Rook polynomials

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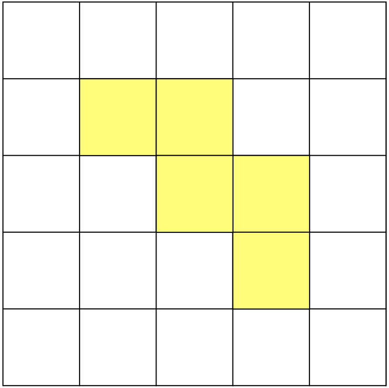

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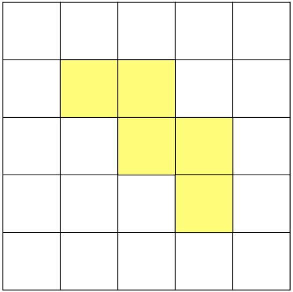
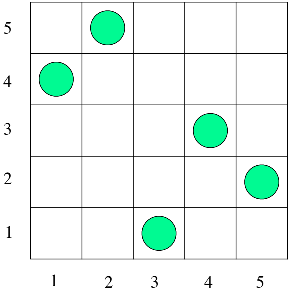


In our example, $r_0 = 1$, $r_1 = 5$, $r_2 = 6$, $r_3 = 1$, $r_4 = r_5 = 0$.

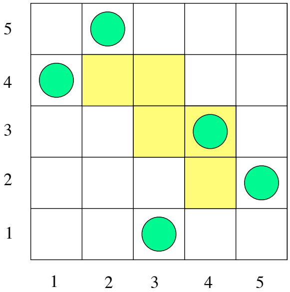
Hit numbers

We can identify a permutation π of $[n] = \{1, 2, \ldots, n\}$ with the set of ordered pairs $\{\, (i, \pi(i)) : i \in [n] \,\} \subseteq [n] \times [n]$, and we can represent such a set of ordered pairs as a set of n squares from $[n] \times [n]$, no two in the same row or column.



This is the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 3 & 2 \end{pmatrix}$. (The rows are i and the columns are $\pi(i)$.)

The squares of a permutation that are on the board are called hits of the permutation. So this permutation has just one hit:



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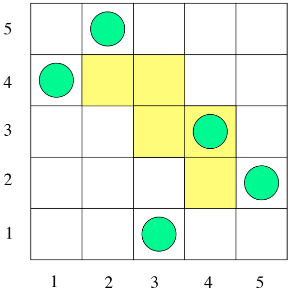

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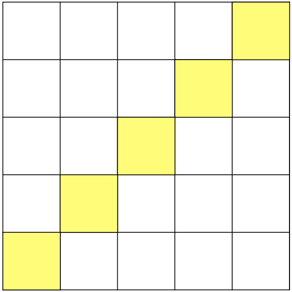
Basic problem: Compute the hit numbers. Sometimes we just want h_0, the number of permutations that avoid the board.

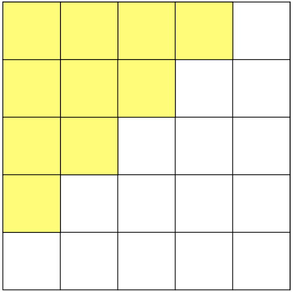
Examples

For the board



h_k is the number of permutations with k fixed points, and in particular, h_0 is the number of derangements.

For the board



h_k is the number of permutations with k excedances, an Eulerian number.

The fundamental identity

$$\sum_i h_i \binom{i}{j} = r_j(n-j)!.$$

Proof: Count pairs (π, H) where H is a j-subset of the set of hits of π. Picking π first gives the left side. Picking H first gives the right side, since a choice of j nonattacking rooks can be extended to a permutation of $[n]$ in $(n-j)!$ ways.

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Multiplying by t^j and summing on j gives

$$\sum_i h_i (1+t)^i = \sum_j t^j r_j (n-j)!.$$

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Multiplying by t^j and summing on j gives

$$\sum_i h_i (1+t)^i = \sum_j t^j r_j (n-j)!.$$

so setting $t = -1$ gives

$$h_0 = \sum_j (-1)^j r_j (n-j)!.$$

Inclusion-Exclusion

Another way to look the formula $h_0 = \sum_k (-1)^k r_k (n-k)!$ is through inclusion-exclusion. We want to count permutations π of $[n]$ satisfying none of the properties $\pi(i) = j$ for $(i, j) \in B$, where B is the board.

