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A Simple Proof of the Hook Length Formula

Kenneth Glass and Chi-Keung Ng

In this note, we give a simple and direct proof for the "Hook Length Formula." The simplicity of our proof relies on the usage of the residue theorem as a short cut.

The number of standard Young tableaux for a given Ferrers diagram λ is the dimension of the irreducible representation of the symmetric group corresponding to λ. The hook length formula (which was first proved in **[1]**) is a method for calculating this number and is a surprisingly beautiful formula (because the problem looks complex but the formula looks naive). Let us begin by giving the precise definitions of Ferrers diagrams, standard Young tableaux, and the hook length formula.

Suppose that N and m are positive integers and that $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a sequence in $\mathbb{N} \cup \{0\}$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ and $\sum_{i=1}^{m} \lambda_i = N$. We can think of this as an array of boxes in which the number of boxes in the top row is λ_1, the number of boxes in the second row is λ_2, and so on. Such a diagram is called a *Ferrers diagram*. For example, in the case when $N = 12, m = 4$, and $\lambda = (5, 3, 3, 1)$, we have the following picture:

$$\begin{array}{ccccc} \square & \square & \square & \square & \square \\ \square & \square & \square & & \\ \square & \square & \square & & \\ \square & & & & \end{array}$$

Given such a Ferrers diagram, a *standard Young tableau* is a bijection $T : [\lambda] \to \{1, \ldots, N\}$ (where $[\lambda] = \{(i, j) : 1 \leq i \leq m, 1 \leq j \leq \lambda_i\}$) such that $T(i, j) \leq T(i', j')$ if $i \leq i'$ and $j \leq j'$. In terms of a picture, we can think of a standard Young tableau as a distribution of $\{1, 2, \ldots, N\}$ into the boxes of the Ferrers diagram such that the numbers in each row are increasing from left to right and the numbers in each column are increasing from top to bottom. The following is an example of a standard Young tableau for the Ferrers diagram $(5, 3, 3, 1)$:

$$\begin{array}{ccccc} 1 & 3 & 4 & 6 & 12 \\ 2 & 5 & 9 & & \\ 7 & 8 & 11 & & \\ 10 & & & & \end{array}$$

An interesting question is: *How many standard Young tableaux are there for a given Ferrers diagram?* As noted, the hook length formula is an answer to this question. Suppose that $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a Ferrers diagram. For any (i, j) in $[\lambda]$, we let $\lambda'_j = \max\{k : j \leq \lambda_k\}$ (i.e., λ'_j is the length of the jth column (from the left)). The *hook length* for the (i, j)-node (i.e., the box in the ith row and jth column) is the number $\lambda'_j - i + \lambda_i - j + 1$ and will be denoted by h_{ij}. In pictorial terms, the hook length for the (i, j)-node is 1 plus the number of boxes below and the number of boxes to the

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right of that node (that's why these sums are called hook lengths). So, for the Ferrers diagram (5, 3, 3, 1) we have $h_{22} = 3$:

$$\begin{array}{ccccc} \square & \square & \square & \square & \square \\ \square & * & * & & \\ \square & * & \square & & \\ \square & & & & \end{array}$$

and $h_{21} = 5$:

$$\begin{array}{ccccc} \square & \square & \square & \square & \square \\ * & * & * & & \\ * & \square & \square & & \\ * & & & & \end{array}$$

For any Ferrers diagram $\lambda = (\lambda_1, \dots, \lambda_m)$, we let $f^{\lambda} = f^{(\lambda_1, \dots, \lambda_m)}$ be the number of standard Young tableaux for λ (note that we use the convention that $f^{(0,0,\dots,0)} = 1$). The *hook length formula* states:

$$f^{\lambda} = \frac{N!}{\prod_{(i,j)\in[\lambda]} h_{ij}}. \tag{1}$$

As an example, for $\lambda = (5, 3, 3, 1)$, we have $f^{\lambda} = 4158$.

The function $\lambda \mapsto f^{\lambda}$ satisfies a recursion relation. In order to present this relation clearly, we first establish the following notation. Let

$$\Lambda_0 := \left\{(\lambda_1, \dots, \lambda_m) : \lambda_i \in \mathbb{N} \cup \{0\} \text{ for } i = 1, 2, \dots, m \text{ and } \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m\right\}$$

and

$$\Lambda := \Lambda_0 \cup \bigcup_{k=1}^{m} \left\{(\lambda_1, \dots, \lambda_{k-1}, \lambda_k - 1, \lambda_{k+1}, \dots, \lambda_m) : (\lambda_1, \dots, \lambda_m) \in \Lambda_0\right\}.$$

In other words, $(\lambda_1, \dots, \lambda_m)$ belongs to $\Lambda \setminus \Lambda_0$ if and only if $\lambda_i \geq \lambda_{i+1}$ (we put $\lambda_{m+1} = 0$) except for one i in $\{1, \dots, m\}$ where $\lambda_i = \lambda_{i+1} - 1$ (in this case λ_i could be -1). For any $\lambda = (\lambda_1, \dots, \lambda_m)$ in $\Lambda \setminus \Lambda_0$, we set $f^{\lambda} = 0$.

