ON THE REPRESENTATIONS OF THE SYMMETRIC GROUP.*

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Introduction. In the study of the irreducible representations of the symmetric group two methods are available. The *first* is an application of the Frobenius-Schur theory of the characters which is valid for any group, the *second* is the 'substitutional analysis' of Young. Neither of these methods tells the whole story, and they should be used in conjunction.[1]

Here we propose to show that the two problems dealt with by Murnaghan [2] in his paper "On the Representations of the Symmetric Group," lend themselves to a treatment by Young's methods. An answer to the first problem may be obtained from a formula given by Young; [3] we shall clarify this somewhat embodying the result in the rule Y or Y'. Littlewood and Richardson [4] have given a theorem involving a rule LR which, if accompanied by a satisfactory proof, would provide an answer to Murnaghan's second problem. We propose to supply a proof, basing it directly on Y. The chief advantage of these methods is in the simplicity of the final expression of the result. It is unnecessary to refer to any tables, and the irreducible components appear explicitly,—no cancellation is necessary. On the other hand if we are concerned with the characters [5] the Frobenius-Schur theory is essential. In the last section of the paper we give illustrations of the application of these rules.

I must express my thanks to Mr. D. E. Littlewood for suggestions which led to the revision of the original draft of § 5 on lattice permutations. A specific acknowledgment is made in the text.

1. The product and power representations of the full linear group. The theory of the representations of the symmetric group S_n on n symbols is very closely associated with that of the rational representations of the full linear group L, whose degree we shall take to be l. There is an infinity of such irreducible representations but those of *order* n are to found amongst the irreducible components of the Kronecker product

$$\Pi_n(L) = L \times L \times L \times \cdots n \text{ factors.} \tag{1.1}$$

* Received February 4, 1938; Revised April 18, 1938.

[1] [20], chapter V.

[2] [5], p. 469 and p. 478; cf. also [8].

[3] [21], Part VI, p. 199.

[4] [3], p. 119.

[5] [6].

The degree of $\Pi_n(L)$, known as the *product* representation, is l^n. A very elegant reduction of $\Pi_n(L)$ has been given by Schur,[6] who shows that with every irreducible representation [7] (λ) of S_n of degree f_λ is associated an irreducible representation $T^{(\lambda)}(L)$, or as we shall write $\{\lambda\}$ of L. This reduction is accomplished by constructing matrices of degree l^n permutable with $\Pi_n(L)$, which interchange the n sets of variables [8] according to the permutations of S_n. These $n!$ matrices yield a representation of S_n, and from Schur's Lemma it follows that $\{\lambda\}$ appears in $\Pi_n(L)$ with multiplicity f_λ.

If we suppose that all the factors in (1.1) operate on the same set of variables the resulting representation is known as the n-th *power* representation of L and denoted $P_n(L)$. Corresponding to a given partition $(\alpha_1, \alpha_2, \cdots, \alpha_\nu)$ or (α) of n where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_\nu$, we may construct the representation

$$P_{\alpha_1}(L) \times P_{\alpha_2}(L) \times \cdots \times P_{\alpha_\nu}(L).$$

Let us denote by P_α the sub-group of S_n of order $\alpha_1!\,\alpha_2!\cdots\alpha_\nu!$, which is the direct product of the sub-groups S_{α_1} on the first α_1 symbols, S_{α_2} on the next α_2 symbols, and so on. This sub-group gives rise to a permutation representation [9] of S_n of degree

$$\frac{n!}{\alpha_1!\,\alpha_2!\cdots\alpha_\nu!} = \binom{n}{\alpha},$$

which we may denote $\Delta(\alpha)$, extending the notation to write

$$\Delta_{(\alpha)}(L) = P_{\alpha_1}(L) \times P_{\alpha_2}(L) \times \cdots \times P_{\alpha_\nu}(L). \tag{1.2}$$

In particular $\Pi_n(L) = \Delta_{(1^n)}(L)$. Evidently $\Delta_{(n)}(L) = P_n(L) = \{n\}$, so that we may write (1.2) in the form

$$\Delta_{(\alpha)}(L) = \{\alpha_1\} \times \{\alpha_2\} \times \cdots \times \{\alpha_\nu\}. \tag{1.3}$$

Clearly also we may write

$$\Delta_{(\alpha)}(L) = \Delta_{(\beta)}(L) \times \Delta_{(\gamma)}(L), \tag{1.4}$$

where the numbers $\beta_1, \beta_2, \cdots, \beta_\lambda;\ \gamma_1, \gamma_2, \cdots, \gamma_\mu$ are the α's possibly rearranged, so that $\beta_i \geq \beta_{i+1}$ and $\gamma_j \geq \gamma_{j+1}$ for all i, j, $\lambda + \mu = \nu$, $\sum_{i=1}^{\lambda} \beta_i = l$, $\sum_{j=1}^{\mu} \gamma_j = m$, and $l + m = n$.

[6] [13], §§ 1 and 2.

[7] No confusion will result if we use the same symbol (λ) to denote the corresponding conjugate set of S_n.

[8] Cf. [19] and [1].

[9] [10].

By identifying the characters[10] *it follows that* $\{\lambda\}$ *appears in* $\Delta_{(\alpha)}(L)$ *with the same multiplicity as does* (λ) *in* $\Delta(\alpha)$.