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If a set of k properties is consistent (corresponding to nonattacking rooks) then the number of permutations satisfying all these properties is $(n-k)!$; otherwise the number is 0. Thus the sum over all sets of k properties of the number of permutations satisfying these properties is $r_k(n-k)!$.

Rook polynomials

We define the rook polynomial for a board $B \subseteq [n] \times [n]$ by

$$r_B(x) = \sum_k (-1)^k r_k x^{n-k}$$

Now let Φ be the linear functional on polynomials in x defined by

$$\Phi(x^n) = n!.$$

(Then $\Phi(p(x)) = \int_0^\infty e^{-x} p(x)\, dx$.) Thus $h_0(B) = \Phi(r_B(x))$.

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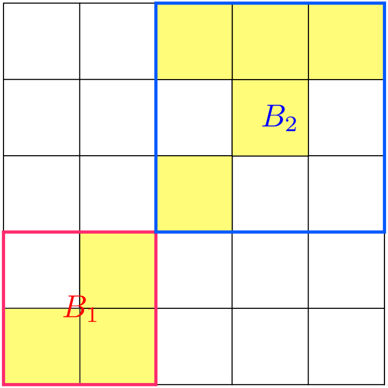
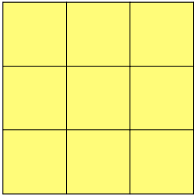
What good are rook polynomials?

They have a multiplicative property: $r_B(x) = r_{B_1}(x) r_{B_2}(x)$.



Of special interest are the rook polynomials of complete boards: Let $l_n(x)$ be the rook polynomial for a board consisting of all of $[n] \times [n]$.



So $l_3(x) = x^3 - 9x^2 + 18x - 6$, and in general

$$l_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 k!\, x^{n-k}.$$

These polynomials are essentially Laguerre polynomials and they are orthogonal with respect to Φ:

$$\Phi(l_m(x)l_n(x)) = \begin{cases} m!^2, & \text{if } m = n \\ 0, & \text{otherwise} \end{cases}$$

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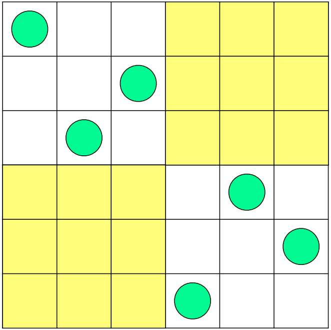

More generally, $\Phi(l_{n_1}(x) l_{n_2}(x) \cdots l_{n_j}(x))$ counts “generalized derangements": permutations of n_1 objects of color 1, n_2 of color 2, ..., such that i and $\pi(i)$ always have different colors.

More generally, $\Phi(l_{n_1}(x)l_{n_2}(x)\cdots l_{n_j}(x))$ counts “generalized derangements": permutations of n_1 objects of color 1, n_2 of color 2, $\dots$, such that i and $\pi(i)$ always have different colors.

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We would like to generalize this to other orthogonal polynomials.

Basic idea: We have a sequence of sets $S_0, S_1, \ldots$ with cardinalities $M_0, M_1, \ldots$. For each n, there is a set of properties that the elements of S_n might have. If a set P of properties is "incompatible" then there is no element of S_n with all these properties. Otherwise, there is some number $\rho(P)$ such that the number of elements of S_n with all the properties in P is $M_{n-\rho(P)}$.

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In our example, S_n is the set of permutations of $[n]$, $M_n = n!$, the properties that a permutation π might have are $\pi(i) = j$ for each possible i and j. A set of properties is compatible if and only if it corresponds to a nonattacking configuration of rooks, and for a set P of k compatible properties, $\rho(P) = k$.

