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Binomial Determinants, Paths, and Hook Length Formulae

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We give a combinatorial interpretation for any minor (or binomial determinant) of the matrix of binomial coefficients. This interpretation involves configurations of nonintersecting paths, and is related to Young tableaux and hook length formulae.

1. THE FUNDAMENTAL THEOREM

Let $A=(a_{ij})_{i,j\geqslant 0}$ be the infinite matrix defined by $a_{ij}=\binom{i}{j}$ for $j\leqslant i$ and $a_{ij}=0$ for $j>i$. A *binomial determinant* is any minor of A. The minor corresponding to rows $0\leqslant a_1<a_2<\cdots<a_k$ and columns $0\leqslant b_1<b_2<\cdots<b_k$ will be denoted by

$$\binom{a_1,\dots,a_k}{b_1,\dots,b_k}=\binom{\mathscr{A}}{\mathscr{B}}=\det\left(\binom{a_i}{b_j}\right)_{1\leqslant i,j\leqslant k},\tag{1}$$

where $\mathscr{A}=\{a_1,\dots,a_k\}$ and $\mathscr{B}=\{b_1,\dots,b_k\}$.

Binomial determinants appear in algebraic geometry as coefficients in the Chern class calculus for the tensor product of two fiber bundles (see Lascoux [16] and Macdonald [19, pp. 30–31]). In particular, all these coefficients are nonnegative, as we shall see from the combinatorial interpretation.

Let $\Pi=\mathbb{N}\times\mathbb{N}$. A *path* of Π is a sequence $w=(s_0,s_1,\dots,s_n)$ of points in Π such that if $s_i=(x,y)$ then s_{i+1} is either $(x,y-1)$ (a vertical step) or $(x+1,y)$ (a horizontal step). See Fig. 1.

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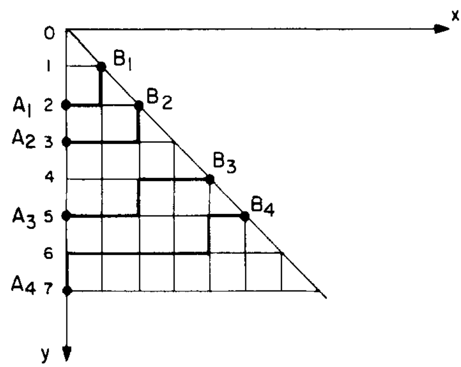

FIG. 1. The binomial determinant $\binom{2,3,5,7}{1,2,4,5}$.

THEOREM 1. *Let* $0 \leqslant a_1 < \cdots < a_k$ *and* $0 \leqslant b_1 < \cdots < b_k$ *be strictly increasing sequences of nonnegative integers. Let* $A_i = (0, a_i)$ *and* $B_i = (b_i, b_i)$. *Then the binomial determinant*

$$\binom{a_1,..., a_k}{b_1,..., b_k}$$

is the number of k-tuples $(w_1,..., w_k)$ *of paths of* Π *such that*:

(i) *For each i,* w_i *is a path from* A_i *to* B_i.

(ii) *The paths* w_i *are pairwise disjoint.*

An example is given in Fig. 1.

Before proving the theorem, we give three immediate corollaries:

COROLLARY 2. *The binomial determinant*

$$\binom{a_1,..., a_k}{b_1,..., b_k}$$

is nonnegative, and is positive iff $b_i \leqslant a_i$ *for each i.*

COROLLARY 3. *If* $a_j = b_j$ *then*

$$\binom{a_1,..., a_k}{b_1,..., b_k} = \binom{a_1,..., a_{j-1}}{b_1,..., b_{j-1}}\binom{a_{j+1},..., a_k}{b_{j+1},..., b_k}.$$

COROLLARY 4. *The number of nonzero minors (including the empty minor) of the matrix* $A_n = (\binom{i}{j})_{0 \leqslant i,j \leqslant n}$ *is the Catalan number*

$$c_{n+2} = \frac{1}{n+3}\binom{2n+4}{n+2}.$$

2. BIJECTIVE PROOF OF THEOREM 1

Let $\mathfrak{S}_k$ be the set of permutations of $[k] = \{1, 2,..., k\}$. Let E be the set of pairs $(\sigma; (w_1,..., w_k))$, where $\sigma \in \mathfrak{S}_k$ and $(w_1,..., w_k)$ is a k-tuple of paths of Π such that w_i goes from A_i to $B_{\sigma(i)}$. Let $NI \subset E$ be the subset of pairs for which σ is the identity permutation and $w_1,..., w_k$ are pairwise disjoint. The number of paths from A_i to B_j is

$$\binom{a_i}{b_j},$$

so

$$\binom{a_1,..., a_k}{b_1,..., b_k} = \sum_{\xi \in E} (-1)^{\mathrm{Inv}(\sigma)}, \tag{2}$$

where $\xi = (\sigma; (w_1,..., w_k))$ and $\mathrm{Inv}(\sigma)$ is the number of inversions of σ.

The theorem is a consequence of the following lemma:

LEMMA 5. *There exists an involution* $\phi: E - NI \to E - NI$ *such that if* $\phi(\sigma; (w_1,..., w_k)) = (\sigma'; (w'_1,..., w'_k))$ *then* $(-1)^{\mathrm{Inv}(\sigma')} = -(-1)^{\mathrm{Inv}(\sigma)}$.

Proof. We define ϕ as follows. Let $\xi = (\sigma; (w_1,..., w_k))$ be an element of $E - NI$. If σ is not the identity, then there exist integers i, j with $1 \leqslant i < j \leqslant k$ and $\sigma(i) > \sigma(j)$. It is easy to see that the paths w_i and w_j have a common vertex. By the definition of NI, if σ is the identity then there also exist two distinct paths with a common vertex. Thus we can define i_0 as the smallest i, $1 \leqslant i \leqslant k$, such that w_i intersects another path. We then define $C = C(\xi)$ as the first vertex of w_{i_0} which is also a vertex of another path and we define j_0 as the smallest j greater than i_0 such that C is a vertex of w_j.