2. A formula of Young applied to the reduction of $\Delta(\alpha)$. A method for determining the irreducible components of $\Delta(\alpha)$, or of $\Delta_{(\alpha)}(L)$, has been given by Murnaghan.[11] We shall make use of a formula[3] of Young's which is applicable to the situation:

$$\frac{1}{n!}\binom{n}{\alpha}\Gamma P_\alpha = \Sigma[\Pi S_{rs}^{\lambda_{rs}}]\frac{n!}{f_\alpha}T_\alpha. \tag{2.1}$$

It is not necessary here to go into any detailed account of Young's analysis, we shall merely give enough to explain the symbols involved. Corresponding to a conjugate set (α) of S_n we may construct a *tableau*

$$[\alpha]: \begin{matrix} a_{11}\ a_{12} & \cdots & a_{1\alpha_1} \\ a_{21}\ a_{22} & \cdots & a_{2\alpha_2} \\ \cdot & \cdots & \\ a_{\nu 1}\ a_{\nu 2} & \cdot & a_{\nu\alpha_\nu} \end{matrix}$$

where, as before, $\alpha_i \geq \alpha_{i+1}$ for all i and $\sum_{i=1}^{\nu}\alpha_i = n$. From the rows of $[\alpha]$ we construct substitutional expressions $\{a_{i1}\, a_{i2} \cdots a_{i\alpha_i}\}$ representing the sum of all the operations of S_{α_i}; multiplying these together we obtain

$$P_\alpha = \{a_{11}\, a_{12} \cdots a_{1\alpha_1}\}\{a_{21}\, a_{22} \cdots a_{2\alpha_2}\} \cdots \{a_{\nu 1} \cdots a_{\nu\alpha_\nu}\}.$$

The brackets $\{\ \}$ and their product P_α as well as other expressions N_α, T_α which we shall form are specially chosen members of the group-ring to which S_n gives rise. The members of this group-ring may be thought of as operators but we shall not stress this interpretation. The relation with our former P_α is so close we need not distinguish between them.

Similarly from the j-th column of $[\alpha]$ we construct $\{a_{1j}\, a_{2j} \cdots\}'$, where now every odd permutation has coefficient -1. Such an expression Young calls a 'negative symmetric group,' and from the columns we obtain

$$N_\alpha = \{a_{11}\, a_{21} \cdots a_{\nu 1}\}'\{a_{12}\, a_{22} \cdots\}' \cdots \{a_{1\alpha_1} \cdots\}'.$$

Thus with any given arrangement of the n letters in the tableau form $[\alpha]$ we may associate the product $P_\alpha N_\alpha$, and denoting summation over all possible $n!$ arrangements by Γ we may write

[10] [13]; [5], pp. 444-448; and [8], p. 45. We use $\Delta(\alpha)$ with the same meaning as does Murnaghan, while our $\{\lambda\}$ has a different significance.

[11] [5]. The results are tabulated up to $n = 9$.

$$T_\alpha = \left(\frac{f_\alpha}{n!}\right)^2 \Gamma P_\alpha N_\alpha .$$

We may define the tableau $[\alpha]$ as *standard* if the letters in each row and column appear in the order of some given sequence. It can be shown that just f_α of the $n!$ are standard, where

$$f_\alpha = n!\,\frac{\prod\limits_{r,s}(\alpha_r - \alpha_s - r + s)}{\prod\limits_{r}(\alpha_r + \nu - r)!},$$

and that T_α may be expressed in terms of them only:

$$T_\alpha = \frac{f_\alpha}{n!}\,(P_1 N'_1 + P_2 N'_2 + \cdots + P_{f_\alpha} N'_{f_\alpha}),$$

where $N'_r = N_r M_r$. The introduction of the multiplicative factor M_r is necessary to obtain orthogonality, but we need not go into this. For us the important fact is that corresponding to each conjugate set (α) we have a tableau $[\alpha]$ and a resulting T_α which leads directly to the irreducible representation [12] of S_n.

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[12] It may be worth while at this point to relate some recent work by Specht [14] and [16] with this analysis of Young. Following Schur [12], if we replace the symbols a_{ij}, in dictionary order, by x_i $(i = 1, 2, \cdots, n)$, we may uniquely associate with each tableau $[\alpha]$ the product of powers

$$\delta(x) = (x_1 x_2 \cdots x_{\alpha_1})^0 (x_{\alpha_1+1} x_{\alpha_1+2} \cdots x_{\alpha_1+\alpha_2})^1 \cdots (x_{n-\alpha_\nu+1} x_{n-\alpha_\nu+2} \cdots x_n)^{\nu-1}.$$

A permutation P of S_n will leave $\delta(x)$ unaltered or transform it into $\delta^P(x)$ according as P is, or is not, contained in P_α. Instead of treating the group ring directly Specht constructs functions

$$d_{(\alpha)}(x) = \sum_Q \zeta(Q)\,\delta Q(x)$$
$$N_\alpha = \sum_Q \zeta(Q)\,Q, \text{ as above,}$$

and $\zeta(Q) = \pm 1$ as Q is even or odd, which under the permutations of S_n yield a modul $M(S_n; d_{(\alpha)}(x))$. Confining our attention to standard tableaux this modul leads to the irreducible representation (α) of S_n. Specht's method of constructing the actual matrices is the same as Young's and the results are identical (cf. [21] Part IV, p. 253; for a résumé of Young's theory cf. Part III, pp. 258-269. In Part VI the theory is further developed to give the actual matrices of the representation (α) in *orthogonal form* according to a very simple rule contained in Theorems 4 and 5, pp. 217 and 218).

