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Arc permutations

Sergi Elizalde · Yuval Roichman

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Abstract Arc permutations and unimodal permutations were introduced in the study of triangulations and characters. This paper studies combinatorial properties and structures on these permutations. First, both sets are characterized by pattern avoidance. It is also shown that arc permutations carry a natural affine Weyl group action, and that the number of geodesics between a distinguished pair of antipodes in the associated Schreier graph, and the number of maximal chains in the weak order on unimodal permutations, are both equal to twice the number of standard Young tableaux of shifted staircase shape. Finally, a bijection from non-unimodal arc permutations to Young tableaux of certain shapes, which preserves the descent set, is described and applied to deduce a conjectured character formula of Regev.

Keywords Arc permutation · Unimodal permutation · Pattern avoidance · Affine Weyl group · Descent set · Shifted staircase · Weak order

1 Introduction

A permutation in the symmetric group $\mathcal{S}_n$ is an *arc permutation* if every prefix forms an interval in $\mathbb{Z}_n$. It was found recently that arc permutations play an important role in the study of graphs of triangulations of a polygon [3]. A familiar subset of arc permutations is that of unimodal arc permutations, which are the permutations whose inverses have one local maximum or one local minimum. These permutations appear

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in the study of Hecke algebra characters [4, 13]. Their cycle structure was studied by Thibon [16] and others.

In this paper we study combinatorial properties and structures on these sets of permutations.

In Sect. 3 it is shown that both arc and unimodal permutations may be characterized by pattern avoidance, as described in Theorem 1 and Proposition 1.

In Sect. 4 we describe a bijection between unimodal permutations and certain shifted shapes. The shifted shape corresponding to a unimodal permutation π has the property that standard Young tableaux of that shape encode all reduced words of π. It follows that

- Domination in the weak order on unimodal permutations is characterized by inclusion of the corresponding shapes (Theorem 3). Hence, this partially ordered set is a modular lattice (Proposition 4).
- The number of maximal chains in this order is equal to twice the number of staircase shifted Young tableaux, that is, $2\binom{n}{2}! \cdot \prod_{i=0}^{n-2} \frac{i!}{(2i+1)!}$ (Corollary 6).

The above formula is analogous to a well-known result of Richard Stanley [15], stating that the number of maximal chains in the weak order on $\mathcal{S}_n$ is equal to the number of standard Young tableaux of triangular shape.

In Sect. 6 we study a graph on arc permutations, where adjacency is defined by multiplication by a simple reflection. It is shown that this graph has the following property: an arc permutation is unimodal if and only if it appears in a geodesic between two distinguished antipodes. Hence the number of geodesics between these antipodes is, again, $2\binom{n}{2}! \cdot \prod_{i=0}^{n-2} \frac{i!}{(2i+1)!}$. This result is analogous to Ref. [3, Theorem 9.9], and related to Ref. [10, Theorem 2].

The set of non-unimodal arc permutations is not a union of Knuth classes. However, it carries surprising Knuth-like properties, which are described in Sect. 7. A bijection between non-unimodal arc permutations and standard Young tableaux of hook shapes plus one box is presented, and shown to preserve the descent set. This implies that for $n \geq 4$,

$$\sum_{T \in \mathcal{T}_n} \mathbf{x}^{\mathrm{Des}(T)} = \sum_{\pi \in \mathcal{Z}_n} \mathbf{x}^{\mathrm{Des}(\pi)},$$

where $\mathcal{Z}_n$ denotes the set of non-unimodal arc permutations in $\mathcal{S}_n$, $\mathcal{T}_n$ denotes the set of standard Young tableaux of shape $(k, 2, 1^{n-k-2})$ for some $2 \leq k \leq n-2$, and $\mathrm{Des}(\pi)$ is the descent set of π (see Theorem 5). Further enumerative results on arc permutations by descent sets appear in Sect. 8. These enumerative results are then applied to prove a conjectured character formula of Amitai Regev in Sect. 9.

Interactions with other mathematical objects are discussed in the last two sections: close relations to shuffle permutations are pointed out in Sect. 10.2; further representation theoretic aspects are discussed in Sect. 10. In particular, Sect. 10.1 studies a transitive affine Weyl group action on the set of arc permutations, whose resulting Schreier graph is the graph studied in Sect. 6.

2 Basic concepts

In the following definitions, an interval in $\mathbb{Z}$ is a subset $[a,b]=\{a,a+1,\dots,b\}$ for some $a\le b$, and an interval in $\mathbb{Z}_n$ is a subset of the form $[a,b]$ or $[b,n]\cup[1,a]$ for some $1\le a\le b\le n$.

2.1 Unimodal permutations

Definition 1 A permutation $\pi\in\mathcal{S}_n$ is *left-unimodal* if, for every $1\le j\le n$, the first j letters in π form an interval in $\mathbb{Z}$. Denote by $\mathcal{L}_n$ the set of left-unimodal permutations in $\mathcal{S}_n$.

Example 1 The permutation 342561 is left-unimodal, but 3412 is not.

Claim 1 $|\mathcal{L}_n|=2^{n-1}$.

Proof A left-unimodal permutation π is uniquely determined by the subset of values $i\in\{2,\dots,n\}$ such that $\pi(i)>\pi(1)$. There are 2^{n-1} such subsets. □

We denote by $\mathrm{Des}(\pi)$ the descent set of a permutation π, and by $\mathrm{RSK}(\pi)=(P,Q)$ the pair of standard Young tableaux associated with π by the RSK correspondence. For a standard Young tableau T, its descent set $\mathrm{Des}(T)$ is defined as the set of entries i that lie strictly above the row where $i+1$ lies. It is well known that if $\mathrm{RSK}(\pi)=(P,Q)$, then $\mathrm{Des}(\pi)=\mathrm{Des}(Q)$ and $\mathrm{Des}(\pi^{-1})=\mathrm{Des}(P)$.

Remark 1 A permutation π is left-unimodal if and only if $\mathrm{Des}(\pi^{-1})=\{1,2,\dots,i\}$ for some $0\le i\le n-1$. In other words $\pi\in\mathcal{L}_n$ if and only if $\mathrm{RSK}(\pi)=(P,Q)$, where P is a hook with entries $1,2,\dots,i+1$ in the first column, and Q is any hook with the same shape as P. It follows that left-unimodal permutations are a union of Knuth classes.

Definition 2 A permutation $\pi\in\mathcal{S}_n$ is *unimodal* if one of the following holds:

(i) every prefix forms an interval in $\mathbb{Z}$; or
(ii) every suffix forms an interval in $\mathbb{Z}$.

Denote by $\mathcal{U}_n$ the set of unimodal permutations in $\mathcal{S}_n$.

We remark that our definition of unimodal permutations is slightly different from the one given in [4, 13], where unimodal permutations are those whose inverse is left-unimodal in this paper, and in [16], where unimodal permutations are those whose inverse is right-unimodal in our terminology.

Example 2 The permutation 165243 is unimodal.

Claim 2 *For* $n\ge 2$, $|\mathcal{U}_n|=2^n-2$.

Proof A permutation $\pi \in \mathcal{S}_n$ is unimodal if either π or its reversal $\pi^R = \pi(n)\ldots\pi(2)\pi(1)$ is left-unimodal. The only permutations for which both π and π^R are left-unimodal are $12\ldots n$ and $n\ldots 21$. The formula now follows from Claim 1. □

Remark 2 A permutation $\pi \in \mathcal{S}_n$ is unimodal if and only if

$$\mathrm{Des}(\pi^{-1}) = \begin{cases} \{1, 2, \ldots, i\} & \text{or} \\ \{i+1, i+2, \ldots, n-1\} \end{cases}$$

for some $1 \le i \le n-1$. This happens if and only if $\mathrm{RSK}(\pi) = (P, Q)$, where P is a hook with entries $1, 2, \ldots, i+1$ in the first column or in the first row, and Q is any hook with the same shape as P. Thus unimodal permutations are a union of Knuth classes.

2.2 Arc permutations

Definition 3 A permutation $\pi \in \mathcal{S}_n$ is an *arc permutation* if, for every $1 \le j \le n$, the first j letters in π form an interval in $\mathbb{Z}_n$ (where the letter n is identified with zero). Denote by $\mathcal{A}_n$ the set of arc permutations in $\mathcal{S}_n$.

Example 3 The permutation 12543 is an arc permutation in $\mathcal{S}_5$, but 125436 is not an arc permutation in $\mathcal{S}_6$, since $\{1, 2, 5\}$ is an interval in $\mathbb{Z}_5$ but not in $\mathbb{Z}_6$.

Claim 3 *For* $n \ge 2$, $|\mathcal{A}_n| = n2^{n-2}$.

Proof To build $\pi \in \mathcal{A}_n$, there are n choices for $\pi(1)$ and two choices for every other letter except the last one. □

Remark 3 Arc permutations are not a union of Knuth classes. Note, however, that arc permutations may be characterized in terms of descent sets as follows. A permutation $\pi \in \mathcal{S}_n$ is an arc permutation if and only if

$$\mathrm{Des}(\pi^{-1}) = \begin{cases} \{1, 2, \ldots, i, j+1, j+2, \ldots, n-1\} & \text{and} \quad \pi^{-1}(1) < \pi^{-1}(n), \quad \text{or} \\ \{i+1, i+2, \ldots, j\} & \text{and} \quad \pi^{-1}(1) > \pi^{-1}(n) \end{cases}$$

for some $i \le j$.

It is clear from the definition that the sets of left-unimodal, unimodal and arc permutations satisfy $\mathcal{L}_n \subset \mathcal{U}_n \subset \mathcal{A}_n$. We denote by $\mathcal{Z}_n = \mathcal{A}_n \setminus \mathcal{U}_n$ the set of non-unimodal arc permutations. It follows from Remarks 2 and 3 that $\mathcal{Z}_n$ is not a union of Knuth classes. However, $\mathcal{Z}_n$ has some surprising Knuth-like properties, which will be described in Sect. 7.

3 Characterization by pattern avoidance

In this section the sets of left-unimodal permutations, arc permutations, and unimodal permutations are characterized in terms of pattern avoidance. Given a set of patterns

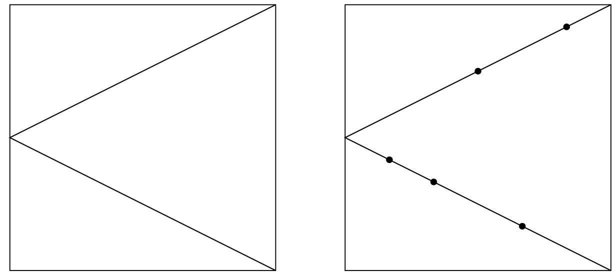

Fig. 1 The grid for left-unimodal permutations, and a drawing of the permutation 32415

$\tau_1, \tau_2, \dots$, denote by $\mathcal{S}_n(\tau_1, \tau_2, \dots)$ the set of permutations in $\mathcal{S}_n$ that avoid all of the τ_i, that is, that do not contain a subsequence whose entries are in the same relative order as those of τ_i. Define $\mathcal{A}_n(\tau_1, \tau_2, \dots)$ analogously.

3.1 Left-unimodal permutations

It will be convenient to use terminology from *geometric grid classes*. Studied by Albert et al. [5], a geometric grid class consists of those permutations that can be drawn on a specified set of line segments of slope ± 1, whose locations are determined by the positions of the corresponding entries in a matrix M with entries in $\{0, 1, -1\}$. More precisely, $\mathcal{G}(M)$ is the set of permutations that can be obtained by placing n dots on the segments in such a way that there are no two dots on the same vertical or horizontal line, labeling the dots with $1, 2, \dots, n$ by increasing y-coordinate, and then reading them by increasing x-coordinate. All the geometric grid classes that we consider in this paper are also profile classes in the sense of Murphy and Vatter [9].

Left-unimodal permutations are those that can be drawn on the picture on the left of Fig. 1, which consists of a segment of slope 1 above a segment of slope -1. The picture on the right shows a drawing of the permutation 32415. The grid class of permutations that can be drawn on this picture is denoted by

$$\mathcal{G}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

so that we have

$$\mathcal{L}_n = \mathcal{G}_n\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \mathcal{G}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cap \mathcal{S}_n.$$

It is clear from the description that geometric grid classes are always closed under pattern containment, so they are characterized by the set of minimal forbidden patterns. In the case of left-unimodal permutations, we get the following description.

Claim 4 $\mathcal{L}_n = \mathcal{S}_n(132, 312)$.

Proof The condition that every prefix of π is an interval in $\mathbb{Z}$ is equivalent to the condition that there is no pattern $\pi(i)\pi(j)\pi(k)$ (with $i < j < k$) where the value of $\pi(k)$ is between $\pi(i)$ and $\pi(j)$, that is, π avoids 132 and 312. □

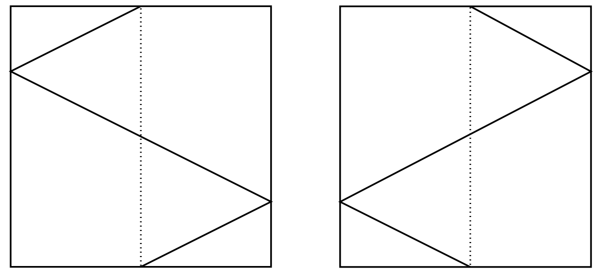

Fig. 2 Grids for arc permutations

3.2 Arc permutations

Arc permutations can be characterized in terms of pattern avoidance, as those permutations avoiding the eight patterns $\tau \in \mathcal{S}_4$ with $|\tau(1) - \tau(2)| = 2$.