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We'd like to count the number of elements of S_n with none of the properties in P. By inclusion-exclusion this is

$$\sum_{\substack{A \subseteq P \\ A \text{ compatible}}} (-1)^{|A|} M_{n-\rho(A)}$$

Now let us define the generalized rook polynomial or characteristic polynomial of P to be

$$r_P(x) = \sum_{\substack{A \subseteq P \\ A \text{ compatible}}} (-1)^{|A|} x^{n-\rho(A)}$$

Then the number of elements of S_n with none of the properties in P is $\Phi(r_B(x))$, where Φ is the linear functional defined by $\Phi(x^n) = M_n$.

A simple example: matching polynomials

Let us take S_n to be the set of complete matchings of $[n]$: partitions of $[n]$ into blocks of size 2. Then $M_n = 0$ if n is odd and if $n = 2k$ then

$$M_n = (n-1)!! = (n-1)(n-3)\ldots 1 = (2k)!/2^k k!.$$

The properties that we consider are of the form "$\{i, j\}$ is a block." Here if A is a set of compatible properties then $\rho(A) = 2|A|$, and the linear functional function Φ has the integral representation

$$\Phi(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} f(x)\, dx,$$

The matching polynomials for “complete boards” are the Hermite polynomials

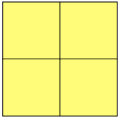
$$H_n(x) = \sum_{k=0}^{n} (-1)^k \frac{n!}{2^k k!\,(n-2k)!} x^{n-k},$$

and these are easily seen to be orthogonal combinatorially.

Let us return to permutations, but add in a parameter to keep track of cycles: we weight each cycle by α. Then the sum of the weights of all permutations of $[n]$ is

$$\alpha^{\overline{n}} = \alpha(\alpha+1)\cdots(\alpha+n-1),$$

which reduces to $n!$ for $\alpha = 1$. Everything works as before, with $\Phi(x^n) = \alpha^{\overline{n}}$. Our “rook numbers” $r_n(\alpha)$ are now polynomials in α. For example, the cycle rook polynomial for the board



is $x^2 - (2+2\alpha)x + (\alpha+\alpha^2)$.

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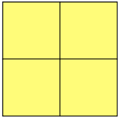

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The cycle rook polynomials for complete boards are general Laguerre polynomials.

Partition polynomials

Now let S_n be the set of partitions of $[n]$, so $M_n = |S_n| = B_n$, the nth Bell number. The linear functional Φ for which $\Phi(x^n) = B_n$ can be represented by

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(Dobiński's formula.) More generally, we could keep track of the number of parts (Stirling numbers of the second kind).

We consider properties

$$P_{ij} : i \text{ and } j \text{ are in the same block.}$$

Then the number of partitions of $[n]$ satisfying P_{ij} is B_{n-1}. The number of partitions with any two of these properties is B_{n-2}.

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B_{n-2}, because P_{13} is implied by P_{12} and P_{23}. So the rank $\rho(\{P_{12}, P_{23}, P_{13}\})$ is 2.

Then the partition polynomial (generalized rook polynomial) of the set $\{P_{12}, P_{23}, P_{12}\}$ (taking $n = 3$) is

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In general, the partition polynomial $r_G(x)$ for a graph G (adjacent vertices in G are not allowed in the same block) is the same as the chromatic polynomial of G.

Why is this? There are two ways to prove this.

(1) Any set of edges corresponding to the same contraction of G will give equivalent conditions. By collecting equivalent terms in the inclusion-exclusion formula for $r_G(x)$, we can write it as a sum over the lattice of contractions of G, and the coefficients will be values of the Möbius function of the lattice of contractions. This sum is known to be equal to the chromatic polynomial.

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(The lattice of contractions of G is the lattice of partitions of the vertex set of G in which every block is connected.)

Alternatively, we could use Möbius inversion directly.

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But it's easy to see that the chromatic polynomial of G can be expressed as

$$P_G(x) = \sum_i u_i x^{\underline{i}},$$

where $x^{\underline{i}} = x(x-1)(x-2)\cdots(x-i+1)$ and u_i is the number of partitions of $[n]$ with i blocks in which vertices adjacent in G are in different blocks. It's well known that $\Phi(x^{\underline{i}}) = 1$ for all i. (*G.-C. Rota suggested that this should be taken as the definition of the Bell numbers!*)

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So $\Phi(P_G(x)) = \sum_i u_i = \Phi(r_G(x))$.