In [18] Specht generalizes Young's T_α to apply to any permutation group P_n. It can be shown that

$$T_\alpha \cdot I = (f_\alpha/n!) \sum_{P \subset S_n} \chi^{(\alpha)}_{P^{-1}} P,$$

(cf. [21] Part IV, p. 256) where $\chi^{(\alpha)}_{P^{-1}}$ is the characteristic of P^{-1} in (α). Specht writes

We may now write (2.1) in the following manner:[13]

$$\Delta(\alpha) = \Sigma[\Pi S_{rs}^{\lambda_{rs}}](\alpha). \tag{2.2}$$

In this form it gives the reduction of $\Delta(\alpha)$ which we are seeking.

Young[14] defines the operation S_{rs} as follows:

Y: "S_{rs} where $r < s$ *represents the operation of moving one letter from the s-th row up to the r-th row, and the resulting term is regarded as zero, whenever any row becomes less than a row below it, or when letters from the same row overlap,—as, for instance, happens when* $\alpha_1 = \alpha_2$ *in the case of* $S_{13}S_{23}$."

As an illustration we have

$$\begin{aligned}\Delta(3,2,1) &= [1 + S_{23} + S_{13} + S_{12} + S_{12}S_{23} + S_{12}S_{13} + S_{12}{}^2S_{23} + S_{12}{}^2S_{13}](3,2,1) \\ &= (3,2,1) + (3^2) + (4,2) + (4,1^2) + (4,2) + (5,1) + (5,1) + (6).\end{aligned} \tag{2.3}$$

3. The Littlewood and Richardson rule for the reduction of $\{\beta\} \times \{\gamma\}$. The second problem treated by Murnaghan[15] is the reduction of $\{\beta\} \times \{\gamma\}$ into its irreducible components $\{\alpha\}$ of order n. His method is based on Schur's expression of the characters as determinants or as quotients of alternants. This method has been used by Specht.[16]

Littlewood and Richardson have also studied this reduction. Their means

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$$X_\xi \cdot f(x) = (g_\xi/h) \sum_{P \subset P_n} \chi_{P^{-1}}^{(\alpha)} f_P(x),$$

where $f(x)$ is a rational integral homogeneous function of the x_i, g_ξ is the degree of the irreducible representation of P_n, and h is the order of P_n. Corresponding to the relations amongst the T's

$$T_\alpha \cdot T_\alpha = T_\alpha,$$
$$T_\alpha \cdot T_\beta = 0,$$
$$1 = \sum_\alpha T_\alpha,$$

we have

$$X_\xi(X_\xi f(x)) = X_\xi f(x), \qquad X_\xi(X_\lambda f(x)) = 0, \qquad M(P_n; f(x)) = \sum_\xi M(P_n; X_\xi f(x)).$$

The function $f(x)$ is $d_{(\alpha)}(x)$ in the case of the symmetric group, and is similarly obtainable from the tableaux in the case of the alternating and hyper-octahedral group (cf. [15] with [21] Part V),—otherwise how actually to construct it is unknown.

[13] Cf. [9].

[14] [21] Part VI, p. 199. For a changed interpretation cf. Y' at the end of § 4, which clarifies somewhat the application of Y, and is applied to the example (2.3) at the beginning of § 7.

[15] Actually he considers the corresponding problem for finite groups.

[16] [17], p. 155.

of approach is through what they call Schur or S-functions, which are none other than the characters of the $\{\alpha\}$. Their reduction of the product of two S-functions of degrees l and m into a sum of S-functions of degree n, corresponds exactly to the reduction of $\{\beta\} \times \{\gamma\}$ into its irreducible components $\{\alpha\}$. Their chief contribution to the theory is the following theorem:[17]

To every tableau which may be constructed according to the following rule there corresponds an irreducible component $\{\alpha\}$ of $\{\beta\} \times \{\gamma\}$, and all such components are thereby obtained.

LR_1: *"Take the tableau $[\beta]$ intact and add to it the letters of the first row of $[\gamma]$. These may be added to one row of $[\beta]$, or the symbols may be divided without disturbing their order into any number of sets, the first set being added to one row of $[\beta]$, the second set to a subsequent row, the third to a row subsequent to this, and so on. After the addition no row must contain more symbols than a preceding row, and no two added symbols may be in the same column.*

Next add the second row of $[\gamma]$, according to the same rules followed by the remaining rows in succession until all the symbols of $[\gamma]$ have been used.

LR_2: *These additions shall be such that each symbol of a given row of $[\gamma]$ in the compound tableau must appear in a later row than the letter in the same column from the preceding row of $[\gamma]$."*

In what follows we shall establish a connection with Young's equation (2. 2) which will enable us to extend the methods used by Littlewood and Richardson to give a proof of their theorem.[18]

4. A proof of the Littlewood and Richardson rule. As a first step it will be convenient to modify somewhat Young's tableau $[\alpha]$ on which the S_{rs} of (2. 2) are supposed to operate. If we interchange a pair of rows leaving the letters in the same columns as before P_α remains unaltered, and the only change induced in (2. 2) is in the interpretation of the operators S_{rs};—others amongst their products will yield the components of the right-hand side. In particular we may rearrange the rows of $[\alpha]$ so that those of $[\beta]$ come first, followed by those of $[\gamma]$, thus:

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[17] [3], p. 119.