Theorem 1

$$\mathcal{A}_n = \mathcal{S}_n(1324, 1342, 2413, 2431, 3124, 3142, 4213, 4231).$$

Proof For an integer m, denote by $\overline{m}$ the element of $\{1, 2, \ldots, n\}$ that is congruent with m mod n. Let $\pi \in \mathcal{S}_n$, and suppose that $\pi \notin \mathcal{A}_n$. Let $i > 1$ be the smallest number with the property that $\{\pi(1), \pi(2), \ldots, \pi(i)\}$ is not an interval in $\mathbb{Z}_n$. By minimality of i, the set $\{\pi(1), \pi(2), \ldots, \pi(i-1)\}$ contains neither $\overline{\pi(i)+1}$ nor $\overline{\pi(i)-1}$. Letting $j < k$ be such that $\{\pi(j), \pi(k)\} = \{\overline{\pi(i)+1}, \overline{\pi(i)-1}\}$, it follows that $\pi(i-1)\pi(i)\pi(j)\pi(k)$ is an occurrence of one of the eight patterns above.

Conversely, if $\pi \in \mathcal{S}_n$ contains one of the eight patterns, let $\pi(h)\pi(i)\pi(j)\pi(k)$ be such an occurrence, where $h < i < j < k$. Then $\{\pi(1), \pi(2), \ldots, \pi(i)\}$ is not an interval in $\mathbb{Z}_n$. □

Corollary 1 $|\mathcal{S}_n(1324, 1342, 2413, 2431, 3124, 3142, 4213, 4231)| = n2^{n-2}$ *for* $n \geq 2$.

Arc permutations can also be described in terms of grid classes, as those permutations that can be drawn on one of the two pictures in Fig. 2. We write

$$\mathcal{A}_n = \mathcal{G}_n \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} \cup \mathcal{G}_n \begin{pmatrix} 0 & -1 \\ 0 & 1 \\ 1 & 0 \\ -1 & 0 \end{pmatrix}.$$

3.3 Unimodal permutations

In terms of grid classes, unimodal permutations are those that can be drawn on one of the two pictures in Fig. 3, that is,

$$\mathcal{U}_n = \mathcal{G}_n \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cup \mathcal{G}_n \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Next we characterize unimodal permutations in terms of pattern avoidance.

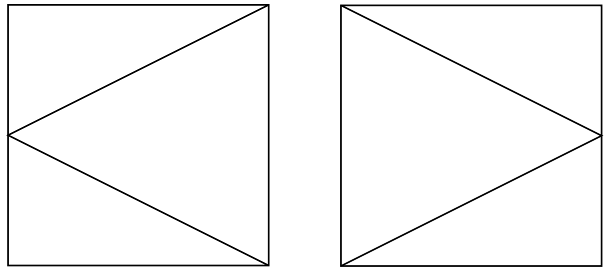

Fig. 3 Grids for unimodal arc permutations

Proposition 1

$$\begin{aligned}\mathcal{U}_n &= \mathcal{A}_n(2143, 3412)\\ &= \mathcal{S}_n(1324, 1342, 2143, 2413, 2431, 3124, 3142, 3412, 4213, 4231).\end{aligned}$$

Proof If π contains 2143 or 3412, then it is clear that π is not unimodal. For the converse, we show that every arc permutation $\pi \in \mathcal{A}_n$ that is not unimodal must contain one of the patterns 2143 or 3412. Since $\pi \in \mathcal{A}_n$, it can be drawn on one of the two pictures in Fig. 2. Suppose it can be drawn on the left picture. Since π is not unimodal, any drawing of π on the left picture requires some element $\pi(i)$ with $i > 1$ to be on the first increasing slope, and some element $\pi(j)$ with $j < n$ to be on the second increasing slope. Then $\pi(1)\pi(i)\pi(j)\pi(n)$ is an occurrence of 3412. An analogous argument shows that if π can be drawn on the right picture in Fig. 2 but it is not unimodal, then it contains 2143. □

Corollary 2 *$|\mathcal{S}_n(1324, 1342, 2143, 2413, 2431, 3124, 3142, 3412, 4213, 4231)| = 2^n - 2$ for $n \geq 2$.*

4 Prefixes associated with the shifted staircase shape

Consider the shifted staircase shape Δ_n with rows labeled $1, 2, \ldots, n-1$ from top to bottom, and columns labeled $2, 3, \ldots, n$ from left to right. Given a filling with the numbers from 1 to $n(n-1)/2$, with increasing entries in each row and column, erase the numbers greater than k, for some k, obtaining a *partial filling* of Δ_n. For each of the remaining entries $1 \leq r \leq k$, if r lies in row i and column j, let t_r be the transposition (i, j). Associate with the partial filling the permutation $\pi = t_1 t_2 \ldots t_k$, with multiplication from the right.

Example 4 The partial filling

	2	3	4	5	6	7
1	1	2	3	6	8	
2		4	5	9	10	
3			7			
4						
5						
6						

corresponds to the product of transpositions

$$(1,2)(1,3)(1,4)(2,3)(2,4)(1,5)(3,4)(1,6)(2,5)(2,6) = 4356217.$$

Theorem 2 *The set of permutations obtained as products of transpositions associated with a partial filling of the shifted staircase shape Δ_n is exactly $\mathcal{L}_n$.*

Proof The first observation is that if two boxes in the tableau are in different rows and columns, the associated transpositions commute. It follows that the resulting permutation depends only on what boxes of the tableaux are filled, but not on the order in which they were filled. For example, the partial filling

	2	3	4	5	6	7
1	1	2	3	4	5	
2		6	7	8	9	
3			10			
4						
5						
6						

yields again the permutation 4356217, just as the partial filling in the above example, since both have the same set of filled boxes.

We claim that, from the set of filled boxes, the corresponding permutation can be read as follows. Let i be the largest such that the box $(i, i+1)$ is filled. Then, starting at the bottom-left corner of that box, consider the path with north and east steps (along the edges of the boxes of the tableau) that separates the filled and unfilled boxes, ending at the top-right corner. At each east step, read the label of the corresponding column, and at each north step, read the label of the corresponding row. This claim can be easily proved by induction on the number of filled boxes. The permutations obtained by reading the labels of such paths are precisely the left-unimodal permutations. $\square$

The above proof gives a bijection between $\mathcal{L}_n$ and the set of shifted shapes of size at most $\binom{n}{2}$, which consist of the filled boxes in partial fillings.

Definition 4 For $\pi \in \mathcal{L}_n$, denote by shape(π) the shifted shape corresponding to any partial filling of Δ_n associated with π.

5 The weak order on $\mathcal{U}_n$

5.1 A criterion for domination

Let $\ell(\cdot)$ be the length function on the symmetric group $\mathcal{S}_n$ with respect to the Coxeter generating set $\mathrm{S} := \{\sigma_i : 1 \le i \le n-1\}$, where σ_i is identified with the adjacent transposition $(i, i+1)$. Recall the definition of the (right) weak order on $\mathcal{S}_n$: for every pair $\pi, \tau \in \mathcal{S}_n$, $\pi \le \tau$ if and only if $\ell(\pi) + \ell(\pi^{-1}\tau) = \ell(\tau)$. Denote this poset by $\mathrm{Weak}(\mathcal{S}_n)$. Recall that $\mathrm{Weak}(\mathcal{S}_n)$ is a lattice, which is not modular. First, we give a combinatorial criterion for weak domination of unimodal permutations.

The concept of shifted shape from Definition 4 can be extended to all unimodal permutations as follows: for $\pi \in \mathcal{U}_n \setminus \mathcal{L}_n$ let $\mathrm{shape}(\pi) := \mathrm{shape}(w_0 \pi w_0)$, where w_0 denotes the longest permutation $n \dots 21$, which is the maximum in $\mathrm{Weak}(\mathcal{U}_n)$. Denote by $e = 12 \dots n$ the identity permutation, which is the minimum in $\mathrm{Weak}(\mathcal{U}_n)$. Note that $w_0 \mathcal{L}_n w_0 = \mathcal{U}_n \setminus \mathcal{L}_n \cup \{e, w_0\}$.

Theorem 3 *For every pair $\pi, \tau \in \mathcal{U}_n$, $\pi \le \tau$ in* $\mathrm{Weak}(\mathcal{S}_n)$ *if and only if*

(i) *either $\pi, \tau \in \mathcal{L}_n$ or $\pi, \tau \in w_0 \mathcal{L}_n w_0$, and*
(ii) $\mathrm{shape}(\pi) \subseteq \mathrm{shape}(\tau)$.

Proof By [6, Corollary 1.5.2, proposition 3.1.3], if $\pi \le \tau$ in $\mathrm{Weak}(\mathcal{S}_n)$, then the corresponding descent sets satisfy $\mathrm{Des}(\pi^{-1}) \subseteq \mathrm{Des}(\tau^{-1})$. Combining this with the characterizations of left-unimodal and unimodal permutations by descent sets, given in Remarks 1 and 2, condition (i) follows.

Now we may assume, without loss of generality, that $\pi, \tau \in \mathcal{L}_n$ (for $\pi, \tau \in w_0 \mathcal{L}_n w_0$, the same proof holds by symmetry, by conjugation by w_0). To complete the proof it suffices to show that for two left-unimodal permutations, domination in weak order is equivalent to inclusion of the corresponding shapes. Indeed, recall the bijection from $\mathcal{L}_n$ to the set of shifted shapes of size at most $\binom{n}{2}$, described in Sect. 4. By this bijection, for any $\pi \in \mathcal{L}_n \setminus \{w_0\}$, the addition of a box in the border of $\mathrm{shape}(\pi)$ corresponds to a switch of two adjacent increasing letters in π giving a permutation in $\mathcal{L}_n$. This is precisely the covering relation in $\mathrm{Weak}(\mathcal{U}_n)$. Thus, for two left-unimodal permutations, the covering relation in $\mathrm{Weak}(\mathcal{U}_n)$ is equivalent to the covering relation in the poset of shifted shapes inside Δ_n ordered by inclusion, and hence domination is equivalent. $\square$

Corollary 3 *For every $\pi \in \mathcal{U}_n$*

$$\ell(\pi) = \left|\mathrm{shape}(\pi)\right|,$$

where $|\mathrm{shape}(\pi)|$ denotes the size of the shape.

5.2 Enumeration of maximal chains

Denote by $\mathrm{Weak}(\mathcal{U}_n)$ the subposet of $\mathrm{Weak}(\mathcal{S}_n)$ which is induced by $\mathcal{U}_n$. Theorem 3 implies the following nice properties of this poset.

Corollary 4 $\mathrm{Weak}(\mathcal{U}_n)$ *is a graded self-dual modular lattice.*

Corollary 5 *For every $\pi \in \mathcal{U}_n \setminus \{w_0\}$, the number of maximal chains in the interval $[e, \pi]$ is equal to the number of standard Young tableaux of shifted shape* $\mathrm{shape}(\pi)$, *hence given by a hook formula.*

Proof By Theorem 2 together with Theorem 3, the statement holds for every $\pi \in \mathcal{L}_n \setminus \{w_0\}$. By conjugation by w_0, it holds for all elements in $\mathcal{U}_n \setminus \mathcal{L}_n$ as well. □

Corollary 6 *For $n > 2$, the number of maximal chains in* $\mathrm{Weak}(\mathcal{U}_n)$ *is equal to twice the number of standard Young tableaux of shifted staircase shape, hence equal to*

$$2\binom{n}{2}! \cdot \prod_{i=0}^{n-2} \frac{i!}{(2i+1)!}.$$

Proof The maximum w_0 covers the two elements $w_0\sigma_1$ and $w_0\sigma_{n-1}$. Thus the number of maximal chains in $\mathrm{Weak}(\mathcal{U}_n)$ is the sum of the numbers of maximal chains in $[e, w_0\sigma_1]$ and $[e, w_0\sigma_{n-1}]$. By Corollary 5, this equals the number of standard Young tableaux of shape $\mathrm{shape}(w_0\sigma_1)$ plus number of standard Young tableaux of shape $\mathrm{shape}(w_0\sigma_{n-1})$. Since $w_0(w_0\sigma_{n-1})w_0 = w_0\sigma_1$, these two shapes are the same, namely Δ_n with the box in row $n-1$ (the bottommost row) removed. By Schur's Formula [14], [8, p. 267 (2)], the number of standard Young tableaux of this shape is $\binom{n}{2}! \cdot \prod_{i=0}^{n-2} \frac{i!}{(2i+1)!}$, completing the proof. □

5.3 The Hasse diagram

Let Γ_n be the undirected Hasse diagram of $\mathrm{Weak}(\mathcal{U}_n)$. A drawing of Γ_4 is given by the black vertices and solid edges in Fig. 4.