We then set $\phi(\xi) = (\tau; (w'_1,..., w'_k))$, where τ is the product of σ by the transposition (i, j), $w'_l = w_l$ for $l \neq i, j$, and w'_i (resp. w'_j) is the path obtained by following w_i (resp. w_j) up to $C(\xi)$ and then following w_j (resp. w_i) after $C(\xi)$. Clearly $\phi(\xi)$ is in $E - NI$ and satisfies the condition of the lemma.

If we repeat the construction of ϕ for $\phi(\xi)$, we obtain $C(\phi(\xi)) = C(\xi)$ and

$\phi(\phi(\xi)) = \xi$. This completes the proof of the lemma, and hence of Theorem 1.

Arguments similar to the one of Lemma 5 have been used by Chaundy [3], Karlin and MacGregor [14], and Lindström [18]. See also Gessel and Viennot [8, 9], Goulden and Jackson [10, pp. 322–328], and Sulanke [27].

3. Permutations with Given Descent Set

Let σ be a permutation in $\mathfrak{S}_n$. The *descent set* of σ is $\mathrm{DES}(\sigma) = \{i \mid \sigma(i) > \sigma(i+1), \text{ for } 1 \leqslant i < n\}$. Theorem 1 gives as a corollary MacMahon's formula [20] (rediscovered by Carlitz [2], Niven [21], and Gupta [11]) for the number of permutations with a given descent set.

Corollary 6. *If* $\mathscr{C} = \{c_1 < c_2 < \cdots < c_k\} \subseteq [n-1]$ *then the number of permutations* $\sigma \in \mathfrak{S}_n$ *with* $\mathrm{DES}(\sigma) = \mathscr{C}$ *is the binomial determinant*

$$\begin{pmatrix} c_1, c_2, \ldots, c_k, n \\ 0, c_1, \ldots, c_{k-1}, c_k \end{pmatrix}. \tag{3}$$

Proof. We code a permutation $\sigma \in \mathfrak{S}_n$ by a configuration of nonintersecting paths which is counted according to Theorem 1 by the binomial determinant.

Let $[n]_0 = \{0\} \cup [n]$. For $\sigma \in \mathfrak{S}_n$ define the function $f\colon [n]_0 \to \mathbb{N}$ such that $f(j)$ is the number of indices i, $1 \leqslant i < j$ with $\sigma(i) < \sigma(j)$. (Then f is a modified inversion table of σ; see Knuth [15, p. 12].) It is easy to see that σ can be recovered from f and that $i \in \mathrm{DES}(\sigma)$ iff $f(i) \geqslant f(i+1)$. Moreover, a function $f\colon [n]_0 \to \mathbb{N}$ comes from some permutation iff f satisfies $0 \leqslant f(i) \leqslant i-1$ for $1 \leqslant i \leqslant n$.

Such a function f can be coded by a configuration of nonintersecting paths counted by (3), where $\mathscr{C}$ is the set of indices i for which $f(i) \geqslant f(i+1)$. (See Fig. 2.) For such a configuration there exists exactly one vertical step between the lines $y = i$ and $y = i-1$, for $i = 1, \ldots, n$. If this vertical step is $((i, j), (i-1, j))$ then we define $f(i) = i - j - 1$. For any configuration (not necessarily nonintersecting) of paths from A_i to B_i, the function f obtained in this way will satisfy $0 \leqslant f(i) \leqslant i-1$ for $1 \leqslant i \leqslant n$ and $f(i) < f(i+1)$ for $i \notin \mathscr{C}$. The nonintersecting condition is equivalent to $f(i) \geqslant f(i+1)$ for $i \in \mathscr{C}$. The correspondence is easily seen to be a bijection.

In Section 8 we will give generating functions for some binomial determinants of the form of (3).

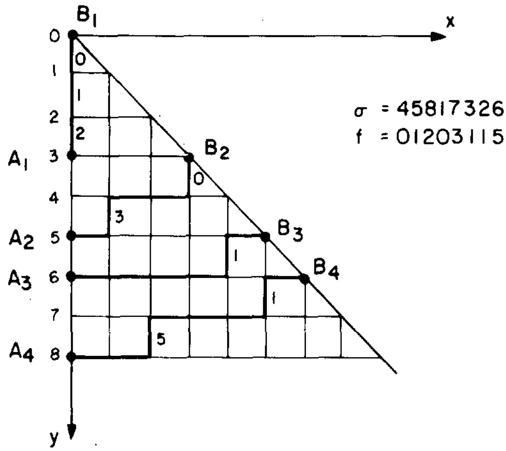

FIGURE 2

4. A DUALITY THEOREM

Let M be a matrix with rows and columns indexed 0, 1,..., n. If $\mathscr{A}$ and $\mathscr{B}$ are subsets of $[n]_0 = \{0, 1,..., n\}$ of the same size, let $M[\mathscr{A} \mid \mathscr{B}]$ denote the minor of M corresponding to the rows in $\mathscr{A}$ and the columns in $\mathscr{B}$. Then by a theorem of Jacobi [13; 17, pp. 153–156], if $L = M^{-1}$ then

$$M[\mathscr{A} \mid \mathscr{B}] = (-1)^{\varepsilon} (\det M)\, L[\bar{\mathscr{B}} \mid \bar{\mathscr{A}}],$$

where $\bar{\mathscr{A}} = [n]_0 - \mathscr{A}$, $\bar{\mathscr{B}} = [n]_0 - \mathscr{B}$, and

$$\varepsilon = \sum_{a \in \mathscr{A}} a + \sum_{b \in \mathscr{B}} b.$$

We may restate Jacobi's theorem in a more convenient form. If L is the inverse of a matrix M, let us call the matrix $((-1)^{i+j} L_{ij})$ the *sign-inverse* of M. Then Jacobi's theorem says that if N is the sign-inverse of M,

$$M[\mathscr{A} \mid \mathscr{B}] = (\det M) \cdot N[\bar{\mathscr{B}} \mid \bar{\mathscr{A}}].$$

Since the binomial coefficient matrix $(\binom{i}{j})_{0 \leqslant i,j \leqslant n}$ is its own sign-inverse, Jacobi's theorem yields the following duality theorem for binomial determinants:

PROPOSITION 7. *For $\mathscr{A}, \mathscr{B} \subseteq [n]_0$, with $|\mathscr{A}| = |\mathscr{B}|$, we have*

$$\binom{\mathscr{A}}{\mathscr{B}} = \binom{[n]_0 - \mathscr{B}}{[n]_0 - \mathscr{A}}.$$

We now sketch a combinatorial proof of the duality theorem. Given a configuration ξ of nonintersecting paths counted by $\binom{\mathscr{A}}{\mathscr{B}}$, we shall construct a configuration $\Phi(\xi)$ of noncrossing paths counted by

$$\binom{[n]_0-\mathscr{B}}{[n]_0-\mathscr{A}},$$

and Φ will be an involution.