[18] There are some slips in the application of the theorem to the reduction of $\{4, 3, 1\} \times \{2^2, 1\}$, pointed out by Murnaghan [7].

$$[\beta;\gamma]: \begin{array}{l} b_{11}\, b_{12} \cdot \cdot \cdot \, b_{1\beta_1} \\ b_{21} \cdot \cdot \, b_{2\beta_2} \\ \cdot \quad \cdot \, \cdot \\ b_{\lambda 1} \cdot b_{\lambda\beta_\lambda} \\ c_{11}\, c_{12} \cdot \cdot \cdot \, c_{1\gamma_1} \\ c_{21} \cdot \cdot \, c_{2\gamma_2} \\ \cdot \quad \cdot \, \cdot \\ c_{\mu 1} \cdot c_{\mu\gamma_\mu} \end{array} .$$

The corresponding representation of the full linear group L remains the same, i. e. $\{\alpha\} = \{\beta;\gamma\}$, and $(\alpha) = (\beta;\gamma)$. If we generalize the S_{rs} so that r may be greater than s, we may describe the passage from $[\alpha]$ to $[\beta;\gamma]$ by means of a product S_0 of S's. In particular $S_0 = S_{32}{}^2 S_{23}$ transforms

$$[3,2,1]: \begin{array}{lll} a & a & a \\ b & b & \\ c & & \end{array} \quad \text{into} \quad [3,1;2]: \begin{array}{lll} a & a & a \\ c & & \\ b & b & \end{array} .$$

To pass from an operator as applied to $[\alpha]$ to that as applied to $[\beta;\gamma]$ it is only necessary to multiply by $S_0{}^{-1}$ and keep track of the letters involved, which we may do by an appropriate prefix. E. g. we may write $S_0{}^{-1} = {}_bS_{23}{}^2\, {}_cS_{32}$ and the operator S_{23} as applied to $[3,2,1]$ leads to

$${}_bS_{23}{}^2\, {}_cS_{32} \cdot {}_cS_{23} = {}_bS_{23}{}^2$$

or simply $S_{23}{}^2$ as applied to $[3,1;2]$. If *after* the multiplication an operator S_{rs} remains where $r > s$, as in the case of 1 operating on $[3,2,1]$, we may *first* combine it with the other members of $S_0{}^{-1}$, ignoring the prefixes, and remultiply. The resulting tableaux will in this case not be identical with those derived from $[3,2,1]$, certain letters in the same columns being interchanged, but *the correspondence between the two sets of tablaux is unique.*[19] Corresponding to (2. 3) we have

$$(4.1) \quad \Delta(3,1;2) = [S_{23} + S_{23}{}^2 + S_{23}{}^2 S_{12} + S_{13} + S_{13}S_{23} + S_{23}S_{13}S_{12} + S_{13}{}^2 + S_{13}{}^2 S_{12}](3,1;2).$$

In order to avoid confusion we shall write these operators S_{rs} as applied to this modification $[\beta;\gamma]$ of $[\alpha]$ as A_{rs}, the operators S_{rs} as applied to $[\beta]$ as B_{rs} and as applied to $[\gamma]$ as C_{rs}. Combining (1. 4) and (2. 2) we may write

[19] A more complicated example would be 1 operating on $[4,3,2,1]$ leading to $S_{23}{}^3 S_{34}{}^2 S_{42} = S_{23}{}^2 S_{34}$ operating on $[4,1;\ 3,2]$.

$$(4.2)\qquad \Sigma[\Pi A_{rs}^{\lambda_{rs}}]\{\beta;\gamma\} = [\Sigma[\Pi B_{rs}^{\lambda_{rs}}]\{\beta\}] \times [\Sigma[\Pi C_{rs}^{\lambda_{rs}}]\{\gamma\}] = \{\beta\} \times [\Sigma[\Pi C_{rs}^{\lambda_{rs}}]\{\gamma\}] + \cdots + \{l\} \times [\Sigma[\Pi C_{rs}^{\lambda_{rs}}]\{\gamma\}].$$

From the above interpretation of S_{rs} operating on $[\beta;\gamma]$ it is clear that we may identify A_{rs} for $s \leq \lambda$ with B_{rs}; e. g. in (4. 1) $S_{12} = A_{12} = B_{12}$.

Young's original restrictions Y on the S_{rs} as applied to $[\alpha]$ still hold since they are not affected by the above correspondence. Thus we may think of the tableaux arising on the left of (4. 2) under the operations A_{rs} as falling into sets representative of the irreducible components [20] of

$$\{\bar{\beta}\} \times \Delta_{(\gamma)}(L),$$

and *built on* $[\bar{\beta}]$, derivable from $[\beta]$ by the B_{rs} operating according to Y. The rule of operation of the $A_{i,\lambda+j}$ we write as:

Y_1: *Take the tableaux $[\beta]$ intact and add to it the letters of the first row of $[\gamma]$ under $A_{i,\lambda+1}$. These may be added to one row of $[\beta]$, or the symbols may be divided (without disturbing their order) into any number of sets, the first set being added to one row of $[\beta]$, the second to a subsequent row, the third to a row subsequent to this, and so on. After the addition no row must contain more symbols than a preceding row, and no two added symbols may be in the same column.*

Next add the second row of $[\gamma]$, according to the same rules followed by the remaining rows in succession, until all the symbols of $[\gamma]$ have been used.