Proposition 2

(i) *The diameter of Γ_n is $\binom{n}{2}$.*
(ii) *The vertices e and w_0 are antipodes in Γ_n.*
(iii) *The number of geodesics between e and w_0 is $2\binom{n}{2}! \cdot \prod_{i=0}^{n-2} \frac{i!}{(2i+1)!}$.*

Proof Since $\mathrm{Weak}(\mathcal{U}_n)$ is a modular lattice, the distance between any two vertices is equal to the difference between the ranks of their join and their meet (see [1, Lemma 5.2]). Hence, the diameter is equal to the maximum rank. This proves (i) and (ii). Part (iii) then follows from Corollary 6. □

6 A graph structure on arc permutations

6.1 The graph X_n

Let X_n be the subgraph of the Cayley graph $X(\mathcal{S}_n, \mathtt{S})$ induced by $\mathcal{A}_n$. In other words, the vertex set of X_n is $\mathcal{A}_n$, and two elements $u, v \in \mathcal{A}_n$ are adjacent if and only if there exists a simple reflection $\sigma_i \in \mathtt{S}$, such that $u = v\sigma_i$. The graph X_4 is drawn in Fig. 4. The following theorem shows that X_n and Γ_n share similar properties.

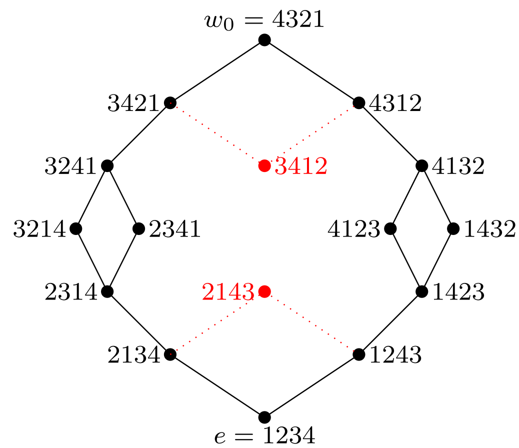

Fig. 4 The graph X_4. The vertices not lying in a geodesic between e and w_0 are drawn in *red with dotted edges*, and they correspond to non-unimodal permutations by Lemma 2

Theorem 4

(i) *The diameter of* X_n *is* $\binom{n}{2}$.
(ii) *The vertices* e *and* w_0 *are antipodes in* X_n.
(iii) *The number of vertices in geodesics between* e *and* w_0 *is* $2^n - 2$.
(iv) *The number of geodesics between* e *and* w_0 *is* $2\binom{n}{2}! \cdot \prod_{i=0}^{n-2} \frac{i!}{(2i+1)!}$.

This theorem will be proved in Sects. 6.2 and 6.3.

6.2 The diameter of X_n

In this subsection we show that the diameter of X_n is $\binom{n}{2}$, proving Theorem 4(i). To see that this is a lower bound, note that the inversion number does not change by more than 1 along each edge of X_n. It follows that the diameter of X_n is at least $\mathrm{inv}(w_0) - \mathrm{inv}(e) = \binom{n}{2}$. This argument also shows that part (ii) of Theorem 4 will follow once we prove part (i), since the distance between e and w_0 is at least $\binom{n}{2}$.

The proof that this is also an upper bound on the diameter is more involved, and it is similar to the proof in [1, Theorem 5.1]. Consider the encoding $\psi : \mathcal{A}_n \to \{0, 1, \dots, n-1\} \times \{0, 1\}^{n-2}$ given by $\psi(\pi) = (\psi_0, \psi_1, \dots, \psi_{n-2})$, where

$$\psi_0 := \pi(1) - 1$$

and, for $1 \le i \le n-2$,

$$\psi_i := \begin{cases} 1 & \text{if } \overline{\pi(i+1) - 1} \in \{\pi(1), \pi(2), \dots, \pi(i)\}, \\ 0 & \text{if } \overline{\pi(i+1) + 1} \in \{\pi(1), \pi(2), \dots, \pi(i)\}, \end{cases}$$

where $\overline{m}$ denotes the element of $\{1, 2, \dots, n\}$ that is congruent with m mod n. Note that exactly one of the two above conditions holds, because $\{\pi(1), \pi(2), \dots, \pi(i)\}$ forms an interval in $\mathbb{Z}_n$.

Example 5 For $\pi = 4352176 \in \mathcal{S}_7$, $\psi(\pi) = (3, 0, 1, 0, 0, 0)$.

The encoding of the vertices of X_4 is given in Fig. 5. The following observation is clear from the definition of X_n and the encoding ψ.

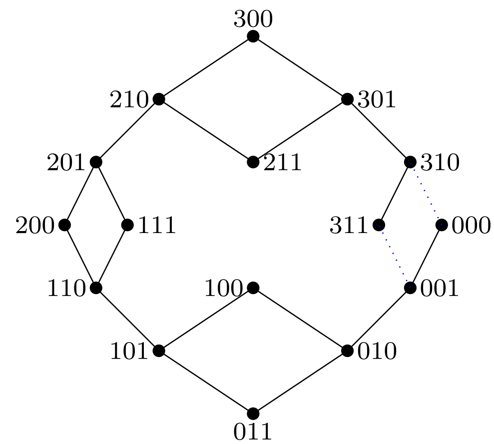

Fig. 5 The graph X_4 with its vertices encoded by ψ (commas and parentheses have been removed). Deleting the *two dotted blue edges* gives the undirected Hasse diagram of the dominance order on $\{0, 1, 2, 3\} \times \{0, 1\}^2 \subset \mathbb{Z}^3$

Lemma 1 *Two arc permutations $\pi, \tau \in \mathcal{A}_n$ with $\pi \neq \tau$ are adjacent in X_n if and only if exactly one of the following holds*:

(i) $\psi(\tau)$ *is obtained from* $\psi(\pi)$ *by switching two adjacent entries* ψ_i *and* ψ_{i+1} *for some* $1 \leq i < n-2$;
(ii) $\psi(\pi)_i = \psi(\tau)_i$ *for all* $0 \leq i < n-2$;
(iii) $\psi(\pi)_0 + \psi(\pi)_1 = \psi(\tau)_0 + \psi(\tau)_1$ *mod* n, *and* $\psi(\tau)_i = \psi(\pi)_i$ *for all* $2 \leq i \leq n-2$.

The set $\{0, 1, \dots, n-1\} \times \{0, 1\}^{n-2}$ of possible encodings inherits the dominance order from $\mathbb{Z}^{n-1}$, that is, $v \leq u$ if and only if for every $1 \leq i \leq n-1$,

$$\sum_{j=1}^{i} v_j \leq \sum_{j=1}^{i} u_j.$$

The covering relations in this poset are almost identical to those described by Lemma 1. More precisely, we have the following result.

Proposition 3 *Through the encoding ψ, the graph X_n is isomorphic to the undirected Hasse diagram of the dominance order on $\{0, 1, \dots, n-1\} \times \{0, 1\}^{n-2}$ with the 2^{n-3} additional edges arising from Lemma 1(iii) with $\{\psi(\tau)_0, \psi(\pi)_0\} = \{n-1, 0\}$.*

Denote by d_{dom} the distance function in the undirected Hasse diagram of the dominance order. To compute $d_{\mathrm{dom}}(\psi(\pi), \psi(\tau))$, let us first recall some basic facts. The dominance order on $\mathbb{Z}^{n-1}$ is a ranked poset where

$$\operatorname{rank}(v_1, \dots, v_{n-1}) = \sum_{j=1}^{n-1} (n-j) v_j = \sum_{j=1}^{n-1} \sum_{k=1}^{j} v_k.$$

This poset is a modular lattice, with

$$(v_1, \dots, v_{n-1}) \wedge (u_1, \dots, u_{n-1}) = (\alpha_1, \dots, \alpha_{n-1})$$

where $\alpha_k = \min\{\sum_{i=1}^{k} v_i, \sum_{i=1}^{k} u_i\} - \min\{\sum_{i=1}^{k-1} v_i, \sum_{i=1}^{k-1} u_i\}$ for every $1 \le k \le n-1$, and

$$(v_1, \dots, v_{n-1}) \vee (u_1, \dots, u_{n-1}) = (\beta_1, \dots, \beta_{n-1})$$

where $\beta_k = \max\{\sum_{i=1}^{k} v_i, \sum_{i=1}^{k} u_i\} - \max\{\sum_{i=1}^{k-1} v_i, \sum_{i=1}^{k-1} u_i\}$ for every $1 \le k \le n-1$. Finally, recall that the distance between two elements in the undirected Hasse diagram of a modular lattice is equal to the difference between the ranks of their join and their meet, see e.g. [1, Lemma 5.2].

Combining these facts implies that

$$\begin{aligned}
&d_{\mathrm{dom}}\bigl(\psi(\pi), \psi(\tau)\bigr) \\
&\quad = \mathrm{rank}\bigl(\psi(\pi) \vee \psi(\tau)\bigr) - \mathrm{rank}\bigl(\psi(\pi) \wedge \psi(\tau)\bigr) \\
&\quad = \sum_{j=0}^{n-2} \sum_{k=0}^{j} \left(\max\left\{ \sum_{i=0}^{k} \psi(\pi)_i, \sum_{i=0}^{k} \psi(\tau)_i \right\} - \max\left\{ \sum_{i=0}^{k-1} \psi(\pi)_i, \sum_{i=0}^{k-1} \psi(\tau)_i \right\} \right) \\
&\qquad - \sum_{j=0}^{n-2} \sum_{k=0}^{j} \left(\min\left\{ \sum_{i=0}^{k} \psi(\pi)_i, \sum_{i=0}^{k} \psi(\tau)_i \right\} - \min\left\{ \sum_{i=0}^{k-1} \psi(\pi)_i, \sum_{i=0}^{k-1} \psi(\tau)_i \right\} \right) \\
&\quad = \sum_{j=0}^{n-2} \left| \sum_{i=0}^{j} \bigl(\psi(\pi)_i - \psi(\tau)_i\bigr) \right|. \qquad (1)
\end{aligned}$$

Now we are ready to prove the upper bound on diameter of X_n. Denoting by d_{X_n} the distance function in X_n, we will show that for any $\pi, \tau \in \mathcal{A}_n$, $d_{X_n}(\pi, \tau) \le \binom{n}{2}$.

Let γ be the n-cycle $(1, \dots, n)$. Clearly, $\mathcal{A}_n$ is invariant under left multiplication by γ. Moreover, left multiplication by γ is an automorphism of X_n. Thus, for any integer k,

$$d_{X_n}(\pi, \tau) = d_{X_n}\bigl(\gamma^k \pi, \gamma^k \tau\bigr) \le d_{\mathrm{dom}}\bigl(\psi\bigl(\gamma^k \pi\bigr), \psi\bigl(\gamma^k \tau\bigr)\bigr), \qquad (2)$$

where the last inequality follows from Proposition 3. Let $x_0 = \psi(\pi)_0 - \psi(\tau)_0$. By (1),

$$d_{\mathrm{dom}}\bigl(\psi(\pi), \psi(\tau)\bigr) = \sum_{j=0}^{n-2} \left| x_0 + \sum_{i=1}^{j} \bigl(\psi(\pi)_i - \psi(\tau)_i\bigr) \right| = \sum_{j=0}^{n-2} |x_j|, \qquad (3)$$

where $x_j = x_0 + \sum_{i=1}^{j} (\psi(\pi)_i - \psi(\tau)_i)$ for $1 \le j \le n-2$. Note that $|x_j - x_{j-1}| \le 1$ for every j. If $x_0 = 0$, then

$$d_{X_n}(\pi, \tau) \le d_{\mathrm{dom}}\bigl(\psi(\pi), \psi(\tau)\bigr) = \sum_{j=0}^{n-2} |x_j| \le 0 + 1 + \cdots + (n-2) = \binom{n-1}{2}$$

and we are done.

Otherwise, we can assume without loss of generality that $1 \le x_0 \le n-1$. Let $k = -\tau(1)$, so that $\psi(\gamma^k \pi)_0 - \psi(\gamma^k \tau)_0 = x_0 - n$. Note that for $1 \le i \le n-2$, we have $\psi(\gamma^k \pi)_i = \psi(\pi)_i$ and $\psi(\gamma^k \tau)_i = \psi(\tau)_i$. Thus, by (1),

$$d_{\mathrm{dom}}\bigl(\psi\bigl(\gamma^k \pi\bigr), \psi\bigl(\gamma^k \tau\bigr)\bigr) = \sum_{j=0}^{n-2} \left| x_0 - n + \sum_{i=1}^{j} \bigl(\psi(\pi)_i - \psi(\tau)_i\bigr) \right| = \sum_{j=0}^{n-2} |x_j - n|.$$

Combining this formula with (2) and (3), we get

$$d_{X_n}(\pi, \tau) \le \min\left\{ \sum_{j=0}^{n-2} |x_j|, \sum_{j=0}^{n-2} |n - x_j| \right\}.$$

If $0 \le x_j \le n$ for all j, then

$$d_{X_n}(\pi, \tau) \le \frac{1}{2}\left(\sum_{j=0}^{n-2} |x_j| + \sum_{j=0}^{n-2} |n - x_j| \right) = \frac{1}{2} \sum_{j=0}^{n-2} (x_j + n - x_j) = \binom{n}{2}.$$

Otherwise, since $|x_j - x_{j-1}| \le 1$ for all j, there must be some i such that $x_i = 0$ or $x_i = n$. If $x_i = 0$ for some i, then $|x_j| \le |j - i|$ for all j, so $\sum_{j=0}^{n-2} |x_j| \le \binom{n}{2}$. Similarly, if $x_j = n$ for some j, then $\sum_{j=0}^{n-2} |n - x_j| \le \binom{n}{2}$, completing the proof of Theorem 4(i).

6.3 Geodesics of X_n

To prove parts (ii) and (iii) of Theorem 4 we need the following lemma.