By the same reasoning, for any m, $\Phi(x^m P_G(x)) = \Phi(x^m r_G(x))$, and this implies that $P_G(x) = r_G(x)$.

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If we take all restrictions on $[n]$, we get orthogonal polynomials $C_n(x)$, called Charlier polynomials.

They are orthogonal because $\Phi(C_m(x)C_n(x))$ counts partitions of $\{1, 2, \ldots, m\} \cup \{1, 2, \ldots, n\}$ in which $1, \ldots, m$ are all in different blocks, $1, \ldots, n$ are in different blocks, and there are no singletons. The only way this can happen is if every block consists of a red number and a blue number, and this requires $m = n$.

Factorial rook polynomials

Let's return to ordinary rook numbers. Recall that we defined the rook polynomial of a board B in $[n] \times [n]$ to be $\sum_k (-1)^k r_k x^{n-k}$.

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Goldman, Joichi, and White (1975) defined the factorial rook polynomial of B to be

$$F_B(x) = \sum_k r_k x(x-1)\cdots(x-(n-k)+1) = \sum_k r_k x^{\underline{n-k}}.$$

Why is it useful?

From the fundamental identity $\sum_i h_i \binom{i}{j} = r_j (n-j)!$ and Vandermonde's theorem, we get

$$F_B(x) = \sum_i h_i \binom{x+i}{n}.$$

So the coefficients of $F_B(x)$ in the basis $\{x^{\underline{k}}\}$ for polynomials are the rook numbers for B, and the coefficients of $F_B(x)$ in the basis $\{\binom{x+i}{n}\}_{0 \leq i \leq n}$ for polynomials of degree at most n are the hit numbers for B.

Equivalently,

$$\sum_{m=0}^{\infty} F_B(m)t^m = \frac{\sum_i h_{n-i}t^i}{(1-t)^{n+1}}.$$

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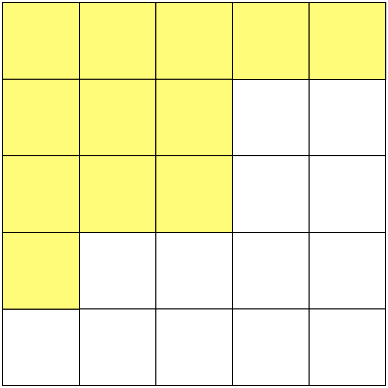
As a consequence of the last formula, we have the reciprocity theorem for factorial rook polynomials:

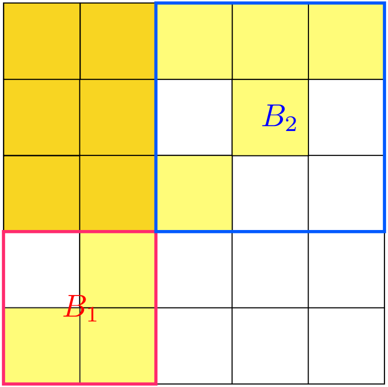
$$F_{\overline{B}}(x) = (-1)^n F_B(-x-1),$$

where $\overline{B}$ is the complement of B in $[n] \times [n]$.

Goldman, Joichi, and White showed that for Ferrers boards:

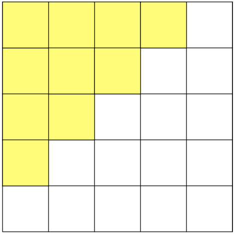


the factorial rook polynomial factors nicely into linear factors and they also proved a factorization theorem for factorial rook polynomials:



$$F_B(x) = F_{B_1}(x)F_{B_2}(x).$$

A simple example: the factorial rook polynomial for the 1×1 empty board  is x. So by the factorization theorem, the factorial rook polynomial for the upper triangular board



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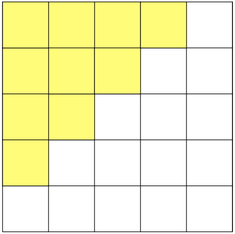

is $F_B(x) = x^n$.