Y_2: *To obtain tableau built on $[\bar{\beta}]$ replace $[\beta]$ by $[\bar{\beta}]$ in Y_1.*

The parenthesis in Y_1 is unnecessary at this stage, in fact all the letters in a given row of $[\gamma]$ may be taken to be the same (cf. the example at the beginning of § 7), but it will be needed shortly to add definiteness to the resulting tableaux. *It is important to remark that LR_1 and Y_1 are identical.*

If we let $\beta_1 = \alpha_1$ and $\gamma_1 = \alpha_2, \cdots, \gamma_{\nu-1} = \alpha_\nu$, Y_2 becomes unnecessary and Y_1 may be written Y', which is equivalent to Y in view of the equation

$$(4.3)\qquad \Delta_{(\alpha)}(L) = \{\alpha_1\} \times \Delta_{(\alpha_2\alpha_3\ldots\alpha_\nu)}(L).$$

This change of viewpoint seems to clarify the application of Y and makes it a very simple matter to write down the tableaux representative of the irreducible components of $\Delta(\alpha)$.

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[20] By suppressing the c's we arrive at a tableau representative of $\{\bar{\beta}\}$ and we may think of $[\bar{\beta}]$ as occupying the upper left hand corner of the compound tableau, leading to the idea that this is *built on* $[\bar{\beta}]$.

5. Lattice permutations. Permutations of the m letters

(5.1) $$c_1^{\gamma_1} c_2^{\gamma_2} \cdots c_\mu^{\gamma_\mu}$$

have been studied at some length, and a particular class called *lattice permutations* have been given prominence by MacMahon.[21] The definition of a lattice permutation is that amongst the first r terms of it the number of c_1's $\geq$ the number of c_2's $\geq \cdots \geq$ the number of c_μ's for all r. If we add a second suffix to the c_i's according to the order of their appearance, for each i, we may define a set of numbers which may be called *indices* of the permutation. Considering first only the c_1's and the c_2's, if c_{2s} follow c_{1t} and precede $c_{1,t+1}$ its index is defined as $s - t$ and we write

$$i_{12s} = s - t,$$

which may be positive, zero, or negative. As pointed out by Littlewood and Richardson [22] the resulting permutation of $c_1^{\gamma_1} c_2^{\gamma_2}$ is a lattice permutation if and only if no $i_{12s} > 0$. Similarly we may define indices i_{23s}, i_{34s}, etc. and *any permutation of the letters* (5.1) *is lattice if and only if no* $i_{x,x+1,s} > 0$ $(x = 1, 2, \cdots, \mu - 1)$. An important property of a lattice permutation is that by comparing it with the natural arrangement, or identical permutation, of the symbols we may uniquely associate it with a standard tableau, and conversely. E. g. with the permutation

$$1231 \text{ we associate the tableau } \begin{matrix} 1\,4 \\ 2 \\ 3 \end{matrix}\,,$$

the lattice permutation indicating in which row the corresponding symbol is to be placed. Thus the number of distinct lattice permutations [23] of the letters (5.1) is just f_γ.

We now show how any non-lattice permutation may be associated with a lattice permutation. The steps in the process are as follows:

(a) Considering only the c_1's and the c_2's in the permutation, take the first c_2 with the greatest positive i_{12s} and change it into a c_1. Reallocating the second suffixes repeat the process, continuing until the c_1's and the c_2's are all lattice.

(b) Considering only the c_2's and the c_3's in the permutation so modified, take the first c_3 with greatest positive i_{23s} and change it into a c_2.

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[21] [4], vol. I, p. 124.

[22] [3], p. 121. Dr. A. Young has drawn my attention to the fact that the index $i_{r,r+1,s}$ is almost identical with a number used by him ([21] Part VI, § 15). In his notation $\gamma_{r+1,s,r} = -i_{r,r+1,s} + 2$ and the condition that $i_{r,r+1,s} \leqq 0$ is the same as $\gamma_{r+1,s,r} \geqq 2$, or that his *second tableau function* $\Pi(\gamma_{r,s,t} - 1) > 0$.

[23] Any two such we may speak of as belonging to the same *class*.

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If this change upsets the 1–2 lattice property correct for it by changing a c_2 into a c_1 according to (a); *this may or may not be the new* c_2. Re-allocating the second suffixes repeat the process, continuing until the c_1's, c_2's and c_3's are all lattice.

(c) Making use of the indices $i_{34s}, i_{45s}, \cdots, i_{\mu-1,\mu s}$ proceed as above, continuing until all the c_1's, c_2's, $\cdots$, c_μ's are lattice.

This [24] we shall refer to as the *association I.*

Let us think of these changes in the light of Y' as applied to $[\gamma]$, and associate them with the operators C_{ij} in the following manner.

(a′) Changing a c_2 into a c_1 we associate with the operator C_{12}.

(b′) If changing a c_3 into a c_2 *does not* spoil the 1–2 lattice property we associate it with the operator C_{23}.

(b″) If changing a c_3 into a c_2 *does* spoil the 1–2 lattice property and we must change a c_2 into a c_1, we associate it with the operator C_{13}.

(c′) Similarly changing a c_4 is associated with C_{34}, C_{24}, or C_{14} as further changes are necessary; etc. Finally changing a c_μ is associated with $C_{\mu-1,\mu}$, $C_{\mu-2,\mu}$, $\cdots$ or $C_{1\mu}$.