Now we can explain the recursion relation. For any standard Young tableau T for a Ferrers diagram λ it is clear that there exists i satisfying $1 \leq i \leq m$ such that $T(i, \lambda_i) = N$. Therefore, we see that:

(i) $f^{(\lambda_1,\dots,\lambda_m)} = 0$ for any $(\lambda_1, \dots, \lambda_m)$ in $\Lambda \setminus \Lambda_0$;

(ii) $f^{(0,0,\dots,0,0)} = 1$;

(iii) $f^{(\lambda_1,\dots,\lambda_m)} = \sum_{k=1}^{m} f^{(\lambda_1,\dots,\lambda_{k-1},\lambda_k-1,\lambda_{k+1},\dots,\lambda_m)}$ for any $(\lambda_1, \dots, \lambda_m)$ in Λ_0 with $\lambda_1 \geq 1$.

It is easily seen that these conditions uniquely determine $f^{(\lambda_1,\dots,\lambda_m)}$. Using this fact, we can establish the following formula:

Theorem 1. *Let* $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$. *If* $N = \sum_{k=1}^{m} \lambda_k$, *then*

$$f^{(\lambda_1,\dots,\lambda_m)} = \frac{N! \prod_{j>i}(\lambda_i - \lambda_j + j - i)}{\prod_{l=0}^{m-1}(\lambda_{m-l} + l)!}. \tag{2}$$

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Proof. For any $(\lambda_1, \dots, \lambda_m)$ in Λ, we let

$$\langle \lambda_1 : \cdots : \lambda_m \rangle := \begin{cases} \dfrac{(\sum_{k=1}^m \lambda_k)! \prod_{j>i}(\lambda_i - \lambda_j + j - i)}{\prod_{l=0}^{m-1}(\lambda_{m-l} + l)!} & \text{if } \lambda_k \neq -1 \text{ for all } k, \\ 0 & \text{if } \lambda_k = -1 \text{ for some } k \end{cases}$$

(here we use the convention that empty products equal one). It is clear that $\langle 0 : 0 : \cdots : 0 : 0 \rangle = 1$. Moreover, if $\lambda_i = \lambda_{i+1} - 1$ for some i with $1 \le i \le m-1$, then $\langle \lambda_1 : \cdots : \lambda_m \rangle = 0$ because $\lambda_i - \lambda_{i+1} + i - (i-1) = 0$. Therefore, in order to show $f^{(\lambda_1, \dots, \lambda_m)} = \langle \lambda_1 : \cdots : \lambda_m \rangle$ for any $(\lambda_1, \dots, \lambda_m)$ in Λ_0, we need to prove only that, if $(\lambda_1, \dots, \lambda_m)$ belongs to Λ_0 with $\lambda_1 \ge 1$, then

$$\langle \lambda_1 : \cdots : \lambda_m \rangle = \sum_{k=1}^m \langle \lambda_1 : \cdots : \lambda_{k-1} : \lambda_k - 1 : \lambda_{k+1} : \cdots : \lambda_m \rangle. \tag{3}$$

Suppose that $(\lambda_1, \dots, \lambda_m)$ in Λ_0 has $\lambda_1 \ge 1$. Notice first of all that

$$\begin{aligned} &\langle \lambda_1 : \cdots : \lambda_{m-1} : 0 \rangle \\ &= \frac{(\sum_{k=1}^{m-1} \lambda_k)! \prod_{m-1 \ge j > i \ge 1}(\lambda_i - \lambda_j + j - i) \prod_{p=1}^{m-1}(\lambda_p + m - p)}{\prod_{l=1}^{m-1}(\lambda_{m-l} + l)!} \\ &= \frac{(\sum_{k=1}^{m-1} \lambda_k)! \prod_{m-1 \ge j > i \ge 1}(\lambda_i - \lambda_j + j - i) \prod_{l=0}^{m-2}(\lambda_{(m-1)-l} + l + 1)}{\prod_{l=0}^{m-2}(\lambda_{(m-1)-l} + l + 1)!} \\ &= \langle \lambda_1 : \cdots : \lambda_{m-1} \rangle. \end{aligned}$$

Thus, we can assume that $\lambda_m \ge 1$ as well. Observe also that

$$\begin{aligned} &\langle \lambda_1 : \cdots : \lambda_{k-1} : \lambda_k - 1 : \lambda_{k+1} : \cdots : \lambda_m \rangle \\ &= \frac{((\sum_{k=1}^m \lambda_k) - 1)! \prod_{k \neq j > i \neq k}(\lambda_i - \lambda_j + j - i) \prod_{j>k}(\lambda_k - \lambda_j + j - k - 1) \prod_{i<k}(\lambda_i - \lambda_k + k - i + 1)}{(\lambda_k + m - k - 1)! \prod_{l \neq k}(\lambda_{m-l} + l)!} \\ &= \langle \lambda_1 : \cdots : \lambda_m \rangle \frac{\lambda_k + m - k}{\sum_{l=1}^m \lambda_l} \prod_{j>k} \frac{\lambda_k - \lambda_j + j - k - 1}{\lambda_k - \lambda_j + j - k} \prod_{i<k} \frac{\lambda_i - \lambda_k + k - i + 1}{\lambda_i - \lambda_k + k - i}. \end{aligned}$$