Lemma 2 *A permutation in $\mathcal{A}_n$ lies in a geodesic between e and w_0 if and only if it is unimodal.*

Proof By Corollary 4, all unimodal permutations lie in geodesics between e and w_0 in the undirected Hasse diagram of Weak$(\mathcal{U}_n)$. By Proposition 2 and Theorem 4(i), the distance between e and w_0 in this Hasse diagram is the same as in X_n, thus the geodesics between these vertices in this Hasse diagram are also geodesics in X_n.

It remains to show that for every non-unimodal arc permutations $\pi \in \mathcal{Z}_n$, π is not in a geodesic between e and w_0. It suffices to prove that for every such π, either $d_{X_n}(e, \pi) > \ell(\pi)$, or $d_{X_n}(w_0, \pi) > \binom{n}{2} - \ell(\pi)$. These two cases are analogous to the dichotomy in Remark 3 and Fig. 2.

If $\pi^{-1}(1) > \pi^{-1}(n)$, then $\pi^{-1}(n-1) < \pi^{-1}(n) < \pi^{-1}(1) < \pi^{-1}(2)$, since otherwise π would be unimodal. Let $\ell = \ell(\pi)$, and suppose for contradiction that $d_{X_n}(e, \pi) = \ell$. Then there is a sequence of arc permutations $\pi = \pi_\ell, \pi_{\ell-1}, \dots, \pi_1 = e$ where each π_i is obtained from π_{i+1} by switching two adjacent letters at a descent, decreasing the number of inversions by one. In particular, in every π_i, the entry $n-1$ is to the left of n, and 2 is to the right of 1. In order to remove the inversion created by the pair $(1, n)$ in π we would have to switch 1 and n, which would create a permutation containing 3142, thus not in $\mathcal{A}_n$ by Theorem 1. This shows that $d_{X_n}(e, \pi) > \ell(\pi)$.

Similarly, if $\pi^{-1}(1) < \pi^{-1}(n)$, then $\pi^{-1}(2) < \pi^{-1}(1) < \pi^{-1}(n) < \pi^{-1}(n-1)$. Let $k = \binom{n}{2} - \ell(\pi)$, and suppose for contradiction that $d_{X_n}(w_0, \pi) = k$. Then there is a sequence of arc permutations $\pi = \pi_k, \pi_{k-1}, \dots, \pi_1 = w_0$ where each π_i is obtained from π_{i+1} by switching two adjacent letters at an ascent, increasing the number of inversions by one. Again, this is impossible because after switching the pair $(1, n)$, the entries $2n1(n-1)$ would form an occurrence of 2413, so the permutation would not be in $\mathcal{A}_n$ by Theorem 1. We conclude that $d_{X_n}(w_0, \pi) > \binom{n}{2} - \ell(\pi)$. □

Proof of Theorem 4 Parts (i) and (ii) were proved in Sect. 6.2. To prove part (iii), combine Lemma 2 with Claim 22. Finally, (iv) follows from Lemma 2 together with Corollary 6. □

7 Equidistribution

In this section we show that the descent set is equidistributed on arc permutations that are not unimodal and on the set of standard Young tableaux obtained from hooks by adding one box in position $(2, 2)$.

7.1 Enumeration of arc permutations by descent set

For a set $D = \{i_1, \dots, i_k\}$, define $\mathbf{x}^D = x_{i_1} \dots x_{i_k}$.

Proposition 4 *For $n \geq 2$,*

$$\sum_{\pi \in \mathcal{A}_n} \mathbf{x}^{\mathrm{Des}(\pi)} = (1 + x_1) \dots (1 + x_{n-1}) \left(1 + \sum_{i=1}^{n-2} \frac{x_i + x_{i+1}}{(1 + x_i)(1 + x_{i+1})} \right).$$

Proof Let $\pi \in \mathcal{A}_n$, and let $i = \max\{\pi^{-1}(1), \pi^{-1}(n)\} - 1$.

If $i = n-1$, then π can be drawn on the picture on the left of Fig. 3. The generating function for these permutations with respect to the descent set is $(1 + x_1) \dots (1 + x_{n-1})$. Indeed, each $\pi(j)$ for $2 \leq j \leq n$ is either larger or smaller than all the previous entries, and causes a descent with $\pi(j-1)$ only in the second case (this is when $\pi(j)$ corresponds to a dot on the descending slope in the picture). So, $\pi(j)$ contributes a factor $1 + x_{j-1}$ to the generating function.

Let us now consider permutations with fixed i, with $1 \leq i \leq n-2$. Since $\pi(1) \dots \pi(i)$ can be drawn on the picture on the left of Fig. 3, the same reasoning as above shows that the contribution of the descents of $\pi(1) \dots \pi(i)$ to the generating function is $(1 + x_1) \dots (1 + x_{i-1})$. Now we have $\pi(i+1) \in \{1, n\}$, by the choice of i. If $\pi(i+1) = 1$ (resp. $\pi(i+1) = n$), then we can draw π on the picture on the left (resp. right) of Fig. 2, with $\pi(i+1)$ being the first entry to the right of the vertical dotted line. In this case, the descent $\pi(i)\pi(i+1)$ contributes x_i (resp. the descent $\pi(i+1)\pi(i+2)$ contributes x_{i+1}) to the generating function. In both cases, each one of the entries $\pi(j)$ with $i+2 \leq j \leq n-1$ will produce a descent $\pi(j)\pi(j+1)$ iff the corresponding dot is on the descending slope to the right of the dotted line. Thus, $\pi(j)$ contributes a factor $1 + x_j$ for each $i+2 \leq j \leq n-1$.

Combining all these contributions we get the generating function

$$\begin{aligned}\sum_{\pi\in\mathcal{A}_n}\mathbf{x}^{\mathrm{Des}(\pi)} &= (1+x_1)\dots(1+x_{n-1})\\ &\quad+\sum_{i=1}^{n-2}(1+x_1)\dots(1+x_{i-1})(x_i+x_{i+1})(1+x_{i+2})\dots(1+x_{n-1})\\ &= (1+x_1)\dots(1+x_{n-1})\left(1+\sum_{i=1}^{n-2}\frac{x_i+x_{i+1}}{(1+x_i)(1+x_{i+1})}\right).\end{aligned}$$

□

Corollary 7 *Let* maj(π) *denote the major index of* π. *For* $n\geq 2$,

$$\sum_{\pi\in\mathcal{A}_n}q^{\mathrm{maj}(\pi)}=(1+q)\dots\left(1+q^{n-2}\right)[n]_q.$$

Proof The generating function for the major index is obtained by replacing x_i with q^i for each $1\leq i\leq n-1$ in the formula from Proposition 4:

$$\sum_{\pi\in\mathcal{A}_n}q^{\mathrm{maj}(\pi)}=(1+q)\dots\left(1+q^{n-1}\right)\left(1+(1+q)\sum_{i=1}^{n-2}\frac{q^i}{(1+q^i)(1+q^{i+1})}\right).$$

The summation inside the parentheses can be simplified as

$$\begin{aligned}\sum_{i=1}^{n-2}\frac{q^i}{(1+q^i)(1+q^{i+1})} &= \frac{1}{q-1}\sum_{i=1}^{n-2}\left(\frac{1}{1+q^i}-\frac{1}{1+q^{i+1}}\right)\\ &= \frac{1}{q-1}\left(\frac{1}{1+q}-\frac{1}{1+q^{n-1}}\right)\\ &= \frac{q-q^{n-1}}{(1-q)(1+q)(1+q^{n-1})}.\end{aligned}$$

Putting it back in the original equation,

$$\begin{aligned}\sum_{\pi\in\mathcal{A}_n}q^{\mathrm{maj}(\pi)} &= (1+q)\dots\left(1+q^{n-1}\right)\left(1+\frac{q-q^{n-1}}{(1-q)(1+q^{n-1})}\right)\\ &= (1+q)\dots\left(1+q^{n-2}\right)\left(1+q^{n-1}+q\frac{1-q^{n-2}}{1-q}\right)\\ &= (1+q)\dots\left(1+q^{n-2}\right)[n]_q.\end{aligned}$$

□

7.2 Non-unimodal arc permutations

Recall that $\mathcal{Z}_n$ denotes the set of arc permutations that are not unimodal. Let f^λ denote the number of standard Young tableaux of shape λ.

Proposition 5 *For $n \geq 4$,*

$$|\mathcal{Z}_n| = \sum_{k=2}^{n-2} f^{(k,2,1^{n-k-2})}.$$

Proof By Claims 2 and 3, it is clear that

$$|\mathcal{Z}_n| = n2^{n-2} - 2^n + 2 = 2^{n-2}(n-4) + 2.$$

On the other hand, using the hook-length formula, we obtain

$$\sum_{k=2}^{n-2} f^{(k,2,1^{n-k-2})} = \sum_{k=2}^{n-2} \frac{(k-1)(n-k-1)}{n-1}\binom{n}{k} = 2^{n-2}(n-4) + 2,$$

where the last step follows from easy manipulations of binomial coefficients. □

Proposition 6 *For $n \geq 2$,*

$$\sum_{\pi\in\mathcal{Z}_n} \mathbf{x}^{\mathrm{Des}(\pi)} = \sum_{\pi\in\mathcal{A}_n} \mathbf{x}^{\mathrm{Des}(\pi)} - 2(1+x_1)\ldots(1+x_{n-1}) + 1 + x_1\ldots x_{n-1}.$$

Proof Since $\mathcal{Z}_n = \mathcal{A}_n \setminus \mathcal{U}_n$, the statement to be proved is equivalent to

$$\sum_{\pi\in\mathcal{U}_n} \mathbf{x}^{\mathrm{Des}(\pi)} = 2(1+x_1)\ldots(1+x_{n-1}) - 1 - x_1\ldots x_{n-1}.$$

Unimodal arc permutations are those that can be drawn on one of the pictures in Fig. 3. We have shown in the proof of Proposition 4 that for permutations that can be drawn on the left picture, the generating function for the descent set is $(1+x_1)\ldots(1+x_{n-1})$. We obtain the same generating function for permutations that can be drawn on the right picture, since each $\pi(j)$ for $1 \leq j \leq n-1$ causes a descent with $\pi(j+1)$ iff it is drawn on the descending slope of the grid, thus contributing a factor $1 + x_j$. Finally, the we have to subtract the contribution of the only two permutations that can be drawn on both grids, which are $12\ldots n$ and $n\ldots 21$. □

Corollary 8 *For $n \geq 2$,*

$$\sum_{\pi\in\mathcal{Z}_n} q^{\mathrm{maj}(\pi)} = (1+q)\ldots\bigl(1+q^{n-2}\bigr)[n]_q - 2(1+q)\ldots\bigl(1+q^{n-1}\bigr) + 1 + q^{\binom{n}{2}}.$$

7.3 Standard Young tableaux of shape $(k, 2, 1^{n-k-2})$

Let $\mathcal{H}_n$ be the set of standard Young tableaux of shape $(k, 1^{n-k})$ (a hook) for some $1 \leq k \leq n$. Let $\mathcal{T}_n$ be the set of standard Young tableaux of shape $(k, 2, 1^{n-k-2})$ for some $2 \leq k \leq n-2$.

Lemma 3

$$\sum_{T\in\mathcal{H}_n} \mathbf{x}^{\mathrm{Des}(T)} = (1+x_1)\dots(1+x_{n-1}).$$

Proof Any $T \in \mathcal{H}_n$ has a 1 in the upper-left corner. T is now determined by the set of entries j with $2 \le j \le n$ that are in the first column, since the rest have to be in the first row. Each such j creates a descent with $j-1$ iff it is in the first column, which gives the contribution $1+x_{j-1}$ to the generating function. □

Theorem 5 *For* $n \ge 4$,

$$\sum_{T\in\mathcal{T}_n} \mathbf{x}^{\mathrm{Des}(T)} = \sum_{\pi\in\mathcal{Z}_n} \mathbf{x}^{\mathrm{Des}(\pi)}.$$

Proof Given $T \in \mathcal{T}_n$, let $i+2$ be the element in the box in the second row and second column. Note that $2 \le i \le n-2$. There are two possibilities for $i+1$: it is either in the first row or in the first column.

