Recounting the Odds of an Even Derangement

ARTHUR T. BENJAMIN
Harvey Mudd College
Claremont, CA 91711
benjamin@hmc.edu

CURTIS T. BENNETT
Loyola Marymount University
Los Angeles, CA 90045
cbennett@lmu.edu

FLORENCE NEWBERGER
California State University
Long Beach, CA 90840-1001
fnewberg@csulb.edu

Odd as it may sound, when n exams are randomly returned to n students, the probability that no student receives his or her own exam is almost exactly $1/e$ (approximately 0.368), for all $n \geq 4$. We call a permutation with no fixed points, a *derangement*, and we let $D(n)$ denote the number of derangements of n elements. For $n \geq 1$, it can be shown that $D(n) = \sum_{k=0}^{n} (-1)^k n!/k!$, and hence the *odds* that a random permutation of n elements has no fixed points is $D(n)/n!$, which is within $1/(n+1)!$ of $1/e$ [**1**].

Permutations come in two varieties: even and odd. A permutation is even if it can be achieved by making an even number of swaps; otherwise it is odd. Thus, one might *even* be interested to know that if we let $E(n)$ and $O(n)$ respectively denote the number of even and odd derangements of n elements, then (oddly enough),

$$E(n) = \frac{D(n) + (n-1)(-1)^{n-1}}{2}$$

and

$$O(n) = \frac{D(n) - (n-1)(-1)^{n-1}}{2}.$$

The above formulas are an immediate consequence of the equation $E(n) + O(n) = D(n)$, which is obvious, and the following theorem, which is the focus of this note.

THEOREM. *For $n \geq 1$,*

$$E(n) - O(n) = (-1)^{n-1}(n-1). \tag{1}$$

Proof 1: Determining a Determinant The fastest way to derive equation (1), as is done in [**3**], is to compute a determinant. Recall that an n-by-n matrix $A = [a_{ij}]_{i,j=1}^{n}$ has determinant

$$\det(A) = \sum_{\pi \in S_n} a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)} \operatorname{sgn}(\pi), \tag{2}$$

where S_n is the set of all permutations of $\{1, \ldots, n\}$, $\operatorname{sgn}(\pi) = 1$ when π is even, and $\operatorname{sgn}(\pi) = -1$ when π is odd. Let A_n denote the n-by-n matrix whose nondiagonal entries are $a_{ij} = 1$ (for $i \neq j$), with zeroes on the diagonal. For example, when $n = 4$,

$$A_4 = J_4 - I_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

By (2), every permutation that is not a derangement will contribute 0 to the sum (since it uses at least one of the diagonal entries), every even derangement will contribute 1 to the sum, and every odd derangement will contribute -1 to the sum. Consequently, $\det(A_n) = E(n) - O(n)$. To see that $\det(A_n) = (-1)^{n-1}(n-1)$, observe that $A_n = J_n - I_n$, where J_n is the matrix of all ones and I_n is the identity matrix. Since J_n has rank one, zero is an eigenvalue of J_n, with multiplicity $n-1$, and its other eigenvalue is n (with an eigenvector of all 1s). Apply $J_n - I_n$ to the eigenvectors of J_n to find the eigenvalues of A_n: -1 with multiplicity $n-1$ and $n-1$ with multiplicity 1. Multiplying the eigenvalues gives us $\det(A_n) = (-1)^{n-1}(n-1)$, as desired. ■

A 1996 Note in the MAGAZINE [**2**] gave *even odder* ways to determine the determinant of A_n.

Although the proof by determinants is quick, the form of (1) suggests that there should also exist an *almost* one-to-one correspondence between the set of even derangements and the set of odd derangements.

Proof 2: Involving an Involution Let D_n denote the set of derangements of $\{1, \ldots, n\}$, and let X_n be a set of $n-1$ *exceptional* derangements (that we specify later), each with sign $(-1)^{n-1}$. We exhibit a *sign reversing involution* on $D_n - X_n$. That is, letting $T_n = D_n - X_n$, we find an invertible function $f : T_n \to T_n$ such that for π in T_n, π and $f(\pi)$ have opposite signs, and $f(f(\pi)) = \pi$. In other words, except for the $n-1$ exceptional derangements, every even derangement "holds hands" with an odd derangement, and vice versa. From this, it immediately follows that $|E_n| - |O_n| = (-1)^{n-1}(n-1)$.

Before describing f, we establish some notation. We express each π in D_n as the product of k disjoint cycles $C_1, \ldots, C_k$ with respective lengths $m_1, \ldots, m_k$ for some

$k \geq 1$. We follow the convention that each cycle begins with its smallest element, and the cycles are listed from left to right in increasing order of the first element. In particular, $C_1 = (1\ a_2\ \cdots\ a_{m_1})$ and, if $k \geq 2$, C_2 begins with the smallest element that does not appear in C_1. Since π is a derangement on n elements, we must have $m_i \geq 2$ for all i, and $\sum_{i=1}^{k} m_i = n$. Finally, since a cycle of length m has sign $(-1)^{m-1}$, it follows that π has sign $(-1)^{\sum_{i=1}^{k}(m_i-1)} = (-1)^{n-k}$.