Then $\sum_{m=0}^{\infty} m^n t^m = A_n(t)/(1-t)^{n+1}$, where $A_n(t)$ is the Eulerian polynomial, and by the reciprocity theorem, $F_{\overline{B}}(x) = (x+1)^n$.

The Cover Polynomial

Just as with ordinary rook polynomials, we can introduce a parameter α to keep track of cycles. The “cycle factorial rook polynomial” is defined by

$$F_B(x, \alpha) = \sum_k r_k(\alpha) x^{\underline{n-k}}.$$

It was introduced by Chung and Graham in 1995 under the name cover polynomial.

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The polynomials

$$\frac{(x+\alpha)^{\bar{i}} x^{\underline{n-i}}}{\alpha^{\overline{n}}} = \frac{(x+i+\alpha-1)\cdots(x+\alpha)x(x-1)\cdots(x+i-n+1)}{\alpha(\alpha+1)\cdots(\alpha+n-1)} \quad (1)$$

are a new basis for polynomials of degree at most n that reduce to $\binom{x+i}{n}$ for $\alpha = 1$.

We have the generating function

$$\sum_{m=0}^{\infty} \binom{m+\alpha-1}{m} F_B(m,\alpha) t^m = \frac{\sum_i h_{n-i}(\alpha) t^i}{(1-t)^{n+\alpha}}.$$

The Goldman-Joichi-White result on factorization of $F(x)$ for Ferrers boards extends directly.

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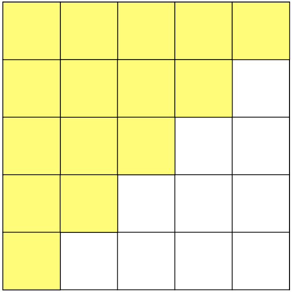
There is a beautiful result of Morris Dworkin giving a sufficient condition for the cover polynomial of a permuted Ferrers board to factor nicely.

Let T_n be the staircase board $\{ \{i, j\} : 1 \leq i \leq j \leq n \}$.



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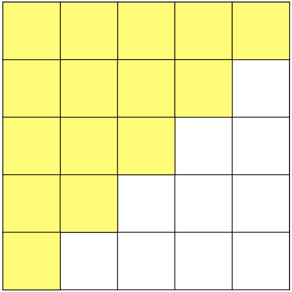

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For a permutation σ, let $\sigma(T_n)$ be T_n with its rows permuted by σ, so there are $\sigma(i)$ squares in row i.

Dworkin's theorem: If σ is a noncrossing permutation with c cycles, then $F_{\sigma(T_n)} = (x+\alpha)^c(x+1)^{n-c}$.

As a consequence, the generating polynomial $A_{n,c}(t,\alpha)$ for permutations π of $[n]$ according to the cycles of π and excedances of $\sigma \circ \pi$ is given by

$$\frac{A_{n,c}(t,\alpha)}{(1-t)^{n+\alpha}} = \sum_{m=0}^{\infty} \binom{m+\alpha-1}{m} (m+\alpha)^c (m+1)^{n-c} t^m.$$

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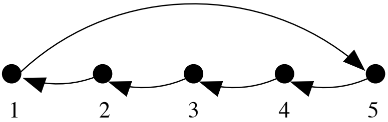
A noncrossing permutation with one cycle looks like this:



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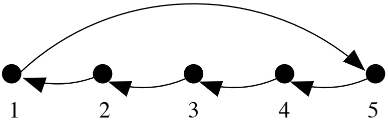
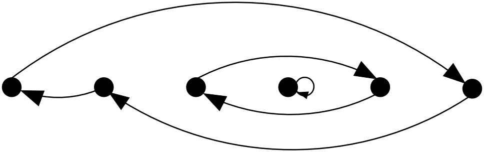


In generally, a noncrossing permutation is made from a noncrossing partition by making each block into a cycle of this type:



So the number of noncrossing permutations of $[n]$ is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ and the number of noncrossing permutations of $[n]$ with c cycles is the Narayana number $\frac{1}{n}\binom{n}{c}\binom{n}{c-1}$.

q-analogs of factorial rook and cover polynomials have been studied by Dworkin, Garsia, Remmel, Haglund, Butler, and others.