Thus with each non-lattice permutation of the letters (5.1) we may also associate an operator

$$C_{12}^{\lambda_{12}}\, C_{13}^{\lambda_{13}} \cdots C_{\mu-1,\mu}^{\lambda_{\mu-1,\mu}} = \Pi C_{rs}^{\lambda_{rs}}, \tag{5.2}$$

which is one of those [25] applied to $[\gamma]$ under Y'. This we shall describe as the *association II.*

We are now ready to pass on to the conclusion of the proof of the Littlewood and Richardson theorem, but before doing so it will be worth while to consider in greater detail these two associations I and II in the case $\gamma_1 = \gamma_2 = \cdots = \gamma_m = 1$. The tableau $[\gamma]$ is in this case

$$\begin{matrix} 1 \\ 2 \\ 3 \\ \cdot \\ \cdot \\ m \end{matrix},$$

and each of the operators (5.2), which we shall denote $\boldsymbol{L}_2$, leads to a standard

[24] I am indebted for this association I to Mr. D. E. Littlewood.

[25] Clearly the lattice condition assures at each stage that the number of letters in any row is not less than the number in a succeeding row of the corresponding tableau. Changing the "*first* c_{r+1} *with greatest positive* $i_{r,r+1,s}$" precludes the possibility of two letters from the same row appearing in the same column, as will be clear from the following example. The permutation $c_3c_3c_1c_2c_1c_2$ leads to $c_1c_1c_1c_2c_1c_2$ under the association

tableau and a corresponding lattice permutation L_2. Thus with each of the $m!$ permutations of the letters we can associate under I a lattice permutation L_1, and under II a lattice permutation L_2, and L_1 and L_2 belong to the same class. Conversely, by reversing the steps (a), (b) and (c) we may pass backward to the permutation which [26] we may denote (S). This passage may be indicated thus:

$$1\,2\,3 \cdot \cdot \cdot m \longrightarrow L_1 \xrightarrow[(\boldsymbol{L}_2)^{-1}]{} (S).$$

If we interchange the rôles of the two lattice permutations L_1 and L_2 it is not difficult to see that

$$1\,2\,3 \cdot \cdot \cdot m \longrightarrow L_2 \xrightarrow[(\boldsymbol{L}_1)^{-1}]{} (S^{-1}).$$

Clearly if $L_1 = L_2$ then $S^2 = 1$. This remarkable duality enables us to construct a square table having Σf_γ rows and columns which is symmetrical about its leading diagonal, down which appear the Σf_γ solutions [27] of $S^2 = 1$. The remaining substitutions of S_m appear in blocks of f_γ^2, and $\Sigma f_\gamma^2 = m!$. There follows this table constructed for $m = 4$.

	1234	1231	1213	1123	1122	1212	1112	1121	1211	1111
	1	S_{14}	$S_{13}S_{34}$	$S_{12}S_{23}S_{34}$	$S_{12}S_{23}S_{24}$	$S_{13}S_{24}$	$S_{12}S_{13}S_{24}$	$S_{12}S_{23}S_{14}$	$S_{13}S_{14}$	$S_{12}S_{13}S_{14}$
1234	1234									
1231		1243 (34)	1342 (234)	2341 (1234)						
1213		1423 (243)	1324 (23)	2314 (123)						
1123		4123 (1432)	3124 (132)	2134 (12)						
1122					2143 (12) (34)	3142 (1342)				
1212					2413 (1243)	3412 (13) (24)				
1112							3214 (13)	4213 (143)	4312 (1423)	
1121							3241 (134)	4231 (14)	4132 (142)	
1211							3421 (1324)	2431 (124)	1432 (24)	
1111										4321 (14) (23)

I and to the operator $C_{13}{}^2$ under II, *not* to $c_1c_2c_1c_2c_1c_2$ and the operator $C_{13}C_{23}$ (*cf.* the end of the third paragraph on p. 122 of [3]).

[26] Assuming that the identical permutation is transformed by S into the given permutation.

[27] [2], p. 197, since all the irreducible representations of S_m are real.

6. Conclusion of the proof. We have now reached the final stage of our argument and may confine our attention to those tableaux built on $[\beta]$ which are representative of the irreducible components of

$$(6.1) \qquad \{\beta\} \times [\Sigma[\Pi C_{rs}^{\lambda_{rs}}]\{\gamma\}] = \{\beta\} \times \{\gamma\} \cdots + \{\beta\} \times \{m\}.$$

We follow Littlewood and Richardson [28] and from any such tableau read the c_{ij}'s from the right omitting the second suffix, beginning at the first row and taking the remaining rows in succession. Written in this order we have a permutation of the letters (5.1). A little consideration will show that *if a tableau built on* $[\beta]$ *according to* Y_1 (*or* LR_1) *is to satisfy* LR_2 *it is necessary and sufficient that the permutation of the c's obtained as above described should be a lattice permutation.*

We assume the Theorem to be true for all products $\{\beta\} \times \{\bar{\gamma}\}$, where $[\bar{\gamma}]$ is derivable from $[\gamma]$ under the C_{rs}, and apply an induction to prove it for $\{\beta\} \times \{\gamma\}$. That is, we assume that all tableaux satisfying the appropriate LR_2 yield the irreducible components of $\{\beta\} \times \{\bar{\gamma}\}$, since as we have seen LR_1 is automatically satisfied; this is equivalent to saying that the corresponding permutation is a lattice permutation. But clearly this is necessarily so in the case of $\{\beta\} \times \{m\}$, where all the letters of $[m]$ belong to the same row. Each non-lattice permutation of $c_1^{\gamma_1} c_2^{\gamma_2} \cdots c_\mu^{\gamma_\mu}$ is associated with an operator $\Pi C_{rs}^{\lambda_{rs}}$ under the association II, and conversely with each such operator is associated a set of tableaux built on $[\beta]$ according to LR_1 and LR_2. Thus those which remain, namely the lattice permutations, represent tableaux built on $[\beta]$ according to LR_2, and yield the irreducible components of $\{\beta\} \times \{\gamma\}$.