First we observe that any configuration of noncrossing paths is determined by its endpoints and its horizontal steps.

The endpoints of $\Phi(\xi)$ are of course the points $\{(0,i) \mid i \notin \mathscr{B}\}$ and $\{(i,i) \mid i \notin \mathscr{A}\}$. To describe the set of horizontal steps of $\Phi(\xi)$, we first define the set H_1 of all horizontal steps $h = \langle (x,y), (x+1,y) \rangle$ with $0 \leqslant x \leqslant y-1$, $0 \leqslant y \leqslant n$ such that h is not a horizontal step of ξ and $\langle (x,y), (x,y-1) \rangle$ is not a vertical step of ξ. Then we define the set H of horizontal steps of $\Phi(\xi)$ to be the set of all steps $\langle (y-x-1,y), (y-x,y) \rangle$ for which $\langle (x,y), (x+1,y) \rangle \in H_1$.

As an illustration, we describe geometrically how $\Phi(\xi)$ is constructed, where ξ is the configuration of Fig. 1. First we shift Fig. 1 so that the boundary becomes an equilateral triangle (Fig. 3a). Then we construct the set H_1 of horizontal edges (Fig. 3b) and mark the points $(0,i)$ and (i,i) which are not endpoints of paths in ξ. Next we remove the "northeast" lines and replace them with "northwest" lines, and fill in vertical steps to make a configuration of nonintersecting paths with steps "west" and "northwest" (Fig. 4a). Finally, we reflect our configuration left to right. (Fig. 4b). We leave it to the reader to verify that a nonintersecting configuration is always obtained and that Φ is an involution.

We will discuss some other applications of Jacobi's theorem in Section 8.

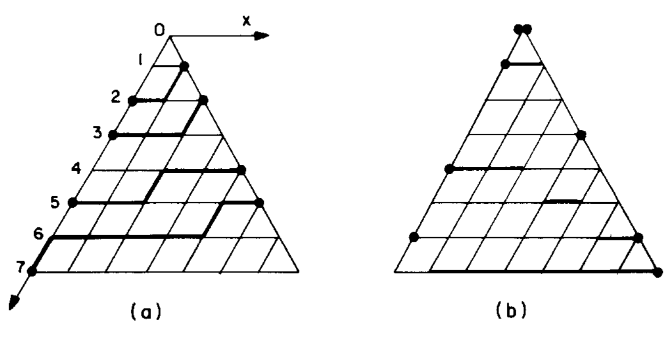

FIGURE 3

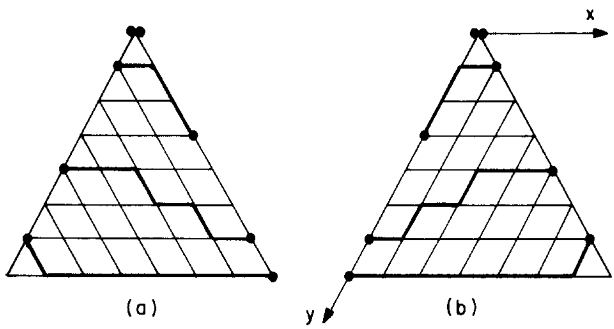

FIGURE 4

5. EVALUATION OF BINOMIAL DETERMINANTS

In this section we show that the binomial determinant

$$\binom{a_1, a_2, \dots, a_k}{b_1, b_2, \dots, b_k}$$

can be evaluated in closed form if either the a's or the b's are consecutive integers.

LEMMA 8. *If* $b_1 \neq 0$, *then*

$$\binom{a_1, \dots, a_k}{b_1, \dots, b_k} = \frac{a_1 \cdots a_k}{b_1 \cdots b_k} \binom{a_1 - 1, \dots, a_k - 1}{b_1 - 1, \dots, b_k - 1}.$$

Proof. The lemma follows immediately from the formula

$$\binom{a_i}{b_j} = \frac{a_i}{b_j} \binom{a_i - 1}{b_j - 1}.$$

LEMMA 9.

$$\binom{a, a+1, \dots, a+k-1}{0, b_2, \dots, b_k} = \binom{a, a+1, \dots, a+k-2}{b_2 - 1, b_3 - 1, \dots, b_k - 1}.$$

Proof. Remove the path from A_1 to B_1 and the first horizontal step of each other path.

A *partition* is an increasing sequence $\lambda = \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k$ of nonnegative integers. To any set $\mathscr{A} = \{a_1 < \cdots < a_k\}$ of nonnegative integers we may associate a partition $\lambda = p(\mathscr{A})$ by $\lambda_i = a_i - i + 1$. Thus $p(0, 1, 3, 5) = 0012$.

The *Ferrers diagram* of a partition $\lambda = (\lambda_1 \leqslant \cdots \leqslant \lambda_k)$ is an array of k rows of cells, with λ_i cells, left justified, in the ith row. (Zero parts are ignored). The rows are numbered from top to bottom. The *hook length* h_x of a cell x in a Ferrers diagram is the number of cells to the right of it plus the number of cells above it plus one. The *content* c_x of a cell x is the number of cells to the left of it minus the number of cells below it. Figure 5a shows the Ferrers diagram of 431 with hook lengths indicated; Fig. 5b shows contents.

The *conjugate* λ^* of a partition λ is the partition (with all nonzero parts) whose Ferrers diagram is the transpose of that of λ. We denote by $H(\lambda)$ the product of the hook lengths of λ and we denote by $C_a(\lambda)$ the product $\prod_x (a + c_x)$ over all cells x in the Ferrers diagram of λ. Note that $H(\lambda) = H(\lambda^*)$.

PROPOSITION 10.

$$\binom{a, a+1, \ldots, a+k-1}{b_1, b_2, \ldots, b_k} = \frac{C_a(\mu)}{H(\mu)},$$

where $\mu = [p(b_1, \ldots, b_k)]^*$.

