattice, with $r(X) = \operatorname{codim}(X)$, and the atoms of L are the hyperplanes.

As in Sect. 2 let $\mathscr{E}$ be the exterior algebra of the vector space with basis consisting of elements e_A in one to one correspondence with the hyperplanes $A \in \mathbf{A}$. Let $\mathscr{A} = \mathscr{E}/\mathscr{I}$, let φ: $\mathscr{E} \to \mathscr{A}$ be the natural homomorphism, let $\mathscr{A}_X = \varphi\mathscr{E}_X$ and let $\alpha_A = \varphi e_A$. Let $\mathscr{R}_X = \sum \mathbb{C}\omega_{A_1} \ldots \omega_{A_p}$ where the sum is over all independent $S = (A_1, \ldots, A_p)$ with $\bigcap_{i=1}^{p} A_i = X$. Note that in the present context the notion of independence defined in Sect. 2 means that $\operatorname{codim} X = p$ so that the hyperplanes A_i are in general position. We have $\mathscr{R}_p = \sum_{r(X)=p} \mathscr{R}_X$.

(5.1) **Lemma.** *There exists a surjective homomorphism* γ: $\mathscr{A} \to \mathscr{R}$ *of graded algebras such that* $\gamma(\alpha_A) = \omega_A$ *and* $\gamma(\mathscr{A}_X) = \mathscr{R}_X$.

Proof. Define an algebra homomorphism v: $\mathscr{E} \to \mathscr{R}$ by $v(e_A) = \omega_A$. To show that $v(\mathscr{I}) = 0$ we need to show that if $S = (A_1, \ldots, A_p)$ is dependent then $v(\partial e_S) = 0$. Let $\varphi_i = \varphi_{A_i}$ and $\omega_i = \omega_{A_i}$. Since S is dependent $r(A_1 \vee \ldots \vee A_p) < p$. Thus $\operatorname{codim}(A_1 \cap \ldots \cap A_p) < p$ so the forms $\varphi_1, \ldots, \varphi_p$ are linearly dependent. Thus $v(e_S) = \omega_1 \ldots \omega_p = 0$. Let $S_k = (A_1, \ldots, \hat{A}_k, \ldots, A_p)$. If S_k is dependent for some k, it follows from (2.3) that $(-1)^k \partial e_S = e_{A_k} \partial e_{S_k} - e_{S_k}$ so we are done by induction. Thus we may assume that no proper subset of $\{\varphi_1, \ldots, \varphi_p\}$ is linearly dependent, so that there exist $c_i \in \mathbb{C}$, all nonzero, with $\sum_{i=1}^{p} c_i \varphi_i = 0$. If we replace φ_i by $c_i \varphi_i$ then ω_i is unchanged. Thus we may assume that $\sum_{i=1}^{p} \varphi_i = 0$. Suppose $j = 1, \ldots, p-1$. Then $0 = \sum_{i=1}^{p} d\varphi_i$ implies $0 = \left(\sum_{i=1}^{p} d\varphi_i\right)(d\varphi_1 \ldots \widehat{d\varphi_j}\widehat{d\varphi_{j+1}} \ldots d\varphi_p)$ so $d\varphi_1 \ldots d\hat{\varphi}_{j+1} \ldots d\varphi_p = -d\varphi_1 \ldots d\hat{\varphi}_j \ldots d\varphi_p$. Define η_j by $\varphi_j \eta_j = (-1)^{j-1} \omega_1 \ldots \hat{\omega}_j \ldots \omega_p$. Then $\varphi_1 \ldots \varphi_p \eta_{j+1} = (-1)^j d\varphi_1 \ldots d\hat{\varphi}_{j+1} \ldots d\varphi_p = (-1)^{j-1} d\varphi_1 \ldots d\hat{\varphi}_j \ldots d\varphi_p = \varphi_1 \ldots \varphi_p \eta_j$. Let η be the common value of the η_j. Then

$$\begin{aligned} v(\partial e_S) &= \sum_{i=1}^{p} (-1)^{i-1} \omega_1 \ldots \hat{\omega}_i \ldots \omega_p \\ &= \sum_{i=1}^{p} \varphi_i \eta_i = \left(\sum_{i=1}^{p} \varphi_i\right) \eta = 0. \quad \square \end{aligned}$$

(5.2) **Theorem.** *Let* $\mathbf{A}$ *be a finite set of hyperplanes in* $\mathbb{C}^\ell$. *There exists an isomorphism* $\mathscr{A} \simeq H^*(M)$ *of graded algebras such that* $\alpha_A \to [\omega_A]$ *for all* $A \in \mathbf{A}$. *In particular the Poincaré polynomial of* M *is given by*

$$P_M(t) = \sum_{X \in L} \mu(X)(-t)^{r(X)}.$$

Proof. We know from Brieskorn's work that the natural map $\mathscr{R} \to H^*(M)$ is an isomorphism sending $\omega_A \to [\omega_A]$. Thus it will suffice to show that the map $\gamma\colon \mathscr{A} \to \mathscr{R}$ is an isomorphism. Since $\dim \mathscr{A} = \sum_{X \in L} (-1)^{r(X)} \mu(X)$ it will suffice to show that

(5.3) $\dim H^*(M) = \sum_{X \in L} (-1)^{r(X)} \mu(X)$.

