Math 561: Foundations of Combinatorics
Lecturer: Prof. Sara Billey

Problem Set #4

due Wednesday, October 23, 2019

Reading: In Chapter 1, read Sections 1.6 - 1.7.

Recommended Problems: Give each of these problems careful consideration before reading the solutions: Chapter 1: 71, 72, 74, 155, 158.

Homework Problems: For each of the problems below, explain your answer fully. No credit will be given for a simple statement of the answer. Each problem is worth 10 points unless otherwise specified.

1. Let $w = [w_1, w_2, \ldots, w_n] \in S_n$ and let $w_0 = [n, n-1, \ldots, 1] \in S_n$. Express $inv(ww_0)$, $maj(ww_0)$ and $des(ww_0)$ in terms of $inv(w)$, $maj(w)$ and $des(w)$.

2. Give a bijective proof showing $\sum_{w \in S_n} q^{maj(w)} = (1 + q + q^2 + \ldots + q^{n-1}) \sum_{v \in S_{n-1}} q^{maj(v)}$.

3. Show that the number of permutations in S_n with k exceedences equals the number of permutations in S_n with $k+1$ weak exceedences.

4. Prove the following identity where x is an indeterminate, n, k are nonnegative integers:

$$x^n = \sum_k A(n,k) \binom{x+k-1}{n}$$

   where $A(n,k)$ is the Eulerian number.

5. If $w \in \mathfrak{S}_n$, let $m(w)$ be the number of left-to-right minima in w and let $inv(w)$ be the number of inversions in w. Find a "nice" way to present the generating function

$$G(q,t) = \sum_{w \in \mathfrak{S}_n} t^{m(w)} q^{inv(w)}.$$

6. Let $k, n \in \mathbb{P}$ with $k \leq n$. Let $V(n,k)$ be the volume of the region in $\mathbb{R}^n$ defined by

$$0 \leq x_i \leq 1 \text{ for all } 1 \leq i \leq n$$
$$k-1 \leq x_1 + x_2 + \cdots + x_n \leq k.$$

   Show that $V(n,k) = A(n,k)/n!$ where $A(n,k)$ is an Eulerian number.

7. What well-known sequence of numbers has

$$\sum_{n=0}^{\infty} \prod_{k=1}^{k} \frac{1}{1-kt}$$

as it's ordinary generating function? Prove your answer.

8. Let $f(w)$ be the number of fixed points of a permutation $w \in S_n$. Conjecture and prove a nice formula for

$$f_n^2 = \sum_{w \in S_n} f(w)^2.$$

9. (Bonus: 10pts) Let $f(w)$ be the number of fixed points of a permutation $w \in S_n$. What can you say about

$$f_n^k = \sum_{w \in S_n} f(w)^k?$$