Proof. Let us write $B(\lambda, a)$ for the binomial determinant in question, where $\lambda = p(b_1, \ldots, b_k)$, so $b_i = \lambda_i + i - 1$. We proceed by induction on b_k. The case $b_k = 0$ is trivial. Now suppose $\lambda_1 = 0$, so that $b_1 = 0$. Then by Lemma 9, $B(\lambda, a) = B(\lambda', a)$ where $\lambda' = (\lambda_2, \ldots, \lambda_k)$ and the formula holds by induction. Finally, suppose $\lambda_1 > 0$. Let $\bar{\lambda} = (\lambda_1 - 1, \ldots, \lambda_k - 1)$ and let $\bar{\mu} = \bar{\lambda}^*$ (so that $\bar{\mu}$ is μ with its largest part removed). Then by Lemma 8,

$$\begin{aligned} B(\lambda, a) &= \frac{a(a+1)\cdots(a+k-1)}{b_1 b_2 \cdots b_k} B(\bar{\lambda}, a-1) \\ &= \left[\prod_x \frac{a + c_x}{h_x}\right] \frac{C_{a-1}(\bar{\mu})}{H(\bar{\mu})} = \frac{C_a(\mu)}{H(\mu)}, \end{aligned}$$

where the product is over all cells x in the first row of the Ferrers diagram of μ.

(a) (b)

FIGURE 5

The duality theorem allows us to evaluate from Proposition 9 binomial determinants of the form

$$\binom{a_1, \ldots, a_k}{0, \ldots, k-1}.$$

In fact we can evaluate the seemingly more general binomial determinant

$$\binom{a_1, a_2, \ldots, a_k}{b, b+1, \ldots, b+k-1}$$

since (as long as $a_1 \geqslant b$) the latter expression is equal to

$$\binom{0, 1, \ldots, b-1, a_1, \ldots, a_k}{0, 1, \ldots, b-1, b, \ldots, b+k-1}.$$

The value of the binomial determinant can be expressed most simply if we consider instead the form

$$D = \binom{a_1+b-k, a_2+b-k, \ldots, a_k+b-k}{b-k, b-k+1, \ldots, b-1}.$$

Let $\lambda = [p(a_1, a_2, \ldots, a_k)]^*$, let $j = a_k - k + 1$, and define the partition μ by $\mu_i + \lambda_{j-i+1} = b$, $i = 1, \ldots, j$. Then the duality theorem applied to Proposition 10 yields $D = C_b(\mu^*)/H(\mu^*)$. It turns out, however, that the simpler formula $D = C_b(\lambda^*)/H(\lambda^*)$ holds. We shall give bijective derivations of both of these formulae from Proposition 10 in the next section.

We note the alternate expression

$$\binom{a_1, a_2, \ldots, a_k}{b, b+1, \ldots, b+k-1} = \frac{(a_1)_b \cdots (a_k)_b}{b! \cdots (b+k-1)!} \Delta(a_1, \ldots, a_k),$$

where

$$\Delta(a_1, \ldots, a_k) = \prod_{1 \leqslant i < j \leqslant k} (a_j - a_i)$$

and $(a)_m = a(a-1) \cdots (a-m+1)$. This formula is easily proved by induction from Lemma 8 and the formula

$$\binom{a_1, \ldots, a_k}{0, \ldots, k-1} = \binom{a_2 - a_1, a_3 - a_1, \ldots, a_k - a.}{1, 2, \ldots, k-1},$$

which is analogous to Lemma 9.

6. Paths and Young Tableaux

Configurations of nonintersecting paths can be used in several ways to encode Young tableaux.

To $\beta=(0\leqslant b_1<\cdots<b_k)$ we associate a "truncated Ferrers diagram" of $b=b_1+\cdots+b_k$ cells: row i with b_i cells is shifted one cell to the left with respect to row $i-1$. A *shifted Young tableau* of shape β is a filling of the truncated Ferrers diagram with positive integers which are (weakly) decreasing from left to right and strictly decreasing from bottom to top. A shifted Young tableau of shape 5421 is shown in Fig. 6.

Given a configuration $\Omega=(w_1,..., w_k)$ of paths satisfying condition (i) of Theorem 1, we can fill in the truncated Ferrers diagram of shape β by writing in the ith row the ordinates of the horizontal steps of path w_i, in decreasing order. It is clear that we obtain a shifted Young tableau iff the configuration satisfies condition (ii) of Theorem 1. For example, the configuration of Fig. 1 corresponds to the tableau of Fig. 6. We deduce:

Corollary 11. *For any sequences* $\alpha=(0\leqslant a_1<\cdots<a_k)$ *and* $\beta=(0\leqslant b_1<\cdots b_k)$, *the binomial determinant*

$$\binom{a_1,..., a_k}{b_1,..., b_k}$$

is the number of shifted Young tableaux of shape β *such that if* f_i *and* l_i *are the first and last elements of row i,* $a_{i-1}\leqslant f_i\leqslant a_i$ *and* $l_i\geqslant b_i$.

Remark. By adding a_i at the beginning and b_i at the end of row i, we see that the tableaux of Corollary 11 correspond to shifted Young tableaux with fixed shape and fixed first and last elements in each row.

George Andrews (unpublished) has given a more elegant interpretation to

$$\binom{a_1,..., a_k}{b_1,..., b_k}$$

which is easily shown to be equivalent to Corollary 10: it is the number of row-strict shifted Young tableaux, with nonnegative integer entries, of

			2	
		3	3	
	5	5	4	4
6	6	6	6	5

Figure 6

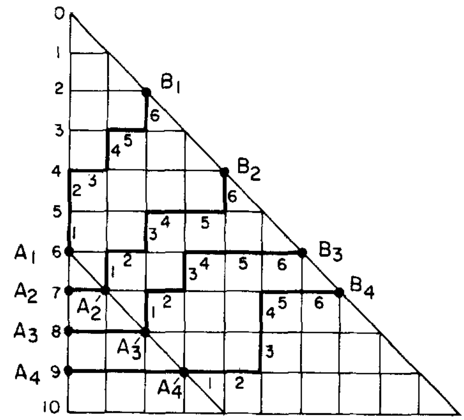

FIGURE 7

shape $(b_1+1,..., b_k+1)$ in which the largest part in row i is a_i. (In a row-strict shifted Young tableau, entries are strictly decreasing from left to right and weakly decreasing from bottom to top.)