If $X \in L$ let $\mathbf{A}_X = \{A \in \mathbf{A} \mid A \supseteq X\}$ and let L_X be the geometric lattice consisting of $V = \mathbb{C}^\ell$ and intersections of elements of $\mathbf{A}_X$. Let $M_X = V - \bigcup_{A \in \mathbf{A}_X} A$. The inclusion $i_X\colon M \to M_X$ induces a map $i_X^*\colon H^{r(X)}(M_X) \to H^{r(X)}(M)$. Let H_X be the image of i_X^*. Brieskorn [5, Lemmas 3, 5] showed that: (i) i_X^* is a monomorphism and $H^p(M) = \bigoplus_{r(X)=p} H_X$ and (ii) if $r(X) > 0$ then the Euler characteristic $e(M_X) = 0$. Assuming (i) and (ii) we prove

(5.4) $\dim H^q(M) = (-1)^q \mu(\bigcap_{A \in \mathbf{A}} A)$

where $q = r(\bigcap_{A \in \mathbf{A}} A)$ is the rank of L. We use induction on q; the argument is similar to the one used in (2.17). If $q = 0$ then both sides of (5.4) are equal to one. If $X \in L$ and $r(X) < q$ then the induction hypothesis shows that

(5.5) $\dim H^{r(X)}(M_X) = (-1)^{r(X)} \mu(X)$.

Note that the rank and Möbius functions of L_X are the restrictions of the corresponding functions of L. Using Brieskorn's (i), (ii) and (5.5) we have

$$\begin{aligned} 0 &= \sum_{p=0}^{q} (-1)^p \dim H^p(M) \\ &= \sum_{p=0}^{q-1} (-1)^p \sum_{r(X)=p} \dim H_X + (-1)^q \dim H^q(M) \\ &= \sum_{r(X)<q} \mu(X) + (-1)^q \dim H^q(M) \\ &= -\mu(\bigcap_{A \in \mathbf{A}} A) + (-1)^q \dim H^q(M). \end{aligned}$$

This proves (5.4), and hence (5.5) for all X. Now using (i) again we obtain (5.3). □

(5.6) **Corollary.** *Let* $\mathbf{A}$ *be a finite set of hyperplanes in* $V = \mathbb{C}^\ell$. *Let* L *be the corresponding lattice and let* $M = V - \bigcup_{A \in \mathbf{A}} A$. *Then the deRham cohomology groups of* M *and the Baclawski (co)homology groups of* L *are isomorphic vector spaces.*

Proof. This follows at once from (5.2) and (3.14). □

(5.7) **Corollary.** *Let G be a finite subgroup of* $\mathbf{GL}(V)$ *which permutes a set of hyperplanes. If $g \in G$ let $L^g = \{X \in L \mid gX \subseteq X\}$ and let μ_g be the Möbius function of the poset L^g. Then*

$$\sum_{p \geq 0} \operatorname{tr}(g \mid H^p(M)) t^p = \sum_{X \in L^g} \mu_g(X)(-t)^{r(X)}$$

where r is the rank function of L. The Poincaré polynomial for the orbit space M/G is

$$P_{M/G}(t) = \frac{1}{|G|} \sum_{g \in G} \sum_{X \in L^g} \mu_g(X)(-t)^{r(X)}.$$

Proof. The first assertion follows at once from Theorem 4.8 since the isomorphism in Theorem 5.2 is G-equivariant. The second assertion follows since $H^*(M/G) \simeq H^*(M)^G$, [4, p. 120]. □

We may use (4.7) to compute the Poincaré polynomial of M/G without the trace formulas. Since each induced module $\mathbb{C}[G/G_{X,\sigma}]$ contains the trivial module with multiplicity one (4.7) gives:

(5.8) **Corollary.** *The Betti numbers of M/G are $b_0 = 1$, $b_1 = |T_1|$ and for $p = 2, \ldots, \ell$*

$$b_p = (-1)^p \sum_{X \in T_p} \sum_{\sigma \in U_X} (-1)^{d(\sigma)}.$$

(5.9) *Remark.* Let G be a finite subgroup of $\mathbf{GL}(\mathbb{R}^\ell)$ generated by reflections in hyperplanes $V_1, \ldots, V_n$. Let G act on $\mathbb{C}^\ell$ and let $A_j = \mathbb{C} \otimes V_j \subseteq \mathbb{C}^\ell$. Brieskorn [5] has computed the Betti numbers b_p explicitly in this case. Since one knows from Brieskorn and Deligne [5, 7] that M and M/G are $K(\pi, 1)$ spaces the Betti numbers b_p give the ranks of the cohomology groups of the corresponding generalized braid groups.