- If $i+1$ is in the first row, the entries $1, 2, \dots, i$ form an arbitrary hook with more than one row. As in the above lemma, the descents of these entries contribute $(1+x_1)\dots(1+x_{i-1})-1$ to the generating function, with the -1 corresponding to the invalid one-row tableau. Now i is not a descent of T but $i+1$ is, producing a factor x_{i+1}. Each one of the remaining entries j with $i+3 \le j \le n$ can be in the first row or in the first column, and it creates a descent with $j-1$ iff it is in the first column, which gives the contribution $1+x_{j-1}$ to the generating function. Thus, this case gives a summand
$$\bigl[(1+x_1)\dots(1+x_{i-1})-1\bigr]x_{i+1}(1+x_{i+2})\dots(1+x_{n-1}).$$
- If $i+1$ is in the first column, the entries $1, 2, \dots, i$ form an arbitrary hook with more than one column. The descents of these entries contribute $(1+x_1)\dots(1+x_{i-1})-x_1\dots x_{i-1}$ to the generating function, subtracting the invalid one-column tableau. Now i is a descent of T but $i+1$ is not, producing a factor x_i. As in the previous case, each one of the remaining entries j with $i+3 \le j \le n$ contributes a factor $1+x_{j-1}$ to the generating function. Thus, this case gives a summand
$$\bigl[(1+x_1)\dots(1+x_{i-1})-x_1\dots x_{i-1}\bigr]x_i(1+x_{i+2})\dots(1+x_{n-1}).$$

We have proved that

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_n} \mathbf{x}^{\mathrm{Des}(T)}\\
&\quad= \sum_{i=2}^{n-2}\bigl[(1+x_1)\dots(1+x_{i-1})-1\bigr]x_{i+1}(1+x_{i+2})\dots(1+x_{n-1})\\
&\qquad+ \sum_{i=2}^{n-2}\bigl[(1+x_1)\dots(1+x_{i-1})-x_1\dots x_{i-1}\bigr]x_i(1+x_{i+2})\dots(1+x_{n-1})
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n-2} (1+x_1)\dots(1+x_{i-1})(x_i+x_{i+1})(1+x_{i+2})\dots(1+x_{n-1}) \\
&\quad - \sum_{i=1}^{n-2} (x_{i+1}+x_1\dots x_i)(1+x_{i+2})\dots(1+x_{n-1}).
\end{aligned} \tag{4}
$$

The last sum above can be simplified using the following two telescopic sums:

$$
\begin{aligned}
&\sum_{i=1}^{n-2} x^{i+1}\big(1+x^{i+2}\big)\dots\big(1+x^{n-1}\big) \\
&\quad = \sum_{i=1}^{n-2} \big(1+x^{i+1}\big)\big(1+x^{i+2}\big)\dots\big(1+x^{n-1}\big) - \big(1+x^{i+2}\big)\dots\big(1+x^{n-1}\big) \\
&\quad = (1+x_2)\dots(1+x_{n-1}) - 1, \\
&\sum_{i=1}^{n-2} x_1\dots x_i(1+x_{i+2})\dots(1+x_{n-1}) \\
&\quad = \sum_{i=1}^{n-2} x_1\dots x_i(1+x_{i+1})(1+x_{i+2})\dots(1+x_{n-1}) \\
&\qquad - x_1\dots x_i x_{i+1}(1+x_{i+2})\dots(1+x_{n-1}) \\
&\quad = x_1(1+x_2)\dots(1+x_{n-1}) - x_1\dots x_{n-1}.
\end{aligned}
$$

Plugging these formulas back into (4) we get

$$
\begin{aligned}
\sum_{T\in\mathcal{T}_n} \mathbf{x}^{\mathrm{Des}(T)} &= \sum_{i=1}^{n-2} (1+x_1)\dots(1+x_{i-1})(x_i+x_{i+1})(1+x_{i+2})\dots(1+x_{n-1}) \\
&\quad - (1+x_1)(1+x_2)\dots(1+x_{n-1}) + 1 + x_1\dots x_{n-1} \\
&= \sum_{\pi\in\mathcal{A}_n} \mathbf{x}^{\mathrm{Des}(\pi)} - 2(1+x_1)(1+x_2)\dots(1+x_{n-1}) + 1 + x_1\dots x_{n-1} \\
&= \sum_{\pi\in\mathcal{Z}_n} \mathbf{x}^{\mathrm{Des}(\pi)}.
\end{aligned}
$$

□

7.4 A bijective proof

We now give a bijection ϕ between $\mathcal{Z}_n$ and $\mathcal{T}_n$ that preserves the descent set, providing an alternative proof of Theorem 5.

Given $\pi \in \mathcal{Z}_n$ with $n \geq 4$, consider two cases. If $\pi^{-1}(1) > \pi^{-1}(n)$ (this happens iff π can be drawn on the picture on the left of Fig. 2, and also iff $\pi(1) > \pi(n)$), let

$j=\pi^{-1}(1)$, and let

$$I=\bigl\{i:\pi(i)\geq\pi(1)\bigr\}\cup\bigl\{i:i>j+1 \text{ and } \pi(i-1)<\pi(n)\bigr\}.$$

Then $\phi(\pi)\in\mathcal{T}_n$ is the tableau having the elements of I in the first row, having $j+1$ in the box in the second row and column, and having the rest of the elements of $[n]=\{1,2,\dots,n\}$ in the first column.

If $\pi^{-1}(1)<\pi^{-1}(n)$ (this happens iff π can be drawn on the picture on the right of Fig. 2, and also iff $\pi(1)<\pi(n)$), let $j=\pi^{-1}(n)$, and let

$$I=\bigl\{i:\pi(i)\leq\pi(1)\bigr\}\cup\bigl\{i:i>j+1 \text{ and } \pi(i-1)>\pi(n)\bigr\}.$$

Then $\phi(\pi)\in\mathcal{T}_n$ is the tableau having the elements of I in the first column, having $j+1$ in the box in the second row and column, and having the rest of the elements of $[n]$ in the first row.

For example, if

$$\pi=\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11\\8&9&10&7&11&1&2&6&5&3&4\end{pmatrix},$$

then $j=6=\pi^{-1}(1)>\pi^{-1}(n)=5$, $I=\{1,2,3,5,8,11\}$, and $\phi(\pi)$ is the tableau

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&2&3&5&8&11\\\hline 4&7\\\cline{1-2} 6\\\cline{1-1} 9\\\cline{1-1} 10\\\cline{1-1}\end{array}$$

If

$$\pi=\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12\\2&3&1&4&5&6&12&7&11&10&8&9\end{pmatrix},$$

then $j=7=\pi^{-1}(n)>\pi^{-1}(1)=3$, $I=\{1,3,10,11\}$, and $\phi(\pi)$ is the tableau

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&2&4&5&6&7&9&12\\\hline 3&8\\\cline{1-2} 10\\\cline{1-1} 11\\\cline{1-1}\end{array}$$

In the case $\pi^{-1}(1)>\pi^{-1}(n)$, the descent set of both π and $\phi(\pi)$ equals $\{i: i+1\notin I, i\neq j\}$. In the case $\pi^{-1}(1)<\pi^{-1}(n)$, the descent set of π and $\phi(\pi)$ equals $\{i:i+1\notin I\}\cup\{j\}$. To see that ϕ is a bijection, note that given a tableau in $T\in\mathcal{T}_n$ with entry $j+1$ in the box in the second row and column, we can distinguish the two cases by checking whether j is in the first row (case $\pi^{-1}(1)<\pi^{-1}(n)$) or not (case $\pi^{-1}(1)>\pi^{-1}(n)$). In both cases, the set I can be immediately recovered from the tableau, and together with the value of j, it uniquely determines the permutation $\phi^{-1}(T)$.

7.5 A shape-preserving bijection

Here we give another bijection ψ between $\mathcal{Z}_n$ and $\mathcal{T}_n$. It has the property that $\psi(\pi)$ has the same shape as $\phi(\pi)$, although it does not preserve the descent set. Another property is that the entries in the tableau $\psi(\pi)$ that are weakly north and strictly east of 2 are the elements in the set $C(\pi)$, defined as follows.

Definition 5 For $\pi \in \mathcal{A}_n$, let $C(\pi)$ be the set of values $i \in \{3, 4, \ldots, n\}$ such that $\pi(i-1) - 1 \in \{\pi(1), \pi(2), \ldots, \pi(i-2)\}$.

Let $\pi \in \mathcal{Z}_n$ with $n \geq 4$. If $\pi^{-1}(1) > \pi^{-1}(n)$, let $j = \pi^{-1}(1)$, and let

$$S = \{1\} \cup \bigl\{i+1 : \pi(1) \geq \pi(i) > \pi(n)\bigr\} = [n] \setminus C(\pi).$$

Then $\psi(\pi)$ is the tableau whose first column is S, having $j+1$ in the box in the second row and column, and having the rest of the elements in $[n]$ in the first row.

If $\pi^{-1}(1) > \pi^{-1}(n)$, let $j = \pi^{-1}(n)$, and let

$$S = \{1\} \cup \bigl\{i+1 : \pi(1) \leq \pi(i) < \pi(n)\bigr\} = \{1, 2\} \cup C(\pi).$$

Then $\psi(\pi)$ is the tableau whose first row is S, having $j+1$ in the box in the second row and column, and having the rest of the elements in $[n]$ in the first column.

For example, if

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 10 & 9 & 11 & 8 & 7 & 12 & 1 & 2 & 6 & 5 & 3 & 4 \end{pmatrix},$$

then $j = 7 = \pi^{-1}(1)$, $S = \{1, 2, 3, 5, 6, 10, 11\}$, and $\psi(\pi)$ is the tableau

$$\begin{array}{lllll} 1 & 4 & 7 & 9 & 12 \\ 2 & 8 & & & \\ 3 & & & & \\ 5 & & & & \\ 6 & & & & \\ 10 & & & & \\ 11 & & & & \end{array}$$

If

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 2 & 4 & 1 & 5 & 6 & 11 & 7 & 10 & 9 & 8 \end{pmatrix},$$

then $j = 7 = \pi^{-1}(n)$, $S = \{1, 2, 4, 6, 7, 9\}$, and $\psi(\pi)$ is the tableau

$$\begin{array}{llllll} 1 & 2 & 4 & 6 & 7 & 9 \\ 3 & 8 & & & & \\ 5 & & & & & \\ 10 & & & & & \\ 11 & & & & & \end{array}$$

The map ψ is clearly invertible, since given a tableau $T \in \mathcal{T}_n$, the position of the entry 2 determines which of the cases $\pi^{-1}(1) > \pi^{-1}(n)$ or $\pi^{-1}(1) > \pi^{-1}(n)$ we are in. In both cases, the set S and the value of j, which are immediately recovered from the tableau, uniquely determine the permutation $\psi^{-1}(T)$. The fact that $\phi(\pi)$ and $\psi(\pi)$ have the same shape follows by noting that in both cases $|I| + |S| = n$, with I as defined in Sect. 7.4, and thus the tableaux $\phi(\pi)$ and $\psi(\pi)$, which consist of a hook plus a box, have the same number of rows.

8 Encoding by descents

In this section we present an encoding of arc permutations that is different from the one used in Sect. 6. This encoding, which keeps track of the positions of the descents, is applied to prove further enumerative results, some of which will be used in Sect. 9.

8.1 The encoding

Let $\mathcal{W}_n$ be the set of words $\mathbf{w} = w_1 w_2 \dots w_{n-1}$ over the alphabet $\{A, D\}$ where at most one adjacent pair AD or DA may be underlined. We encode arc permutations by words in $\mathcal{W}_n$ via the following bijection ν.

Lemma 4 *There is a bijection $\nu : \mathcal{A}_n \longrightarrow \mathcal{W}_n$ such that if $\nu(\pi) = \mathbf{w}$, then*

$$\mathbf{w}_i = D \quad \Longleftrightarrow \quad i \in \mathrm{Des}(\pi).$$

Proof Given $\pi \in \mathcal{A}_n$, first let $w_i = D$ if $i \in \mathrm{Des}(\pi)$, and $w_i = A$ otherwise, for each $1 \le i \le n-1$. If $\pi \in \mathcal{L}_n$, then $\nu(\pi)$ is the word $\mathbf{w}$ with no underlined pair. Otherwise, let k be the smallest such that $\{\pi(1), \dots, \pi(k)\}$ is not an interval in $\mathbb{Z}$, and note that $k < n$. If $w_{k-1} = D$, then $\pi(k) = 1 < \pi(k+1)$, so $w_k = A$. Similarly, if $w_k = A$, then $\pi(k) = n > \pi(k+1)$, so $w_k = D$. In both cases, $\nu(\pi)$ is the word $\mathbf{w}$ with the pair $w_{k-1} w_k$ underlined.

To show that ν is a bijection between $\mathcal{A}_n$ and $\mathcal{W}_n$, we describe the inverse map. Given $\mathbf{w} \in \mathcal{W}_n$, let $w_{k-1} w_k$ be the underlined pair in $\mathbf{w}$ if there is one, and let $k = n+1$ otherwise. It is easy to verify that the unique $\pi \in \mathcal{A}_n$ with encoding $\nu(\pi) = \mathbf{w}$ can be recovered as follows.

If $w_{k-1} w_k = \underline{DA}$, let $\delta = n+1$, otherwise let $\delta = k$. Then, for $1 \le i < k$,

$$\pi(i) = \begin{cases} \delta - 1 - |\{j \in [i, k-2] : w_j = A\}| & \text{if } i = 1 \text{ or } w_{i-1} = A, \\ \delta - k + 1 + |\{j \in [i, k-2] : w_j = D\}| & \text{if } w_{i-1} = D. \end{cases}$$

Now let $\delta' = \delta \mod n$. For $k \le i \le n$,

$$\pi(i) = \begin{cases} \delta' + |\{j \in [k, i-1] : w_j = A\}| & \text{if } i = 1 \text{ or } w_i = A, \\ \delta' + n - k - |\{j \in [k, i-1] : w_j = D\}| & \text{if } w_i = D. \end{cases} \qquad \square$$

For example, $\nu(342561) = ADAAD$, $\nu(12543) = A\underline{AD}D$, and $\nu(65781423) = DAA\underline{DA}DA$. The following is an immediate consequence of our encoding of arc permutations.