If λ is a partition, a *Young tableau* of shape λ is a filling of the (ordinary) Ferrers diagram of λ with positive integers which are increasing from left to right and strictly increasing from bottom to top.

Suppose that $\Omega=(w_1,..., w_k)$ satisfies conditions (i) and (ii) of Theorem 1, where $a_i=a+i-1$, $1\leqslant i\leqslant k$. Let $\lambda=p(b_1,..., b_k)$ and define μ by $\mu_{k-i+1}+\lambda_i=a$, $i=1,..., k$. Let A_i' be the point $(i-1, a_i)$. Then each path w_i must go from A_i to A_i' with horizontal steps. Let us label the steps from A_i' to B_i as $1, 2,..., a$. (There are always a such steps.) Note that path w_i' consists of $b_i-i+1=\lambda_i$ horizontal steps and $a-b_i+i-1=\mu_{k-i+1}$ vertical steps. We now define two Young tableaux $T_1(\Omega)$ and $T_2(\Omega)$: column i of $T_1(\Omega)$ is the sequence of horizontal labels of path w_{k-i+1}' and column i of $T_2(\Omega)$ is the sequence of vertical labels of path w_i'. (See Fig. 7 and 8.) It is clear that T_1 is a bijection from configurations of nonintersecting paths counted by the binomial determinant

$$\binom{a,\ a+1,..., a+k-1}{b_1, b_2, b_k}$$

$T_1(\Omega)$ =

6	6		
5	5	5	
2	4	4	5
1	2	2	3

$T_2(\Omega)$ =

6			
4	6		
2	3	3	4
1	1	1	3

FIGURE 8

to Young tableaux of shape λ^* with entries in $[a]$ and that T_2 is a bijection from these configurations to Young tableaux of shape μ^* with entries in $[a]$.

Thus from Proposition 10 we obtain the following hook length formula (see Stanley [25]):

COROLLARY 12. *Let $\lambda=(\lambda_1\leqslant\cdots\leqslant\lambda_k)$ be a partition and suppose $a\geqslant\lambda_k$. Define the partition μ by $\mu_i+\lambda_{k-i+1}=a$, $i=1,\ldots,k$. Then the number of Young tableaux of shape λ^* with entries in $[a]$ and the number of Young tableaux of shape μ^* with entries in $[a]$ are both equal to $C_a(\lambda^*)/H(\lambda^*)$.*

We note that the bijection from $T_1(\Omega)$ to $T_2(\Omega)$ was given by Stanley [26]. It follows immediately from Corollary 11 that $C_a(\lambda^*)/H(\lambda^*)=C_a(\mu^*)/H(\mu^*)$.

Now let us suppose that $\Omega=(w_1,\ldots,w_k)$ satisfies conditions (i) and (ii) of Theorem 1, with $a_i=c_i+b-k$, $b_i=b-k+i-1$; $i=1,\ldots,k$, where $b\geqslant k$. Let $\lambda=p(c_1,\ldots,c_k)$. Each path w_i contains $a_i-b_i=c_i-i+1=\lambda_i$ vertical steps. Let us assign to a vertical step with abscissa i the label $b-i$. (See Fig. 9.) Then we construct a Young tableau $T(\Omega)$ in which row i of $T(\Omega)$ consists of the labels of path w_{k-i+1}, in increasing order. It is easy to see that T gives a bijection between configurations of nonintersecting paths counted by the binomial determinant

$$\binom{c_1+b-k,\, c_2+b-k,\ldots,\, c_k+b-k}{b-k,\, b-k+1,\ldots,\, b-1}$$

and Young tableaux of shape λ with parts in $[b]$. Thus from Proposition 10 and Corollary 12 we have:

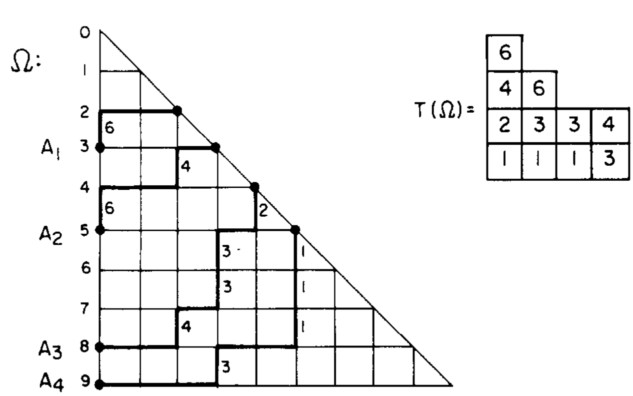

FIGURE 9

COROLLARY 13. *If* $0 \leqslant c_1 < \cdots < c_k$ *and* $b \geqslant k$ *then*

$$\binom{c_1+b-k,\ c_2+b-k,\dots,\ c_k+b-k}{b-k,\ b-k+1,\dots,\ b-1} = \frac{C_b(\lambda)}{H(\lambda)},$$

where $\lambda = p(c_1,\dots, c_k)$.

7. BIJECTIVE PROOFS FOR HOOK LENGTH FORMULAE

It is natural to ask whether a bijective proof can be given for the hook length formula of Proposition 10. Everything we have done is bijective except for Lemma 8, which is trivial algebraically. To construct a bijective proof of Proposition 10 we need only find a bijective proof of Lemma 8.

The case $k=1$ of Lemma 8 is the formula

$$b\binom{a}{b} = a\binom{a-1}{b-1}$$

which is easily explained bijectively: since $\binom{a}{b}$ counts paths from $(0, a)$ to (b, b), and these paths each have b horizontal steps, $b\binom{a}{b}$ counts paths from $(0, a)$ to (b, b) in which one of the horizontal steps is "marked." Similarly $a\binom{a-1}{b-1}$ counts paths from $(0, a-1)$ to $(b-1, b-1)$ in which one of the vertices is marked. To go from one to the other, we contract a marked step to a marked vertex, or expand a marked vertex to a marked horizontal step.