(5.10) *Example.* Let $\mathbf{A}$ be the set of hyperplanes in $\mathbb{C}^4$ defined by the linear forms $z_i - z_j$, $1 \leq i < j \leq 4$. The corresponding lattice L is the lattice of partitions of $\{1, 2, 3, 4\}$. Let $G = \mathrm{Sym}(4)$. In view of Corollary 5.7 the polynomials $P_{\mathscr{A}}(t, g)$ computed in (4.10) give the trace for the G-action on $H^*(M)$. According to (5.7) the Poincaré polynomial of M/G is the average over G of the polynomials $P_{\mathscr{A}}(t, g)$. This turns out to be $1 + t$, which agrees with Brieskorn. We may also compute this Poincaré polynomial directly from (5.8) as follows. Since G acts transitively on the hyperplanes we have $b_1 = 1$. Let $A_1 = \ker(z_1 - z_2)$, $A_2 = \ker(z_2 - z_3)$, $A_3 = \ker(z_3 - z_4)$ and $A_4 = \ker(z_1 - z_3)$. For $p = 2$ we may choose $T_2 = \{A_1 \cap A_2, A_1 \cap A_3\}$. For $p = 3$ there is a unique representative element so $T_3 = \{A_1 \cap A_2 \cap A_3\}$. The table

X	$\tilde{K}(L_X)$	U_X
$A_1 \cap A_2$	ϕ, A_1, A_2, A_4	ϕ, A_1
$A_1 \cap A_3$	ϕ, A_1, A_3	ϕ, A_1
$A_1 \cap A_2 \cap A_3$	$\tilde{K}(L)$	$\phi, A_1, A_1 \cap A_2, A_1 \cap A_3, (A_1, A_1 \cap A_2), (A_1, A_1 \cap A_3)$

yields $b_2 = b_3 = 0$. Here ϕ denotes the simplex of dimension -1. □

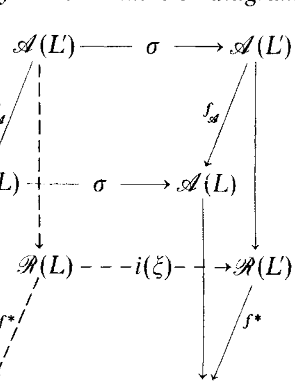
The isomorphism of Theorem 5.2 allows us to give geometric interpretations to some of our combinatorially defined maps, which we summarize in Proposition 5.11. To interpret the map $\sigma\colon \mathscr{A} \to \mathscr{A}$ consider the space $\mathscr{X}(V)$ of holomorphic vector fields on V. There is a natural map $V^* \otimes V \to \mathscr{X}(V)$ given by $h \otimes v \to hD_v$ where D_v is directional derivative in the direction v. Let $e_1, \ldots, e_\ell$ be any basis for V and let $z_1, \ldots, z_\ell$ be the dual basis for V^*. The (Casimir) element $\sum_{j=1}^{\ell} z_j \otimes e_j$ is independent of basis and thus defines a holomorphic vector field $\xi = \sum_{j=1}^{\ell} z_j D_{e_j}$, which is independent of basis. We may view $\xi \in \mathscr{X}(M)$ as a holomorphic vector field on M. Let $\Omega^p(M)$ be the space of holomorphic p-forms on M. Recall that any vector field ξ on M induces an interior multiplication $i(\xi)$: $\Omega^p(M) \to \Omega^{p-1}(M)$ defined by

$$(i(\xi)\eta)(\xi_1, \ldots, \xi_{p-1}) = \eta(\xi, \xi_1, \ldots, \xi_{p-1})$$

where $\eta \in \Omega^p(M)$ and $\xi_j \in \mathscr{X}(M)$. Define $i(\xi)^*\colon \Omega^{p-1}(M) \to \Omega^p(M)$ by $i(\xi)^*\eta = (\sum_{A \in \mathbf{A}} \omega_A)\eta$. Note that both $i(\xi)$ and $i(\xi)^*$ map $\mathscr{R} \to \mathscr{R}$.