7. Examples of the application of the rules Y, Y′, *LR*. We may obtain the irreducible components appearing in

$$(2.3) \quad \Delta(3,2,1) = [1 + S_{23} + S_{13} + S_{12} + S_{12}S_{23} + S_{12}S_{13} + S_{12}^2 S_{23} + S_{12}^2 S_{13}](3,2,1)$$
$$= (3,2,1) + (3^2) + (4,2) + (4,1^2) + (4,2) + (5,1) + (5,1) + (6),$$

by the more systematic rule Y'. Taking the tableau $[3,2,1]$ to be

$$\begin{matrix} a & a & a \\ b & b & \\ c & & \end{matrix}$$

we write down the first row intact, and add the letters of the second row according to Y_1. To the resulting tableaux, namely

[28] [3], p. 121.

$$\begin{array}{l} a\ a\ a\ b\ b, \end{array} \qquad \begin{array}{l} a\ a\ a\ b, \\ b \end{array} \qquad \begin{array}{l} a\ a\ a, \\ b\ b \end{array}$$

we must add the letter c, obtaining from the first

$$\begin{array}{l} a\ a\ a\ b\ b\ c = S_{12}{}^2 S_{13}[3, 2, 1], \end{array} \qquad \begin{array}{l} a\ a\ a\ b\ b = S_{12}{}^2 S_{23}[3, 2, 1]; \\ c \end{array}$$

from the second,

$$\begin{array}{l} a\,a\,a\,b\ c = S_{12}S_{13}[3, 2, 1], \\ b \end{array} \quad \begin{array}{l} a\,a\,a\,b = S_{12}S_{23}[3, 2, 1], \\ b\ c \end{array} \quad \begin{array}{l} a\,a\,a\,b = S_{12}[3, 2, 1]; \\ b \\ c \end{array}$$

and from the third,

$$\begin{array}{l} a\ a\ a\ c = S_{13}[3, 2, 1], \\ b\ b \end{array} \quad \begin{array}{l} a\ a\ a = S_{23}[3, 2, 1], \\ b\ b\ c \end{array} \quad \begin{array}{l} a\ a\ a = [3, 2, 1]. \\ b\ b \\ c \end{array}$$

These tableaux yield the required components.

As an illustration of LR we shall write down the tableaux representative of the irreducible components of $\{4, 2^2, 1\} \times \{2^3\}$. It will be easier to build on $[4, 2^2, 1]$, and since this tableau remains unaltered we shall represent its elements by $\bullet$'s. We write

$$[4, 2^2, 1] : \begin{array}{l} \bullet\ \bullet\ \bullet\ \bullet, \\ \bullet\ \bullet \\ \bullet\ \bullet \\ \bullet \end{array} \qquad [2^3] : \begin{array}{l} a_1\, a_2, \\ b_1\, b_2 \\ c_1\, c_2 \end{array}$$

and begin by adding the letters of the first row of $[2^3]$ to $[4, 2^2, 1]$ according to LR_1 $(= Y_1)$. Then to these tableaux we similarly add the letters of the second and third rows of $[2^3]$, *subject at each stage to* LR_2.

$$\begin{array}{l} \bullet\ \bullet\ \bullet\ \bullet\ a_1\, a_2, \\ \bullet\ \bullet\ b_1\, b_2 \\ \bullet\ \bullet\ c_1\ c_2 \\ \bullet \end{array} \quad \begin{array}{l} \bullet\ \bullet\ \bullet\ \bullet\ a_1, a_2, \\ \bullet\ \bullet\ b_1\, b_2 \\ \bullet\ \bullet\ c_1 \\ \bullet\ c_2 \end{array} \quad \begin{array}{l} \bullet\ \bullet\ \bullet\ \bullet\ a_1\, a_2, \\ \bullet\ \bullet\ b_1\, b_2 \\ \bullet\ \bullet\ c_1 \\ \bullet \\ c_2 \end{array} \quad \begin{array}{l} \bullet\ \bullet\ \bullet\ \bullet\ a_1\, a_2; \\ \bullet\ \bullet\ b_1\, b_2 \\ \bullet\ \bullet \\ \bullet\ c_1 \\ c_2 \end{array}$$

• • • • $a_1\, a_2$,	• • • • $a_1\, a_2$;	• • • • $a_1\, a_2$,	• • • • $a_1\, a_2$;	• • • • $a_1\, a_2$:
• • b_1	• • b_1	• • b_1	• • b_1	• •
• • c_1	• •	• • c_1	• •	• •
• b_2	• b_2	•	• c_1	• b_1
c_2	$c_1\, c_2$	b_2	b_2	$b_2\, c_1$
		c_2	c_2	c_2