This idea can be applied to a configuration of nonintersecting paths: we mark a horizontal step in each path ($b_1 b_2,\dots, b_k$ choices) or a vertex in each path ($a_1 a_2,\dots, a_k$ choices). Unfortunately, the contracted or expanded paths need not be nonintersecting. However, the correspondence θ which we obtain can be applied to any marked element $(\sigma; (w_1,\dots, w_k))$ of E (as defined in Section 2), and θ preserves $(-1)^{\mathrm{Inv}(\sigma)}$. Moreover, the involution ϕ of Section 2 can easily be modified to apply to marked paths. Thus if $B=[b_1]\times\cdots\times[b_k]$ and $A=[a_1]\times\cdots\times[a_k]$, we have the diagram of bijections shown in Fig. 10. It follows that $B\times NI$ and $A\times NI'$ have the

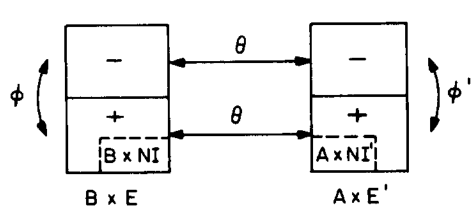

FIGURE 10

same cardinality. An explicit bijection between $B \times NI$ and $A \times NI'$ can be obtained by applying the "involution principle" of Garsia and Milne [7]: given ξ in $B \times NI$ there is a unique ξ' in $A \times NI'$ which may be expressed as $(\theta\phi\theta\phi')^n\theta(\xi)$ for some $n \geqslant 0$.

This "bijective proof" of Lemma 8 can then be combined with the bijective proof of Lemma 9 to yield a bijective proof of Proposition 10. The program we have just sketched has been carried out in detail by Remmel and Whitney [23]; see also related work in [22, 24].

A disadvantage of the bijections obtained this way are that they are not "canonical" because the involution ϕ of Section 2 depends on the choice of a point $C(\xi)$ for any configuration ξ in $E - NI$, and there are many different choices that will work. It may be hoped that by the proper choice of $C(\xi)$ a bijection can be obtained with a more direct description, or at least with some interesting properties.

For other bijective proofs of hook length formulae, see [4–6, 12].

8. Applications of Generating Functions

In this section we give some noncombinatorial methods for expressing certain binomial determinants in terms of coefficients of quotients of the power series

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{mk+r}}{(mk+r)!}.$$

First we prove an elementary transformation formula.

Proposition 14. *For any $m \geqslant a_k$,*

$$\binom{a_1, \ldots, a_k}{b_1, \ldots, b_k} = \frac{\binom{m}{b_1} \cdots \binom{m}{b_k}}{\binom{m}{a_1} \cdots \binom{m}{a_k}} \binom{m-b_k, m-b_{k-1}, \ldots, m-b_1}{m-a_k, m-a_{k-1}, \ldots, m-a_1}.$$

Proof. If we transpose the determinant

$$\det\left(\binom{a_i}{b_j}\right),$$

and then reverse the order of the rows and columns, we obtain

$$\binom{a_1,\dots,a_k}{b_1,\dots,b_k} = \det\left(\binom{a_{k-j+1}}{b_{k-i+1}}\right) = \det\left(\frac{\binom{m}{b_{k-i+1}}}{\binom{m}{a_{k-j+1}}}\binom{m-b_{k-i+1}}{m-a_{k-j+1}}\right)$$

As a special case, we have

$$\binom{c_1, c_2,\dots, c_k, n}{0, c_1,\dots, c_{k-1}, c_k} = \binom{n-c_k, n-c_{k-1},\dots, n-c_1, n}{0, n-c_k,\dots, n-c_2, n-c_1},$$

which is easily proved combinatorially via Corollary 6.

PROPOSITION 15. *Let m and r be integers, with $0 \leqslant r < m$. Define numbers P_i by*

$$\sum_{i=0}^{\infty} P_i \frac{x^{mi}}{(mi)!} = \frac{x^r}{r!}\Bigg/\left[\sum_{i=0}^{\infty}(-1)^i \frac{x^{mi+r}}{(mi+r)!}\right],$$

with $P_i = 0$ for $i < 0$.

Let $\mathscr{A} = \{a_1 < \cdots < a_k\}$ and $\mathscr{B} = \{b_1 < \cdots < b_k\}$ be subsets of $[n]_0$ and let M be the matrix, $(\binom{mi}{mj}P_{i-j})_{0 \leqslant i,j \leqslant n}$. Then

$$\binom{ma_1+r,\dots, ma_k+r}{mb_1,\dots, mb_k} = \left[\prod_{j\in A}\binom{mj+r}{r}\right] M[\bar{\mathscr{B}} \mid \bar{\mathscr{A}}].$$

Proof. Let $W = (W_{ij})_{0\leqslant i,j\leqslant n}$ be defined by

$$W_{ij} = \frac{\binom{mi}{mj}}{\binom{mj+r}{r}} P_{i-j}.$$

It is easily verified that the matrices W and $(\binom{mi+r}{mj})_{0\leqslant i,j\leqslant n}$ are sign-inverse. Then Jacobi's theorem (as stated in Section 4) yields

$$\begin{aligned}\binom{ma_1+r,\dots, ma_k+r}{mb_1,\dots, mb_k} &= \left[\prod_{j=0}^{n}\binom{mj+r}{r}\right] W[\bar{\mathscr{B}} \mid \bar{\mathscr{A}}] \\ &= \left[\prod_{j=0}^{n}\binom{mj+r}{r}\right]\left[\prod_{j\in\bar{\mathscr{A}}}\binom{mj+r}{r}\right]^{-1} M[\bar{\mathscr{B}} \mid \bar{\mathscr{A}}].\end{aligned}$$

The special case $\mathscr{A} = [n]$, $\mathscr{B} = [n-1]_0$ of Proposition 13 yields

$$\binom{m+r,\, 2m+r,\ldots,\, nm+r}{0,\, m,\ldots,\, (n-1)\,m} = \left[\prod_{j=1}^{n} \binom{mj+r}{r}\right] P_n.$$

If $r=0$, this becomes

$$\binom{m,\, 2m,\ldots,\, nm}{0,\, m,\ldots,\, (n-1)\,m} = P_n,$$

which is equivalent, via Corollary 6, to a result of Carlitz [2].