Therefore, in order to prove equation (3) it suffices to establish the following equation:

$$\sum_{l=1}^m \lambda_l = \sum_{k=1}^m (\lambda_k + m - k) \prod_{j>k} \frac{\lambda_k - \lambda_j + j - k - 1}{\lambda_k - \lambda_j + j - k} \prod_{i<k} \frac{\lambda_i - \lambda_k + k - i + 1}{\lambda_i - \lambda_k + k - i}. \tag{4}$$

This can be obtained by putting $z_i = \lambda_i + m - i$ in the following lemma. ■

Lemma 2. *For any distinct points $z_1, \dots, z_m$ in $\mathbb{C}$,*

$$\left(\sum_{l=1}^m z_l \right) - \frac{m(m-1)}{2} = \sum_{k=1}^m z_k \prod_{j \neq k} \left(1 + \frac{1}{z_j - z_k} \right)$$

Proof. Consider the function

$$f(z) = z \prod_{i=1}^m \left(1 + \frac{1}{z_i - z} \right).$$

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It is clear that

$$\mathrm{Res}(f, z_k) = -z_k \prod_{j \neq k} (1 + \frac{1}{z_j - z_k}).$$

Let $g(z) = f(1/z)/z^2$. Then

$$g(z) = z^{-3} \prod_{i=1}^{m} (1 - z - z_i z^2 + \text{“higher order terms”})$$

(if $|z_i z| < 1$ for $i = 1, \ldots, m$). Hence $\mathrm{Res}(f, \infty) = -\mathrm{Res}(g, 0) = (\sum_{l=1}^{m} z_l) - m(m-1)/2$. Since $\mathrm{Res}(f, \infty) = -\sum_{k=1}^{m} \mathrm{Res}(f, z_k)$, the result follows. ■

It is not hard to see that equation (2) is the same as equation (1). More precisely, for any Ferrers diagram $\lambda = (\lambda_1, \ldots, \lambda_m)$ it is easily seen that the product of the hook lengths of the boxes in the kth row is equal to

$$\begin{aligned}[1 \cdot 2 \cdots (\lambda_k - \lambda_{k+1})] &\cdot \big[(\lambda_k - \lambda_{k+1} + 2) \cdots (\lambda_k - \lambda_{k+2} + 1)\big] \\ &\cdot \big[(\lambda_k - \lambda_{k+2} + 3) \cdots (\lambda_k - \lambda_{k+3} + 2)\big] \\ &\cdots \big[(\lambda_k - \lambda_m + m - k + 1) \cdots (\lambda_k + m - k)\big]\end{aligned}$$

(starting from the right-hand end of the row), which is exactly

$$\frac{(\lambda_k + m - k)!}{\prod_{l>k}(\lambda_k - \lambda_l - k + l)}.$$

Thus, we obtain the hook length formula.

Corollary 3. *For a Ferrers diagram $\lambda = (\lambda_1, \ldots, \lambda_m)$ with $\sum_{i=1}^{m} \lambda_i = N$, the number of standard Young tableaux for λ is*

$$\frac{N!}{\prod_{(i,j)\in[\lambda]} h_{ij}}$$

(where h_{ij} is the hook length of the (i, j)-node).

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Playing Catch-Up with Iterated Exponentials

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1. INTRODUCTION. Suppose that we have two animals that make the same number of strides per minute, but that one of them takes larger strides than the other. If the strides of the smaller animal (the prey) have length a and those of the larger animal (the predator) have length b, it is easy to see that a persistent predator will always be able to catch up with its prey. Let us assume that the prey starts one step ahead of the predator. After n steps the distance between the two is

$$nb - (n+1)a = n(b-a) - a$$

and consequently, if $n > a/(b-a)$, the predator will have overtaken its prey.

Let us now imagine a planet on which creatures move by jumps of increasing length. A creature on such a planet is at a distance a from where it started after one jump, a distance a^2 after two jumps, and a distance a^n after n jumps. Let us also assume that $a > 1$ so that creatures move away from their starting points. We can again ask whether a small creature that starts one step ahead of a predator can escape from it. We assume that the initial step of the predator is of size $b > a > 1$, so that if $b^n > a^{n+1}$, the smaller creature is in the maw (or the extraterrestrial equivalent) of its predator. A simple calculation shows that this happens if the predator is sufficiently persistent to make n jumps, where

$$n > (\log a)\left(\log \frac{b}{a}\right)^{-1}.$$

Of course, we can imagine an even stranger planet on which a creature makes an initial jump of size a, followed by a jump that moves it at distance a^a from its starting place, and another that brings it to a distance a^{a^a}, and so on. Thus the distance that such creatures travel is determined by *towers of a*.

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