Let π be a derangement in D_n with first cycle $C_1 = (1\ a_2\ \cdots\ a_m)$ for some $m \geq 2$. We say that π has *extraction point* $e \geq 2$ if e is the smallest number in the set $\{2, \ldots, n\} - \{a_2\}$ for which C_1 does *not* end with the numbers of $\{2, \ldots, e\} - \{a_2\}$ written in decreasing order. Note that π will have extraction point $e = 2$ unless the number 2 appears as the second term or last term of C_1. We illustrate this definition with some pairs of examples from D_9. Notice that in each pair below, the number of cycles of π and π' differ by one, and the extraction point e occurs in the first cycle of π and is the leading element of the second cycle of π'.

$\pi = (1\,9\,7\,2\,8)(3\,6)(4\,5)$	and	$\pi' = (1\,9\,7)(2\,8)(3\,6)(4\,5)$	have $e = 2$.
$\pi = (1\,2\,9\,7\,3\,8\,5)(4\,6)$	and	$\pi' = (1\,2\,9\,7)(3\,8\,5)(4\,6)$	have $e = 3$.
$\pi = (1\,9\,7\,3\,8\,5\,2)(4\,6)$	and	$\pi' = (1\,9\,7\,2)(3\,8\,5)(4\,6)$	have $e = 3$.
$\pi = (1\,9\,4\,8\,5\,3\,2)(6\,7)$	and	$\pi' = (1\,9\,3\,2)(4\,8\,5)(6\,7)$	have $e = 4$.
$\pi = (1\,4\,9\,5\,8\,3\,2)(6\,7)$	and	$\pi' = (1\,4\,9\,3\,2)(5\,8)(6\,7)$	have $e = 5$.
$\pi = (1\,3\,8\,6\,9\,7\,5\,4\,2)$	and	$\pi' = (1\,3\,8\,5\,4\,2)(6\,9\,7)$	have $e = 6$.

Observe that every derangement π in D_n contains an extraction point unless π consists of a single cycle of the form $\pi = (1\ a_2\ Z)$, where Z is the ordered set $\{2, 3, \ldots, n-1, n\} - \{a_2\}$, written in decreasing order. For example, the 9-element derangement $(1\ 5\ 9\ 8\ 7\ 6\ 4\ 3\ 2)$ has no extraction point. Since a_2 can be any element of $\{2, \ldots, n\}$, there are exactly $n-1$ derangements of this type, all of which have sign $(-1)^{n-1}$. We let X_n denote the set of derangements of this form. Our problem reduces to finding a sign reversing involution f over $T_n = D_n - X_n$.

Suppose π in T_n has extraction point e. Then the first cycle C_1 of π ends with the (possibly empty) ordered subset Z consisting of the elements of $\{2, \ldots, e-1\} - \{a_2\}$ written in decreasing order. Our sign reversing involution $f : T_n \to T_n$ can then be succinctly described as follows:

$$(1\ a_2\ X\ e\ Y\ Z)\sigma \overset{f}{\longleftrightarrow} (1\ a_2\ X\ Z)(e\ Y)\sigma, \tag{3}$$

where X and Y are ordered subsets, Y is nonempty, and σ is the rest of the derangement π.

Notice that since the number of cycles of π and $f(\pi)$ differ by one, they must be of opposite signs. The derangements on the left side of (3) are those for which the extraction point e is in the first cycle. In this case, Y must be nonempty, since otherwise "$e\ Z$" would be a longer decreasing sequence and e would not be the extraction point. The derangements on the right side of (3) are those for which the extraction point e is not in the first cycle (and must therefore be the leading element of the second cycle). In this case, Y is nonempty since π is a derangement. Thus for any derangement π, the derangement $f(\pi)$ is also written in standard form, with the same extraction point e and with the same associated ordered subset Z. Another way to see that π and $f(\pi)$ have opposite signs is to notice that $f(\pi) = (xy)\pi$ (multiplying from left to right), where x is the last element of X ($x = a_2$ when X is empty), and y is the last element of Y. Either way, $f(f(\pi)) = \pi$, and f is a well-defined, sign-reversing involution, as desired. ■

In summary, we have shown combinatorially that for all values of n, there are almost as many even derangements as odd derangements of n elements. Or to put it another way, when randomly choosing a derangement with at least five elements, the *odds* of having an even derangement are nearly *even*.

Acknowledgment. We are indebted to Don Rawlings for bringing this problem to our attention and we thank Magnhild Lien, Will Murray, and the referees for many helpful ideas.

REFERENCES

1. R. A. Brualdi, *Introductory Combinatorics*, 3rd ed., Prentice-Hall, New Jersey, 1999.
2. C. A. McCarthy and A. T. Benjamin, Determinants of the tournaments, this MAGAZINE, **69** (1996), 133–135.
3. C. D. Olds, Odd and even derangements, Solution E907, *Amer. Math. Monthly*, **57** (1950), 687–688.