For $m=2$, the generating functions involve trigonometric functions. For $m=2$ and $r=0$, P_n is the *secant number* S_{2n} *given by*

$$\sec x = \sum_{i=0}^{\infty} S_{2i} \frac{x^{2i}}{(2i)!}.$$

Thus

$$\binom{2,\, 4,\ldots,\, 2n}{0,\, 2,\ldots,\, 2n-2} = S_{2n},$$

and by duality,

$$\binom{1,\, 3,\ldots,\, 2n-1,\, 2n}{0,\, 1,\ldots,\, 2n-3,\, 2n-1} = S_{2n}.$$

For $m=2$ and $r=1$, we have

$$\sum_{i=0}^{\infty} P_i \frac{x^{2i}}{(2i)!} = \frac{x}{\sin x}.$$

It is not hard to show that here $P_i = (-1)^{i+1}(2^{2i}-2)\, B_{2i}$, where B_{2i} is the Bernoulli number given by

$$\sum_{j=0}^{\infty} B_j \frac{x^j}{j!} = x/(e^x - 1).$$

Thus

$$\binom{3,\, 5,\ldots,\, 2n+1}{0,\, 2,\ldots,\, 2n-2} = 1 \cdot 3 \cdots (2n+1)(-1)^{n+1}(2^{2n}-2)\, B_{2n}.$$

The next simplest case of Proposition 15 is $\mathscr{A} = \{2, 3,\ldots, n\}$, $\mathscr{B} = [n-2]_0$, which yields

$$\binom{2m+r,\,2m+r,\ldots,\,nm+r}{0,\,m,\ldots,\,(n-2)\,m}=\left[\prod_{j=2}^{n}\binom{mj+r}{r}\right]\begin{vmatrix} P_{n-1} & \binom{(n-1)\,m}{m} & P_{n-2} \\ P_n & \binom{nm}{m} & P_{n-1} \end{vmatrix}.$$

In particular, for $m=2$, $r=0$, we get

$$\binom{4,\,6,\ldots,\,2n}{0,\,2,\ldots,\,2n-4}=\binom{2n}{2}S_{2n-2}^2-\binom{2n-2}{2}S_{2n}S_{2n-4}.$$

PROPOSITION 16. *Let m, r, and P_i be as in Proposition 15. Let s be an integer with $s\geqslant r$. Define numbers Q_i by*

$$\sum_{j=0}^{\infty}Q_i\frac{x^{mi+s-r}}{(mi+s-r)!}=\left[\sum_{i=0}^{\infty}(-1)^i\frac{x^{mi+s}}{(mi+s)!}\right]\Bigg/\left[\sum_{i=0}^{\infty}(-1)^i\frac{x^{mi+r}}{(mi+r)!}\right].$$

Let $M=(M_{ij})_{0\leqslant i\leqslant n,\,-1\leqslant j\leqslant n}$ by defined by

$$M_{i,-1}=Q_i,$$

$$M_{i,j}=\binom{mi+s-r}{mj+s-r}P_{i-j},\qquad j\geqslant 0.$$

Let $\mathscr{A}=\{a_1<\cdots<a_k\}$ and $\mathscr{B}=\{b_2<b_3<\cdots<b_k\}$ be subsets of $[n]_0$. Then

$$\binom{a_1m+s,\,a_2m+s,\ldots,\,a_km+s}{0,\,b_2m+s-r,\ldots,\,b_km+s-r}=\left[\prod_{j\in\mathscr{A}}\binom{mj+s}{r}\right]M[\bar{\mathscr{B}}\mid\bar{\mathscr{A}}\cup\{-1\}].$$

Proof. We define matrices $U=(U_{ij})_{-1\leqslant i,j\leqslant n}$ and $V=(V_{ij})_{-1\leqslant i,j\leqslant n}$ by

$$U_{i,-1}=1=\binom{mi+s}{0},\qquad U_{-1,j}=0\qquad \text{for } j\geqslant 0,$$

$$U_{ij}=\binom{mi+s}{mj+s-r}\qquad\qquad \text{for } i,j\geqslant 0,$$

$$V_{-1,-1}=1,\qquad V_{-1,j}=0\qquad\qquad \text{for } j\geqslant 0,$$

$$V_{i,-1}=Q_i\qquad\qquad \text{for } i\geqslant 0,$$

$$V_{i,j}=\frac{\binom{mi+s-r}{mj+s-r}}{\binom{mj+s}{r}}P_{i-j}\qquad\qquad \text{for } i,j\geqslant 0.$$

It is easily verified that the matrices U and V are sign-inverse. Then the proposition follows by applying Jacobi's theorem to the minor $U[\mathscr{A} \mid \mathscr{B} \cup \{-1\}]$ and simplifying.

The special case $\mathscr{A} = [n]_0$, $\mathscr{B} = [n-1]_0$ of Proposition 16 yields

$$\binom{s, m+s, ..., nm+s}{0, s-r, ..., (n-1)m+s-r} = \left[\prod_{j=0}^{n} \binom{mj+s}{r}\right] Q_n. \tag{4}$$

If $r=0$, this becomes

$$\binom{s, m+s, ..., nm+s}{0, s, ..., (n-1)m+s} = Q_n,$$

which is equivalent, via Corollary 6, to a result of Carlitz [2].

For $m=1$ and $r=0$ we have $P_i = 1$ and

$$\sum_{i=0}^{\infty} Q_i \frac{x^{i+s}}{(i+s)!} = e^x \left[\sum_{i=0}^{\infty} (-1)^i \frac{x^{i+s}}{(i+s)!}\right],$$

so

$$Q_n = \sum_{i=0}^{n} (-1)^i \binom{n+s}{n-i} = \binom{n+s-1}{n} = \binom{s, s+1, ..., n+s}{0, s, ..., n+s-1},$$

which is a special case of Proposition 10.

For $m=2$, $r=0$, and $s=1$, Q_n is the *tangent number* T_{2n+1} defined by

$$\tan x = \sum_{i=0}^{\infty} T_{2i+1} \frac{x^{2i+1}}{(2i+1)!}.$$

Thus (using duality)

$$\binom{1, 3, ..., 2n+1}{0, 1, ..., 2n-1} = T_{2n+1} = \binom{2, 4, ..., 2n, 2n+1}{0, 2, ..., 2n-2, 2n}.$$

The combinatorial interpretations of the secant and tangent numbers were first found by D. André [1].