• • • • a_1,	• • • • a_1,	• • • • a_1,	• • • • a_1;	• • • • a_1,	• • • • a_1;
• • $a_2\, b_1$	• • $a_2\, b_1$	• • $a_2\, b_1$	• • $a_2\, b_1$	• • $a_2\, b_1$	• • $a_2\, b_1$
• • $b_2\, c_1$	• • $b_2\, c_1$	• • b_2	• • b_2	• • c_1	• •
• c_2	•	• $c_1\, c_2$	• c_1	• b_2	• b_2
	c_2		c_2	c_2	$c_1\, c_2$

• • • • a_1,	• • • • a_1;	• • • • a_1,	• • • • a_1;	• • • • a_1;	• • • • a_1:
• • $a_2\, b_1$	• • $a_2\, b_1$	• • a_2	• • a_2	• • a_2	• • a_2
• • c_1	• •	• • b_1	• • b_1	• • b_1	• •
•	• c_1	• $b_2\, c_1$	• b_2	• c_1	• b_1
b_2	b_2	c_2	$c_1\, c_2$	b_2	$b_2\, c_1$
c_2	c_2			c_2	c_2

• • • • a_1,	• • • • a_1;	• • • • a_1:	• • • • a_1,	• • • • a_1;
• • b_1	• • b_1	• •	• • b_1	• • b_1
• • c_1	• •	• •	• • c_1	• •
• a_2	• a_2	• a_2	•	• c_1
b_2	$b_2\, c_1$	$b_1\, b_2$	a_2	a_2
c_2	c_2	$c_1\, c_2$	b_2	b_2
			c_2	c_2

• • • • a_1:	• • • •,	• • • •;	• • • •,	• • • •;
• •	• • $a_1\, a_2$	• • $a_1\, a_2$	• • $a_1\, a_2$	• • $a_1\, a_2$
• •	• • $b_1\, b_2$	• • $b_1\, b_2$	• • b_1	• • b_1
• b_1	• $c_1\, c_2$	• c_1	• $b_2\, c_1$	• b_2
$a_2\, c_1$		c_2	c_2	$c_1\, c_2$
b_2				
c_2				

• • • •;	• • • •:	• • • •,	• • • •;	• • • •:
• • $a_1\, a_2$	• • $a_1\, a_2$	• • a_1	• • a_1	• • a_1
• • b_1	• •	• • b_1	• • b_1	• •
• c_1	• b_1	• $a_2\, c_1$	• a_2	• a_2
b_2	$b_2\, c_1$	b_2	$b_2\, c_1$	$b_1\, b_2$
c_2	c_2	c_2	c_2	$c_1\, c_2$

$$
\begin{array}{llll}
\cdot & \cdot & \cdot & \cdot\ ; \\
\cdot & \cdot & a_1 & \\
\cdot & \cdot & b_1 & \\
\cdot & c_1 & & \\
a_2 & & & \\
b_2 & & & \\
c_2 & & &
\end{array}
\qquad
\begin{array}{llll}
\cdot & \cdot & \cdot & \cdot\ : \\
\cdot & \cdot & a_1 & \\
\cdot & \cdot & & \\
\cdot & b_1 & & \\
a_2 & c_1 & & \\
b_2 & & & \\
c_2 & & &
\end{array}
\qquad
\begin{array}{llll}
\cdot & \cdot & \cdot & \cdot\ . \\
\cdot & \cdot & & \\
\cdot & \cdot & & \\
\cdot & a_1 & & \\
a_2 & b_1 & & \\
b_2 & c_1 & & \\
c_2. & & &
\end{array}
$$

While the number of tableaux which it is necessary to construct in a given product may not be small, nevertheless after a little practice the application of LR becomes quite mechanical, and is entirely elementary. Each tableau representing an irreducible component, we have the equation

$$
\begin{aligned}
\{4, 2^2, 1\} \times \{2^3\} = {} & (6, 4^2, 1) + (6, 4, 3, 2) + (6, 4, 3, 1^2) + (6, 4, 2^2, 1) \\
& + (6, 3^2, 2, 1) + (6, 3, 2^3) + (6, 3^2, 1^3) + (6, 3, 2^2, 1^2) \\
& + (6, 2^4, 1) + (5, 4^2, 2) + (5, 4^2, 1^2) + (5, 4, 3^2) \\
& + 2(5, 4, 3, 2, 1) + (5, 4, 2^3) + (5, 4, 3, 1^3) \\
& + (5, 4, 2^2, 1^2) + (5, 3^3, 1) + (5, 3^2, 2^2) + (5, 3^2, 2, 1^2) \\
& + (5, 3, 2^3, 1) + (5, 3^2, 2, 1^2) + (5, 3, 2^3, 1) + (5, 2^5) \\
& + (5, 3^2, 1^4) + (5, 3, 2^2, 1^3) + (5, 2^4, 1^2) + (4^3, 3) \\
& + (4^3, 2, 1) + (4^2, 3^2, 1) + (4^2, 3, 2^2) + (4^2, 3, 2, 1^2) \\
& + (4^2, 2^3, 1) + (4, 3^3, 1^2) + (4, 3^2, 2^2, 1) + (4, 3, 2^4) \\
& + (4, 3^2, 2, 1^3) + (4, 3, 2^3, 1^2) + (4, 2^5, 1).
\end{aligned}
$$

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For other references cf. [5].