Proposition 7 *Let $B \subseteq [n-1]$. Then*

$$\left|\left\{\pi \in \mathcal{A}_n : \mathrm{Des}(\pi) = B\right\}\right| = 1 + \left|\left\{i \in [n-2] : |B \cap \{i, i+1\}| = 1\right\}\right|.$$

Proof Permutations in $\mathcal{A}_n$ with descent set B correspond via ν to encodings $\mathbf{w} \in \mathcal{W}_n$ such that $w_i = D$ if and only if $i \in B$. There is one such encoding with no underlined pairs, and one encoding where the pair $w_i w_{i+1}$ is underlined for each i for which this pair equals AD or DA. □

8.2 μ-Left-unimodal permutations

Here we define a generalization of left-unimodal permutations. Let $\mu = (\mu_1, \ldots, \mu_t)$ be a partition of n with t nonzero parts. For $1 \le i \le t$, denote

$$\mu_{(i)} := \sum_{j=1}^{i} \mu_j$$

and

$$S(\mu) := (\mu_{(1)}, \ldots, \mu_{(t)}).$$

Let $\mu_{(0)} := 0$. A permutation $\pi \in \mathcal{S}_n$ is *μ-left-unimodal* if for every $0 \le i < t$ there exists $0 \le j_i \le \mu_{i+1}$ such that

$$\pi^{-1}(\mu_{(i)} + 1) > \pi^{-1}(\mu_{(i)} + 2) > \cdots > \pi^{-1}(\mu_{(i)} + j_i)$$
$$< \pi^{-1}(\mu_{(i)} + j_i + 1) < \cdots < \pi^{-1}(\mu_{(i+1)}).$$

Denote the set of μ-left-unimodal permutations by $\mathcal{L}_\mu$. Note that for the partition with one part $\mu = (n)$, we have $\mathcal{L}_{(n)} = \mathcal{L}_n$, the set of left-unimodal permutations in $\mathcal{S}_n$.

We will consider the set $\mathcal{L}_\mu^{-1}$, consisting of those permutations whose inverse is μ-left-unimodal. Translating the above definition, we see that $\pi \in \mathcal{L}_\mu^{-1}$ if for every $0 \le i < t$, the sequence $\pi(\mu_{(i)} + 1), \pi(\mu_{(i)} + 2), \ldots, \pi(\mu_{(i+1)})$ first decreases and then increases; we say that this sequence is *V-shaped*.

For example, if $\mu = (\mu_1, \mu_2, \mu_3) = (4, 3, 1)$, then $S(\mu) = (\mu_{(1)}, \mu_{(2)}, \mu_{(3)}) = (4, 7, 8)$. In this case $53687142, 35687412 \in \mathcal{L}_\mu^{-1}$ because the sequences 5368, 714, 8, 3568, 741, 2 are V-shaped. However, $53867142, 53681742 \notin \mathcal{L}_\mu^{-1}$ because the sequences 5286 and 174 are not V-shaped.

8.3 Enumeration of arc permutations whose inverse is μ-left-unimodal

Proposition 8 *For every partition $\mu = (\mu_1, \ldots, \mu_r, 1^s)$ of n with $\mu_r > 1$,*

$$\left|\mathcal{A}_n \cap \mathcal{L}_\mu^{-1}\right| = \mu_1 \ldots \mu_r \, 2^{r+s} \left(r + \frac{s}{4} - \sum_{i=1}^{r} \frac{1}{\mu_i}\right).$$

Note that when $\mu = (1^n)$, Proposition 8 gives Claim 3. Indeed, in this case $\mathcal{L}^{-1}_{(1^n)} = \mathcal{S}_n$, $r = 0$ and $s = n$, so

$$|\mathcal{A}_n| = \left|\mathcal{A}_n \cap \mathcal{L}^{-1}_{(1^n)}\right| = 2^n \binom{n}{4} = n2^{n-2}.$$

Proof We use the encoding ν of permutations in $\mathcal{A}_n$ by words in $\mathcal{W}_n$, defined in Lemma 4. Recall that $\pi \in \mathcal{A}_n \cap \mathcal{L}^{-1}_\mu$ if $\pi(\mu_{(i-1)}+1), \pi(\mu_{(i-1)}+2), \ldots, \pi(\mu_{(i)})$ is V-shaped for all $1 \le i \le r+s$. By Lemma 4, this property is equivalent to the fact that for all $1 \le i \le r$, the subword $\mathbf{w}^{(i)} := w_{\mu_{(i-1)}+1} w_{\mu_{(i-1)}+2} \ldots w_{\mu_{(i)}-1}$ has no A followed by a D.

To find $|\mathcal{A}_n \cap \mathcal{L}^{-1}_\mu|$ we will enumerate the words $\mathbf{w} \in \mathcal{W}_n$ where each block $\mathbf{w}^{(i)}$ satisfies this condition. Note that the length of $\mathbf{w}^{(i)}$ is $\mu_i - 1$, and that $\mathbf{w}$ has one letter between each block and the next, and s letters to the right of the rightmost block $\mathbf{w}^{(r)}$. Consider four cases as follows.

(i) If $\mathbf{w}$ has no underlined pair, then $\mathbf{w}^{(i)} = D^k A^{\mu_i - k - 1}$ for some $0 \le k \le \mu_i - 1$, so there are μ_i choices for each $\mathbf{w}^{(i)}$, times 2^{r+s-1} choices for the remaining letters, for a total of $\mu_1\mu_2 \ldots \mu_r 2^{r+s-1}$ words of this form.

(ii) If $\mathbf{w}$ has an underlined pair contained inside a block, say $\mathbf{w}^{(i)}$, then $\mathbf{w}^{(i)} = D^k \underline{DA} A^{\mu_i - k - 3}$ for some $0 \le k \le \mu_i - 3$, so there are $\mu_i - 2$ choices for this block. The remaining choices are as in case (i), so the number of words of this form is

$$\sum_{i=1}^{r} \mu_1 \ldots \mu_{i-1}(\mu_i - 2)\mu_{i+1} \ldots \mu_r 2^{r+s-1} = \mu_1\mu_2 \ldots \mu_r 2^{r+s-1}\left(r - 2\sum_{i=1}^{r} \frac{1}{\mu_i}\right).$$

(iii) If $\mathbf{w}$ has an underlined pair outside of the blocks $\mathbf{w}^{(i)}$, then there are $s-1$ choices for the location of the pair (assuming $s \ge 1$), 2 choices for whether it is $\underline{AD}$ or $\underline{DA}$, and 2^{r+s-3} choices for the remaining letters outside the blocks, giving a total of $\mu_1\mu_2 \ldots \mu_r (s-1)2^{r+s-2}$ words of this form. When $s = 0$ there are no words of this form.

(iv) If $\mathbf{w}$ has an underlined pair that is partly inside a block $\mathbf{w}^{(i)}$ and partly outside, the number of choices for the location of the underlined pair is $2r - 1$ if $s \ge 1$, since it can be at the beginning or at the end of any block but not at the beginning of $\mathbf{w}^{(1)}$, and $2r - 2$ if $s = 0$, since it cannot be at the end of block $\mathbf{w}^{(r)}$ either. Afterwards, the letters inside the blocks can be chosen in $\mu_1\mu_2 \ldots \mu_r$ ways as before, with the understanding that the choice of the underlined letter forces the other underlined letter. Now we have 2^{r+s-2} choices for the not underlined letters outside the blocks, for a total of $\mu_1\mu_2 \ldots \mu_r (2r-1)2^{r+s-2}$ words if $s \ge 1$, or $\mu_1\mu_2 \ldots \mu_r (2r-2)2^{r+s-2}$ if $s = 0$.

Adding the four contributions we obtain the stated formula. □

Next we give a signed version of Proposition 8, that is, a formula for signed enumeration of arc permutations whose inverse is μ-left-unimodal. This formula will be used in the proof of Regev's character formula in Sect. 9.

Proposition 9 *For every partition $\mu = (\mu_1, \ldots, \mu_r, 1^s)$ of n with $\mu_r > 1$,*

$$\sum_{\pi \in \mathcal{A}_n \cap \mathcal{L}_\mu^{-1}} (-1)^{|\operatorname{Des}(\pi) \setminus S(\mu)|} = \frac{1}{4} \cdot s \cdot \prod_{j=1}^{r+s} \bigl(1 + (-1)^{\mu_j - 1}\bigr). \tag{5}$$

Before proving this proposition, we give an example for $n = 4$ and $\mu = (3, 1)$. In this case,

$$\mathcal{A}_n \cap \mathcal{L}_\mu^{-1} = \{1234, 1243, 2134, 2143, 2341, 3214, 3241, 4123, 4132, 4312, 4321\}.$$

Since $|\operatorname{Des}(\pi) \setminus S(\mu)| = |\operatorname{Des}(\pi) \setminus \{3\}|$, the left-hand side of (5) becomes

$$\begin{aligned}&(-1)^0 + (-1)^0 + (-1)^1 + (-1)^1 + (-1)^0 + (-1)^2 + (-1)^1 + (-1)^1 \\ &\quad + (-1)^1 + (-1)^2 + (-1)^2 = 1.\end{aligned}$$

Proof of Proposition 9 We use the encoding ν from Lemma 4. Defining the sign of a permutation π to be $(-1)^{|\operatorname{Des}(\pi) \setminus S(\mu)|}$, we will construct a sign-reversing involution on the set $\mathcal{A}_n \cap \mathcal{L}_\mu^{-1}$. If μ has some part of even size, then this involution will have no fixed points, so all the terms on the left-hand side of (5) cancel with each other. If all the parts of μ have odd size, then some permutations will not be canceled by the involution, but their contribution to the sum is easy to compute.

Recall that $\pi \in \mathcal{A}_n \cap \mathcal{L}_\mu^{-1}$ if and only if $\mathbf{w}^{(i)} = w_{\mu_{(i-1)}+1} w_{\mu_{(i-1)}+2} \ldots w_{\mu_{(i)}-1}$ has no A followed by a D for all $1 \le i \le r$.

Suppose first that μ has some part of even size, and let μ_e be the first such part. We define an involution φ on the set of encodings $\mathbf{w}$ of permutations in $\mathcal{A}_n \cap \mathcal{L}_\mu^{-1}$ by changing only the subword $\mathbf{w}^{(e)}$, and possibly the letters immediately preceding and following $\mathbf{w}^{(e)}$, but leaving the rest of the word $\mathbf{w}$ unchanged. Note that $\mathbf{w}^{(e)}$ has odd length, which we write as $\mu_e - 1 = 2a + 1$. If the underlined pair is completely inside or outside of $\mathbf{w}^{(e)}$ (or there is no underlined pair), define φ on $\mathbf{w}^{(e)}$ as follows:

$$\begin{aligned} &D^{2i+1} A^{2(a-i)} \overset{\varphi}{\longleftrightarrow} D^{2i} A^{2(a-i)+1} \quad \text{for } 0 \le i \le a, \\ &D^{2i+1} \underline{DA} A^{2(a-i-1)} \overset{\varphi}{\longleftrightarrow} D^{2i} \underline{DA} A^{2(a-i-1)+1} \quad \text{for } 0 \le i \le a-1. \end{aligned}$$

If the pair $w_{\mu_{(e)}-1} w_{\mu_{(e)}}$ is underlined, define φ on $\mathbf{w}^{(e)} w_{\mu_{(e)}}$ as follows:

$$\begin{aligned} &D^{2a} \underline{DA} \overset{\varphi}{\longleftrightarrow} D^{2a} \underline{AD}, \\ &D^{2i+1} A^{2(a-i)-1} \underline{AD} \overset{\varphi}{\longleftrightarrow} D^{2i} A^{2(a-i)} \underline{AD} \quad \text{for } 0 \le i \le a-1. \end{aligned}$$

Finally, if the pair $w_{\mu_{(e-1)}} w_{\mu_{(e-1)}+1}$ is underlined, define φ on $w_{\mu_{(e-1)}} \mathbf{w}^{(e)}$ as follows:

$$\begin{aligned} &\underline{AD} A^{2a} \overset{\varphi}{\longleftrightarrow} \underline{DA} A^{2a}, \\ &\underline{AD} D^{2i} A^{2(a-i)} \overset{\varphi}{\longleftrightarrow} \underline{AD} D^{2i-1} A^{2(a-i)+1} \quad \text{for } 1 \le i \le a. \end{aligned}$$

Since the above definitions cover all the possibilities, the map φ is an involution on the set $\nu(\mathcal{A}_n \cap \mathcal{L}_\mu^{-1})$. It is also clear that the sign of the permutation encoded by a word changes when applying φ to it, since the parity of the number of Ds in $\mathbf{w}^{(e)}$ changes (note that the letters $w_{\mu_{(e-1)}}$ and $w_{\mu_{(e)}}$ do not contribute to the sign). It follows that

$$\sum_{\pi \in \mathcal{A}_n \cap \mathcal{L}_\mu^{-1}} (-1)^{|\operatorname{Des}(\pi) \setminus S(\mu)|} = 0$$

when μ has some part of even size, which agrees with the right-hand side of (5) in this case.