For $m=2$, $r=0$, $s=2$, we get the secant numbers again. For $m=2$, $r=0$, $s=3$ we have

$$\sum_{i=0}^{\infty} Q_i \frac{x^{2i+3}}{(2i+3)!} = \frac{x - \sin x}{\cos x} = x \sec x - \tan x,$$

so

$$Q_n = (2n+3)\,S_{2n+2} - T_{2n+3} = \begin{pmatrix} 3, 5, \dots, 2n+3 \\ 0, 3, \dots, 2n+1 \end{pmatrix}.$$

In general,

$$\begin{pmatrix} s, 2+s, \dots, 2n+s \\ 0, s, \dots, 2n+s-2 \end{pmatrix}$$

can be expressed in terms of tangent and secant numbers.

For $m=2$, $r=1$, $s=2$, we have

$$\sum_{i=0}^{\infty} Q_i \frac{x^{2i+1}}{(2i+1)!} = \frac{1-\cos x}{\sin x} = \tan\frac{x}{2},$$

so $Q_i = 2^{-2n-1}\,T_{2n+1}$ and we have

$$\begin{pmatrix} 3, 5, \dots, 2n+3 \\ 0, 3, \dots, 2n+1 \end{pmatrix} = 2^{-n}(n+1)!\,T_{2n+1}$$
$$= 2^n n!\,G_{2n+2},$$

where G_{2n+2} is the *Genocchi number* defined by

$$\sum_{n=1}^{\infty} G_{2n} \frac{x^{2n}}{(2n)!} = x\tan(x/2).$$

The case $\mathscr{A} = [n]$, $\mathscr{B} = [n-2]_0$ of Proposition 16 gives

$$\begin{pmatrix} m+s, 2m+s, \dots, n+s \\ 0, s-r, \dots, (n-2)m+s-r \end{pmatrix} = \left[\prod_{j=1}^{n} \binom{mj+s}{r}\right] \begin{vmatrix} Q_{n-1} & \binom{m(n-1)+sr}{s-r} & P_{n-1} \\ Q_n & \binom{mn+s-r}{s-r} & P_n \end{vmatrix}.$$

Thus, for $m=2$, $r=0$, $s=1$, we get

$$\begin{pmatrix} 3, 5, \dots, 2n+1 \\ 0, 1, \dots, 2n-3 \end{pmatrix} = (2n+1)\,T_{2n-1}\,S_{2n} - (2n-1)\,T_{2n+1}\,S_{2n-2}.$$

There is an analog of Proposition 16 for the case $0 \leqslant s \leqslant r$; however, we shall give here only the formula corresponding to (4).

PROPOSITION 17. *Suppose that* $0 \leqslant s < r < m$. *Define numbers* R_i *by*

$$\sum_{i=0}^{\infty} R_i \frac{x^{mi}}{(mi)!} = -\frac{x^{r-s}}{(r-s)!}\left[\sum_{i=0}^{\infty} (-1)^i \frac{x^{mi+s}}{(mi+s)!}\right] \Big/ \left[\sum_{i=0}^{\infty} (-1)^i \frac{x^{mi+r}}{(mi+r)!}\right].$$

Then for $n \geqslant 1$,

$$\begin{pmatrix} r, m+r, 2m+r, 3m+r, \ldots, nm+r \\ 0, r-s, m, 2m, \ldots, (n-1)\,m \end{pmatrix} = \left[\prod_{j=0}^{n} \binom{mj+r}{r}\right] R_n.$$

Proof. Let $Y_i = (-1)^{i-1} R_i$. Then Y_i satisfies the recurrence

$$\sum_{i=0}^{n} \binom{mn+r}{mi} Y_i = \binom{mn+r}{r-s}.$$

Thus we have the system of equations

$$\binom{r}{r-s} = \binom{r}{0} Y_0,$$

$$\binom{m+r}{r-s} = \binom{m+r}{0} Y_0 + \binom{m+r}{m} Y_1,$$

$$\cdots$$

$$\binom{nm+r}{r-s} = \binom{nm+r}{0} Y_0 + \binom{nm+r}{m} Y_1 + \cdots + \binom{nm+r}{nm} Y_n.$$

Solving for Y_n, we obtain

$$Y_n = \left[\prod_{j=0}^{n} \binom{mj+r}{r}\right]^{-} \begin{vmatrix} \binom{r}{0} & \binom{r}{m} & \cdots \binom{r}{(n-1)\,m} & \binom{r}{r-s} \\ \binom{m+r}{0} & \binom{m+r}{m} & \cdots \binom{m+r}{(n-1)\,m} & \binom{m+r}{r-s} \\ \vdots & \vdots & & \vdots \\ \binom{nm+r}{0} & \binom{nm+r}{m} & \cdots \binom{nm+r}{(n+1)\,m} & \binom{nm+r}{r-s} \end{vmatrix}.$$

Moving the last column $(n-1)$ columns to the left gives

$$Y_n = (-1)^{n-1} \left[\prod_{j=0}^{n} \binom{mj+r}{r}\right]^{-1} \begin{pmatrix} r, m+r, \ldots, nm+r \\ 0, r-s, \ldots, (n-1)\,m \end{pmatrix}.$$

The most interesting case of Proposition 17 is $m=2$, $s=0$, $r=1$. Here we have

$$\sum_{i=0}^{\infty} R_i \frac{x^{2i}}{(2i)!} = -x\cos x/\sin x = -x\cot x$$
$$= \sum_{i=0}^{\infty} (-1)^{i+1} 2^{2i} B_{2i} \frac{x^{2i}}{(2i)!},$$

so $R_n = (-1)^{n+1} 2^{2n} B_{2n}$ and thus

$$\begin{pmatrix} 1, 3, 5, 7, \ldots, 2n+1 \\ 0, 1, 2, 4, \ldots, 2n-2 \end{pmatrix} = (-1)^{n+1}\, 1\cdot 3 \cdots (2n+1)\cdot 2^{2n} B_{2n}.$$

This seems to be the first example of a combinatorial interpretation for $c_n B_{2n}$, where c_n is a simple product.

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