To interpret the map $\varphi\colon \mathscr{E} \to \mathscr{A}$ geometrically we imbed M in a complex n-torus $M' = (\mathbb{C}^*)^n$ as follows. Write $\mathbf{A} = \{A_1, \ldots, A_n\}$ in some chosen order. Let $V' = \mathbb{C}^n$. Define a $\mathbb{C}$-linear map $f\colon V \to V'$ by $f(v) = (\varphi_1(v), \ldots, \varphi_n(v))$, where $\varphi_i = \varphi_{A_i}$. Note that $f\colon M \to M'$. Let $z'_1, \ldots, z'_n$ be the coordinate functions in V' and let $A'_i = \ker z'_i$. Since M' is the complement of the set $\mathbf{A}' = \{A'_1, \ldots, A'_n\}$ of coordinate hyperplanes in $\mathbb{C}^n$, all our constructions may be applied to it. Let L' be the corresponding lattice. Since L' is a Boolean algebra the map $f\colon V \to V'$ induces a map, again called $f\colon L' \to L$, defined by $f(A'_i) = A_i$. This map satisfies (2.4). In view of (2.5) $\varphi\colon \mathscr{E}(L) \to \mathscr{A}(L)$ may be identified with $f_{\mathscr{A}}\colon \mathscr{A}(L') \to \mathscr{A}(L)$.

(5.11) **Proposition.** *The connection between the combinatorics and the topology is given by the commutative diagram:*



where the vertical maps are γ and f^* is the induced map on differential forms. There is a similar diagram with σ, $i(\xi)$ replaced by σ^*, $i(\xi)^*$ and we have $i(\xi)\,i(\xi)^* + i(\xi)^*\,i(\xi) = n \cdot id_{\mathscr{R}}$.

Proof. The top and bottom faces commute by functoriality. Let $A_1, \ldots, A_p \in \mathbf{A}$ and let $\alpha_j = \alpha_{A_j}$. Since $\partial\colon \mathscr{E} \to \mathscr{E}$ induces $\sigma\colon \mathscr{A} \to \mathscr{A}$, we have

$$\sigma(\alpha_1 \ldots \alpha_p) = \sum_{k=1}^{p} (-1)^{k-1} \alpha_1 \ldots \hat{\alpha}_k \ldots \alpha_p$$

and there is a corresponding formula for the map $\gamma\sigma\gamma^{-1}$ where α_j is replaced by $\omega_j = \omega_{A_j} = d\varphi_j/2\pi i\varphi_j$. To show that the front face is commutative it suffices to check the equality $i(\xi)\eta = \gamma\sigma\gamma^{-1}\eta$ on a set of elements η which span $\mathscr{R}(L)$. Thus take $\eta = \omega_1 \ldots \omega_p$ where $\varphi_1, \ldots, \varphi_p$ are independent. We must prove that

$$(5.12) \quad (\omega_1 \ldots \omega_p)(\xi, \xi_1, \ldots, \xi_{p-1}) = \sum_{k=1}^{p} (-1)^{k-1} (\omega_1 \ldots \hat{\omega}_k \ldots \omega_p)(\xi_1, \ldots, \xi_{p-1})$$

for all $\xi_j \in \mathscr{X}(M)$. Let $D_j = D_{e_j}$. We may assume $(\xi_1, \ldots, \xi_{p-1}) = (D_{j_1}, \ldots, D_{j_{p-1}})$ where $j_1 < \ldots < j_{p-1}$. Since

$$(dz_1 \wedge \ldots \wedge dz_p)(D_k, D_{j_1}, \ldots, D_{j_{p-1}}) = (-1)^{k-1}$$

if $(j_1, \ldots, j_{p-1}) = (1, \ldots, \hat{k}, \ldots, p)$ and is zero otherwise, both sides of (5.12) are equal to $(-1)^{k-1}(z_1 \ldots \hat{z}_k \ldots z_p)^{-1}$. This proves the commutativity of the front face, and hence the back face. Since $f^*(z'_j) = \varphi_j$ we have $f^*(\omega'_j) = \omega_j$ where $\omega'_j = dz'_j/2\pi i z'_j$. This proves the commutativity of the two sides. The last assertion follows from the formula $\sigma\sigma^* + \sigma^*\sigma = n \cdot \mathrm{id}_{\mathscr{A}}$. □

Define $f\colon V \to \mathbb{C}$ by $f = \prod_{A \in \mathbf{A}} \varphi_A$. Then f has a critical point at the origin which is not in general isolated. It follows from Milnor's fibration theorem [12, p. 5] that the map $f\colon M \to \mathbb{C}^*$ is the projection map of a smooth fiber bundle. Thus our computations yield the cohomology of the total space of this fibration but they do not yield the cohomology of the Milnor fiber.

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Received March 20, 1979