Consider now the case where all the parts of μ have odd size. The case $r = 0$ is trivial because then $\mathcal{A}_n \cap \mathcal{L}_\mu^{-1} = \mathcal{A}_n$ and $|\operatorname{Des}(\pi) \setminus S(\mu)| = 0$, so (5) becomes $|\mathcal{A}_n| = n2^{n-2}$, which is true by Claim 3. Suppose in what follows that $r \geq 1$. For each fixed $1 \leq i \leq r$, we first define an operation φ_i on $\mathbf{w}^{(i)}$ and its surrounding letters that can only be applied if this subword has a certain form. Write the length of $\mathbf{w}^{(i)}$, which is even, as $\mu_i - 1 = 2b$. If the underlined pair is completely inside or outside of $\mathbf{w}^{(i)}$, or there is no underlined pair, define φ_i on $\mathbf{w}^{(i)}$ as follows:

$$D^{2i+1}A^{2(b-i)-1} \overset{\varphi_i}{\longleftrightarrow} D^{2i}A^{2(b-i)} \quad \text{for } 0 \leq i \leq b-1,$$

$$D^{2i+1}\underline{DA}A^{2(b-i-1)-1} \overset{\varphi_i}{\longleftrightarrow} D^{2i}\underline{DA}A^{2(b-i-1)} \quad \text{for } 0 \leq i \leq b-2.$$

If $w_{\mu_{(i)}-1}w_{\mu_{(i)}} = \underline{AD}$, define φ_i on $\mathbf{w}^{(i)}w_{\mu_{(i)}}$ by

$$D^{2i+1}A^{2(b-i-1)}\underline{AD} \overset{\varphi_i}{\longleftrightarrow} D^{2i}A^{2(b-i-1)+1}\underline{AD} \quad \text{for } 0 \leq i \leq b-1.$$

If $w_{\mu_{(i-1)}}w_{\mu_{(i-1)}+1} = \underline{AD}$, define φ_i on $w_{\mu_{(i-1)}}\mathbf{w}^{(i)}$ by

$$\underline{AD}D^{2i}A^{2(b-i-1)+1} \overset{\varphi_i}{\longleftrightarrow} \underline{AD}D^{2i+1}A^{2(b-i-1)} \quad \text{for } 0 \leq i \leq b-1.$$

The operation φ_i is defined for all words $\mathbf{w} \in \mathcal{W}_n$ except when $\mathbf{w}^{(i)} = D^{2b}$, $\mathbf{w}^{(i)} = D^{2b-2}\underline{DA}$, $\mathbf{w}^{(i)}w_{\mu_{(i)}} = D^{2b-1}\underline{DA}$, or $w_{\mu_{(i-1)}}\mathbf{w}^{(i)} = \underline{DA}A^{2b-1}$.

Now we are ready to define the involution φ when all the parts of μ are odd. To compute $\varphi(\mathbf{w})$ for a given $\mathbf{w} \in \nu(\mathcal{A}_n \cap \mathcal{L}_\mu^{-1})$, if there is some i with $1 \leq i \leq r$ for which the operation φ_i is defined, take the smallest such i and apply φ_i to $\mathbf{w}^{(i)}$ and its surrounding letters, leaving the rest of $\mathbf{w}$ unchanged. If there is no such i, then φ does not change $\mathbf{w}$, that is, $\mathbf{w}$ is a fixed point of φ. It is clear from the construction that φ is a sign-reversing involution, since φ_i changes the parity of the number of Ds in $\mathbf{w}^{(i)}$, while the other $\mathbf{w}^{(j)}$ for $j \neq i$ are unchanged. However, unlike in the case of parts of even size, here φ has some fixed points: those words $\mathbf{w} \in \mathcal{W}_n$ for which the operation φ_i is not defined for any $1 \leq i \leq r$. For this to happen, $\mathbf{w}^{(i)}$ has to equal either D^{μ_i-1}, $D^{\mu_i-3}\underline{DA}$, $D^{\mu_i-2}\underline{D}$, or $\underline{A}A^{\mu_i-2}$, for all $1 \leq i \leq r$.

Let us enumerate the words $\mathbf{w} \in \mathcal{W}_n$ having this property, separating them in four cases. Note that $\mathbf{w}$ has one letter between $\mathbf{w}^{(i)}$ and $\mathbf{w}^{(i+1)}$ for each $1 \leq i \leq r-1$, and s letters to the right of $\mathbf{w}^{(r)}$. Suppose first that $s \geq 1$.

(i) If $\mathbf{w}$ has no underlined pair, then $\mathbf{w}^{(i)} = D^{\mu_i - 1}$ for all i, and there are 2^{r+s-1} choices for the remaining letters of $\mathbf{w}$. The corresponding permutations have positive sign, since every block $\mathbf{w}^{(i)}$ has an even number of Ds.

(ii) If $\mathbf{w}$ has an underlined pair contained inside a block, the above number of choices has to be multiplied by the r choices of the block for which $\mathbf{w}^{(i)} = D^{\mu_i - 3}\underline{DA}$. The $r\,2^{r+s-1}$ corresponding permutations have now negative sign, since exactly one block $\mathbf{w}^{(i)}$ has an odd number of Ds.

(iii) If $\mathbf{w}$ has an underlined pair outside of all the blocks $\mathbf{w}^{(i)}$, then again $\mathbf{w}^{(i)} = D^{\mu_i - 1}$ for all i. Since the underlined pair has to be contained in the last s letters of $\mathbf{w}$, we have $s-1$ choices for its location, times 2 choices for whether it is $\underline{AD}$ or $\underline{DA}$, and finally 2^{r+s-3} choices for the remaining letters in $\mathbf{w}$. The $(s-1)2^{r+s-2}$ corresponding permutations have positive sign.

(iv) If $\mathbf{w}$ has an underlined pair that is partly inside a block $\mathbf{w}^{(i)}$ and partly outside, we have to choose the index i such that $\mathbf{w}^{(i)}$ equals $D^{\mu_i-2}\underline{D}$ or $\underline{A}A^{\mu_i-2}$. For each possible i we have two choices, except for $i=1$, in which case $\mathbf{w}^{(1)} = D^{\mu_1-2}\underline{D}$ is the only possibility because there is no letter to the left of $\mathbf{w}^{(1)}$. This gives $2r-1$ choices for the underlined pair, which also forces the entries in all the blocks $\mathbf{w}^{(j)}$, leaving 2^{r+s-2} choices for the remaining entries of $\mathbf{w}$. The $(2r-1)2^{r+s-2}$ corresponding permutations have again positive sign.

Adding the contributions in the four cases, the sum of the signs of the permutations fixed by the involution is

$$2^{r+s-1} - r\,2^{r+s-1} + (s-1)2^{r+s-2} + (2r-1)2^{r+s-2} = s\,2^{r+s-2}, \tag{6}$$

which agrees with the right-hand side of (5) when all the parts of μ have odd size. Formula (6) also holds when $s=0$. The only change in the argument is that there are no permutations in case (iii), but in case (iv), when choosing the index i such that $\mathbf{w}^{(i)}$ equals $D^{\mu_i-2}\underline{D}$ or $\underline{A}A^{\mu_i-2}$, the choice $i=r$ forces $\mathbf{w}^{(r)} = \underline{A}A^{\mu_r-2}$, giving $2r-2$ choices for the underlined pair, times 2^{r-2} choices for the remaining entries of $\mathbf{w}$. □

As an example of the involution in the above proof, let $n=10$ and $\mu=(5,3,1,1)$. The words fixed by φ of each of the above types are those of the following form:

(i) $DDDD\,w_5\,DD\,w_8\,w_9$,

(ii) $DD\underline{DA}\,w_5\,DD\,w_8\,w_9$, $DDDD\,w_5\,\underline{DA}\,w_8\,w_9$,

(iii) $DDDD\,w_5\,DD\,\underline{AD}$, $DDDD\,w_5\,DD\,\underline{DA}$,

(iv) $DDD\underline{DA}\,DD\,w_6\,w_7$, $DDDD\,\underline{DA}A\,w_6\,w_7$, $DDDD\,w_5\,D\underline{DA}\,w_7$.

The sum of the signs of the corresponding permutations is $8-16+4+12=8=s\,2^{r+s-2}$. The contributions of all the other words are canceled by the sign-reversing involution. For example,

$$\varphi(DDAA\,w_5w_7\,w_8\,w_9) = DDDA\,w_5w_7\,w_8\,w_9 \quad \text{and}$$
$$\varphi(DDDD\,w_5\,D\underline{AD}A) = DDDD\,w_5\,A\underline{AD}A.$$

9 A character formula of Regev

In this section we apply Theorem 5 and Proposition 9 to prove a conjectured character formula of Amitai Regev.

Let V be an $(n-1)$-dimensional vector space over $\mathbb{C}$, and let $\wedge V$ be its exterior algebra. Consider the natural action of the symmetric group $\mathcal{S}_{n-1}$ on $\wedge V$, and denote the character of the induced $\mathcal{S}_n$-module $\wedge V\uparrow^{\mathcal{S}_n}$ by χ_n. Regev conjectured the following character formula (personal communication, 2011).

Theorem 6 (Regev's formula) *Let* $\mu=(\mu_1,\dots,\mu_r,1^s)$ *be a partition of* n *with* $\mu_r>1$. *Then*

$$\chi_n(\mu)=\frac{1}{4}\cdot s\cdot\prod_{j=1}^{r+s}\bigl(1+(-1)^{\mu_j-1}\bigr).$$

Let us introduce some notation for the proof of Regev's Formula. Given a partition $\mu=(\mu_1,\dots,\mu_t)$ of n, recall from Sect. 8.2 that $\mathcal{L}_\mu^{-1}$ is the set of permutations whose inverse is μ-left-unimodal. If $\mathrm{RSK}(\pi)=(P,Q)$, then the fact that $\pi\in\mathcal{L}_\mu^{-1}$ translates into the following condition on the descent set of Q: for every $1\le i<t$ there exists $0\le j_i\le\mu_{i+1}$ such that $\mu_{(i)}+k\in\mathrm{Des}(Q)$ for $1\le k<j_i$, and $\mu_{(i)}+k\notin\mathrm{Des}(Q)$ for $j_i\le k<\mu_{i+1}$. A standard Young tableau Q satisfying this condition is called μ-*unimodal*. Denote the set of μ-unimodal standard Young tableaux of shape λ by SYT_μ^λ.

Proof of Theorem 6 It is well known that the exterior algebra $\wedge V$ is equivalent as an $\mathcal{S}_{n-1}$-module to a direct sum of all Specht modules indexed by hooks, see e.g. [7, Ex. 4.6]. By the branching rule, the decomposition of the induced exterior algebra $\wedge V\uparrow^{\mathcal{S}_n}$ into irreducibles is then given by

$$\chi_n=\sum_{k=1}^{n}\chi^{(k,1^{n-k})}+\sum_{k=2}^{n-1}\chi^{(k,1^{n-k})}+\sum_{k=2}^{n-2}\chi^{(k,2,1^{n-k-2})}. \tag{7}$$

Now we use the following character formula for symmetric group irreducible characters, which is a special case of Ref. [12, Theorem 4] (see also [11, 13]):

$$\chi^\lambda(\mu)=\sum_{T\in\mathrm{SYT}_\mu^\lambda}(-1)^{|\mathrm{Des}(T)\setminus S(\mu)|}.$$

Applying this formula to (7), we get the following expression for the character of χ_n at a conjugacy class of cycle type μ:

$$\chi_n(\mu) = \sum_{k=1}^{n} \sum_{T \in \mathrm{SYT}_\mu^{(k,1^{n-k})}} (-1)^{|\mathrm{Des}(T)\setminus S(\mu)|} + \sum_{k=2}^{n-1} \sum_{T \in \mathrm{SYT}_\mu^{(k,1^{n-k})}} (-1)^{|\mathrm{Des}(T)\setminus S(\mu)|}$$
$$+ \sum_{k=2}^{n-2} \sum_{T \in \mathrm{SYT}_\mu^{(k,2,1^{n-k-2})}} (-1)^{|\mathrm{Des}(T)\setminus S(\mu)|}. \tag{8}$$

By Remark 1, the RSK correspondence gives a descent-set-preserving bijection between permutations $\pi \in \mathcal{L}_n$ and standard Young tableaux Q of hook shape. Since $\pi \in \mathcal{L}_\mu^{-1}$ if and only if Q is μ-unimodal, we get

$$\sum_{k=1}^{n} \sum_{T \in \mathrm{SYT}_\mu^{(k,1^{n-k})}} (-1)^{|\mathrm{Des}(T)\setminus S(\mu)|} = \sum_{\pi \in \mathcal{L}_n \cap \mathcal{L}_\mu^{-1}} (-1)^{|\mathrm{Des}(\pi)\setminus S(\mu)|}.$$

Combining Remarks 1 and 2, the RSK correspondence also gives a descent-set-preserving bijection between permutations $\pi \in \mathcal{U}_n \setminus \mathcal{L}_n$ and standard Young tableaux Q of hook shape having at least two rows or two columns. It follows that

$$\sum_{k=2}^{n-1} \sum_{T \in \mathrm{SYT}_\mu^{(k,1^{n-k})}} (-1)^{|\mathrm{Des}(T)\setminus S(\mu)|} = \sum_{\pi \in (\mathcal{U}_n \setminus \mathcal{L}_n) \cap \mathcal{L}_\mu^{-1}} (-1)^{|\mathrm{Des}(\pi)\setminus S(\mu)|}.$$

For the third sum in (8), instead of the RSK correspondence, we use Theorem 5, which was proved in Sect. 7.4 via a different descent-set-preserving bijection. Standard Young tableaux in $\mathcal{T}_n$ that are μ-unimodal correspond to permutations in $\mathcal{Z}_n$ whose inverse is μ-left-unimodal, and so

$$\sum_{k=2}^{n-2} \sum_{T \in \mathrm{SYT}_\mu^{(k,2,1^{n-k-2})}} (-1)^{|\mathrm{Des}(T)\setminus S(\mu)|} = \sum_{\pi \in \mathcal{Z}_n \cap \mathcal{L}_\mu^{-1}} (-1)^{|\mathrm{Des}(\pi)\setminus S(\mu)|}.$$

