Math 561: Foundations of Combinatorics
Lecturer: Prof. Sara Billey

Problem Set #5

due Wednesday, October 30, 2019

Reading: In Chapter 1, read Section 1.8-1.9. Plus, read "Juggling card sequences" by Steve Butler, Fan Chung, Jay Cummings, and Ron Graham, published in Journal of Combinatorics, vol. 8 (2017).

Recommended Problems: Give each of these problems careful consideration before reading the solutions: Chapter 1: 143, 154, 169, 190.

Homework Problems: For each of the problems below, explain your answer fully. No credit will be given for a simple statement of the answer. Each problem is worth 10 points unless otherwise specified.

1. (Second try!) What well-known sequence of numbers has
$$1 + \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{t}{1-kt}$$
as it's ordinary generating function? Prove your answer.

2. Describe the set of permutations in S_n with exactly 2 commutivity classes. In this special case, can you refine the bounds from "Enumerations relating braid and commutation classes" by Fishel, Milićević, Patrias and Tenner ?

3. Give a combinatorial proof of the following identity:
$$k^n = \binom{k}{1}\, 1!\, S(n,1) + \binom{k}{2}\, 2!\, S(n,2) + \cdots + \binom{k}{n}\, n!\, S(n,n)$$
where $S(n,k)$ is the Stirling number of the second kind.

4. Show $\sum_{k=0}^{n} S(n,k)x^k = e^{-x} \sum_{k=0}^{\infty} k^n x^k / k!$.

5. Let $f(n)$ be the number of non-isomorphic ways one can color a $1 \times n$ rectangular map of countries in a row such that no two adjacent countries gets the same color. Note, two colorings are isomorphic if there is a permutation on the colors that takes one coloring to the other. Find the exponential generating function for this sequence.

6. The Euler number E_n is the number of alternating permutations in S_n. Defining sec and tan in terms of the even and odd Euler numbers respectively, prove the identity $sec^2(x) = tan^2(x) + 1$.

7. Use the basic recurrence relations to extend the definitions of $c(n,k)$ and $S(n,k)$ to all $n,k \in \mathbb{Z}$ with the base cases $c(0,k) = S(0,k) = \delta_{k=0}$ and $c(n,0) = S(n,0) = \delta_{n=0}$. Show that for all integers k,n

$$c(n,k) = S(-k,-n).$$

8. Given a set partition π of $[n]$, let $x^{\pi} = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ where b_i equals the number of blocks of size i in π for $1 \leq i \leq n$. Let $B_{n,k}(x_1, x_2, \ldots, x_n) = \sum x^{\pi}$ where the sum is over all partitions of $[n]$ into exactly k blocks.

   (a) Show $B_{n,k}(x_1, x_2, \ldots, x_n)$ is the partial Bell polynomial defined in class (also on Wikipedia).

   (b) Let $f(x) = \sum_{n \geq 1} a_n x^n/n!$ and $g(x) = \sum_{n \geq 1} b_n x^n/n!$. Show that the coefficients of the composition are determined by

$$g(f(x)) = \sum_{n \geq 0} \left(\sum_{k=1}^{n} b_k B_{n,k}(a_1, a_2, \ldots, a_n) \right) x^n/n!.$$

9. (Bonus) Are the Bell numbers ever divisible by 8? If so, for which n? If not, prove it.

10. (Bonus) Are there an infinite number of Bell numbers which are also prime numbers?