Combining the last four equations and using that $\mathcal{A}_n$ is the disjoint union of $\mathcal{L}_n$, $\mathcal{U}_n \setminus \mathcal{L}_n$ and $\mathcal{Z}_n = \mathcal{A}_n \setminus \mathcal{U}_n$, we get

$$\begin{aligned}\chi_n(\mu) &= \sum_{\pi \in \mathcal{L}_n \cap \mathcal{L}_\mu^{-1}} (-1)^{|\mathrm{Des}(\pi)\setminus S(\mu)|} + \sum_{\pi \in (\mathcal{U}_n \setminus \mathcal{L}_n) \cap \mathcal{L}_\mu^{-1}} (-1)^{|\mathrm{Des}(\pi)\setminus S(\mu)|} \\ &\quad + \sum_{\pi \in \mathcal{Z}_n \cap \mathcal{L}_\mu^{-1}} (-1)^{|\mathrm{Des}(\pi)\setminus S(\mu)|} \\ &= \sum_{\pi \in \mathcal{A}_n \cap \mathcal{L}_\mu^{-1}} (-1)^{|\mathrm{Des}(\pi)\setminus S(\mu)|}.\end{aligned}$$

Proposition 9 now completes the proof. □

10 Further representation-theoretic aspects

10.1 An affine Weyl group action

Recall that the affine Weyl group $\widetilde{C}_{n-2}$ is generated by

$$S = \{s_0, s_1, \dots, s_{n-2}\}$$

subject to the Coxeter relations

$$\begin{aligned} s_i^2 &= 1 \quad \forall i, \\ (s_i s_j)^2 &= 1 \quad \text{for } |j-i| > 1, \\ (s_i s_{i+1})^3 &= 1 \quad \text{for } 1 \le i < n-3, \\ (s_i s_{i+1})^4 &= 1 \quad \text{for } i = 0,\ n-3. \end{aligned}$$

We now describe a natural action of the group $\widetilde{C}_{n-2}$ on the set of arc permutations $\mathcal{A}_n$. Recall that σ_i denotes the adjacent transposition $(i, i+1)$.

Definition 6 For every $0 \le i \le n-2$, define a map $\rho_i : \mathcal{A}_n \to \mathcal{A}_n$ as follows:

$$\rho_i(\pi) = \begin{cases} \pi\sigma_{i+1}, & \text{if } \pi\sigma_{i+1} \in \mathcal{A}_n; \\ \pi, & \text{otherwise.} \end{cases}$$

Proposition 10 *The maps ρ_i, $0 \le i \le n-2$, when extended multiplicatively, determine a well-defined $\widetilde{C}_{n-2}$-action on the set of arc permutations $\mathcal{A}_n$.*

Proof To prove that the operation is a well-defined $\widetilde{C}_{n-2}$-action, it suffices to show that it is consistent with the defining Coxeter relations of $\widetilde{C}_{n-2}$. For every i and $\pi \in \mathcal{A}_n$, we have $\rho_i^2(\pi) = \rho_i(\pi) = \pi$ if $\pi\sigma_{i+1} \notin \mathcal{A}_n$, and $\rho_i^2(\pi) = \rho_i(\pi\sigma_{i+1}) = \pi\sigma_{i+1}^2 = \pi$ otherwise. Also, if $|i-j| > 1$, then ρ_i and ρ_j commute, so $(\rho_i\rho_j)^2 = 1$.

To verify the other two braid relations recall the encoding $\psi : \mathcal{A}_n \to \{0, 1, \dots, n-1\} \times \{0, 1\}^{n-2}$ from Sect. 6.2. For every $1 \le i \le n-3$, if $\psi(\pi)_i = \psi(\pi)_{i+1}$ then $\pi\sigma_{i+1} \notin \mathcal{A}_n$, thus $\rho_i(\pi) = \pi$; if $\psi(\pi)_i \ne \psi(\pi)_{i+1}$ then $\pi\sigma_{i+1} \in \mathcal{A}_n$, thus $\rho_i(\pi) = \pi\sigma_{i+1}$. One concludes that, in both cases, the effect of ρ_i on $\psi(\pi)$ is to switch the entries $\psi(\pi)_i$ and $\psi(\pi)_{i+1}$. It follows that for every $1 \le i < n-3$, $\rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1}$.

Finally, note that for every $\pi \in \mathcal{A}_n$, $\pi\sigma_{n-1} \in \mathcal{A}_n$, thus $\rho_{n-2}(\pi) = \pi\sigma_{n-1}$ and

$$\psi\big(\rho_{n-2}(\pi)\big)_{n-2} = 1 - \psi(\pi)_{n-2},$$

that is, ρ_{n-2} flips the value of $\psi(\pi)_{n-2}$. On the other hand, as shown above, ρ_{n-3} switches the entries $\psi(\pi)_{n-3}$ and $\psi(\pi)_{n-2}$. One concludes that $(\rho_{n-3}\rho_{n-2})^4 = 1$. Since right multiplication by $w_0 = n \dots 21$ is an involution on $\mathcal{A}_n$ and $\rho_i(\pi) = \rho_{n-2-i}(\pi w_0)w_0$, it follows by symmetry that $(\rho_0\rho_1)^4 = 1$ as well. □

Given a group G with a generating set R, and an action of G on a set $\mathcal{C}$, the associated Schreier graph is the graph with vertex set $\mathcal{C}$ and edge set $\{(x, rx) : x \in \mathcal{C}, r \in R\}$. Recall the graph X_n from Sect. 6. The following is clear by definition.

Remark 4 The graph X_n is isomorphic (up to loops) to the Schreier graph determined by the above $\widetilde{C}_{n-2}$-action on $\mathcal{A}_n$.

Corollary 9 *The above $\widetilde{C}_{n-2}$-action on $\mathcal{A}_n$ is transitive.*

Proof An action is transitive if and only if the associated Schreier graph is connected. The result now follows from Theorem 4(i) together with Remark 4. □

Let $J := S \setminus \{s_0\} = \{s_1, \dots, s_{n-3}, s_{n-2}\}$. Recall that the maximal parabolic subgroup W_J is isomorphic to the hyperoctahedral group B_{n-2}.

Proposition 11

(i) *The maps ρ_i, $0 < i \le n-2$, when extended multiplicatively, determine a well-defined B_{n-2}-action on $\mathcal{A}_n$.*
(ii) *The orbits of this action are $\{\pi \in \mathcal{A}_n : \pi(1) = k\}$, for $1 \le k \le n$.*
(iii) *The B_{n-2}-action on each of these orbits is multiplicity-free.*

Proof Part (i) follows from Proposition 10.

For $i > 0$, it is clear that $\rho_i(\pi)(1) = \pi(1)$, hence the sets of arc permutations with fixed first letter are invariant under this B_{n-2} action. On the other hand, for each $1 \le k \le n$, the map ψ defined in Sect. 6.2 determines a bijection from $\{\pi \in \mathcal{A}_n : \pi(1) = k\}$ to $0{-}1$ vectors of length $n-2$. The restricted B_{n-2} action on $\{\pi \in \mathcal{A}_n : \pi(1) = k\}$ may thus be identified with the natural B_{n-2}-action on all subsets of $[n-2]$, which is transitive, implying (ii).

To prove (iii) recall that the B_{n-2}-representation induced from the trivial representation of S_{n-2} is multiplicity-free, see e.g. [2, Lemma 2.2(a)], and notice that this representation is isomorphic to the B_{n-2}-action on all subsets of $[n-2]$. □

The following question was posed by David Vogan (personal communication, 2010).

Question 1 Is the $\widetilde{C}_{n-2}$-module determined by its action on $\mathcal{A}_n$ multiplicity-free?

10.2 Representation-theoretic proofs

By Remarks 1 and 2, the sets $\mathcal{L}_n$ and $\mathcal{U}_n$ are unions of Knuth classes, hence they carry the associated symmetric group representation. As noted below Remark 3, the set $\mathcal{Z}_n$ of non-unimodal arc permutations is not a union of Knuth classes. However, by Proposition 5, its size is equal to the number of standard Young tableaux of hook shape plus one box. Here is a short representation-theoretic proof of Proposition 5.

Proof of Proposition 5 By the decomposition of the induced exterior algebra $\wedge V \uparrow^{\mathcal{S}_n}$ into $\mathcal{S}_n$-irreducible characters, which is described in (7),

$$f^{(n)} + f^{(1^n)} + 2\sum_{k=2}^{n-1} f^{(k,1^{n-k})} + \sum_{k=2}^{n-2} f^{(k,2,1^{n-k-2})}$$
$$= \dim \wedge V \uparrow^{\mathcal{S}_n} = n \cdot \dim \wedge V = n2^{n-2} = |\mathcal{A}_n|.$$

On the other hand, by Remark 2, $|\mathcal{U}_n| = f^{(n)} + f^{(1^n)} + 2\sum_{k=2}^{n-1} f^{(k,1^{n-k})}$. Since $\mathcal{U}_n \subseteq \mathcal{A}_n$, one concludes

$$|\mathcal{Z}_n| = |\mathcal{A}_n \setminus \mathcal{U}_n| = \sum_{k=2}^{n-2} f^{(k,2,1^{n-k-2})}.$$

□

Question 2 Find representation-theoretic proofs of Theorem 5 and other results in Sect. 7.

By Corollary 6, the number of maximal chains in any interval of Weak($\mathcal{U}_n$) is equal to twice the number of standard Young tableaux of shifted staircase shape. It is well known that this is a dimension of a projective $\mathcal{S}_n$ representation.

Question 3 Determine a projective $\mathcal{S}_n$ representation on the set of maximal chains in Weak($\mathcal{U}_n$).

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Appendix: Shuffles

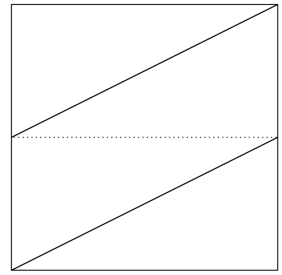
The purpose of this section is to point out that permutations obtained as shuffles of two increasing sequences have properties similar to those of unimodal and arc permutations. In analogy to Theorem 2 for $\mathcal{L}_n$, shuffles are obtained as partial fillings of certain shapes. As a consequence, the weak order restricted to these shuffles has properties analogous to those given in Sect. 5 for Weak($\mathcal{U}_n$).

A.1 Prefixes associated with a rectangle

Similarly to the partial fillings of the shifted staircase in Sect. 4, we will now consider partial fillings of a $k \times m$ rectangle with rows labeled $k, \dots, 2, 1$ from top to bottom, and columns labeled $k+1, k+2, \dots, k+m$ from left to right. Again, with each of the entries ℓ in the partial filling, if ℓ lies in row i and column j, we associate the transposition (i, j). The product of these transpositions gives a permutation, and the set of permutations obtained in this way is denoted by $\mathcal{F}_{k,m}$.

It is easy to see that $\mathcal{F}_{k,m}$ is also the set of permutations $\pi \in \mathcal{S}_{m+k}$ which are a shuffle of the two sequences $1, 2, \dots, k$ and $k+1, k+2, \dots, k+m$ (that is, both appear as subsequences of π from left to right).

Fig. 6 Grid for shuffles



As is the case for $\mathcal{L}_n, \mathcal{U}_n$ and $\mathcal{A}_n$, the set $\bar{\mathcal{F}}_n = \bigcup_{k+m=n} \mathcal{F}_{k,m}$ can be characterized in terms of pattern avoidance.

Proposition 12 $\bar{\mathcal{F}}_n = \mathcal{S}_n(321, 2143, 2413)$.

Shuffles can be easily enumerated, obtaining that $|\mathcal{S}_n(321, 2143, 2413)| = |\bar{\mathcal{F}}_n| = 2^n - n$ for $n \geq 2$. As in Sect. 3, it is also the case here that shuffles are a grid class, consisting of those permutations that can be drawn on the picture in Fig. 6. We write

$$\bar{\mathcal{F}}_n = \mathcal{G}_n \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Shuffles can be characterized as those permutations π with $\mathrm{Des}(\pi^{-1}) = \{k\}$ for some k. For $\pi \in \bar{\mathcal{F}}_n$, if $\mathrm{RSK}(\pi) = (P, Q)$, then P is a two-row tableau with consecutive entries $k+1, k+2, \ldots, k+\ell$ in the second row, and Q is any two-row tableau with the same shape as P. It follows that shuffles are a union of Knuth classes.

A.2 Weak order and enumeration of maximal chains

Let $\mathrm{Weak}(\bar{\mathcal{F}}_n)$ be the subposet of $\mathrm{Weak}(\mathcal{S}_n)$ induced by the subset $\bar{\mathcal{F}}_n = \mathcal{S}_n(321, 2143, 2413)$. The following result follows from arguments analogous to the ones used in Sect. 5.

Proposition 13 *The poset* $\mathrm{Weak}(\bar{\mathcal{F}}_n)$ *has the following properties.*

(i) *The local maxima are exactly the permutations* $\pi_k := (k+1)(k+2)\ldots n12\ldots k$ *for some* k.
(ii) *A permutation is in the interval* $[e, \pi_k]$ *if and only if it belongs to* $\mathcal{F}_{k,n-k}$; *hence, the number of elements in this interval is* $\binom{n}{k}$.
(iii) *The number of maximal chains in* $[e, \pi_k]$ *is equal to the number of standard Young tableaux of rectangular shape* $k \times (n-k)$.

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